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# Representations of Lie superalgebras, II Unitary representations of Lie superalgebras of type $A(n, 0)$

By

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## Introduction

In this paper we introduce a new method of constructing irreducible unitary representations (=IURs) of a classical Lie superalgebra of type $A$. Then we classify all the irreducible unitary representations of real forms of Lie superalgebra $A(1, 0)$ and construct them explicitly by using this method.

In the previous paper [4] we define unitary representations of Lie superalgebras and introduce a general method of constructing irreducible representations of any simple Lie superalgebras. Let us explain it briefly. Let $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$ be a Lie surperalgebra over $\boldsymbol{R}$. Take a representation $(\rho, W)$ of $\mathfrak{g}_0$ and consider a $\mathfrak{g}_0$-equivariant linear map $B$ from $\mathfrak{g}_1 \otimes \mathfrak{g}_1$ to $\mathfrak{gl}(W)$. We gave necessary and sufficient conditions by means of $B$ that there exist an irreducible representation $(\pi, V)$, $V=V_0+V_1$, of $\mathfrak{g}$ such that its even part $V_0$ is isomorphic to $W$ as $\mathfrak{g}_0$-modules. Further, we can construct $(\pi, V)$ from $(\rho, V_0)$ and $B$ cannonically.

Moreover in that paper we classify and construct all the irreducible (unitary) representations of classical Lie superalgebra $\mathfrak{osp}(1, 2)$. Further we gave the similar results for real forms of the Lie superalgebra $\mathfrak{sl}(2, 1)(=A(1, 0))$ exhaustively for the case where $(\rho, W)$ are irreducible, but there remains to study the case where $(\rho, W)$ are reducible.

In this paper a new method is induced, which uses a $\boldsymbol{Z}$-gradation $\mathfrak{g}_C=\mathfrak{g}_C^{-1}\oplus\mathfrak{g}_C^0\oplus\mathfrak{g}_C^{+1}$, with $\mathfrak{g}_C^0=\mathfrak{g}_{0,C}$, of Lie superalgebras $\mathfrak{g}_C=A(n, 0)$ instead of the $\boldsymbol{Z}_2$-gradation $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ of a real form $\mathfrak{g}$ of $\mathfrak{g}_C$. In more detail, (i) first we study the weight distributions for IURs $(\pi, V)$, and see in particular that any IUR must be a highest (or lowest) weight representation because of its unitarity (see Proposition 2.2). (ii) Next we consider induced $\mathfrak{g}_C$-module $\bar{V}(\Lambda)=\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}_C}L(\Lambda)$. Here $L(\Lambda)$ is an irreducible highest weight representation of $\mathfrak{g}_{0,C}$ with highest weight $\Lambda$ and $\mathfrak{p}=\mathfrak{g}_C^0\oplus\mathfrak{g}_C^{+1}$. We extend $L(\Lambda)$ as $\mathfrak{p}$-module by putting $\mathfrak{g}_C^{+1}$-action as trivial. Any irreducible representation $V(\Lambda)$ of $\mathfrak{g}_C$ with highest weight $\Lambda$ is a quotient of $\bar{V}(\Lambda)$. (iii) Therefore we should determine the maximal submodule $I(\Lambda)$ of $\bar{V}(\Lambda)$ to get $V(\Lambda)=\bar{V}(\Lambda)/I(\Lambda)$.

We give in Lemma 2.4 a necessary and sufficient condition for the irreducibility of $\bar{V}(\Lambda)$ by means of its highest weight $\Lambda$. V.G. Kac [6, §2] proved this criterion in case $L(\Lambda)$ is finite-dimensional. In 2.5, we determine branching rules of $\bar{V}(\Lambda)$ res-

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trected to $\mathfrak{g}_{0,C}$ for $\mathfrak{g}_C=\mathfrak{sl}(2, 1)$ and $\mathfrak{sl}(3, 1)$. They are crucial to determine the maximal submodule $I(\Lambda)$.

In § 3, we classify all the IURs of real forms of $\mathfrak{sl}(2, 1)$ using these results. The case of $\mathfrak{g}=\mathfrak{su}(2, 1; 2, 1)$ (see Theorem 3.3) is similar as the case of $\mathfrak{g}=\mathfrak{su}(2, 1; 1, 1)$ (see Theorem 3.6). So we show here our results in the case of $\mathfrak{g}=\mathfrak{su}(2, 1; 1, 1)$. In this case $\mathfrak{g}_0=\mathfrak{u}(1, 1)$. Let $\{H, C\}$ be a basis of a Cartan subalgebra $\mathfrak{h}_C$ of $\mathfrak{g}_C=\mathfrak{sl}(2, 1)$ given as $H=\mathrm{diag}(1, -1, 0)$ and $C=\mathrm{diag}(1, 1, 2)$. And let $\alpha, \beta, \gamma\in\mathfrak{h}_C^*$ be positive roots of $\mathfrak{g}_C$ such that $\alpha(H)=2$, $\alpha(C)=0$, $\beta(H)=\beta(C)=-1$, $\gamma=\alpha+\beta$, and put $\delta=\beta+\gamma\in\mathfrak{h}_C^*$. Then we get a complete result for the case of $\mathfrak{g}=\mathfrak{su}(2, 1; 1, 1)$ as follows.

**Theorem 3.6.** (1) *Any irreducible unitary representation $V$ of Lie superalgebra $\mathfrak{g}=\mathfrak{su}(2, 1; 1, 1)$ is a highest or lowest weight representation. If $V$ is a highest weight IUR, then $V$ is isomorphic to one of the representations $V(\Lambda)$ with highest weight $\Lambda$ such that $\Lambda(H)\leqq\Lambda(C)\leqq-\Lambda(H)-2$ or $\Lambda(H)=\Lambda(C)=0$.*

(2) *As $\mathfrak{g}_0$-module, the above $V(\Lambda)$ is decomposed into $\mathfrak{g}_0$-irreducible components as follows:*

(i) $V(\Lambda)=L(\Lambda)$ *for* $\Lambda(C)=\Lambda(H)=0$,

(ii) $V(\Lambda)=L(\Lambda)\oplus L(\Lambda-\gamma)$ *for* $\Lambda(C)=\Lambda(H)\leqq-1$,

(iii) $V(\Lambda)=L(\Lambda)\oplus L(\Lambda-\beta)$ *for* $\Lambda(C)=-\Lambda(H)-2\geqq0$,

(iv) $V(\Lambda)=L(\Lambda)\oplus L(\Lambda-\beta)\oplus L(\Lambda-\gamma)\oplus L(\Lambda-\delta)$ *otherwise.*

Realizations of the IURs of type (i) (ii) (iii) were given in the preveous paper, and those for type (iv) are given in 3.4.2 of this paper.

This paper is organized as follows. In § 1, first we recall the definition of unitary representations of Lie superalgebras, given in [4, § 1], and introduce the basic classical Lie superalgebras of type $A(m, n)$ after Kac [5, § 2] and give their structure in 1.2. Then we list up all the real forms of Lie superalgebras of type $A(n-1, 0)=\mathfrak{sl}(n, 1)$. There are two types of them: (i) $\mathfrak{sl}(n, 1; \boldsymbol{R})$ and (ii) $\mathfrak{su}(n, 1; p, 1)([n-1/2]\leqq p\leqq n)$ up to isomorphims and transition to their duals.

In § 2, we give some properties of representations of Lie superalgebra $\mathfrak{sl}(n, 1)$. In 2.1, the case of $\mathfrak{sl}(n, 1; \boldsymbol{R})$ is mentioned. Then we study irreducible unitary representations of real forms $\mathfrak{g}=\mathfrak{su}(n, 1; p, 1)$ as follows. In 2.2, we give a necessary condition for unitarity of an irreducible representation by means of the set of its weights. Then in 2.3 we introduce $\boldsymbol{Z}$-gradations in Lie superalgebra $\mathfrak{sl}(n, 1)$, in its universal enveloping algebra and also in its representation space, and we get some properties of IURs of $\mathfrak{g}$ with respect to these $\boldsymbol{Z}$-gradations. In 2.4, we introduce a highest weight representation $\overline{V}(\Lambda)$ of $\mathfrak{sl}(n, 1)$ which is induced from $\mathfrak{p}=\mathfrak{g}_C^0+\mathfrak{g}_C^{+1}$ starting from IUR $L(\Lambda)$ of $\mathfrak{g}_0$. Then a necessary and sufficient condition for its irreducibility is given in Lemma 2.4. In 2.5, we give branching rules of $\overline{V}(\Lambda)$ restrected to the even part $\mathfrak{g}_{0,C}$ of $\mathfrak{g}_C$ for $n=2$ and 3, and in 2.6, datailed calculations in some cases are given.

In § 3, we classify all the IURs of real forms of $\mathfrak{sl}(2, 1)$ and give their explicit realizations except the cases already treated in the previous paper [4, § 8]. Applying the branching rules in § 2, we get the complete classification of IURs of $\mathfrak{su}(2, 1; 2, 1)$ in Theorem 3.3, and that for $\mathfrak{su}(2, 1; 1, 1)$ in Theorem 3.6. In 3.4, we give a standard orthonormal basis in each $V(\Lambda)$ in connection with the $\mathfrak{g}$-action, and write down explicitly the action of $\mathfrak{g}_1$ with respect to them. Finally in 3.5, detailed calculations for these results are given.

