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H^{∞}-well-posedness of two-sided problem for Schrödinger equation

By

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§ 0. Introduction

There are many works on the initial boundary value problem in a half-space, but there are few works on the problem in a domain limited by two parallel hyperplanes which we call two-sided problem. For it is generally supposed that if the former problem is solved, then the latter can also be solved. R. Hersh [1] shows that if an operator is "well-behaved" then the above supposition is true, while giving the operator $(\partial/\partial t)-i(\partial/\partial x)^2$, which is not "well-behaved", as a counter-example. However it may not be construed as a counter-example in the sense of H^{∞}-well-posedness that we will consider.

The operator to be discussed is:

$$P(D_t, D_x, D_y)=D_t+D_x^2+\sum_{k=1}^{d} D_{y_k}^2=\frac{1}{i}\frac{\partial}{\partial t}-\Big(\frac{\partial}{\partial x}\Big)^2-\sum_{k=1}^{d}\Big(\frac{\partial}{\partial y_k}\Big)^2$$

(Schrödinger operator)

where $D_t=\frac{1}{i}\frac{\partial}{\partial t}$, $D_x=\frac{1}{i}\frac{\partial}{\partial x}$ and $D_{y_k}=\frac{1}{i}\frac{\partial}{\partial y_k}$.

We will consider the initial boundary value problems in each of the following domains:

$\Omega(0, \infty)=\{(x, y);\ x>0,\ y\in \boldsymbol{R}^d\}$ (right half space),

$\Omega(-\infty, L)=\{(x, y);\ x<L,\ y\in \boldsymbol{R}^d\}$ (left half space)

and

$\Omega(0, L)=\{(x, y);\ 0<x<L,\ y\in \boldsymbol{R}^d\}$ (slab domain),

where we denote $y=(y_1, y_2, \cdots, y_d)$.

Problem $P(0, \infty)$ (in the right half space)

$P(D_t, D_x, D_y)u(t, x, y)=f(t, x, y)$ in $[0, T_0]\times\Omega(0, \infty)$,

$u(0, x, y)=u_0(x, y)$,

$B_1(D_x, D_y)u|_{x=0}=g_1(t, y)$,

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Problem $\boldsymbol{P}(-\infty, \boldsymbol{L})$ (in the left half space)

$$P(D_t, D_x, D_y)u(t, x, y)=f(t, x, y) \quad \text{in } [0, T_0]\times\Omega(-\infty, L),$$

$$u(0, x, y)=u_0(x, y),$$

$$B_2(D_x, D_y)u|_{x=L}=g_2(t, y),$$

Problem $\boldsymbol{P}(0, \boldsymbol{L})$

$$P(D_t, D_x, D_y)u(t, x, y)=f(t, x, y) \quad \text{in } [0, T_0]\times\Omega(0, L),$$

$$u(0, x, y)=u_0(x, y),$$

$$B_1(D_x, D_y)u|_{x=0}=g_1(t, y),$$

$$B_2(D_x, D_y)u|_{x=L}=g_2(t, y),$$

where

$$B_1=1 \quad \text{or} \quad B_1=D_x+a\cdot D_y$$

and

$$B_2=1 \quad \text{or} \quad B_2=-D_x+b\cdot D_y .$$

Here the vectors $a=(a_1, a_2, \cdots, a_d)$ and $b=(b_1, b_2, \cdots, b_d)$ are complex constant vectors and $a\cdot D_y$ denotes $\sum_{k=1}^{d} a_k D_{y_k}$.

The boundary operators B_1 at $x=0$ in $\boldsymbol{P}(0, L)$ and B_2 at $x=L$ in $\boldsymbol{P}(0, L)$ are identical to the boundary operators B_1 in $\boldsymbol{P}(0, \infty)$ and B_2 in $\boldsymbol{P}(-\infty, L)$, respectively.

The present paper attempts to investigate the relation between H^∞-well-posedness of $\boldsymbol{P}(0, \infty)$ and $\boldsymbol{P}(-\infty, \boldsymbol{L})$ and that of $\boldsymbol{P}(0, \boldsymbol{L})$.

In §1 we will mention our results. Theorems 1 and 2 give the necessary and sufficient conditions for H^∞-well-posedness of the problems in the half-spaces and for H^∞-well-posedness of the two-sided problems respectively. Our conclusion is that the two-sided problem $\boldsymbol{P}(0, L)$ is not always H^∞-well-posed even if each of the corresponding problems $\boldsymbol{P}(0, \infty)$ and $\boldsymbol{P}(-\infty, L)$ in the half spaces is H^∞-well-posed. In §2 we will provide preliminary arguments to prove the theorems by making use of Fourier-Laplace transform. In §3 and §4 we will prove the theorems and in §5 and §6 we will prove lemmata used in §3 and §4.

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§1. Definitions and Theorems

We will first define some terminologies and notations.

Definition 1 (Compatibility condition). We assume that $f\in H^\infty([0, T_0]\times\Omega(0, L))$, $u_0\in H^\infty(\Omega(0, L))$, $g_j\in H^\infty([0, T_0]\times\boldsymbol{R}^d)$ $(j=1, 2)$. Then a data $\{f, u_0, g_1, g_2\}$ is said to satisfy the compatibility condition for Problem $\boldsymbol{P}(0, L)$, if the following two conditions (1.1) and (1.2) are met:

(1.1) $$B_1 u_j|_{x=0}=D_t^j g_1|_{t=0} \quad (j=0, 1, 2, 3, \cdots)$$

and

(1.2) $$B_2 u_j|_{x=L}=D_t^j g_2|_{t=0} \quad (j=0, 1, 2, 3, \cdots)$$

where

$$u_j(x, y)\equiv-\left\{D_x^2+\sum_{k=1}^{d} D_{y_k}^2\right\}u_{j-1}(x, y)+D_t^{j-1}f|_{t=0} \quad (j=1, 2, 3, \cdots)$$

Definition 2 (H^∞-well-posedness). We say that the problem $\boldsymbol{P}(0, L)$ is H^∞-well-posed, if there exists a unique solution of $\boldsymbol{P}(0, L)$ for every such data $\{f, u_0, g_1, g_2\}$ ($f\in H^\infty([0, T_0]\times\Omega(0, L))$, $u_0\in H^\infty(\Omega(0, L))$, g_1 and $g_2\in H^\infty([0, T_0]\times\boldsymbol{R}^d)$) that satisfies the compatibility condition.

The compatibility conditions and H^∞-well-posedness for the problems $\boldsymbol{P}(0, \infty)$ and $\boldsymbol{P}(-\infty, L)$ can be defined in similar fashions. We will try to find conditions on B_1 and B_2 under which the problem $\boldsymbol{P}(0, L)$ is H^∞-well-posed. In order to do this we will classify the boundary operators B_1 and B_2 into the following three groups:

type I: $B_1=1$ or $B_1=D_x+a\cdot D_y$ $(\mathrm{Re}\, a=0)$

$B_2=1$ or $B_2=-D_x+b\cdot D_y$ $(\mathrm{Re}\, b=0)$

type II: $B_1=D_x+a\cdot D_y$ ($\mathrm{Re}\, a\neq 0$ and $\mathrm{Im}\,\{(a\cdot\eta)^2\}\leqq 0$ for every $\eta\in\boldsymbol{R}^d$)

$B_2=-D_x+b\cdot D_y$ ($\mathrm{Re}\, b\neq 0$ and $\mathrm{Im}\,\{(b\cdot\eta))^2\}\leqq 0$ for every $\eta\in\boldsymbol{R}^d$)

type III: $B_1=D_x+a\cdot D_y$ ($\mathrm{Im}\,\{(a\cdot\hat{\eta})^2\}>0$ for some $\hat{\eta}\in\boldsymbol{R}^d$)

$B_2=-D_x+b\cdot D_y$ ($\mathrm{Im}\,\{(b\cdot\hat{\eta})^2\}>0$ for some $\hat{\eta}\in\boldsymbol{R}^d$).

