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L-valued 1-jet de Rham complexes and their induced spectral sequences

By

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§ 0. Introduciton

In [U-4], we raised several problems on a system of projective equations of a projective submanifold X. Those problems were heavily concerned with a given section of a given vector bundle E on X and its induced cohomology classes of the vector bundle valued cohomology groups $H^q(X, \Omega_X^p(E))$. Hence, we need a suitable means to study each cohomology class of $H^q(X, \Omega_X^p(E))$.

As is well-known, there are many works on geometric explanations of the cohomology classes of the cohomology groups $H^q(X, \Omega_X^p)$. Nevertheless, in spite of their importance, we do not have much knowledge on the geometric meanings of the cohomology classes of the E-valued cohomology groups $H^q(X, \Omega_X^p(E))$. And this fact has a lot to do with that, though we have Hodge spectral sequence from $H^q(X, \Omega_X^p)$ to $H^{p+q}(X, \mathbb{C})$, we do not have spectral sequences from "E-valued" cohomology groups to some topological cohomology groups. The difficulty for constructiong such a spectral sequence arises from the fact that in general we can not let the following sequence:

$$0 \longrightarrow E \xrightarrow{\nabla} \Omega_X^1 \otimes E \xrightarrow{\nabla} \cdots \cdots \xrightarrow{\nabla} \Omega_X^n \otimes E \longrightarrow 0$$

be a complex by introducing a connection ∇. Roughly speaking, the obstruction in making a complex with a connection ∇ is described by the curvature operator Θ for the connection ∇. This curvature operator Θ determines the Chern classes of E which are invariants of E. Hence, this approach has serious difficulty.

Thus, in view of our original purpose, we intend to construct a double spectral sequence depending on a given section of E by using another method (cf. (4.5) Theorem). The key idea is to use the sheaves of L-valued 1-jet forms on the projective bundle $P(E)$, in making a suitable complex with a differential operator of first order (cf. Added in proof).

We have some applications of our spectral sequences, for which we need complicated calculation, and we shall publish them elsewhere.

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§ 1. Jet sheaves (cf. EGA IV § 16 [G-2])

Let $f: X \to S$ be a smooth morphism of complex manifolds, and F a locally free O_X-module of finite rank. The diagonal embedding $\Delta: X \hookrightarrow X \times_S X$ canonically determines a sheaf of ideals I_Δ. Put $X_{(n)} := (|X|, O_{X \times_S X}/I_\Delta^{n+1})$, $h_{(n)}: X_{(n)} \hookrightarrow X \times_S X$ to be the canonical closed immersion and $p_{i(n)} = p_i \cdot h_{(n)}$, where p_i denotes the i-th projection. Then we define the F-valued n-jet sheaf $J^n_{X/S}(F)$ to be $(p_{1(n)})_*(p_{2(n)})^* F$. The O_X-module structure of $J^n_{X/S}(F)$ and a homomorphism $j^n_{X/S}$: $F \to J^n_{X/S}(F)$ are defined by the structure homomorphism $p_1^\#: p_1^{-1} O_X \to O_{X \times_S X}$ and the identity map of F respectively. The following proposition shows an elementary property of jet sheaves.

(1.1) Proposition. *Let $f: X \to Y$ and $g: Y \to Z$ be smooth morphisms of complex manifolds and F* a locally free *O_X-module of finite rank. Then we have the following exact sequences.*

(1.1.1) $$0 \longrightarrow S^n(\Omega^1_{X/Y}) \otimes F \longrightarrow J^n_{X/Y}(F) \longrightarrow J^{n-1}_{X/Y}(F) \longrightarrow 0,$$

in particular, putting $n = 1$,

(1.1.2) $$0 \longrightarrow \Omega^1_{X/Y} \otimes F \longrightarrow J^1_{X/Y}(F) \longrightarrow F \longrightarrow 0.$$

Moreover,

(1.1.3) $$0 \longrightarrow f^* \Omega^1_{Y/Z} \otimes F \longrightarrow J^1_{X/Z}(F) \longrightarrow J^1_{X/Y}(F) \longrightarrow 0.$$

Proof. For (1.1.1) and (1.1.2), see [G-2]. We have only to prove (1.1.3). Obviously there is a canonical map $J^1_{X/Z}(F) \longrightarrow J^1_{X/Y}(F)$. Using (1.1.2), an easy diagram chasing shows that the sequence (1.1.3) is exact. □

(1.2) Remark. (i) By the definition, the n-jet map $j^n_{X/S}: F \longrightarrow J^n_{X/S}(F)$ obviously possesses O_S-linearity though it does not O_X-linearity.