## § 1. Preliminaries

**1.1. Unitary representations of Lie superalgebras.** Let $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$ be a real Lie superalgebra and $(\pi, V)$ be an irreducible representation of $\mathfrak{g}$ on a $\boldsymbol{Z}_2$-graded complex vector space $V=V_0+V_1$ in the sense of Kac [5, § 1]. On the even part $V_0$ and also on the odd part $V_1$ of $V$, we have naturally representations of the even part $\mathfrak{g}_0$, of which $\pi$ is called an extension. We call $\pi$ *unitary* [4, § 1] if $V$ is equipped with a positive definite inner product $\langle\cdot, \cdot\rangle$ in $V$ satisfying

(i) $V_0 \perp V_1$ (orthogonal) under $\langle\cdot, \cdot\rangle$, and

(ii) $\langle\cdot, \cdot\rangle$ is $\mathfrak{g}$-invariant in the sense that

$$\langle i\pi(X)v, v'\rangle=\langle v, i\pi(X)v'\rangle \ (v, v'\in V, X\in\mathfrak{g}_0),$$

$$\langle j\pi(\xi)v, v'\rangle=\langle v, j\pi(\xi)v'\rangle \ (v, v'\in V, \xi\in\mathfrak{g}_1),$$

where $i=\sqrt{-1}$ and $j$ is a fixed forth root (depending only on $\pi$) of $-1$, i.e., $j^2=\varepsilon i$ with $\varepsilon=\pm 1$ (cf. [4, § 1]). We call $\varepsilon$ *the associated constant for* $\pi$. Here, both $\pi(\mathfrak{g}_0)|V_0$ and $\pi(\mathfrak{g}_0)|V_1$ are usual unitary representations of $\mathfrak{g}_0$.

**1.2. Simple Lie superalgebra** $A(m, n)$**.** Now we define the Lie superalgebra of type $A(m, n)$. We denote by $M(p, q; K)$ the set of all matrices of type $p\otimes q$ with entries in a field $K$. The underlying vector space is $\mathfrak{v}=M(m+n, m+n; \boldsymbol{C})$. Let $E_{i,j}$, $1\leqq i, j\leqq m+n$, be an element of $\mathfrak{v}$ with components 1 at $(i, j)$ and 0 elsewhere. Let $\mathfrak{v}_0$ be a complex subspace of $\mathfrak{v}$ generated by

$$\{E_{i,j};\ 1\leqq i,\ j\leqq m\}\cup\{E_{i,j};\ m+1\leqq i,\ j\leqq m+n\},$$

and further $\mathfrak{v}_{1,+}$(resp. $\mathfrak{v}_{1,-}$) a complex subspace of $\mathfrak{v}$ generated by

$$\{E_{i,j};\ 1\leqq i\leqq m;\ m+1\leqq j\leqq m+n\},$$

$$(\text{resp. } \{E_{i,j};\ m+1\leqq i\leqq m+n;\ 1\leqq j\leqq m\}),$$

and put $\mathfrak{v}_1=\mathfrak{v}_{1,+}+\mathfrak{v}_{1,-}$. The bracket product

$$[X, Y]=XY-(-1)^{st}YX \qquad \text{for } X\in\mathfrak{v}_s, Y\in\mathfrak{v}_t,$$

where $s, t\in\{0, 1\}$, makes $\mathfrak{v}$ a Lie superalgebra, denoted by $\mathfrak{l}(m, n)$, where $\mathfrak{l}(m, n)_s=\mathfrak{v}_s$ $(s=0, 1)$. We put $\mathfrak{l}(m, n)_{1,\pm}=\mathfrak{v}_{1,\pm}$, then $\mathfrak{l}(m, n)=\mathfrak{l}(m, n)_{1,-}+\mathfrak{l}(m, n)_0+\mathfrak{l}(m, n)_{1,+}$ gives a $\boldsymbol{Z}$-gradation of $\mathfrak{l}(m, n)$. For the algebra $\mathfrak{l}(m, n)$, we define the supertrace str, a linear form on it, as follows:

$$\operatorname{str} X=\operatorname{tr} A-\operatorname{tr} D \qquad \text{for } X=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\mathfrak{l}(m, n),$$

where $A$, $B$, $C$ and $D$ is in $M(m, m; \boldsymbol{C})$, $M(m, n; \boldsymbol{C})$, $M(n, m; \boldsymbol{C})$ and $M(n, n; \boldsymbol{C})$ respectively. Define $\mathfrak{sl}(m, n)$ as

$$\mathfrak{sl}(m, n)=\{X\in\mathfrak{l}(m, n);\ \operatorname{str} X=0\},$$

then this is an ideal in $\mathfrak{l}(m, n)$ of codimension 1. In case $m=n$, $\mathfrak{sl}(n, n)$ has one-dimensional center $\mathfrak{z}$ consisting of scalar matrices $\lambda \cdot I_{2n} (\lambda \in \boldsymbol{C})$. We set

$$A(m, n)=\mathfrak{sl}(m+1, n+1) \quad \text{for} \quad m, n \geqq 0, m \neq n,$$

$$A(n, n)=\mathfrak{sl}(n+1, n+1)/\mathfrak{z} \quad \text{for} \quad n>0.$$

We denote by $\mathfrak{g}_C$ the complex algebra $\mathfrak{sl}(n, 1)=A(n-1, 0)$, keeping the symbol $\mathfrak{g}$ to its real forms. For later use, we give two kinds of basis (1.1) and (1.2) of a Cartan subalgebra $\mathfrak{h}_C$ of $\mathfrak{g}_C$:

(1.1) $$H_i=E_{i,i}+E_{n+1,n+1} \quad \text{for} \quad 1 \leqq i \leqq n,$$

and

(1.2) $$\begin{cases} H(i)=E_{i,i}-E_{i+1,i+1} & \text{for} \quad 1 \leqq i \leqq n-1, \\ C=\sum_{1 \leq i \leq n} E_{i,i}+nE_{n+1,n+1}. \end{cases}$$

We also give a basis of $\mathfrak{g}_{1,C}$ as

(1.3) $$\begin{cases} \xi_i=E_{i,n+1} & \text{for} \quad 1 \leqq i \leqq n, \\ \eta_i=E_{n+1,i} & \text{for} \quad 1 \leqq i \leqq n. \end{cases}$$

$\Pi_0=\{\alpha_i ; 1 \leqq i \leqq n-1\} \cup \{\beta\}$ denotes a system of simple roots of $(\mathfrak{g}_C, \mathfrak{h}_C)$ given by

(1.4) $$\alpha_i(H_k)=\begin{cases} 1 & \text{for} \quad i=k, \\ -1 & \text{for} \quad i=k+1, \\ 0 & \text{for} \quad i \neq k, k+1; \end{cases}$$

and

(1.5) $$\beta(H_k)=\begin{cases} -1 & \text{for} \quad 1 \leqq k \leqq n-1, \\ 0 & \text{for} \quad k=n. \end{cases}$$

**1.3. Real forms of $\mathfrak{sl}(n, 1)=A(n-1, 0)$.** Here we list up real forms $\mathfrak{g}$ of $\mathfrak{g}_C=\mathfrak{sl}(n, 1)$ (cf. [5, § 5]). There exist two types of them. A real form of first type is

$$\mathfrak{sl}(n, 1; \boldsymbol{R})=\mathfrak{sl}(n, 1) \cap M(n+1, n+1; \boldsymbol{R}).$$

Real forms of second type are defined as follows. Let $0 \leqq p \leqq n$ and $q \in \{0, 1\}$. For $s=0, 1$, put

$$\mathfrak{su}(n, 1; p, q)_s=\{X \in \mathfrak{sl}(n, 1)_s ; J_{p,q}X+{}^t\bar{X}J_{p,q}=0\},$$

where ${}^tX$ is the transposed matrix of $X$, and

$$J_{p,q}=\text{diag}(1, \cdots, 1, -1, \cdots, -1, -(-1)^q(\sqrt{-1})^s)$$

where the number of 1 is $p$ and the number of $-1$ is $n-p$ and diag$(\cdot, \cdot, \cdots)$ denotes a diagonal matrix. Then $\mathfrak{su}(n, 1; p, q)=\mathfrak{su}(n, 1; p, q)_0 \oplus \mathfrak{su}(n, 1; p, q)_1$ is a real Lie superalgebra for each $(p, q)$.

**Proposition 1.1** (cf. [5, § 5]). *Real forms of $\mathfrak{sl}(n, 1)$ are isomorphic, up to transition to their duals, to one of the following*:

(a) $\mathfrak{sl}(n, 1; \boldsymbol{R})$;
(b) $\mathfrak{su}(n, 1; n, 1)$;
(c) $\mathfrak{su}(n, 1; p, 1)$ *for* $\left[\frac{n+1}{2}\right] \leqq p \leqq n-1$.