Remark 1. The following conditions (i) and (ii) are equivalent.

(i) B_1 is of type II.

(ii) There exists a non-negative number λ such that

$$\mathrm{Re}\, a\neq 0 \quad \text{and} \quad \mathrm{Im}\, a=-\lambda\,\mathrm{Re}\, a.$$

Now for the problems in half-spaces we have

Theorem 1. *$\boldsymbol{P}(0, \infty)$ is H^∞-well-posed if and only if B_1 is of type I or of type II. Similarly $\boldsymbol{P}(-\infty, L)$ is H^∞-well-posed if and only if B_2 is of type I or of type II.*

Our main theorem is as follows:

Theorem 2. *The problem $\boldsymbol{P}(0, L)$ is H^∞-well-posed if and only if one of the following two conditions is satisfied:*

(i) *both B_1 and B_2 are of type I.*

(ii) *both B_1 and B_2 are of type II and $a+b=0$.*

In §3 we will prove that the problems are H^∞-well-posed and we will deal with unsolvable cases in § 4.

Remark 2. From Theorem 1 and Theorem 2 it follows that if B_1 is of type I and B_2 is of type II, then $\boldsymbol{P}(0, \infty)$ and $\boldsymbol{P}(-\infty, L)$ are H^∞-well-posed while $\boldsymbol{P}(0, L)$ is not.

§ 2. Lopatinski determinant $D(\tau, \eta)$

In this section we will provide preliminary arguments to prove the theorems mentioned in the previous section.

We consider the following boundary value problem $\hat{\boldsymbol{P}}(0, L)$ for the ordinary differential equation with parameters $\tau=\sigma-i\gamma$ ($\sigma\in\boldsymbol{R}$ and $\gamma>0$) and $\eta\in\boldsymbol{R}^d$:

Problem $\hat{\boldsymbol{P}}(0, L)$

$$(D_x^2+\tau+|\eta|^2)\hat{u}(x, \tau, \eta)=0\,, \tag{2.1}$$

$$B_1(D_x, \eta)\hat{u}|_{x=0}=\hat{g}_1(\tau, \eta)\,, \tag{2.2}$$

$$B_2(D_x, \eta)\hat{u}|_{x=L}=\hat{g}_2(\tau, \eta)\,. \tag{2.3}$$

We get $\hat{\boldsymbol{P}}(0, L)$ from $\boldsymbol{P}(0, L)$ by extending u, f, g_1 and g_2 to $t>T_0$ and $t<0$ and making use of Fourier-Laplace transform. We do not discuss here in detail how they are extended. $\hat{\boldsymbol{P}}(0, L)$ is obtained by putting $f=0$ for simplicity and deleting the initial condition.

Given a function $v(t, y)$, we denote Fourier-Laplace transform of $v(t, y)$ by

$$\hat{v}(\tau, \eta)=\iint e^{-i(\tau t+\eta\cdot y)}v(t, y)dtdy$$

$$=\iint e^{-i(\sigma t+\eta\cdot y)}\{e^{-\gamma t}v(t, y)\}dtdy\,.$$

The general solution of (2.1) is written as

$$\hat{u}(x, \tau, \eta)=C_1e^{i\xi(\tau,\eta)x}+C_2e^{-i\xi(\tau,\eta)(x-L)}$$

where $\xi(\tau, \eta)=\sqrt{-\tau-|\eta|^2}$ ($\mathrm{Im}\sqrt{-\tau-|\eta|^2}>0$).

From (2.2) and (2.3), C_1 and C_2 are independent of x and are the solutions of

$$\begin{cases} B_1(\xi(\tau, \eta), \eta)C_1+B_1(-\xi(\tau, \eta), \eta)e^{i\xi(\tau,\eta)L}C_2=\hat{g}_1 \\ B_2(\xi(\tau, \eta), \eta)e^{i\xi(\tau,\eta)L}C_1+B_2(-\xi(\tau, \eta), \eta)C_2=\hat{g}_2\,. \end{cases}$$

We set

$$\boldsymbol{B}(\tau, \eta)=\begin{bmatrix} B_1(\xi(\tau, \eta), \eta) & B_1(-\xi(\tau, \eta), \eta)e^{i\xi(\tau,\eta)L} \\ B_2(\xi(\tau, \eta), \eta)e^{i\xi(\tau,\eta)L} & B_2(-\xi(\tau, \eta), \eta) \end{bmatrix}$$

and

$$D(\tau, \eta)=\det\boldsymbol{B}(\tau, \eta)\quad(\tau=\sigma-i\gamma,\ \sigma\in\boldsymbol{R},\ \gamma>0 \text{ and } \eta\in\boldsymbol{R}^d)\,.$$

$D(\tau, \eta)$ is called Lopatinski determinant for the two-sided problem $\boldsymbol{P}(0, L)$. It is a function of (τ, η), where τ and η run over the lower half-plane and $\boldsymbol{R}^d$, respectively.

Incidentally, $D(\tau, \eta)$ plays an important role in proving Theorem 2.

§ 3. Well-posed cases

Our results for solvable cases in Theorem 1 and Theorem 2 are as follows:

Proposition 3.1. *If B_1 is of type I or of type II, then $\boldsymbol{P}(0, \infty)$ is H^∞-well-posed. (Similarly if B_2 is of type I or of type II, then $\boldsymbol{P}(-\infty, L)$ is H^∞-well-posed.)*

Proposition 3.2. *If B_1 and B_2 satisfy* (i) *or* (ii) *in Theorem* 2, *then $\boldsymbol{P}(0, L)$ is H^∞-well-posed.*

For the proof of the above two propositions, we need the following two lemmata:

Lemma 3.1. *If B_1 (B_2) is of type I or of type II, then there exists a positive number c such that*

$$|B_1(\xi(\tau, \eta), \eta)| \geqq \frac{c\gamma}{(|\tau|+|\eta|^2)^{1/2}}$$

$$\left(\textit{respectively, } |B_2(-\xi(\tau, \eta), \eta)| \geqq \frac{c\gamma}{(|\tau|+|\eta|^2)^{1/2}}\right)$$

for any $\tau=\sigma-i\gamma$ with $\gamma>0$ and any $\eta\in\boldsymbol{R}^d$.

Lemma A. *If B_1 and B_2 satisfy* (i) *or* (ii) *in Theorem* 2, *then there exist positive numbers γ_0 and c such that*

$$|D(\tau, \eta)| \geqq \frac{c\gamma^{5/2}}{(|\tau|+|\eta|^2)^{3/2}} \quad \textit{for any } \tau=\sigma-i\gamma \textit{ with } \gamma\geqq\gamma_0 \textit{ and any } \eta\in\boldsymbol{R}^d.$$

Here we will not prove Lemma 3.1, because it is easily verified. Lemma A will be proved in §5.

