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Homology of the Kac-Moody groups I

Dedicated to Professor Masahiro Sugawara on his 60th birthday

By

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§1. Introduction

Let G be a compact, connected, simply connected, simple Lie group of exceptional type and $\mathcal{G}$ its Lie algebra. The homotopy type of the Kac-Moody Lie group $\mathcal{K}(\mathcal{G}^{(1)})$ is $\Omega G\langle 2\rangle\times G$ where $\Omega G\langle 2\rangle$ is the 2-connected cover of the space of loops on G (cf. [5]). The purpose of this paper is to determine $H_*(\mathcal{K}(\mathcal{G}^{(1)}); \boldsymbol{Z})$. The homology $H_*(G; \boldsymbol{Z})$ is known and therefore we need only determine $H_*(\Omega G\langle 2\rangle; \boldsymbol{Z})$. Since $H_*(\Omega G\langle 2\rangle; \boldsymbol{Z})$ is a finitely generated abelian group for any $*$, it is sufficient to determine $H_*(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)})$ for all prime p.

In [8], there exists an integer $d(G, p)$ such that

$$\begin{aligned}&\text{P.S.}\ (H_*(\Omega G\langle 2\rangle; \boldsymbol{F}_p))\\ &= (1+t^{2q^{d(G,p)}-1})(1-t^{2p^{d(G,p)}})^{-1}\prod_{j=2}^{l}(1-t^{2n(j)})^{-1}\end{aligned}$$

where $l=\text{rank } G$ and $1=n(1)<n(2)<\cdots<n(l)$ are the exponents of G (cf. §2). Since $H_*(\Omega G; \boldsymbol{Z})$ is free and $H_{2j-1}(\Omega G; \boldsymbol{Z})=0$, for any ring R the Gysin exact sequence for the fibering

$$S^1 \to \Omega G\langle 2\rangle \xrightarrow{\pi} \Omega G$$

splits as

$$0 \to H_{2j}(\Omega G\langle 2\rangle; R) \xrightarrow{\pi_*} H_{2j}(\Omega G; R) \xrightarrow{\chi} H_{2j-2}(\Omega G; R) \to H_{2j-1}(\Omega G\langle 2\rangle; R) \to 0$$

where χ is a derivation (of degree -2). For $R=\boldsymbol{Q}$, χ is epic. Therefore $H_{2j}(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)})$ is a free $\boldsymbol{Z}_{(p)}$-module and $H_{2j+1}(\Omega G; \boldsymbol{Z}_{(p)})$ is a finite p-group.

In this paper $H_*(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)})$ except $(G, p)=(E_6, 2)$ is determined (cf. Theorem 3.3).
The cases $(E_6, 2)$ and G of classical type are determined in part II.

§2. Mod p cohomology and homology

For the simplicity, we define a set of integers:

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$$E(G) = \{n(j) \mid j>1\}\ .$$

Then $E(G_2) = \{5\}$, $E(F_4) = \{5, 7, 11\}$,
$E(E_6) = \{4, 5, 7, 8, 11\}$,
$E(E_7) = \{5, 7, 9, 11, 13, 17\}$ and
$E(E_8) = \{7, 11, 13, 15, 17, 19, 23, 29\}$.

First, one can observe that

Proposition 2.1 (Borel [1].). $H_*(G; \boldsymbol{Z})$ has non-trivial p-torsion if and only if $p \leq [l/2]+1$.

First, we consider this case.
We define an integer $d(G, p)$ by the following table:

	G_2	F_4	E_6	E_7	E_8
$p=2$	2	2	4	4	4
$p=3$		2	2	3	3
$p=5$					2.

Let $M(G, p)$ be the $\boldsymbol{F}_p$-algebra $E(b_{2j+1} \mid j \in E(G))$ except for $G=E_8$, $p=2$ and let $M(E_8, 2)$ be

$$E(b_{2j+1} \mid j \in E(E_8) \cup \{14\} - \{7, 29\}) \otimes P(b_{15})/(b_{15}^4)$$

where degree b_i=i.