## §2. Generalities for irreducible unitary representations of real forms of $\mathfrak{sl}(n, 1)$.

**2.1. Irreducible unitary representations of $\mathfrak{sl}(n, 1; \boldsymbol{R})$.** Let $\mathfrak{g}=\mathfrak{sl}(n, 1; \boldsymbol{R})$. Then there exist no irreducible unitary representations (=IURs) except trivial one. More generally, we have a similar situation as above for this type of real form $\mathfrak{sl}(m, n; \boldsymbol{R})$ of $\mathfrak{sl}(m, n)$:

$$\mathfrak{sl}(m, n; \boldsymbol{R}) \equiv \mathfrak{sl}(m, n) \cap M(m+n, m+n; \boldsymbol{R}).$$

**Theorem 2.1** [4, Th. 6.2]. *Let* $\mathfrak{g}=\mathfrak{sl}(m, n; \boldsymbol{R})$, $m, n \geqq 1$. *Then it has only a unique irreducible unitary representation, the trivial one.*

**2.2. Weight distributions for IURs of $\mathfrak{su}(n, 1; p, 1)$.** The odd part $\mathfrak{g}_1$ of $\mathfrak{g}=\mathfrak{su}(n, 1; p, 1)$ contains the following elements

$$\tau_k = \begin{cases} \xi_k + \sqrt{-1}\eta_k & \text{for } 1 \leqq k \leqq p, \\ \xi_k - \sqrt{-1}\eta_k & \text{for } p+1 \leqq k \leqq n. \end{cases} \tag{2.1}$$

Applying the positive-definiteness condition to these elements, we get a condition on the distribution of weights $\rho \in \mathfrak{h}_c^*$ for an IUR of $\mathfrak{g}$.

**Proposition 2.2.** *Let* $(\pi, V)$ *be an IUR of a real Lie superalgebra* $\mathfrak{su}(n, 1; p, 1)$. *Then there are* $\{\varepsilon_k\}_{1 \leqq k \leqq n}$, $\varepsilon_k = \pm 1$, *satisfying*
(1) $\varepsilon_1 = \cdots = \varepsilon_p = -\varepsilon_{p+1} = \cdots = -\varepsilon_n$, *and*
(2) *any weight* $\rho$ *of* $V$ *satisfies*

$$\varepsilon_k \rho(H_k) \geqq 0 \quad \textit{for all} \quad k.$$

*Proof.* From the definition of unitarity, we have

$$j^2 \langle \pi(\zeta)\pi(\zeta)v_\rho, v_\rho \rangle \geqq 0,$$

where $\zeta \in \mathfrak{g}_1$ and $v_\rho$ is a non-zero weight vector with weight $\rho \in \mathfrak{h}_c^*$. Put $\zeta = \tau_k = \xi_k + \sqrt{-1}\eta_k (1 \leqq k \leqq p)$, then $\pi(\zeta)\pi(\zeta) = \pi([\xi_k, \sqrt{-1}\eta_k]) = \sqrt{-1}\pi(H_k)$. Therefore

$$-\varepsilon\rho(H_k) \langle v_\rho, v_\rho \rangle \geqq 0,$$

where $\varepsilon$ is given as $j^2 = \varepsilon i$. Similarly, put $\zeta = \tau_k = \xi_k - \sqrt{-1}\eta_k (p+1 \leqq k \leqq n)$, then

$$\varepsilon\rho(H_k) \langle v_\sigma, v_\sigma \rangle \geqq 0.$$

Thus we can set $\varepsilon_k = -\varepsilon$ for $1 \leqq k \leqq p$ and $\varepsilon_k = \varepsilon$ for $p+1 \leqq k \leqq n$. Q.E.D.

**2.3. $\boldsymbol{Z}$-gradations.** Let $\mathfrak{g}_C = \mathfrak{sl}(n, 1)$ and $C' = (1/1-n)C \in \mathfrak{h}_C$, then $\mathfrak{g}_C$ is decomposed into $C'$-eigenspaces as

$$\mathfrak{g}_C = \mathfrak{g}_C^{-1} \oplus \mathfrak{g}_C^0 \oplus \mathfrak{g}_C^{+1},$$

where $\mathfrak{g}_C^{\pm 1}=\mathfrak{g}_{1,\pm}$ and $\mathfrak{g}_C^0=\mathfrak{g}_{0,C}$ corresponds to eigenvalues $\pm 1$ and 0 respectively. By this grading, $\mathfrak{g}_C$ becomes a $\boldsymbol{Z}$-graded algebra, and so is $\mathcal{U}(\mathfrak{g}_C)$ and $\mathcal{U}(\mathfrak{g}_{1,\pm})$, that is,

$$\mathcal{U}(\mathfrak{g}_C)=\oplus_{-n\leq k\leq n}\mathcal{U}^k, \qquad \mathcal{U}(\mathfrak{g}_{1,\pm})=\oplus_{0\leq k\leq n}\mathcal{U}(\pm k), \tag{2.2}$$

where $\mathcal{U}^k$ and $\mathcal{U}(k)$ are $C'$-eigenspaces of $\mathcal{U}(\mathfrak{g}_C)$ and $\mathcal{U}(\mathfrak{g}_{1,\pm})$ with eigenvalue $k$ respectively. We remark that $\mathcal{U}(0)=\boldsymbol{C}$, $\mathcal{U}(\pm 1)=\mathfrak{g}_{1,\pm}$ and $\mathcal{U}(\mathfrak{g}^C)=\mathcal{U}(\mathfrak{g}_{1,-})\mathcal{U}(\mathfrak{g}_{0,C})\mathcal{U}(\mathfrak{g}_{1,+})$.

Let $(\pi, V)$ be an IUR of $\mathfrak{g}$, then $V$ is decomposed into $\pi(C')$-eigenspaces, that is,

$$V=\oplus_{m\in C}V^m,$$

where $V^m$ denotes the $\pi(C')$-eigenspace with eigenvalue $m$. Conserning this eigenspace decomposition, we have the following

**Lemma 2.3.** (1) *There exist a complex number $M$ such that $V^M\neq(0)$, and if $V^m\neq(0)$ then $M-m$ is in $\boldsymbol{Z}_+\cup\{0\}$.*

(2) *$(\pi|\mathfrak{g}_0, V^M)$ is an IUR of $\mathfrak{g}_0$.*

(3) *$V=\mathcal{U}(\mathfrak{g}_{1,-})V^M$.*

(4) *Put $V^{(k)}=V^{M-k}$, then $V=V^{(0)}\oplus V^{(1)}\oplus\cdots\oplus V^{(n)}$. Moreover there exist an integer $k$ such that $V^{(r)}\neq(0)$ for $0\leqq r\leqq k$, $V^{(r)}=(0)$ for $k+1\leqq r\leqq n$, and $(\pi|\mathfrak{g}_0, V^{(k)})$ is an IUR of $\mathfrak{g}_0$.*

(5) *The decomposition of the representation space*

$$V=V^{(0)}\oplus V^{(1)}\oplus\cdots\oplus V^{(k)}$$

*is a $\boldsymbol{Z}$-gradation compatible with that of $\mathfrak{g}_C$.*

*Proof.* For (1) it is enough to remark that $V$ is irreducible and the grading of $\mathcal{U}(\mathfrak{g}_C)$ is finite.

Let us prove (2). Let $v$ be any element of $V^M$, then from the irreducibility of $V$,

$$V=\mathcal{U}(\mathfrak{g}_C)v=\mathcal{U}(\mathfrak{g}_{1,-})\mathcal{U}(\mathfrak{g}_{0,C})\mathcal{U}(\mathfrak{g}_{1,+})v.$$

But eigenvalues of $C'$ on $\mathcal{U}(\mathfrak{g}_{1,+})v$ are of the form $M+s$, $s\in\boldsymbol{Z}_+\cup\{0\}$. From the "maximality" of $M$ in (1), we get $\mathcal{U}(\mathfrak{g}_{1,+})v=\boldsymbol{C}v$ since $\mathcal{U}(0)=\boldsymbol{C}$. Thus we have

$$\begin{aligned} V&=\mathcal{U}(\mathfrak{g}_C)v=\mathcal{U}(\mathfrak{g}_{1,-})\mathcal{U}(\mathfrak{g}_{0,C})\mathcal{U}(\mathfrak{g}_{1,+})v \\ &=\mathcal{U}(\mathfrak{g}_{1,-})\mathcal{U}(\mathfrak{g}_{0,C})\boldsymbol{C}v=\mathcal{U}(\mathfrak{g}_{1,-})V'=\oplus_{0\leq k\leq n}\mathcal{U}(-k)V', \end{aligned} \tag{2.3}$$

where $V'$ denotes a $\mathfrak{g}_{0,C}$-submodule of $V$ generated by $v$. Since $C'$ commutes with $\mathfrak{g}_{0,C}$, each $\mathcal{U}(-k)V'$ is contained in $C'$-eigenspace with eigenvalue $M-k$, hence $V^{M-k}=V^{(k)}=\mathcal{U}(-k)V'$ for $0\leqq k\leqq n$.

Moreover $V^M=V^{(0)}=V'$ is irreducible $\mathfrak{g}_{0,C}$-module because any non-zero $v\in V^M$ generates $V'=V^M$. Thus we get (2).

The assertion (3) follows from (2.3). Changing the role of $+$ and $-$ of grading, we get (4) similarly as (1) and (2).

The assertion (5) is true since all the gradings are defined by eigenvalues of the same operator $C'$. Q.E.D.