(ii) The sequence (1.1.2) does not split in general. The extension class of this sequence determines the total Chern class of F.

(1.3) Convention. In the sequel, we shall treat only holomorphic objects and meromorphic objects otherwise mentioned.

§ 2. The local study of $\wedge^p J^1_{X/S}(L)$

In the sequel, we use local expressions of 1-jet sheaf with coefficients in a line bundle L. Hence, in this section, we shall determine the transformation rules explicitly. Let us take a sufficiently fine open covering of X with systems of local coordinates and local frames $\{e_\lambda\}$ of L:

(2.1.1) $$\mathfrak{A} = \{U_\lambda | \lambda \in A\}, \ s = \dim S, \ n = \dim X - \dim S,$$

$$U_\lambda = (U_\lambda, (z^1_\lambda, \ldots, z^n_\lambda; v^1_\lambda, \ldots, v^s_\lambda), e_\lambda),$$

(such a pair is called "*a C.F.-open set*" of L over S),

where $$f\colon U_\lambda \ni (z_\lambda^1,\dots z_\lambda^n, v_\lambda^1,\dots,v_\lambda^s) \longrightarrow (v_\lambda^1,\dots v_\lambda^s) \in S.$$

Since $J^1_{X/S}(L)$ is a collection of the first order Taylor expansions of the local sections of L to the direction of the fiber, for a given C.F.-open set U_λ, we can uniquely choose a local frame $(G_\lambda^0, G_\lambda^1,\dots,G_\lambda^n)$ of $J^1_{X/S}(L)$ which satisfies the following condition: for a local section σ of L,

$$j^1_{X/S}(\sigma) = \sigma_\lambda G_\lambda^0 + \sum_{i=1}^n (\partial\sigma_\lambda/\partial z_\lambda^i) G_\lambda^i, \tag{2.1.2}$$

$$(\sigma = \sigma_\lambda e_\lambda,\ \sigma_\lambda \in \Gamma(U_\lambda, O_X)).$$

We call this frame "*the canonical local frame*" of $J^1_{X/S}(L)$ on the C.F.-open set U_λ.

(2.1) Remark. In our tensor calculus, G_λ^0 must be distinguished from other $G_\lambda^1,\dots,G_\lambda^n$.

Now we suppose that we have two C.F.-open sets U_λ, U_μ, and the change of local frames $g_{\mu\lambda}\colon U_\lambda \cap U_\mu \to \mathbb{C}^*$ of the line bundle L which satisfies

$$e_\lambda = e_\mu g_{\mu\lambda}, \tag{2.1.3}$$

where e_λ and e_μ are local frames of L determined by the C.F.-open sets U_λ and U_μ respectively. Then the transformation rules of the canonical local frames on the C.F.-open sets are given as follows.

(2.2) Proposition. (Transfomation rules of canonical local frames)

$$G_\lambda^0 = G_\mu^0 g_{\mu\lambda} + \sum_{k=1}^n (\partial g_{\mu\lambda}/\partial z_\mu^k) G_\mu^k \tag{2.2.1}$$

$$G_\lambda^i = \sum_{k=1}^n (\partial z_\lambda^i/\partial z_\mu^k) g_{\mu\lambda} G_\mu^k \qquad (i = 1,\dots,n). \tag{2.2.2}$$