Since Proposition 3.1 can be proved in the same way as proposition 3.2, we will prove only Proposition 3.2 here.

Proof of Proposition 3.2.

Existence of solutions. We referred the present proof to R. Sakamoto [3].

Given a data $\{f, u_0, g_1, g_2\}$ satisfying the compatibility conditions, we can find a function $w_1(t, x, y)\in H^\infty([0, T_0]\times\Omega(0, L))$ such that

$$D_t^j w_1|_{t=0}=u_j(x, y) \quad (j=0, 1, 2, 3, \cdots), \tag{3.1}$$

where $u_j(x, y)$ are the functions in Definition 1. Let us put

$$v_1(t, x, y)=u(t, x, y)-w_1(t, x, y),$$

where u is the solution of $\boldsymbol{P}(0, L)$ which is to be determined. Then

$$Pv_1=f_1=f-Pw_1,$$

$$v_1(0, x, y)=0,$$

$$B_1 v_1|_{x=0} = g_1 - B_1 w_1|_{x=0},$$

$$B_2 v_1|_{x=L} = g_2 - B_2 w_1|_{x=L}.$$

From the definitions of $u_j(x, y)$ and (3.1) we obtain

(3.2) $$D_t^k f_1|_{t=0} = 0 \quad (k=0, 1, 2, 3, \cdots).$$

By virtue of the compatibility condition for $\boldsymbol{P}(0, L)$, it is easily verified that

$$D_t^k(g_1 - B_1 w_1|_{x=0})|_{t=0} = 0 \quad \text{and} \quad D_t^k(g_2 - B_2 w_1|_{x=L})|_{t=0} = 0 \qquad \text{for } k=0, 1, 2, 3, \cdots.$$

Secondly we extend f_1 to $x<0$ and $x>L$ in such a way that f_1 belongs to $H^\infty([0, T_0]\times\boldsymbol{R}^{d+1})$ and satisfies (3.2) in $\boldsymbol{R}^{d+1}$. Then we can find the unique solution $w_2(t, x, y) \in H^\infty([0, T_0]\times\boldsymbol{R}^{d+1})$ of the Cauchy problem

$$\begin{cases} Pw_2 = f_1 & \text{in } [0, T_0]\times\boldsymbol{R}^{d+1}, \\ w_2(0, x, y) = 0 & \text{on } \boldsymbol{R}^{d+1}, \end{cases}$$

by means of Fourier transform with respect to (x, y).

Let us put

$$v(t, x, y) = v_1(t, x, y) - w_2(t, x, y),$$

then

(3.3) $$Pv = 0$$

(3.4) $$v(0, x, y) = 0$$

(3.5) $$B_1 v|_{x=0} = h_1(t, y) = g_1 - B_1 w_1|_{x=0} - B_1 w_2|_{x=0},$$

(3.6) $$B_2 v|_{x=L} = h_2(t, y) = g_2 - B_2 w_1|_{x=L} - B_2 w_2|_{x=L}.$$

We can easily verify that

$$D_t^k h_j|_{t=0} = 0 \qquad \text{for } j=1, 2 \text{ and } k=0, 1, 2, 3, \cdots.$$

Thirdly let us denote by $h_j(t, y)$ the extensions of $h_j(t, y)$ as elements of $H^\infty(\boldsymbol{R}\times\boldsymbol{R}^d)$ with supports in $[0, 2T_0]\times\boldsymbol{R}^d$. Let $\hat{h}_j(\tau, \eta)$ be Fourier transforms of $e^{-\gamma t}h_j(t, y)$.

We are going to find the solutions of

(3.7) $$(D_x^2 + \tau + |\eta|^2)\hat{v} = 0,$$

(3.8) $$B_1(D_x, \eta)\hat{v}|_{x=0} = \hat{h}_1,$$

(3.9) $$B_2(D_x, \eta)\hat{v}|_{x=L} = \hat{h}_2.$$

$\hat{v}$, if exists any, shall be given by

$$\hat{v}(\tau, x, \eta) = C_1(\tau, \eta)e^{i\xi(\tau, \eta)x} + C_2(\tau, \eta)e^{-i\xi(\tau, \eta)(x-L)},$$

where $C_j(\tau, \eta)$ satisfy

(3.10) $$\boldsymbol{B}(\tau, \eta)\begin{bmatrix} C_1(\tau, \eta) \\ C_2(\tau, \eta) \end{bmatrix} = \begin{bmatrix} \hat{h}_1(\tau, \eta) \\ \hat{h}_2(\tau, \eta) \end{bmatrix}.$$

Lemma A guarantees the existence of $C_j(\tau, \eta)$. They satisfy

$$\|(|\tau|+|\eta|^2)^N C_k(\tau, \eta)\|_{L^2_{\sigma,\eta}} \leqq M_{N,k} \tag{3.11}$$

for any positive integer N and $k=1, 2$, where $M_{N,k}$ are positive constant numbers independent of $\gamma \geqq \gamma_0$.

Let us put

$$v(t, x, y)=e^{\gamma t}\iint e^{i(\sigma t+\eta y)}\hat{v}(\tau, \eta)d\sigma d\eta, \tag{3.12}$$

then $v(t, x, y)$ belongs to $H^\infty((-\infty, T_0]\times\Omega(0, L))$ and satisfies

$$\begin{cases} Pv=0 \quad \text{in } (-\infty, T_0]\times\Omega(0, L), \\ B_1 v|_{x=0}=h_1, \\ B_2 v|_{x=L}=h_2. \end{cases}$$

Moreover, from (3.11) and (3.12) we get

$$\operatorname*{supp}_t v\subset\{t; t\geqq 0\},$$

therefore

$$v|_{t=0}=0.$$

Accordingly $v(t, x, y)$ is the solution of (3.3), (3.4), (3.5) and (3.6). Finnaly, $u=v+w_1+w_2$ solves the problem $\boldsymbol{P}(0, L)$.

Uniqueness of solutions. Now let us assume that $u(t, x, y)\in H^\infty([0, T_0]\times\Omega(0, L))$ satisfy

$$Pu=0 \quad \text{in } [0, T_0]\times\Omega(0, L), \tag{3.13}$$

$$u(0, x, y)=0, \tag{3.14}$$

$$B_1 u|_{x=0}=0, \tag{3.15}$$

$$B_2 u|_{x=L}=0. \tag{3.16}$$

For any given function $f(t, x, y)\in C_0^\infty((0, T_0)\times\Omega(0, L))$, we consider the adjoint problem (backward problem):

$$Pv=f(t, x, y) \quad \text{in } [0, T_0]\times\Omega(0, L), \tag{3.17}$$

$$v(T_0, x, y)=0, \tag{3.18}$$

$$(D_x-\bar{a}\cdot D_y)v|_{x=0}=0, \tag{3.19}$$

$$(D_x+\bar{b}\cdot D_y)v|_{x=L}=0. \tag{3.20}$$

We can find a solution of the adjoint problem in the same way as we find a solution of $\boldsymbol{P}(0, L)$. Then from (3.13)~(3.16) and (3.17)~(3.20) we obtain

$$(u, f)_{L^2_{t,x,y}}=(u, Pv)_{L^2_{t,x,y}}=(Pu, v)_{L^2_{t,x,y}}=0.$$

Therefore $u=0$. Thus we have shown the uniqueness of solutions of the problem $\boldsymbol{P}(0, L)$.