**2.4. Induced highest weight modules for $\mathfrak{g}_C$** (the functor $L(\Lambda)\to V(\Lambda)$). Let $\mathfrak{g}_C=\mathfrak{sl}(n, 1)$ and $\mathfrak{p}=\mathfrak{g}_{0,C}+\mathfrak{g}_{1,+}$, then $\mathfrak{p}$ is a subalgebra of $\mathfrak{g}_C$ since $[\mathfrak{g}_{1,+}, \mathfrak{g}_{1,+}]=0$. Let $\mathfrak{b}_0$ be a subalgebra of $\mathfrak{g}_{0,C}$ generated by

$$\{E_{i,j};\ 1\leqq i<j\leqq n\}\cup\mathfrak{h}_C,$$

then $\mathfrak{b}=\mathfrak{b}_0+\mathfrak{g}_{1,+}$ is also a subalgebra of $\mathfrak{g}_C$. For $\Lambda\in\mathfrak{h}_C^*$, denote by $L(\Lambda)$ (resp. $V(\Lambda)$) the irreducible highest weight module of $\mathfrak{g}_{0,C}$(resp. $\mathfrak{g}_C$) with highest weight $\Lambda$ and by $v_\Lambda\in L(\Lambda)$ a non-zero highest weight vector. Then $\boldsymbol{C}v_\Lambda$ is a $\mathfrak{b}_0$-module. We define a $\mathfrak{g}_{1,+}$-action on $\boldsymbol{C}v_\Lambda$ by $\zeta v_\Lambda=0$ for any $\zeta\in\mathfrak{g}_{1,+}$, then $\boldsymbol{C}v_\Lambda$ becomes a $\mathfrak{b}$-module. Now define a $\mathfrak{g}_C$-module $\tilde{V}(\Lambda)$ by

$$\tilde{V}(\Lambda)\equiv\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}_C}\boldsymbol{C}v_\Lambda=\mathfrak{g}_C\otimes_{\mathfrak{b}}(\boldsymbol{C}v_\Lambda).$$

On the other hand, we define a $\mathfrak{g}_{1,+}$-action on $L(\Lambda)$ by $\zeta v=0$ for any $\zeta\in\mathfrak{g}_{1,+}$ and any $v\in L(\Lambda)$. Then $L(\Lambda)$ becomes a $\mathfrak{p}$-module and we get

$$\bar{V}(\Lambda)\equiv\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}_C}L(\Lambda)=\mathfrak{g}_{1,-}\otimes_C L(\Lambda). \tag{2.4}$$

**Lemma 2.4.** *The $\mathfrak{g}_C$-module $\bar{V}(\Lambda)$ is irreducible if and only if*

$$\Pi_{1\leq k\leq n}(\Lambda(H_k)+n-k)\neq 0.$$

*Proof.* We can induce a $\boldsymbol{Z}$-grading of $\bar{V}(\Lambda)$ from that of $\mathcal{U}(\mathfrak{g}_C)$. We can decompose $\bar{V}(\Lambda)$ into $C'$-eigenspaces as

$$\bar{V}(\Lambda)=\bar{V}_0+\bar{V}_{-1}+\cdots+\bar{V}_{-n},$$

where $\bar{V}_{-k}=\mathcal{U}(-k)L(\Lambda)$. Therefore we see that $\bar{V}_{-n}\cong L(\Lambda)$ as $[\mathfrak{g}_{0,C}, \mathfrak{g}_{0,C}]$-modules and that $w_\Lambda=\eta_1\eta_2\cdots\eta_n v_\Lambda$ is a $\mathfrak{g}_{0,C}$-highest weight vector of $\bar{V}_{-n}$. Because of the irreducibility of $\bar{V}_{-n}$, $\bar{V}(\Lambda)$ is irreducible if and only if $w_\Lambda$ generates $\bar{V}(\Lambda)$. And the latter is equivalent to the condition

$$\xi_n\xi_{n-1}\cdots\xi_1 w_\Lambda\neq 0.$$

Calculating $[\xi_i, \eta_i]$, we get finally

$$\xi_n\xi_{n-1}\cdots\xi_1 w_\Lambda=\{\Pi_{1\leq k\leq n}(\Lambda(H_k)+n-k)\}v_\Lambda.$$

This gives the lemma. Q.E.D.

**Remark 2.5.** (1) When $\bar{V}(\Lambda)$ is irreducible, it is called a typical representation. Kac shows the above criterion for the case $L(\Lambda)$ is finite [6, §2].

(2) $V(\Lambda)$ is a unique irreducible quotient of $\bar{V}(\Lambda)$ and we can define a functor $L(\Lambda)\to\bar{V}(\Lambda)\to V(\Lambda)$.

**2.5. Branching rules of induced $\mathfrak{g}$-modules $\bar{V}(\Lambda)$ restrected to $\mathfrak{g}_0$.** Now consider the branching rules of $\bar{V}(\Lambda)$ restrected to $\mathfrak{g}_0$. It is not easy to give an exact rule generally for $\mathfrak{g}=\mathfrak{sl}(n, 1)$. Moreover when $L(\Lambda)$ is infinite-dimensional, $\bar{V}(\Lambda)$ is not necessarily semisimple with respect to $\mathfrak{g}_0$. Here we restrict ourselves to the case of $\mathfrak{g}=\mathfrak{sl}(2, 1)$ and $\mathfrak{sl}(3, 1)$.

If the branching rule is acquired, we can obtain $V(\Lambda)$ as follows:

**2.5.1. Method of constructing $V(\Lambda)$ from $\bar{V}(\Lambda)$.** We construct $V(\Lambda)$ according to the steps (1)～(4).

(1) First we decompose each $\bar{V}_{-k}=\mathcal{U}(-k)L(\Lambda)$ into irreducible representations of $\mathfrak{g}_0$, or determine its subquotient structure, where $\mathcal{U}(-k)$ is as in (2.2).

(2) Check the $\mathfrak{g}_{1,-}$-action on each component.

Notation. We denote $\pi(\mathfrak{g}_{1,-})V_a\cong V_b\oplus V_c$ by a diagram $\begin{array}{ccc} & V_a & \\ \swarrow & & \searrow \\ V_b & \oplus & V_c\end{array}$ where $V_a$, $V_b$ and $V_c$ are $\mathfrak{g}_0$-modules.

In the following we take as $V_a$ a $\mathfrak{g}_0$-submodule of $\bar{V}_{-k}$. Then we see that these branching diagrams are independent of the value $\Lambda(C)$ for the central element $C$.

(3) $\pi(\mathfrak{g}_{1,+})V_a$ depends on the value of $\Lambda(C)$. So we calculate its structure case by case.

(4) Finally, from (2) and (3) we get the unique maximal submodule $I(\Lambda)$ and obtain $V(\Lambda)=\bar{V}(\Lambda)/I(\Lambda)$. (We note that a submodule is understood to be $\boldsymbol{Z}_2$-graded.)

**2.5.2. Case $\mathfrak{g}=\mathfrak{sl}(2,1)$ and** dim $L(\Lambda)<\infty$. In this case $n=\Lambda(H)$ is a non-negative integer and we have two cases.

CASE 1: $n\geqq 1$. In this case $\bar{V}(\Lambda)$ splits into four $\mathfrak{g}_0$-irreducible components as follows:

$$\begin{array}{lccccc}
\bar{V}_0\cong & & & L(\Lambda) & & \\
 & & \swarrow & & \searrow & \\
\oplus & & & & & \\
\bar{V}_{-1}\cong & L(\Lambda-\beta) & & \oplus & & L(\Lambda-\gamma) \\
\oplus & & \searrow & & \swarrow & \\
\bar{V}_{-2}\cong & & & L(\Lambda-\beta-\gamma) & &
\end{array}$$

Here $\beta$ is defined in (1.5) and $\gamma=\alpha_1+\beta$ with $\alpha_1$ in (1.4).

CASE 2: $n=0$. In this case $\bar{V}(\Lambda)$ splits into three $\mathfrak{g}_0$-irreducible components, as

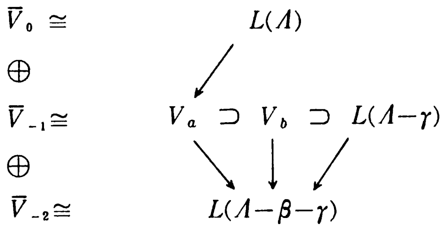
$$\begin{array}{ccccc}
\bar{V}_0 & \oplus & \bar{V}_{-1} & \oplus & \bar{V}_{-2} \\
\wr\Vert & & \Vert\wr & & \wr\Vert \\
L(\Lambda) & \longrightarrow & L(\Lambda-\beta) & \longrightarrow & L(\Lambda-\beta-\gamma)
\end{array}$$

**2.5.3. Case $\mathfrak{g}=\mathfrak{sl}(2,1)$ and** dim $L(\Lambda)=\infty$. In this case $n=\Lambda(H)$ is a negative integer. Then we have two cases.