Proof. For (2.2.1), we consider $j^1_{X/S}(e_\lambda)$. Then (2.1.2) shows:

$$G_\lambda^0 = j^1_{X/S}(e_\lambda) = j^1_{X/S}(e_\mu g_{\mu\lambda}) = g_{\mu\lambda} G_\mu^0 + \sum (\partial g_{\mu\lambda}/\partial z_\mu^k) G_\mu^k.$$

Hence we see that (2.2.1) holds. As for (2.2.2), we put F in the sequence (1.1.2) to be the line bundle L. Then it is easy to see that $\operatorname{Ker}(J^1_{X/S}(L) \to L)$ has a local frame $(G_\lambda^1,\dots,G_\lambda^n)$ on U_λ, which corresponds to the local frame $(dz_\lambda^1 \otimes e_\lambda,\dots,dz_\lambda^n \otimes e_\lambda)$ through the isomorphism $\operatorname{Ker}(J^1_{X/S}(L) \to L) \simeq \Omega^1_{X/S} \otimes L$. Thus we have (2.2.2). $\square$

Next we study the sheaves of L-valued 1-jet p-forms $\wedge^p J^1_{X/S}(L)$. Obviously $\{G_\lambda^{a(1)} \wedge \dots \wedge G_\lambda^{a(p)} \mid 0 \leq a(1) < \dots < a(p) \leq n\}$ is a local frame of $\wedge^p J^1_{X/S}(L)$ on the C.F.-open set U_λ. Hence, every (local) L-valued 1-jet p-form ϕ has the following unique expression on the C.F.-open set.

$$(2.2.3)\qquad \phi = \sum \phi_{\lambda a(1),\dots,a(p)} G_\lambda^{a(1)} \wedge \dots \wedge G_\lambda^{a(p)}$$
$$(\phi_{\lambda a(1),\dots,a(p)} \in \Gamma(U_\lambda, O_X))$$

We also require the transformation rules of the coefficients $\{\phi_{\lambda a(1)\dots a(p)}\}$ and $\{\phi_{\mu b(1)\dots b(p)}\}$ in the sequel. To simplify our computation and the descriptions of the transformation rules, we shall exploit the following convention as usual.

(2.3) Convention. (i) $D(f_1,\dots,f_k)/D(z_\lambda^{a(1)},\dots,z_\lambda^{a(k)}) := \det(\partial f_p/\partial z_\lambda^{a(q)})_{p,q=1,\dots,k}$.
(ii) $\sum T'^{*\dots*}_{*\dots*a(t)a(t+1)\dots a(s)} T''^{*\dots*a(t)a(t+1)\dots a(s)}_{*\dots*} :=$ the summation for all the indices $1 \leqq a(t) < \cdots < a(s) \leqq n$ (when we admit $a(t)=0$, ${}^0\sum$ will be used instead of $\sum$), if there is no other pair of indices up and down.
(iii) $\sum^\# T'^{*\dots*}_{*\dots*a(t)a(t+1)\dots a(s)} T''^{*\dots*a(t)a(t+1)\dots a(s)}_{*\dots*} :=$ the summation for all the indices $1 \leqq a(t),\dots,a(s) \leqq n$ (when we admit $a(i)=0$, ${}^0\sum^\#$ will be used instead of $\sum^\#$), if there is no other pair of indices up and down.

(2.4) Proposition. (Transformation rules of coefficients)

$$(2.4.1)\qquad \phi_{\mu 0 b(2)\dots b(p)} = \sum \{D(z_\lambda^{a(2)}\cdots z_\lambda^{a(p)})/D(z_\mu^{b(2)}\cdots z_\mu^{b(p)})\}\, g_{\mu\lambda}^p\, \phi_{\lambda 0 a(2)\dots a(p)}.$$
$$(1 \leqq b(2) < \cdots < b(p) \leqq n)$$

$$(2.4.2)\qquad \begin{aligned}\phi_{\mu b(1)\dots b(p)} &= \sum \{D(g_{\mu\lambda}, z_\lambda^{a(2)}\cdots z_\lambda^{a(p)})/D(z_\mu^{b(1)}\cdots z_\mu^{b(p)})\}\, g_{\mu\lambda}^{p-1}\, \phi_{\lambda 0 a(2)\dots a(p)}\\ &+ \sum \{D(z_\lambda^{a(1)}\cdots z_\lambda^{a(p)})/D(z_\mu^{b(1)}\cdots z_\mu^{b(p)})\}\, g_{\mu\lambda}^p\, \phi_{\lambda a(1)\dots a(p)},\end{aligned}$$
$$(1 \leqq b(1) < \cdots < b(p) \leqq n).$$