This completes the proof of Proposition 3.2.

§ 4. Unsoluvable cases

Our results for unsolvable cases are divided into the following three propositions:

Proposition 4.1. *If $B_1(B_2)$ is of type III, then $\boldsymbol{P}(0, \infty)$ (resp. $\boldsymbol{P}(-\infty, L)$) is not H^∞-well-posed.*

Proposition 4.2. *If B_1 or B_2 is of type III, then the problem $\boldsymbol{P}(0, L)$ is not H^∞--well-posed.*

Proposition 4.3. *If one of B_1 and B_2 is of type I and the other of type II, then the problem $\boldsymbol{P}(0, L)$ is not H^∞-well-posed.*

Proposition 4.4. *If both B_1 and B_2 are of type II and $a+b\neq 0$, then the problem $\boldsymbol{P}(0, L)$ is not H^∞-well-posed.*

We will present the lemma of the same kind as is often used to show that certain conditions are necessary for well-posedness.

Lemma 4.1. *If the problem $\boldsymbol{P}(0, L)$ is H^∞-well-posed, then there exists a positive integer m and a positive number c such that*

$$(4.1)\qquad \|u(t, x, y)\|_{H^2_{t,x,y}} \leqq c\{\|Pu\|_{H^m_{t,x,y}} + \|u(0, x, y)\|_{H^m_{x,y}} + \|B_1 u|_{x=0}\|_{H^m_{t,y}} + \|B_2 u|_{x=L}\|_{H^m_{t,y}}\}$$

$$\text{for every } u(t, x, y) \in H^\infty([0, T_0]\times\Omega(0, L)).$$

This lemma is a simple consequence of Banach's closed graph theorem. (See Mizohata [2].)

As Proposition 4.1 can be proved in the same way as Proposition 4.2, we will not present the proof of Proposition 4.1.

Proof of Proposition 4.2. Assume that B_1 is of type III. Then there exists a unit vector $\hat{\eta}$ such that $\mathrm{Im}\{(a\cdot\hat{\eta})^2\}>0$ and $\mathrm{Im}(a\cdot\hat{\eta})<0$. (If necessary, $\hat{\eta}$ is replaced by $-\hat{\eta}$.)

Now let $\alpha(\eta)$ be the function such that $\alpha(\eta)\in C_0^\infty(\boldsymbol{R}^d)$, $\alpha(\eta)>0$ in $\{\eta;|\eta|<1\}$, $\mathrm{supp}\{\alpha(\eta)\}=\{\eta;|\eta|\leqq 1\}$ and $\|\alpha(\eta)\|_{L^2_\eta}=1$. Put $\alpha_\rho(\eta)=\rho^{d/2}\alpha(\rho(\eta-\rho\hat{\eta}))$, then we have $\mathrm{supp}\{\alpha_\rho\}=\{\eta;|\eta-\rho\hat{\eta}|\leqq 1/\rho\}$ and $\|\alpha_\rho(\eta)\|_{L^2_\eta}=1$ where ρ is a large parameter.

We set

$$(4.2)\qquad u_\rho(t, x, y)=\int_{R^d} e^{i(-[(a\cdot\eta)^2+|\eta|^2]t-(a\cdot\eta)x+\eta\cdot y)}\alpha_\rho(\eta)d\eta\,.$$

It is easily verified that

$$(4.3)\qquad B_1 u_\rho|_{x=0}=0 \quad\text{and}\quad Pu_\rho=0\,.$$

Set $\mathrm{Im}\{(a\cdot\hat{\eta})^2\}=c_1$ $(c_1>0)$ and $\mathrm{Im}(a\cdot\hat{\eta})=-c_2$ $(c_2>0)$. For $\eta\in\mathrm{supp}\{\alpha_\rho\}$

$$\text{(4.4)}\qquad \operatorname{Im}\{(a\cdot\eta)^2\}=\operatorname{Im}\{(a\cdot\hat{\eta})^2\}\rho^2+2\operatorname{Im}\{(a\cdot\hat{\eta})(a\cdot(\eta-\rho\hat{\eta}))\}\rho+\operatorname{Im}\{(a\cdot(\eta-\rho\hat{\eta}))^2\}$$
$$=c_1\rho^2+O(1)\quad (\text{as } \rho\to\infty),$$

and

$$\text{(4.5)}\qquad \operatorname{Im}(a\cdot\eta)=\operatorname{Im}(a\cdot\hat{\eta})\rho+\operatorname{Im}\{a\cdot(\eta-\rho\hat{\eta})\}$$
$$=-c_2\rho+O\left(\frac{1}{\rho}\right)\quad (\text{as } \rho\to\infty),$$

From now on we denote positive constant numbers by c_j $(j=1, 2, 3, \cdots)$. We have at first

$$\|u_\rho(t, x, y)\|_{H^2_{t,x,y}}\geqq c_3\|u_\rho(T_0, 0, y)\|_{L^2_y}\quad (\text{Sobolev's inequality})$$
$$=c_4\|e^{\operatorname{Im}\{(a\cdot\eta)^2\}T_0}\alpha_\rho(\eta)\|_{L^2_\eta}\quad (\text{Parseval's formula})$$
$$\geqq c_5 e^{(c_1T_0)\rho^2}\quad (\text{by (4.4)}).$$

Secondly it is not difficult to justify

$$\text{(4.7)}\qquad \|u_\rho(0, x, y)\|_{H^m_{x,y}}\leqq c_6(\rho+1)^m$$

Thirdly (4.4) and (4.5) yield

$$\text{(4.8)}\qquad \|B_2u_\rho|_{x=L}\|_{H^m_{t,y}}\leqq c_7(\rho+1)^{2m+1}e^{(c_1T_0)\rho^2-(c_2L)\rho}.$$

From (4.3), (4.6), (4.7), (4.8) and Lemma 4.1, if the problem $\boldsymbol{P}(0, L)$ were H^∞-well-posed, then the inequality

$$e^{(c_1T_0)\rho^2}\leqq c_8\{(\rho+1)^m+(\rho+1)^{2m+1}e^{(c_1T_0)\rho^2-(c_2L)\rho}\}$$

with a certain constant c_8 independent of ρ should hold. However, as ρ tends to the infinity, it cannot hold. Therefore the problem $\boldsymbol{P}(0, L)$ is not H^∞-well-posed. We can also prove analogous results when B_2 is of type III.