CASE 3: $n\leqq -2$. In this case $\bar{V}(\Lambda)$ splits into four $\mathfrak{g}_0$-irreducible components as follows:

$$\begin{array}{lccccc}
\bar{V}_0\cong & & & L(\Lambda) & & \\
 & & \swarrow & & \searrow & \\
\oplus & & & & & \\
\bar{V}_{-1}\cong & L(\Lambda-\beta) & & \oplus & & L(\Lambda-\gamma) \\
\oplus & & \searrow & & \swarrow & \\
\bar{V}_{-2}\cong & & & L(\Lambda-\beta-\gamma) & &
\end{array}$$

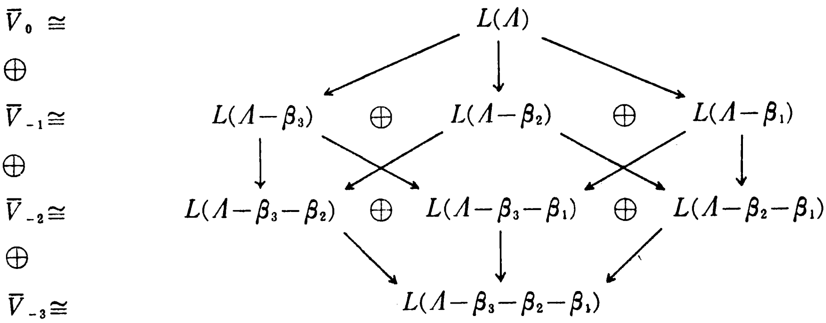
CASE 4: $n=-1$. In this case $\bar{V}(\Lambda)$ is not $\mathfrak{g}_0$-semisimple and




Here the $\mathfrak{g}_0$-module $V_a$ is not semisimple, and $V_a/V_b \cong L(\Lambda-\gamma)$, $V_b/L(\Lambda-\gamma) \cong L(\Lambda-\beta)$ as $\mathfrak{g}_0$-modules.

**2.5.4. Case** $\mathfrak{g}=\mathfrak{sl}(3, 1)$ **and** $\dim L(\Lambda)<\infty$**.** In this case $m=\Lambda(H(1))$ and $n=\Lambda(H(2))$ are non-negative integers. We illustrate the branching rules in four cases separately.

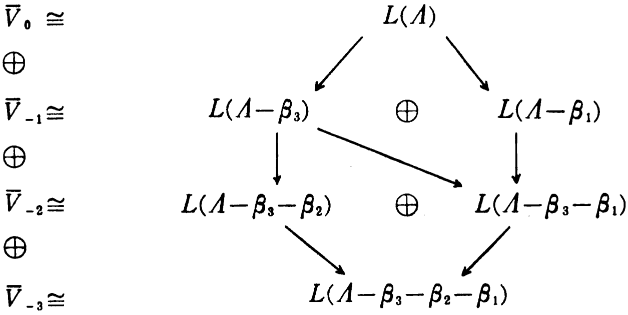
CASE 1: $m \geqq 1$ and $n \geqq 1$. In this case $\bar{V}(\Lambda)$ splits into eight $\mathfrak{g}_0$-irreducible components as follows:



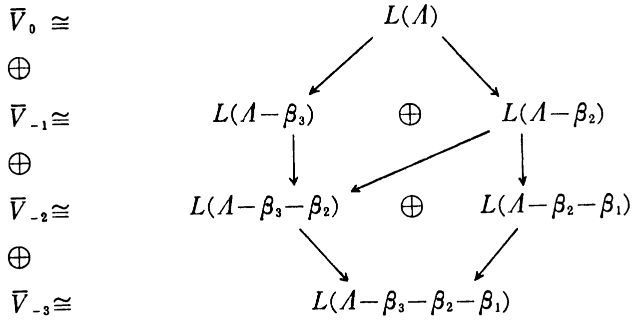

Here $\{\beta_1, \beta_2, \beta_3\}$ are positive odd roots given as

$$\beta_k(H_j)=-1+\delta_{k,j} \qquad \text{for } 1 \leqq k,\ j \leqq 3.$$

CASE 2: $m \geqq 1$ and $n=0$. In this case $\bar{V}(\Lambda)$ splits into six $\mathfrak{g}_0$-irreducible components, as



CASE 3: $m=0$ and $n\geqq 1$. In this case $\bar{V}(\Lambda)$ splits into six $\mathfrak{g}_0$-irreducible components, as




CASE 4: $m=n=0$. In this case $\bar{V}(\Lambda)$ splits into four $\mathfrak{g}_0$-irreducible components, as

$$\begin{array}{ccccccc} \bar{V}_0 & \oplus & \bar{V}_{-1} & \oplus & \bar{V}_{-2} & \oplus & \bar{V}_{-3} \\ \wr\| & & \|\wr & & \wr\| & & \wr\| \\ L(\Lambda) & \longrightarrow & L(\Lambda-\beta_3) & \longrightarrow & L(\Lambda-\beta_3-\beta_2) & \longrightarrow & L(\Lambda-\beta_3-\beta_2-\beta_1) \end{array}$$

**2.6. Calculations for some cases in 2.5.** In this subsection we give a sketch of calculations for the above diagrams in two cases. The other cases can be calculated similarly.

CASE I. Diagrams in 2.5.4 for $\mathfrak{g}=\mathfrak{sl}(3, 1)$.

We use the next lemma.

**Lemma 2.6.** *Let $\mathfrak{g}=\mathfrak{sl}(n; C)$ and $V$ be an irreducible finite-dimensional $\mathfrak{g}$-module with highest weight $\Lambda$. Take $F$ an irreducible finite-dimensional $\mathfrak{g}$-module such that each weight has multiplicity* 1. *Then $\mathfrak{g}$-module $V\otimes F$ is isomorphic to a quotient of the following $\mathfrak{g}$-module*

$$\oplus_{1\leq i\leq m} L(\Lambda-\lambda_i),$$

*where $\{\lambda_i\}_{1\leq i\leq m}$ is the set of weights for $F$, and $L(\lambda)$ is a highest weight $\mathfrak{g}$-module with highest weight $\lambda$.*

*Proof.* Since $V$ and $F$ are irreducible and finite-dimensional, both of them can be assumed to be unitary for $\mathfrak{su}(n)$ and then $V\otimes F$ becomes a unitary representation. Therefore $V\otimes F$ is decomposed into a direct sum of IURs of $\mathfrak{su}(n)$. Each of these representations is a highest weight representation of $\mathfrak{g}$. Thus all we have to do is to find all the highest weight vectors in $V\otimes F$. Each weight vector $w$ of $V\otimes F$ is of the form

$$w=\sum_{1\leq i\leq m} v_i\otimes f_i,$$

where, for $1\leqq i\leqq m$, $f_i$ is a weight vector of $F$ with weight $\lambda_i$ and $v_i$ is a weight vector of $V$. A simple calculation shows that one of $v_i$ is a highest weight vector of $V$ when $w$ is a highest weight vector. Thus we get the result. Q.E.D.

Let $\mathcal{U}(-1)\subset\mathcal{U}(\mathfrak{g}_{1,-})$ be as in (2.2). From the above lemma, $\bar{V}_{-1}=L(\Lambda)\otimes\mathcal{U}(-1)$ is a quotient of

$$L(\Lambda-\beta_1)\oplus L(\Lambda-\beta_2)\oplus L(\Lambda-\beta_3),$$

since $\dim \mathcal{U}(-1)=3$ and its weights are $-\beta_i$, $i=1, 2, 3$. To get the decomposition of $\bar{V}_{-1}$ it is sufficient for us (fortunately in this case) to calculate the dimensions of $L(\Lambda-\beta_i)$ and $\bar{V}_{-1}$ and compare them.

Similarly we can prove that $\bar{V}_{-2}$ is a quotient of

$$L(\Lambda-\beta_1-\beta_2)\oplus L(\Lambda-\beta_1-\beta_3)\oplus L(\Lambda-\beta_2-\beta_3).$$

Now we determine arrows from $\bar{V}_{-1}$ to $\bar{V}_{-2}$. Take a component $L(\Lambda-\beta_i)$ of $\bar{V}_{-1}$ and consider the image $\mathfrak{g}_{1,-}\cdot L(\Lambda-\beta_i)$ in $\bar{V}_{-2}$. Then $\mathfrak{g}_{1,-}\cdot L(\Lambda-\beta_i)$ is a quotient of $\mathfrak{g}_0$-module $L(\Lambda-\beta_i)\otimes\mathfrak{g}_{1,-}$. Thus using Lemma 2.6 again, we see that $\mathfrak{g}_{1,-}\cdot L(\Lambda-\beta_i)$ is a quotient of $\mathfrak{g}_0$-module $\oplus_{1\leq j\leq 3}L(\Lambda-\beta_i-\beta_j)$. On the other hand $\mathfrak{g}_{1,-}\cdot L(\Lambda-\beta_i)$ is also a quotient of

$$L(\Lambda-\beta_1-\beta_2)\oplus L(\Lambda-\beta_1-\beta_3)\oplus L(\Lambda-\beta_2-\beta_3)$$

because $\mathfrak{g}_{1,-}\cdot L(\Lambda-\beta_i)$ is in $\bar{V}_{-2}$. Therefore, taking the common components of the avobe two $\mathfrak{g}_0$-modules, we see that $\mathfrak{g}_{1,-}\cdot L(\Lambda-\beta_i)$ is a quotient of $\oplus_{j\neq i}L(\Lambda-\beta_i-\beta_j)$. Then we check the image under $\mathfrak{g}_{1,-}$ in $\bar{V}_{-2}$ of the highest weight vector of $L(\Lambda-\beta_i)$. Then we get the arrows from $\bar{V}_{-1}$ to $\bar{V}_{-2}$. The arrows in other places are obvious.

CASE II. Diagram of Case 4 in 2.4.3 for $\mathfrak{g}=\mathfrak{sl}(2, 1)$ and $\Lambda(H)=-1$.