Proof. With using (2.2) Proposition, we shall check the above by direct computation. For $1 \leqq a(2) < \cdots < a(p) \leqq n$,

$$\begin{aligned}
&G_\lambda^0 \wedge G_\lambda^{a(2)} \wedge \cdots \wedge G_\lambda^{a(p)}\\
&= \{g_{\mu\lambda} G_\mu^0 + \sum(\partial g_{\mu\lambda}/\partial z_\mu^{b(1)})\, G_\mu^{b(1)}\}\\
&\quad \wedge \{\sum(\partial z_\lambda^{a(2)}/\partial z_\mu^{b(2)}) g_{\mu\lambda} G_\mu^{b(2)}\} \wedge \cdots \wedge \{\sum(\partial z_\lambda^{a(p)}/\partial z_\mu^{b(p)})\, g_{\mu\lambda} G_\mu^{b(p)}\}\\
&= \sum^\# g_{\mu\lambda}^p (\partial z_\lambda^{a(2)}/\partial z_\mu^{b(2)})\cdots(\partial z_\lambda^{a(p)}/\partial z_\mu^{b(p)})\, G_\mu^0 \wedge G_\mu^{b(2)} \wedge \cdots \wedge G_\mu^{b(p)}\\
&\qquad + \sum^\# g_{\mu\lambda}^{p-1} (\partial g_{\mu\lambda}/\partial z_\mu^{b(1)})(\partial z_\lambda^{a(2)}/\partial z_\mu^{b(2)})\cdots(\partial z_\lambda^{a(p)}/\partial z_\mu^{b(p)})\, G_\mu^{b(1)} \wedge \cdots \wedge G_\mu^{b(p)}\\
&= \sum g_{\mu\lambda}^p \{D(z_\lambda^{a(2)}\cdots z_\lambda^{a(p)})/D(z_\mu^{b(2)}\cdots z_\mu^{b(p)})\}\, G_\mu^0 \wedge G_\mu^{b(2)} \wedge \cdots \wedge G_\mu^{b(p)}\\
&\qquad + \sum g_{\mu\lambda}^{p-1} \{D(g_{\mu\lambda}, z_\lambda^{a(2)},\dots,z_\lambda^{a(p)})/D(z_\mu^{b(1)}\cdots z_\mu^{b(p)})\}\, G_\mu^{b(1)} \wedge \cdots \wedge G_\mu^{b(p)}.
\end{aligned}$$

By the same way, for $1 \leqq a(1) < \cdots < a(p) \leqq n$, we have:

$$G_\lambda^{a(1)} \wedge \cdots \wedge G_\lambda^{a(p)} = \sum g_{\mu\lambda}^p \{D(z_\lambda^{a(1)}\cdots z_\lambda^{a(p)})/D(z_\mu^{b(1)}\cdots z_\mu^{b(p)})\}\, G_\mu^{b(1)} \wedge \cdots \wedge G_\mu^{b(p)}.$$

Hence, for a section ϕ of $\wedge^p J^1_{X/S}(L)$,

$$\phi = {}^0\sum \phi_{\lambda a(1)\dots a(p)} G_\lambda^{a(1)} \wedge \cdots \wedge G_\lambda^{a(p)}$$