Proof of Proposition 4.3. *and Proposition* 4.4. We can assume that B_1 is of type II and B_2 is of type I or of type II without losing generality. When B_2 is of type II, we assume that $a+b\neq 0$. Then we have

Lemma B. *There exist positive numbers c_1 and c_2, a sequence $\{\eta_n\}_{n=1}^\infty\subset\boldsymbol{R}^d$, a sequence $\{U_n\}_{n=1}^\infty$ of open neighborhoods of η_n and a sequence $\{\tau_n(\eta)\}_{n=1}^\infty$ of complex valued functions defined in U_n such that*

$$\text{(4.9)}\qquad |\eta_n|\longrightarrow\infty\quad \textit{as } n\to\infty,$$

$$\text{(4.10)}\qquad D(\tau_n(\eta), \eta)=0\quad \textit{in } U_n,$$

$$\text{(4.11)}\qquad c_1|\eta_n|\leqq -\operatorname{Im}\{\tau_n(\eta)\}=\gamma_n(\eta)\quad \textit{in } U_n,$$

$$\text{(4.12)}\qquad |\xi(\tau_n(\eta), \eta)|\leqq c_2|\eta_n|\quad \textit{in } U_n,$$

$$\text{(4.13)}\qquad \left|\frac{B_1(\xi(\tau_n(\eta), \eta), \eta)}{B_1(-\xi(\tau_n(\eta), \eta), \eta)}\cdot\frac{B_2(-\xi(\tau_n(\eta), \eta), \eta)}{B_2(\xi(\tau_n(\eta), \eta), \eta)}\right|\leqq 1-2c_1\quad \textit{in } U_n.$$

This lemma will be proved in § 6.

From (4.13) we can assume that

$$\left|\frac{B_1(\xi(\tau_n(\eta)),\ \eta),\ \eta)}{B_1(-\xi(\tau_n(\eta)),\ \eta),\ \eta)}\right| \leqq 1-c_1 \quad \text{in } U_n \quad \text{for all } n\,, \tag{4.14}$$

or

$$\left|\frac{B_2(-\xi(\tau_n(\eta)),\ \eta),\ \eta)}{B_2(\xi(\tau_n(\eta)),\ \eta),\ \eta)}\right| \leqq 1-c_1 \quad \text{in } U_n \quad \text{for all } n\,, \tag{4.15}$$

if necessary we take subsequences.

Now we assume (4.14). Let $C_{1,n}(\eta)$ be the function defined in U_n such that $\mathrm{supp}\{C_{1,n}(\eta)\}\subset U_n$ and $\|C_{1,n}(\eta)\|_{L^2_\eta}=1$. And set

$$C_{2,n}(\eta) = -\frac{B_2(\xi(\tau_n(\eta)),\ \eta),\ \eta)}{B_2(-\xi(\tau_n(\eta)),\ \eta),\ \eta)} e^{i\xi(\tau_n(\eta),\ \eta)L} C_{1,n}(\eta)$$

then from (4.10) we obtain

$$B(\tau_n(\eta),\ \eta)\begin{bmatrix} C_{1,n}(\eta) \\ C_{2,n}(\eta) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.16}$$

We put

$$u_n(t,\ x,\ y) = \int e^{i(\tau_n(\eta)t+\eta\cdot y)}\{C_{1,n}(\eta)e^{i\xi(\tau_n(\eta),\ \eta)x} + C_{2,n}(\eta)e^{-i\xi(\tau_n(\eta),\ \eta)(x-L)}\}d\eta\,. \tag{4.17}$$

then u_n is a solution of $Pu_n=0$, $B_1 u_n|_{x=0}=0$ and $B_2 u_n|_{x=L}=0$.

Let us show that the inequality (4.1) does not hold for u_n as n tends to the infinity.

From (4.10) we get

$$\begin{aligned} u_n(t_0,\ 0,\ y) &= \int e^{i(\tau_n(\eta)t_0+\eta\cdot y)}\{C_{1,n}(\eta) + C_{2,n}(\eta)e^{i\xi(\tau_n(\eta),\ \eta)L}\}d\eta \\ &= \int e^{i(\tau_n(\eta)t_0+\eta\cdot y)}\left\{1 - \frac{B_2(\xi(\tau_n(\eta)),\ \eta),\ \eta)}{B_2(-\xi(\tau_n(\eta)),\ \eta),\ \eta)} e^{2i\xi(\tau_n(\eta),\ \eta)L}\right\}C_{1,n}(\eta)d\eta \\ &= \int e^{i(\tau_n(\eta)t_0+\eta\cdot y)}\left\{1 - \frac{B_1(\xi(\tau_n(\eta)),\ \eta),\ \eta)}{B_1(-\xi(\tau_n(\eta)),\ \eta),\ \eta)}\right\}C_{1,n}(\eta)d\eta\,, \end{aligned}$$

where $0<t_0\leqq T_0$.

(4.18) can be obtained from (4.11) and (4.14).

$$\begin{aligned} \|u_n(t_0,\ 0,\ y)\|_{L^2_y} &= \frac{1}{(2\pi)^d}\left\| e^{\tau_n(\eta)t_0}\left\{1 - \frac{B_1(\xi(\tau_n(\eta)),\ \eta,\ \eta)}{B_1(-\xi(\tau_n(\eta)),\ \eta,\ \eta)}\right\}C_{1,n}(\eta)\right\|_{L^2_\eta} \\ &\geqq \frac{c_1}{(2\pi)^d} e^{c_1 t_0 |\eta_n|}\,. \end{aligned} \tag{4.18}$$

Moreover, from (4.12) we can derive

$$\|u_n(0,\ x,\ y)\|_{H^m_{x,y}} \leqq c_3(|\eta_n|^2+1)^{m/2}\,. \tag{4.19}$$

If the problem $\boldsymbol{P}(\mathbf{0},\ \boldsymbol{L})$ were H^∞-well-posed, we could have (4.20) from Sobolev's lemma and Proposition 4.1

(4.20) $$\|u_n(t_0, 0, y)\|_{L^2_y} \leqq c_4 \|u_n(0, x, y)\|_{H^m_{x,y}}.$$

However, (4.20) cannot hold for any fixed positive number t_0 due to (4.9), (4.18) and (4.19). An analogous results can be obtained when (4.15) is assumed. Thus Proposition 4.3 and Proposition 4.4 are proved.

§ 5. Proof of Lemma A

In this section we will prove Lemma A used in § 3.

Lemma 5.1. *Let c be a positive number and a be a purely imaginary vector. Then there exists a large positive number* $\gamma_0 = \gamma_0(c, a)$ *such that*

$$\left| \frac{\xi(\tau, \eta) - a\cdot\eta}{\xi(\tau, \eta) + a\cdot\eta} e^{ic\xi(\tau,\eta)} \right| \leqq 1$$

for every $\tau = \sigma - i\gamma$ $(\gamma \geqq \gamma_0)$ *and every* $\eta \in \boldsymbol{R}^d$, *where* $\xi(\tau, \eta) = \sqrt{-\tau - |\eta|^2}$ *and* $\operatorname{Im} \xi(\tau, \eta) > 0$.