In this case $\bar{V}_0$ and $\bar{V}_{-2}$ are irreducible. So we study the structure of $\bar{V}_{-1}$. Let $v_1$ be a non-zero highest weight vector of $\bar{V}_0$, and $\bar{W}$ be a subspace of $\bar{V}_{-1}$ given by $E_{1,2}\in\mathfrak{g}_0$ as

$$\bar{W}=\{v\in\bar{V}_{-1};\ E_{1,2}\cdot v=0\}.$$

Then $\bar{W}$ is two dimensional and we give a basis $\{w_1, w_2\}$ as

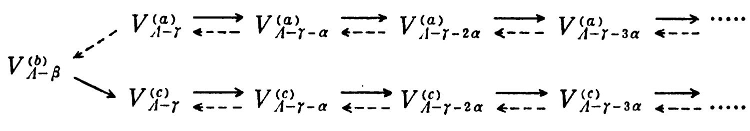
$$w_1=\eta_2\cdot v_1, \qquad w_2=\eta_2 E_{2,1}\cdot v_1-\eta_1\cdot v_1.$$

Then $w_1$ (resp. $w_2$) is a weight vector with weight $\Lambda-\beta$ (resp. $\Lambda-\gamma$). We put $V_a=\bar{V}_{-1}$, $V_b=\mathcal{U}(\mathfrak{g}_0)w_1$ and $V_c=\mathcal{U}(\mathfrak{g}_0)w_2$, then both $V_b$ and $V_c$ are highest weight submodules of $V_a$, and $V_b\supset V_c$ since $E_{2,1}\cdot w_1=w_2$. Now all the weight spaces $V_{a,\lambda}$ with weight $\lambda$ of $V_a$ are two-dimensional except $\dim V_{a,\Lambda-\beta}=1$. Therefore examinig the highest weight vectors and weight distributions for $V_a$, we get the following results:

$$V_c\cong V_a/V_b\cong L(\Lambda-\gamma), \quad\text{and}\quad V_b/V_c\cong L(\Lambda-\beta)\ \text{(one dimensional)}.$$

Thus we get the diagram since the arrows in this case are all obvious.

Diagram of weight distribution for $V_a$:




The arrows $\longrightarrow$ and $\dashleftarrow$ denote $\mathfrak{g}_{1,-}$- and $\mathfrak{g}_{1,+}$-action, and each space $V^{(a)}_\lambda$, $V^{(b)}_\lambda$ and $V^{(c)}_\lambda$ has weight $\lambda$ and dimension 1. Here

$$V_c=\sum V^{(c)}_\lambda, \quad V_b=V_c+V^{(b)}_{\Lambda-\beta} \quad\text{and}\quad V_a=V_b+\sum V^{(a)}_\lambda.$$

## § 3. Classification of irreducible unitary representations of real forms of $\mathfrak{sl}(2, 1)$.

**3.1. Positive roots for** $\mathfrak{sl}(2, 1)$. Let $\mathfrak{g}_C = \mathfrak{sl}(2, 1)$, and $\{\alpha, \beta\}$ be simple roots of $(\mathfrak{g}_C, \mathfrak{h}_C)$ given as

$$\alpha(H)=2, \qquad \alpha(C)=0;$$

$$\beta(H)=-1, \qquad \beta(C)=-1,$$

where $\{H=H(1), C\}$ is a basis of $\mathfrak{h}_C$ in (1.2). Another positive root is given by $\gamma=\alpha+\beta$. We also introduce a notation $\delta=\beta+\gamma$, the sum of positive odd roots.

**3.2. Classification of IURs of** $\mathfrak{su}(2, 1; 2, 1)$. Let $\mathfrak{g}=\mathfrak{su}(2, 1; 2, 1)$, then $\mathfrak{g}_0 \cong \mathfrak{u}(2)$ and any IUR of $\mathfrak{g}_0$ is finite-dimensional. From the definition, any unitary representation of $\mathfrak{g}$ is unitary when it is restricted to $\mathfrak{g}_0$. Therefore an IUR $V$ of $\mathfrak{g}$ is finite-dimensional and so a highest weight module (by Lemma 2.3). Let the highest weight be $\Lambda \in \mathfrak{h}_C^*$, then $L(\Lambda) \hookrightarrow V$ becomes a unitary representation of $\mathfrak{g}_0$, and so $\Lambda$ has to satisfy the following conditions:

(3.1) $\qquad \Lambda(H)$ is a non-negative integer, and

(3.2) $\qquad \Lambda(C)$ is a real number.

For the classification of IURs, first we must check when $\bar{V}(\Lambda)$ is irreducible. From Lemma 2.4, we get,

**Lemma 3.1.** *$\bar{V}(\Lambda)$ is not irreducible if and only if one of the following conditions holds*:

(i) $\qquad \Lambda(C)=\Lambda(H)$,

(ii) $\qquad \Lambda(C)=-\Lambda(H)-2$.

Next we must check the unitarity condition. From Proposition 2.2, we get,

**Lemma 3.2.** *Let $L(\lambda)$ be a $\mathfrak{g}_0$-irreducible component of an IUR $V(\lambda)$ of $\mathfrak{g}$, then $\lambda$ satisfies the similar conditions as* (3.1), (3.2) *and also*

$$0 \leqq \lambda(H) \leqq |\lambda(C)|.$$

Summarizing these results, we get the following

**Theorem 3.3.** (1) *Any irreducible unitary representation of Lie superalgebra $\mathfrak{su}(2, 1; 2, 1)$ is isomorphic to one of the irreducible highest weight representation $V(\Lambda)$ with $\Lambda(C) \leqq -\Lambda(H)-2$ or $\Lambda(H) \leqq \Lambda(C)$.*

(2) *As $\mathfrak{g}_0$-modules, the above $V(\Lambda)$ is decomposed as follows*:

(i) $V(\Lambda)=L(\Lambda)$ *for* $\Lambda(C)=\Lambda(H)=0$,

(ii) $V(\Lambda)=L(\Lambda) \oplus L(\Lambda-\gamma)$ *for* $\Lambda(C)=\Lambda(H) \geqq 1$,

(iii) $V(\Lambda)=L(\Lambda) \oplus L(\Lambda-\beta)$ *for* $\Lambda(C)=-\Lambda(H)-2$,

(iv) $V(\Lambda)=L(\Lambda)\oplus L(\Lambda-\beta)\oplus L(\Lambda-\delta)$ *for* $\Lambda(H)=0$ *and* $\Lambda(C)<-2,\ 0<\Lambda(C)$,

(v) $V(\Lambda)=L(\Lambda)\oplus L(\Lambda-\beta)\oplus L(\Lambda-\gamma)\oplus L(\Lambda-\delta)$ *otherwise.*

Realization of the type (v) will be given in 3.4.1 and those of other types have been given in [4, §8].

**Note 3.4.** The $\boldsymbol{Z}_2$-gradation $V_0\oplus V_1$ of $V(\Lambda)$ in the above theorem is given by

$$V_0=L(\Lambda) \text{ or } L(\Lambda)\oplus L(\Lambda-\delta),$$
$$V_1=(0),\ L(\Lambda-\beta),\ L(\Lambda-\gamma) \text{ or } L(\Lambda-\beta)\oplus L(\Lambda-\gamma).$$

**3.3. Classification of IURs of $\mathfrak{su}(2, 1; 1, 1)$.** Let $\mathfrak{g}=\mathfrak{su}(2, 1; 1, 1)$, then $\mathfrak{g}_0\cong\mathfrak{u}(1, 1)$ and IURs of $\mathfrak{g}_0$ are classified as

(PCS) principal continuous series;
(DS) discrete series;
(LDS) limit of discrete series;
(CS) complementary series;
(T) the trivial representation.

Let $V$ be an IUR of $\mathfrak{g}$, then each $\mathfrak{g}_0$-irreducible component $W$ of $V$ is also unitary for $\mathfrak{g}_0$. By Proposition 2.2, $W$ is a highest or lowest weight $\mathfrak{g}$-module and so is $V$ accordingly. Since the situation is parallel, we may take $V$ a highest weight representation $V(\Lambda)$. Since $L(\Lambda)\hookrightarrow V(\Lambda)$ be unitarizable, $\Lambda$ satisfies the following two conditions:

(3.3) $\Lambda(H)$ is a non-positive integer, and

(3.4) $\Lambda(C)$ is a real number.

In this case, Lemma 3.1 still holds. From Proposition 2.2, we get

**Lemma 3.5.** *Let $L(\lambda)$ be a $\mathfrak{g}_0$-irreducible component of an IUR $V(\Lambda)$ of $\mathfrak{g}$, then $\lambda$ satisfy the similar conditions as* (3.3), (3.4) *and also*

$$\lambda(H)\leqq-|\lambda(C)|.$$

Thus we get

**Theorem 3.6.** (1) *Any irreducible unitary representation $V$ of Lie superalgebra $\mathfrak{su}(2, 1; 1, 1)$ is a highest or lowest weight representation. If $V$ is a highest weight IUR, then $V$ is isomorphic to one of the representations $V(\Lambda)$ with $\Lambda(H)\leqq\Lambda(C)\leqq-\Lambda(H)-2$ or $\Lambda(H)=\Lambda(C)=0$.*

(2) *As a $\mathfrak{g}_0$-module, the above $V(\Lambda)$ is decomposed into $\mathfrak{g}_0$-irreducible componnents as follows:*

(i) $V(\Lambda)=L(\Lambda)$ *for* $\Lambda(C)=\Lambda(H)=0$,

(ii) $V(\Lambda)=L(\Lambda)\oplus L(\Lambda-\gamma)$ *for* $\Lambda(C)=\Lambda(H)\leqq-1$,

(iii) $V(\Lambda)=L(\Lambda)\oplus L(\Lambda-\beta)$ *for* $\Lambda(C)=-\Lambda(H)-2\geqq0$,

(iv) $V(\Lambda)=L(\Lambda)\oplus L(\Lambda-\beta)\oplus L(\Lambda-\gamma)\oplus L(\Lambda-\delta)$ otherwise.