We define also the Hopf algebra over $\boldsymbol{F}_p$ $N(G, p)$ by $P(h_{2j} \mid j \in E(G))$ except for $G=E_8$, $p=2$ and $N(E_8, 2)$ by

$$P(h_{2j} \mid j \in E(G) \cup \{14)\}/(h_{14}^2)$$

where degree $h_{2i}=2i$ and except h_{28}, all h_{2j} are primitive and $\phi(h_{28})=h_{28}\otimes 1+h_{14}\otimes h_{14}+1\otimes h_{14}$.

Lemma 2.2. *Put* $e=2\cdot p^{d(G,p)}$.
(1) $H^*(G\langle 3\rangle; \boldsymbol{F}_p)=P(x_e)\otimes E(x_{e+1})\otimes M(G, p)$ *as an algebra where degree* $x_i=i$.
(2) $H^*(\Omega G\langle 2\rangle; \boldsymbol{F}_p)=P(y_e)\otimes E(y_{e-1})\otimes N(G, p)$ *as a Hopf algebra where degree* $y_i=i$ *and* y_i *are primitive.*
(3) *Let* β_* *be the homology Bockstein operation. Then* $\beta_* h_{2j}=0$ *for all* j *and* $\beta_* y_e=y_{e-1}$.

Proof. (1) is the reformulation of the results in Kachi [7], Kono-Mimura [9] and Mimura [11].

For (2), we consider the Eilenberg-Moore spectral sequence

$$\mathrm{Ext}_{H^*(\Omega G\langle 2\rangle;\, F_p)}(\boldsymbol{F}_p, \boldsymbol{F}_p) \Rightarrow H^*(G\langle 3\rangle; \boldsymbol{F}_p)\ .$$

By the result of Kono [8],

$$\text{P.S.}\,(H_*(\Omega G\langle 2\rangle; \boldsymbol{F}_p)) = \text{P.S.}\,(P(y_e)\otimes E(y_{e-1})\otimes N(G, p))\,.$$

If $H_*(\Omega G\langle 2\rangle; \boldsymbol{F}_p)$ has more relations than the right side of (2), then by the dimentional reason, E_2-term has the permanent cycles which correspond to the relations. One can easily check that no such elements appear in $H^*(G\langle 3\rangle; \boldsymbol{F}_p)$ by (1). The Hopf algebra structure follows from Milnor-Moore [10]. For all $j\in E(G)$, $\beta_* h_{2j}$ is clearly primitive and is not decomposable by the dimensional reason. $\beta_* h_{28}$ is not decomposable by the same reason. So $\beta_* h_{2j}$ is zero by (2). The last statement follows from that $\beta x_e = x_{e+1}$ which is also a result of Kachi [7], Kono-Mimura [9] and Mimura [11].

§3. $\boldsymbol{Z}_{(p)}$-homology

Lemma 3.1. *Let $\rho: \boldsymbol{Z}_{(p)} \to \boldsymbol{F}_p$ be the mod p reduction map.*
(1) *For any $j\in E(G)$, there is an element*

$$a'_{2j}\in H_*(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)}) \quad \text{such that} \quad \rho(a'_{2j}) = h_{2j}\,.$$

(2) *In the case $(G, p)=(E_8, 2)$, a'^2_{14} is divisible by 2. If we put $a'_{28}=\frac{1}{2}\cdot a'^2_{14}$, then $\rho(a'_{28})=h_{28}$.*

Proof. Since $\rho: H_{2j}(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)}) \to H_{2j}(\Omega G\langle 2\rangle; \boldsymbol{F}_p)$ is epic by (3) of (2.2), (1) is clear.

(2) is a result of Kono [8].

Let put $\pi_* a'_j = a_j$.
Let $L(G, p)$ be the $\boldsymbol{Z}_{(p)}$-algebra

$$P(a_{2j}\,|\, j\in E(G)) \quad \text{for} \quad (G, p)\neq(E_8, 2)$$

$$P(a_{2j}\,|\, j\in E(G)\cup\{28\})/(a_{14}^2 - 2\cdot a_{28}) \quad \text{for} \quad (G, p) = (E_8, 2)\,.$$

We define $\boldsymbol{Z}_{(p)}$-algebra $\Gamma(d, p)$ by

$$P(u_{2i}\,|\, i\in\{1, p, \cdots, p^d\})/(u_{2i}^p - p\cdot u_{2pi}\,|\, i\in\{1, p, \cdots, p^{d-1}\})\,.$$