Proof. Let us put $s = ia\cdot\eta$, $q = \operatorname{Im} \xi(\tau, \eta)$ and $K = \left| \frac{\xi(\tau, \eta) - a\cdot\eta}{\xi(\tau, \eta) + a\cdot\eta} e^{ic\xi(\tau,\eta)} \right|^2$. From the equality $2\operatorname{Re}\{\xi(\tau, \eta)\}\operatorname{Im}\{\xi(\tau, \eta)\} = \gamma$, we get

$$\xi(\tau, \eta) = \frac{\gamma}{2q} + iq \quad (q > 0).$$

We also get

$$K = \frac{\left(\frac{\gamma}{2q}\right)^2 + (q+s)^2}{\left(\frac{\gamma}{2q}\right)^2 + (q-s)^2} e^{-2cq}$$

$$= \frac{4q^2(s+q)^2 + \gamma^2}{4q^2(s-q)^2 + \gamma^2} e^{-2cq} = K(q, \gamma, s),$$

leading to

$$K(q, \gamma, s) \leqq 1 \qquad \text{for } s \leqq 0$$

and

$$\frac{\partial K}{\partial s} = \frac{-16q^3(4q^2 s^2 - 4q^4 - \gamma^2)}{\{4q^2(s-q)^2 + \gamma^2\}^2} e^{-2cq}.$$

Then as a function of $s \geqq 0$, K attains maximum at $s = \sqrt{q^2 + \frac{\gamma^2}{4q^2}}$. The maximum $K_1(q, \gamma)$ is as follows:

$$K_1(q, \gamma) = K\left(q, \gamma, \sqrt{q^2 + \frac{\gamma^2}{4q^2}}\right)$$

$$= \frac{\{\sqrt{4q^4 + \gamma^2} + 2q^2\}^2 + \gamma^2}{\{\sqrt{4q^4 + \gamma^2} - 2q^2\}^2 + \gamma^2} e^{-2cq}$$

$$= \gamma^{-2}\{\sqrt{4q^4 + \gamma^2} + 2q^2\}^2 e^{-2cq}.$$

Putting $q = \sqrt{\frac{\gamma}{2}}\, p$, then

$$K_2(p,\gamma)\equiv K_1(q,\gamma)=\{\sqrt{p^4+1}+p^2\}^2 e^{-\sqrt{2\gamma}\,cp},$$

and

$$K(q,\gamma,s)\leqq K_2(p,\gamma)\leqq(2p^2+1)^2 e^{-\sqrt{2\gamma}\,cp}.$$

We then have

$$K(q,\gamma,s)\leqq 1 \qquad \text{for } \gamma\geqq 4c^{-2} \text{ and any real number } s.$$

Lemma 5.1 is proved.

Lemma 5.2. *Let c be a positive number. Then there exists a positive number c_1 such that*

$$|e^{ic\xi(\tau,\eta)}|\leqq 1-\frac{c_1\gamma^{1/2}}{(|\tau|+|\eta|^2)^{1/2}}$$

for every $\tau=\sigma-i\gamma$ ($\gamma\geqq 1$) and every $\eta\in\boldsymbol{R}^d$.

Proof. Note that $|e^{ic\xi(\tau,\eta)}|=e^{-c\,\mathrm{Im}\,\xi(\tau,\eta)}$. The following inequalities hold:

$$1-e^{-x}\geqq x/2 \qquad \text{for } 0\leqq x\leqq \log 2$$

and

$$1-e^{-x}\geqq 1/2 \qquad \text{for } x\geqq \log 2,$$

Therefore, when $0\leqq \mathrm{Im}\,\xi(\tau,\eta)\leqq(1/c)\log 2$,

$$1-|e^{ic\xi(\tau,\eta)}|\geqq(c/2)\,\mathrm{Im}\,\xi(\tau,\eta)$$

and when $\mathrm{Im}\,\xi(\tau,\eta)\geqq(1/c)\log 2$,

$$1-|e^{ic\xi(\tau,\eta)}|\geqq\frac{1}{2}\geqq\frac{\gamma^{1/2}}{2(|\tau|+|\eta|^2)^{1/2}}.$$

We have only to show that $\mathrm{Im}\,\xi(\tau,\eta)\geqq\dfrac{\gamma}{2(|\tau|+|\eta|^2)^{1/2}}$. By the definition of $\xi(\tau,\eta)$ we obtain

$$\mathrm{Im}\,\xi(\tau,\eta)=\sqrt{\frac{\gamma^2}{2\{\sqrt{(\sigma+|\eta|^2)^2+\gamma^2}-(\sigma+|\eta|^2)\}}}$$

$$\geqq\frac{\gamma}{2}\sqrt{\frac{1}{|\sigma|+|\eta|^2+\gamma}}$$

$$\geqq\frac{\gamma}{2(|\sigma|+\gamma+|\eta|^2)^{1/2}}.$$

Lemma 5.2 is proved.

Proof of Lemma A. Let us assume the case (i) in Theorem 2. Consider the subcase where $B_1=D_x+a\cdot D_y$ and $B_2=-D_x+b\cdot D_y$ ($\mathrm{Re}\,a=\mathrm{Re}\,b=0$). Then

$$(5.1)\qquad |D(\tau,\eta)|=|B_1(\xi,\eta)B_2(-\xi,\eta)-B_1(-\xi,\eta)B_2(\xi,\eta)e^{2i\xi L}|$$

$$=|B_1(\xi,\eta)B_2(-\xi,\eta)|\cdot\left|1-\frac{B_1(-\xi,\eta)B_2(\xi,\eta)}{B_1(\xi,\eta)B_2(-\xi,\eta)}e^{2iL\xi}\right|.$$

Because neither B_1 nor B_2 is of type III, we have (5.2) from Lemma 3.1 in § 3.

$$(5.2)\qquad |B_1(\xi, \eta)B_2(-\xi, \eta)| \geqq \frac{c_1\gamma^2}{(|\tau|+|\eta|^2)}$$

On the other hand, Lemma 5.1 and Lemma 5.2 derive

$$(5.3)\qquad \left|1-\frac{B_1(-\xi, \eta)B_2(\xi, \eta)}{B_1(\xi, \eta)B_2(-\xi, \eta)}e^{2iL\xi}\right|$$

$$\geqq 1-\left|\frac{B_1(-\xi, \eta)}{B_1(\xi, \eta)}e^{(1/2)iL\xi}\right|\cdot\left|\frac{B_2(\xi, \eta)}{B_2(-\xi, \eta)}e^{(1/2)iL\xi}\right||e^{iL\xi}|$$

$$\geqq 1-|e^{iL\xi}|$$

$$\geqq \frac{c_2\gamma^{1/2}}{(|\tau|+|\eta|^2)^{1/2}}.$$

By (5.1). (5.2) and (5.3) we get

$$|D(\tau, \eta)| \geqq \frac{c_3\gamma^{5/2}}{(|\tau|+|\eta|^2)^{3/2}}.$$

This completes the proof for this subcase. The remaining subcases of (i) are not difficult.

Now let us assume the case (ii) in Theorem 2. In this case the equality $B_1(-\xi, \eta)B_2(\xi, \eta)=B_1(\xi, \eta)B_2(-\xi, \eta)$ holds. Therefore, Lemma 3.1 and Lemma 5.2 yield

$$|D(\tau, \eta)| = |B_1(\xi, \eta)B_2(-\xi, \eta)|\cdot|1-e^{2i\xi L}|$$

$$\geqq \frac{c_1\gamma^2}{(|\tau|+|\eta|^2)}\frac{c_4\gamma^{1/2}}{(|\tau|+|\eta|^2)^{1/2}}$$

$$\geqq \frac{c_5\gamma^{5/2}}{(|\tau|+|\eta|^2)^{3/2}}.$$

This completes the proof for the case (ii) in Theorem 2.