Realization of the type (iv) will be given in 3.4.2 and those of other types were in [4, §8]. The $\boldsymbol{Z}_2$-gradation of $V(\Lambda)$ is given as in Note 3.4.

**3.4. Realizations of Irreducible unitary representations.** Now we give explicit realizations of IURs classified in 3.2 and 3.3. Except the last types in Theorems 3.3 and 3.6, we have given their realizations in [4, §8]. So we treat here the case where $V(\Lambda)=V_0\oplus V_1$ with

$$V_0=L(\Lambda)\oplus L(\Lambda-\delta), \qquad V_1=L(\Lambda-\beta)\oplus L(\Lambda-\gamma).$$

In this case, the $\boldsymbol{Z}$-gradation in Lemma 2.3 is given as $V^{(0)}=L(\Lambda)$, $V^{(1)}=V_1$ and $V^{(2)}=L(\Lambda-\delta)$. So we must calculate at first the explicit decomposition of $V^{(1)}=V_1$ as $\mathfrak{g}_0$-module. In each irreducible $\mathfrak{g}_0$-submodule the inner product is determined uniquely up to constant multiples, therefore we should determine these multiplicative constants.

Put $n=\Lambda(H)+1$, $m=\Lambda(C)$, $l=-\Lambda(H)$ and $\boldsymbol{Z}_+=\{k\in\boldsymbol{Z};\ k>0\}$.

**3.4.1. The case** $\mathfrak{g}=\mathfrak{su}(2,1;2,1)$. In this case $n=\dim L(\Lambda)$ and $m>n-1$ or $m<-n-1$ from Theorem 3.3(v). Put $\sigma=\operatorname{sgn}(m)$. Let $v_1^0\in L(\Lambda)$ be a unit highest weight vector of $V(\Lambda)$ and $\{v_k^0\}_{1\leq k\leq n}$ be a standard orthonormal basis of $L(\Lambda)$ given by

$$\sqrt{k(n-k)}\,v_{k+1}^0=\pi(E_{2,1})v_k^0 \qquad \text{for } 1\leqq k\leqq n-1\,. \tag{3.5}$$

And let $\{v_k^\delta\}_{1\leq k\leq n}$ be a standard orthonormal basis of $L(\Lambda-\delta)$ given by

$$v_k^\delta=\frac{2}{\sqrt{(m+n+1)(m-n+1)}}\pi(\eta_1)\pi(\eta_2)v_k^0 \qquad \text{for } 1\leqq k\leqq n\,, \tag{3.6}$$

where $\eta_i$'s are as in (1.3). Thus we fixed a basis for $V_0$.

Next we define a basis of $V_1$ by choosing standard orthonormal bases $\{v_k^\beta\}_{1\leq k\leq n+1}$ and $\{v_k^\gamma\}_{1\leq k\leq n-1}$ of $L(\Lambda-\beta)$ and $L(\Lambda-\gamma)$ respectively, given by

$$v_k^\beta=\frac{\sqrt{2}}{\sqrt{n|m-n+1|}}(\sqrt{n-k+1}\,\pi(\eta_2)v_k^0-\sqrt{k-1}\,\pi(\eta_1)v_{k-1}^0)\,, \tag{3.7}$$

$$v_k^\gamma=\frac{\sqrt{2}}{\sqrt{n|m+n+1|}}(\sqrt{k}\,\pi(\eta_2)v_{k+1}^0+\sqrt{n-k}\,\pi(\eta_1)v_k^0)\,. \tag{3.8}$$

Now we write down the operator $\pi(\zeta)$, $\zeta\in\mathfrak{g}_1$, in the form of blockwise matrix $(D_{j,k})_{j,k=0,\beta,\gamma,\delta}$, where

$$D_{j,k}: L(\Lambda-k)\longrightarrow L(\Lambda-j)\,.$$

If $\zeta\in\mathfrak{g}_{1,+}$, then $D_{j,k}=0$ except the cases $(j,k)=(0,\beta)$, $(0,\gamma)$, $(\beta,\delta)$, $(\gamma,\delta)$. If $\zeta\in\mathfrak{g}_{1,-}$, then $D_{j,k}=0$ except the cases $(j,k)=(\beta,0)$, $(\gamma,0)$, $(\delta,\beta)$, $(\delta,\gamma)$. Therefore each $\pi(\zeta)=(D_{j,k})$ is of the following form respectively depending on $\zeta\in\mathfrak{g}_{1,+}$ or $\zeta\in\mathfrak{g}_{1,-}$:

$$(D_{j,k})_{j,k=0,\beta,\gamma,\delta}=\begin{pmatrix}0&*&*&0\\0&0&0&*\\0&0&0&*\\0&0&0&0\end{pmatrix} \text{ or } \begin{pmatrix}0&0&0&0\\ *&0&0&0\\ *&0&0&0\\0&*&*&0\end{pmatrix}. \tag{3.9}$$

And the action of $\mathfrak{g}_{1,c}=\mathfrak{g}_{1,+}\oplus\mathfrak{g}_{1,-}$ is given with respect to these bases as follows:

For $\mathfrak{g}_{1,+}$:

$$D_{0,\beta} \;\&\; D_{0,\gamma}:\ \pi(\xi_1)v_k^\beta=-\sigma C_-a_{k-1}v_{k-1}^0, \qquad \pi(\xi_1)v_k^\gamma=\sigma C_+b_kv_k^0,$$

$$D_{\beta,\delta} \;\&\; D_{\gamma,\delta}:\ \pi(\xi_1)v_k^\delta=\sigma C_+b_{k-1}v_k^\beta+\sigma C_-a_{k-1}v_{k-1}^\gamma,$$

$$D_{0,\beta} \;\&\; D_{0,\gamma}:\ \pi(\xi_2)v_k^\beta=\sigma C_-b_{k-1}v_k^0, \qquad \pi(\xi_2)v_k^\gamma=\sigma C_+a_kv_{k+1}^0,$$

$$D_{\beta,\delta} \;\&\; D_{\gamma,\delta}:\ \pi(\xi_2)v_k^\delta=\sigma C_+a_kv_{k+1}^\beta-\sigma C_-b_kv_k^\gamma,$$

For $\mathfrak{g}_{1,-}$:

$$D_{\beta,0} \;\&\; D_{\gamma,0}:\ \pi(\eta_1)v_k^0=-C_-a_kv_k^\beta+C_+b_kv_k^\gamma,$$

$$D_{\delta,\beta} \;\&\; D_{\delta,\gamma}:\ \pi(\eta_1)v_k^\beta=C_+b_{k-1}v_k^\delta, \qquad \pi(\eta_1)v_k^\gamma=C_-a_kv_{k+1}^\delta,$$

$$D_{\beta,0} \;\&\; D_{\gamma,0}:\ \pi(\eta_2)v_k^0=C_-b_{k-1}v_k^\beta+C_+a_{k-1}v_{k-1}^\gamma,$$

$$D_{\delta,\beta} \;\&\; D_{\delta,\gamma}:\ \pi(\eta_2)v_k^\beta=C_+a_{k-1}v_{k-1}^\delta, \qquad \pi(\eta_2)v_k^\gamma=-C_-b_kv_k^\delta,$$

where

$$a_k=\sqrt{k}, \quad b_k=\sqrt{n-k} \quad \text{and} \quad C_\pm=\frac{\sqrt{|m\pm n+1|}}{\sqrt{2}}. \tag{3.10}$$

**3.4.2. The case** $\mathfrak{g}=\mathfrak{su}(2,1;1,1)$. In this case $l=-\Lambda(H)$ is a positive integer since $L(\Lambda)$ is a holomorphic discrete series or its limit. Further $l\geqq 2$ and $-l<m<l-2$ from Theorem 3.6(iv). Let $v_1^0\in L(\Lambda)$ be a unit highest weight vector of $V(\Lambda)$, and $\{v_k^0\}_{k\in Z_+}$ be a standard orthonormal basis of $L(\Lambda)$ given inductively by

$$v_{k+1}^0=\frac{1}{\sqrt{(k+l-1)k}}\pi(E_{2,1})v_k^0 \qquad \text{for } k\in \boldsymbol{Z}_+.$$

Next let $\{v_k^\delta\}_{k\in Z_+}$ be a standard orthonormal basis of $L(\Lambda-\delta)$ determined by

$$v_k^\delta=\frac{2}{\sqrt{(l+m)(l-m-2)}}\pi(\eta_1)\pi(\eta_2)v_k^0 \qquad \text{for } k\in \boldsymbol{Z}_+.$$

We define standard orthonormal bases $\{v_k^\beta\}_{k\in Z_+}$ and $\{v_k^\gamma\}_{k\in Z_+}$ of $L(\Lambda-\beta)$ and $L(\Lambda-\gamma)$ respectively by

$$v_k^\beta=\frac{\sqrt{2}}{\sqrt{(l-1)(l+m)}}(\sqrt{l+k-2}\,\pi(\eta_2)v_k^0-\sqrt{k-1}\,\pi(\eta_1)v_{k-1}^0),$$

$$v_k^\gamma=\frac{\sqrt{2}}{\sqrt{(l-1)(l-m-2)}}(\sqrt{k}\,\pi(\eta_2)v_{k+1}^0-\sqrt{l+k-1}\,\pi(\eta_1)v_k^0).$$