Theorem 3.2. *If $(G, p)\neq(E_6, 2)$, then*
(1) *$H_*(\Omega G; \boldsymbol{Z}_{(p)}) = \Gamma(d(G, p)-1, p)\otimes_{\boldsymbol{Z}_{(p)}} L(G, p)$ as algebra.*
(2) $\chi(a_{2j})=0$, $\chi(u_2)=1$, $\chi(u_{2p})=u_2^{p-1}$,

$$\chi(u_{2p^2}) = (u_2\, u_{2p})^{p-1}, \cdots,$$
$$\chi(u_{2p^{d(G,p)-1}}) = (u_2\, u_{2p}\cdots u_{2p^{d(G,p)-2}})^{p-1}\,.$$

Proof. The algebra structure is the result of Duckworth [3]. Since $\chi: H_*(\Omega G; \boldsymbol{Z}_{(p)}) \to H_{*-2}(\Omega G; \boldsymbol{Z}_{(p)})$ is epic for $*<2\cdot p^{d(G,p)}$, we can take the elements u_{2p^i} so that the theorem holds.

Let define a graded $\boldsymbol{Z}_{(p)}$-module $C(d, p)$ as follows:

$$C(d,p)_j = \begin{cases} \boldsymbol{Z}_{(p)} & \text{if } j=0 \\ \boldsymbol{Z}/p^{r-d} & \text{if } j+1=2p^r j',\ (j',p)=1 \text{ and } r\geq d \\ 0 & \text{otherwise}. \end{cases}$$

Theorem 3.3.

$$H_*(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)}) = C(d(G,p)-1,p)\otimes_{\boldsymbol{Z}_{(p)}} L(G,p)$$

Proof. For the simplicity, we put $d=d(G,p)-1$. χ induces a derivation of $\Gamma(d,p)$. We denote it as χ_Γ.

We have only to show that $\mathrm{Coker}(\chi_\Gamma)_{j-1}=C(d,p)_j$. $\Gamma(d,p)_j$ is spaned freely by

$$\{u_2^{j_0} u_{2p}^{j_1}\cdots u_{2p^d}^{j_d} \mid 0\leq j_k<p \quad \text{for} \quad k\leq d-1 \quad \text{and} \quad \sum 2p^k j_k=j\}.$$

If j_k is the first non zero integer in $(j_0, j_1, \cdots, j_d)$, then an easy calculation shows that

$$\begin{aligned} &\chi_\Gamma(u_{2p^k}^{j_k}\cdots u_{2p^d}^{j_d}) \\ &= (j_k+p\cdot j_{k+1}+\cdots+p^{d-k} j_d)\, u_2^{p-1}\cdots u_{2p^{k-1}}^{p-1}\, u_{2p^k}^{j_k-1}\, u_{2p^{k+1}}^{j_{k+1}}\cdots u_{2p^d}^{j_d}. \end{aligned}$$

Thus $\mathrm{Coker}(\chi_\Gamma)$ is generated t by $u_2^{p-1}\cdots u_{2p^{d-1}}^{p-1}\, u_{2p^d}^{j_d-1}$ and its order is $p^{\nu_p(j_d)}$.

Now we turn to the other cases.

Theorem 3.4. *If $(G,p)=(G_2,3)$ or $p>n(l)$, then*

$$H_*(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)}) = C(0,p)\otimes_{\boldsymbol{Z}_{(p)}} P(a'_{2j}\mid j\in E(G)).$$

Proof. In the case of $p>n(l)$, by the result of Serre [12], we have a p-equivalence of spaces

$$G\simeq_p S^3\times X.$$

Then the result is clear.

Since $H_*(\Omega G_2; \boldsymbol{Z}_{(3)})=P(a_2, a_{10})$ and we can take a_{10} as $\chi(a_{10})=0$, the same but more easy calculation as in the the proof of the previous theorem deduces the result.

Remark. (1) If $n(2)\leq p\leq n(l)$, then $d(G,p)$ is 2.

(2) $n(2)>p>\left[\frac{l}{2}\right]+1$ if and olny if $(G,p)=(G_2,3)$.

So we cover all cases.

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