§ 6. Proof of Lemma B

In this section we will prove Lemma B used in § 4.

Recall

$$D(\tau, \eta)=B_1(\xi(\tau, \eta), \eta)B_2(-\xi(\tau, \eta), \eta)-B_1(-\xi(\tau, \eta), \eta)B_2(\xi(\tau, \eta), \eta)e^{2i\xi(\tau,\eta)L}.$$

By putting $\lambda=|\eta|$, $z=\frac{1}{|\eta|}\xi(\tau, \eta)=\sqrt{-\frac{\tau}{|\eta|^2}-1}$ and $\omega(\eta)=\frac{\eta}{|\eta|}$, the equation

$$(6.1)\qquad D(\tau, \eta)=0$$

is equivalent to

$$(6.2)\qquad e^{2iL\lambda z}=\frac{B_1(z, \omega(\eta))B_2(-z, \omega(\eta))}{B_1(-z, \omega(\eta))B_2(z, \omega(\eta))}.$$

We will be concerned with some lemmata to prove Lemma B.

Lemma 6.1. *We assume that B_1 is of type II and that B_2 is of type I or of type*

II. Moreover, when B_2 is of type II, we assume that $a+b\neq 0$. Then there exists a positive number z_0 and a non-zero vector $\hat{\eta}\in \boldsymbol{R}^d$ such that

$$\left|\frac{B_1(z_0,\ \omega(\hat{\eta}))B_2(-z_0,\ \omega(\hat{\eta}))}{B_1(-z_0,\ \omega(\hat{\eta}))B_2(z_0,\ \omega(\hat{\eta}))}\right|<1. \tag{6.3}$$

Proof. When B_2 is of type I, we have $|B_2(-z,\ \omega(\eta))/B_2(z,\ \omega(\eta))|=1$ for every $z\in \boldsymbol{R}$ and every $\eta\in \boldsymbol{R}^d$. Then we can easily find a $z_0>0$ and a non-zero $\hat{\eta}\in \boldsymbol{R}^d$ which satisfy $|B_1(z_0,\ \omega(\hat{\eta}))/B_1(-z_0,\ \omega(\hat{\eta}))|<1$.

Now let us assume that both B_1 and B_2 are of type II and that $a+b\neq 0$. We attempt to prove (6.3) by *reductio ad absurdum*. Contrary to (6.3), assume that

$$\left|\frac{B_1(z,\ \omega(\eta))B_2(-z,\ \omega(\eta))}{B_1(-z,\ \omega(\eta))B_2(z,\ \omega(\eta))}\right|\geqq 1 \quad \text{for } z>0 \text{ and every non-zero } \eta\in \boldsymbol{R}^d.$$

Replacing η by $-\eta$, the fraction is inverted

$$\left|\frac{B_1(z,\ \omega(\eta))B_2(-z,\ \omega(\eta))}{B_1(-z,\ \omega(\eta))B_2(z,\ \omega(\eta))}\right|\leqq 1 \quad \text{for } z>0 \text{ and every non-zero } \eta\in \boldsymbol{R}^d.$$

Therefore

$$\left|\frac{B_1(z,\ \omega(\eta))B_2(-z,\ \omega(\eta))}{B_1(-z,\ \omega(\eta))B_2(z,\ \omega(\eta))}\right|=1 \quad \text{for } z>0 \text{ and every non-zero } \eta\in \boldsymbol{R}^d. \tag{6.4}$$

(6.4) holds for every positive number z if and only if

$$\tilde{a}_1\cdot\omega(\eta)+\tilde{b}_1\cdot\omega(\eta)=0 \quad \text{and} \quad (\tilde{a}_2\cdot\omega(\eta))^2=(\tilde{b}_2\cdot\omega(\eta))^2 \quad \text{for every non-zero vector } \eta\in \boldsymbol{R}^d \tag{6.5}$$

where $a=\tilde{a}_1+i\tilde{a}_2$ and $b=\tilde{b}_1+i\tilde{b}_2$.

From (6.5) and the assumption that both B_1 and B_2 are of type II, we have $a+b=0$. Lemma 6.1 is proved.

Lemma 6.2. *Let $f(z)$ be a holomorphic function in a neighborhood V_0 of a point z_0 on the positive real axis. Assume that $|f(z_0)|\neq 0$ and $|f(z_0)|<1$, then there is a sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers, a sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers in V_0 and positive numbers c and M such that*

$$\begin{cases} e^{2iL\lambda_n z_n}=f(z_n) \\ \lambda_n \longrightarrow \infty \quad (\text{as } n\to\infty) \\ \mathrm{Im}\{(z_n)^2\}\geqq \dfrac{c}{\lambda_n} \\ |z_n|\leqq M \\ |f(z_n)|\leqq 1-c \end{cases} \tag{6.6}$$

Proof. For a positive number δ $(\leqq(1/2)z_0)$, let us consider a neighborhood

$$U_\delta=\{z\in \boldsymbol{C}\,;\ |\mathrm{Re}(z-z_0)|<\delta \quad \text{and} \quad 0<\mathrm{Im}\, z<\delta\}.$$

If δ is small enough, U_δ is contained in V_0. Moreover, we can assume that

$$\left|\frac{f(z)-f(z_0)}{f(z_0)}\right| \leqq \frac{1}{2} \quad \text{and} \quad |f(z)| \leqq \frac{1+|f(z_0)|}{2} \qquad \text{in } U_\delta .$$

For $n=1, 2, 3, \cdots$ we put

$$\lambda_n = \frac{\operatorname{Arg} f(z_0)+2n\pi}{2z_0 L} \qquad \text{with } 0 \leqq \operatorname{Arg} f(z_0) < 2\pi$$

and

$$\begin{aligned} z_{n,1} &= \frac{1}{2\lambda_n Li}\{\operatorname{Log} f(z_0)+2n\pi i\} \\ &= \frac{\operatorname{Arg} f(z_0)+2n\pi}{2\lambda_n L} - \frac{i}{2\lambda_n L}\log|f(z_0)| \\ &= z_0 - \frac{i}{2\lambda_n L}\log|f(z_0)|. \end{aligned}$$

We define $z_{n,p}$'s succesively by

$$\begin{aligned} z_{n,p} &= \frac{1}{2\lambda_n Li}\left\{\operatorname{Log} f(z_0)+\operatorname{Log}\left\{1+\frac{f(z_{n,p-1})-f(z_0)}{f(z_0)}\right\}+2n\pi i\right\} \\ &= z_{n,1}+\frac{1}{2\lambda_n Li}\operatorname{Log}\left\{1+\frac{f(z_{n,p-1})-f(z_0)}{f(z_0)}\right\} \quad (p=2, 3, 4, \cdots), \end{aligned} \tag{6.7}$$

where $\operatorname{Log} f(z_0)=\log|f(z^0)|+i \operatorname{Arg} f(z_0)$ and $\operatorname{Log}(1+w)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}w^k$ for $|w|<1$.