As in 3.4.1, we write the operator $\pi(\zeta)$, $\zeta\in\mathfrak{g}_1$, in the form of $(D_{j,k})_{j,k=0,\beta,\gamma,\delta}$, where $D_{j,k}: L(\Lambda-k)\to L(\Lambda-j)$. Then the blockwise matrix $(D_{j,k})$ has the same form as in (3.9) depending on $\zeta\in\mathfrak{g}_{1,+}$ or $\zeta\in\mathfrak{g}_{1,-}$. And the action of $\mathfrak{g}_{1,c}=\mathfrak{g}_{1,+}\oplus\mathfrak{g}_{1,-}$ is given as follows:

For $\mathfrak{g}_{1,+}$:

$$D_{0,\beta} \;\&\; D_{0,\gamma}:\ \pi(\xi_1)v_k^\beta=-\tilde{C}_-a_{k-1}v_{k-1}^0, \qquad \pi(\xi_1)v_k^\gamma=\tilde{C}_+\tilde{b}_kv_k^0,$$

$$D_{\beta,\delta} \;\&\; D_{\gamma,\delta}:\ \pi(\xi_1)v_k^\delta=-\tilde{C}_+\tilde{b}_{k-1}v_k^\beta-\tilde{C}_-a_{k-1}v_{k-1}^\gamma,$$

$D_{0,\beta}$ & $D_{0,\gamma}$: $\pi(\xi_2)v_k^\beta=\tilde{C}_-\tilde{b}_{k-1}v_k^0$, $\quad \pi(\xi_2)v_k^\gamma=-\tilde{C}_+a_kv_{k+1}^0$,

$D_{\beta,\delta}$ & $D_{\gamma,\delta}$: $\pi(\xi_2)v_k^\delta=\tilde{C}_+a_kv_{k+1}^\beta+\tilde{C}_-\tilde{b}_kv_k^\gamma$,

For $\mathfrak{g}_{1,-}$:

$D_{\beta,0}$ & $D_{\gamma,0}$: $\pi(\eta_1)v_k^0=\tilde{C}_-a_kv_{k+1}^\beta-\tilde{C}_+\tilde{b}_kv_k^\gamma$,

$D_{\delta,\beta}$ & $D_{\delta,\gamma}$: $\pi(\eta_1)v_k^\beta=\tilde{C}_+\tilde{b}_{k-1}v_k^\delta$, $\quad \pi(\eta_1)v_k^\gamma=\tilde{C}_-a_kv_{k+1}^\delta$,

$D_{\beta,0}$ & $D_{\gamma,0}$: $\pi(\eta_2)v_k^0=\tilde{C}_-\tilde{b}_{k-1}v_k^\beta-\tilde{C}_+a_{k-1}v_{k-1}^\gamma$,

$D_{\delta,\beta}$ & $D_{\delta,\gamma}$: $\pi(\eta_2)v_k^\beta=\tilde{C}_+a_{k-1}v_{k+1}^\delta$, $\quad \pi(\eta_2)v_k^\gamma=\tilde{C}_-\tilde{b}_kv_k^\delta$,

where

(3.11) $$a_k=\sqrt{k},\qquad \tilde{b}_k=\sqrt{l+k-1}\quad\text{and}\quad \tilde{C}_\pm=\frac{\sqrt{(l-1)\mp(m+1)}}{\sqrt{2(l-1)}}.$$

**3.5. Method of determining the orthonormal basis of $V$ in 3.4.** Both case 3.4.1 and 3.4.2 are similar, and so we discuss here the case in 3.4.1.

Step 1. First we take a unit highest weight vector $v_1^0\in L(\Lambda)\subset V_0$. Then we determine inductively one vector from each weight space so that all $\pi(Z)$, $Z\in\mathfrak{g}_0$, are expressed by hermitian matrices, namely

(3.5) $$\sqrt{k(n-k)}v_{k+1}^0=\pi(E_{2,1})v_k^0\quad\text{for } 1\leqq k\leqq n-1.$$

Then $\{v_k^0\}_{1\leqq k\leqq n}$ is an orthonormal basis of $L(\Lambda)$.

Step 2. Here we give certain orthogonal bases of $L(\Lambda-\beta)$, $L(\Lambda-\gamma)$ and $L(\Lambda-\delta)$ temporarily. Then in Step 3 we normalize them. Since $L(\Lambda-\delta)$ is isomorphic to $L(\Lambda)$ as $[\mathfrak{g}_0,\mathfrak{g}_0]$-modules, we determine

(3.12) $$\tilde{v}_k^\delta=\pi(\eta_1)\pi(\eta_2)v_k^0\quad\text{for}\quad 1\leqq k\leqq n.$$

Then $\{\tilde{v}_k^\delta\}_{1\leqq k\leqq n}$ becomes an orthogonal basis of $L(\Lambda-\delta)$ such that $|\tilde{v}_k^\delta|=$constant.

Next, as the $(\Lambda-\beta)$-weight space $(V_1)_{\Lambda-\beta}$ of $V_1=L(\Lambda-\beta)\oplus L(\Lambda-\gamma)$ is one-dimensional, we can take $\pi(\eta_2)v_1^0\in(V_1)_{\Lambda-\beta}\subset L(\Lambda-\beta)$ as a non-zero highest weight vector $\tilde{v}_1^\beta$ of $L(\Lambda-\beta)$. Then we can define an orthogonal basis $\{\tilde{v}_k^\beta\}_{1\leqq k\leqq n+1}$ of $L(\Lambda-\beta)$ similarly as $\{v_k^0\}$, namely

(3.13) $$\sqrt{k(n+1-k)}\tilde{v}_{k+1}^\beta=\pi(E_{2,1})\tilde{v}_k^\beta\quad\text{for}\quad 1\leqq k\leqq n.$$

Now the $(\Lambda-\gamma)$-weight space $(V_1)_{\Lambda-\gamma}$ of $V_1$ is two-dimensional and $\{\pi(\eta_1)v_1^0, \pi(\eta_2)v_2^0\}$ gives a basis. Simple calculation shows that

$$\pi(E_{1,2})\{\pi(\eta_2)v_2^0+\sqrt{n-1}\pi(\eta_1)v_1^0\}=0.$$

Therefore we can take $\pi(\eta_2)v_2^0+\sqrt{n-1}\pi(\eta_1)v_1^0$ as a non-zero highest weight vector $\tilde{v}_0^\gamma$ of $L(\Lambda-\gamma)$. Similarly as for $L(\Lambda-\beta)$ we can define an orthogonal basis $\{\tilde{v}_k^\gamma\}_{1\leqq k\leqq n-1}$ of $L(\Lambda-\gamma)$ inductively as follows

(3.14) $$\sqrt{k(n-1-k)}\tilde{v}_{k+1}^\gamma=\pi(E_{2,1})\tilde{v}_k^\gamma\quad\text{for}\quad 1\leqq k\leqq n-2.$$

Step 3. From Steps 1 and 2, each operator $\pi(Z)$, $Z\in\mathfrak{g}_0$, are expressed by hermitian matrices with respect to the above determined basis. So we should consider the $\mathfrak{g}_1$-action and normalize the basis in four components. According to the definition of unitarity, we have

$$ij|\tilde{v}_1^\beta|^2=\langle j\pi(\xi_2+i\eta_2)v_1^0,\ \pi(\eta_2)v_1^0\rangle=\langle v_1^0,\ j\pi(\xi_2+i\eta_2)\pi(\eta_2)v_1^0\rangle$$
$$=\bar{j}\langle v_1^0,\ \pi(H_2)v_1^0\rangle=\bar{j}\,\frac{m-n+1}{2}\,|v_1^0|^2.$$

Here

$$|\tilde{v}_1^\beta|^2=-\frac{\varepsilon(m-n+1)}{2}.$$

Therefore we must take $v_1^\beta=c_\beta\tilde{v}_1^\beta$, where $c_\beta=2/\varepsilon(m-n+1)$. Note that this determines the constant $\varepsilon$ uniquely since $-\varepsilon(m-n+1)>0$. Thus we put $v_k^\beta=c_\beta\tilde{v}_k^\beta$ for all $k$ and we get (3.7) giving $v_k^\beta$ by means of $\{v_l^0\}$.

Similary, since

$$|\tilde{v}_1^\gamma|^2=-\frac{\varepsilon n(m+n+1)}{2}.$$

we must take $v_1^\gamma=c_\gamma\tilde{v}_1^\gamma$, where $c_\gamma=2/\varepsilon n(m+n+1)$. We put $v_k^\gamma=c_\gamma\tilde{v}_k^\gamma$ for all $k$ and get (3.8).

Finally we define $v_k^\delta=c_\delta\tilde{v}_k^\delta$ with $c_\delta=\dfrac{2}{\sqrt{(m+n+1)(m-n+1)}}$, and get (3.6).

Thus we get the normalized basis of $V$ as $\{v_k^0\}\cup\{v_k^\beta\}\cup\{v_k^\gamma\}\cup\{v_k^\delta\}$.

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