Then we obtain

$$\operatorname{Re} z_{n,1}=z_0 \quad \text{and} \quad \operatorname{Im} z_{n,1}=\frac{-1}{2\lambda_n L}\log|f(z_0)|.$$

There exists an $n_1=n_1(\delta)$ such that

$$0<\operatorname{Im} z_{n,1} \leqq \frac{\delta}{2} \quad \text{and} \quad z_{n,1}\in U_\delta \qquad \text{for } n \geqq n_1(\delta). \tag{6.8}$$

Then

$$\begin{aligned} |z_{n,2}-z_{n,1}| &= \frac{1}{2\lambda_n L}\left|\operatorname{Log}\left\{1+\frac{f(z_{n,1})-f(z_0)}{f(z_0)}\right\}\right| \\ &\leqq \frac{1}{2\lambda_n L}\cdot 2\left|\frac{f(z_{n,1})-f(z_0)}{f(z_0)}\right| \\ &\leqq \frac{1}{\lambda_n L}\cdot M_1|z_{n,1}-z_0| \\ &\leqq \frac{M_1}{\lambda_n L}\cdot\frac{-1}{2\lambda_n L}\log|f(z_0)|, \end{aligned} \tag{6.9}$$

where $M_1=\sup_{z\in U_\delta}\left|\frac{f'(z)}{f(z_0)}\right|$ and we used the following inequality:

$$|\operatorname{Log}(1+w)-\operatorname{Log}(1+w')| \leqq 2\cdot|w-w'| \qquad \text{for } |w|, |w'| \leqq \frac{1}{2}.$$

Moreover we can choose an $n_2=n_2(\delta)$ $(\geqq n_1)$ such that

$$\frac{M_1}{\lambda_n L}\leqq\frac{M_1}{\lambda_n L-M_1}\leqq\frac{1}{2}\qquad\text{if } n\geqq n_2. \tag{6.10}$$

Then we can prove by induction that

$$\begin{cases} z_{n,p-1}\in U_\delta \\ \mathrm{Im}\, z_{n,p-1}\geqq\frac{1}{2}\,\mathrm{Im}\, z_{n,1} \\ |z_{n,p}-z_{n,p-1}|\leqq\left(\frac{M_1}{\lambda_n L}\right)^{p-1}\cdot\frac{-1}{2\lambda_n L}\log|f(z_0)| \end{cases} \tag{6.11}$$

$$(p=2,\ 3,\ 4,\ \cdots,\ n\geqq n_2).$$

In fact (6.11) holds for $p=2$ from (6.8) and (6.9). Assume that (6.11) holds for $p\leqq k$, then

$$\begin{aligned} |z_{n,k}-z_{n,1}| &\leqq\sum_{p=2}^{k}|z_{n,p}-z_{n,p-1}| \\ &\leqq\sum_{p=2}^{k}\left(\frac{M_1}{\lambda_n L}\right)^{p-1}\cdot\frac{-1}{2\lambda_n L}\cdot\log|f(z_0)| \\ &\leqq\frac{M_1}{\lambda_n L-M_1}\cdot\frac{-1}{2\lambda_n L}\log|f(z_0)| \\ &\leqq\frac{M_1}{\lambda_n L-M_1}\cdot\mathrm{Im}\, z_{n,1} \\ &\leqq\frac{1}{2}\,\mathrm{Im}\, z_{n,1} \\ &\leqq\frac{1}{4}\delta, \end{aligned}$$

therefore

$$\begin{aligned} \mathrm{Im}\, z_{n,k} &=\mathrm{Im}\, z_{n,1}+\mathrm{Im}\,(z_{n,k}-z_{n,1}) \\ &\geqq\mathrm{Im}\, z_{n,1}-|z_{n,k}-z_{n,1}| \\ &\geqq\frac{1}{2}\,\mathrm{Im}\, z_{n,1}, \end{aligned}$$

and

$$\begin{aligned} |z_{n,k}-z_0| &\leqq|z_{n,k}-z_{n,1}|+|z_{n,1}-z_0| \\ &\leqq\frac{1}{4}\delta+\frac{1}{2}\delta=\frac{3}{4}\delta, \end{aligned}$$

proving that $z_{n,k}\in U_\delta$. Moreover

$$\begin{aligned} |z_{n,k+1}-z_{n,k}| &=\frac{1}{2\lambda_n L}\cdot\left|\mathrm{Log}\left\{1+\frac{f(z_{n,k})-f(z_0)}{f(z_0)}\right\}-\mathrm{Log}\left\{1+\frac{f(z_{n,k-1})-f(z_0)}{f(z_0)}\right\}\right| \\ &=\frac{1}{2\lambda_n L}\cdot 2\left|\frac{f(z_{n,k})-f(z_{n,k-1})}{f(z_0)}\right| \\ &\leqq\frac{1}{\lambda_n L}M_1\cdot|z_{n,k}-z_{n,k-1}| \end{aligned}$$

$$\leqq\left(\frac{M_1}{\lambda_n L}\right)^k\cdot\frac{-1}{2\lambda_n L}\log|f(z_0)|$$

Thus (6.11) is proved for all $p\geqq 2$ and $n\geqq n_2$. From (6.10) and (6.11) the sequence $\{z_{n,p}\}_{p=1}^{\infty}$ converges as p tends to ∞. We denote the limit by z_n. It is easily verified that $\{\lambda_n, z_n\}$ satisfies (6.6). Lemma 6.2 is proved.

Proof of Lemma B. We set $f(z, \omega)=B_1(z, \omega)B_2(-z, \omega)/B_1(-z, \omega)B_2(z, \omega)$ ($z\in \boldsymbol{R}$ and $\omega\in S^{d-1}$). Then from Lemma 6.1, there exists a positive number z_0 and a non-zero real vector $\hat{\eta}$ such that $|f(z_0, \omega(\hat{\eta}))|<1$. We can also assume that $f(z_0, \omega(\hat{\eta}))\neq 0$. For $\{\lambda_n, z_n\}$ obtained by Lemma 6.2, we have

$$\frac{\partial}{\partial z}\{e^{2iL\lambda_n z}-f(z, \omega(\hat{\eta}))\}|_{z=z_n}=2iL\lambda_n e^{2iL\lambda_n z_n}-f_z(z_n, \omega(\hat{\eta}))\,.$$

The right handside is non-zero for sufficiently large n, say $n\geqq n_3$. Then from the implicit function theorem, for each n there exists an open neighborhood $\hat{U}_n\subset S^{d-1}$ of $\omega(\hat{\eta})$, an open interval I_n containing λ_n and a continuous function $z_n(\lambda, \omega)$ defined in $I_n\times\hat{U}_n$ such that

$$e^{2iL\lambda z_n(\lambda,\omega)}=f(z_n(\lambda, \omega), \omega)\,,$$

$$\text{Im}\,\{z_n(\lambda, \omega)^2\}\geqq\frac{c}{2\lambda_n}\,,$$

$$|z_n(\lambda, \omega)|\leqq 2M\,,$$

and

$$|f(z_n(\lambda, \omega), \omega)|\leqq 1-\frac{c}{2}$$

for every $(\lambda, \omega)\in I_n\times\hat{U}_n$,

We put $\eta_n=\lambda_n\omega(\hat{\eta})$ and $U_n=I_n\times\hat{U}_n$. Let us put $\tau_n(\eta)=-\{z_n(\eta)^2+1\}\lambda^2$ as a function of $\eta=\lambda\omega$ defined in U_n. Then after renumbering again they satisfy (4.9), (4.10), (4.11), (4.12) and (4.13). This completes the proof.

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