$$= \sum \phi_{\lambda 0 a(2)\dots a(p)} [\sum g^{p}_{\mu\lambda} \{D(z^{a(2)}_{\lambda} \cdots z^{a(p)}_{\lambda})/D(z^{b(2)}_{\mu} \cdots z^{b(p)}_{\mu})\} G^{0}_{\mu} \wedge G^{b(2)}_{\mu} \wedge \cdots \wedge G^{b(p)}_{\mu}$$
$$+ \sum g^{p-1}_{\mu\lambda} \{D(g_{\mu\lambda}, z^{a(2)}_{\lambda} \cdots z^{a(p)}_{\lambda})/D(z^{b(1)}_{\mu} \cdots z^{b(p)}_{\mu})\} G^{b(1)}_{\mu} \wedge \cdots \wedge G^{b(p)}_{\mu}]$$
$$+ \sum \phi_{\lambda a(1)\dots a(p)} [\sum g^{p}_{\mu\lambda} \{D(z^{a(1)}_{\lambda} \cdots z^{a(p)}_{\lambda})/D(z^{b(1)}_{\mu} \cdots z^{b(p)}_{\mu})\} G^{b(1)}_{\mu} \wedge \cdots \wedge G^{b(p)}_{\mu}]$$
$$= \sum [\sum \phi_{\lambda 0 a(2)\dots a(p)} g^{p}_{\mu\lambda} \{D(z^{a(2)}_{\lambda} \cdots z^{a(p)}_{\lambda})/D(z^{b(2)}_{\mu} \cdots z^{b(p)}_{\mu})\}] G^{0}_{\mu} \wedge G^{b(2)}_{\mu} \wedge \cdots \wedge G^{b(p)}_{\mu}$$
$$+ \sum [\sum \phi_{\lambda a(1)\dots a(p)} g^{p}_{\mu\lambda} \{D(z^{a(1)}_{\lambda} \cdots z^{a(p)}_{\lambda})/D(z^{b(1)}_{\mu} \cdots z^{b(p)}_{\mu})\}$$
$$+ \sum \phi_{\lambda 0 a(2)\dots a(p)} g^{p-1}_{\mu\lambda} \{D(g_{\mu\lambda}, z^{a(2)}_{\lambda} \cdots z^{a(p)}_{\lambda})/D(z^{b(1)}_{\mu} \cdots z^{b(p)}_{\mu})\}] G^{b(1)}_{\mu} \wedge \cdots \wedge G^{b(p)}_{\mu}$$

Thus we obtain (2.4.1) and (2.4.2). □

§3. Construction of L-valued 1-jet de Rham complex

Let us take a global section $\tau \in \Gamma(X, L)$ and fix it through out this section. Then the section τ corresponds to an effective (not necessarily reduced) divisor D_τ on X. *The sheaf of meromorphic L-valued* 1*-jet p-forms of order q*: $\{\wedge^p J^1_{X/S}(L)\}(qD_\tau)$ is defined as follows. For an open set V of X,

$$\Gamma(V, \{\wedge^p J^1_{X/S}(L)\}(qD_\tau)) := \{\phi \,|\, \textit{if } \phi = {}^0\textstyle\sum \phi_{\lambda a(1)\dots a(p)} G^{a(1)}_{\lambda} \wedge \cdots \wedge G^{a(p)}_{\lambda}$$
$$\text{on } V \cap U_\lambda, \text{ then } f^q_\lambda \phi_{\lambda a(1)\dots a(p)} \text{ is holomorphic on } V \cap U_\lambda \text{ for } \lambda \in A\},$$

where f_λ denotes the function defined by $\tau = f_\lambda \cdot e_\lambda$.

Our aim is to make those sheaves into a complex with defining a suitable differential operator in three steps. In the first step, we define τ-dervation ∇_τ: $\{\wedge^p J^1_{X/S}(L)\}(qD_\tau) \to \{\wedge^{p+1} J^1_{X/S}(L)\}(qD_\tau)$ as follows. For a local section $\phi = {}^0\sum \phi_{\lambda a(1)\dots a(p)} G^{a(1)}_{\lambda} \wedge \cdots \wedge G^{a(p)}_{\lambda}$ of $\{\wedge^p J^1_{X/S}(L)\}(qD_\tau)$, we put

$$\text{(3.1.1)} \qquad \nabla_\tau(\phi) := {}^0\sum \sum_{k=1}^{n} (\partial \phi_{\lambda a(1)\dots a(p)}/\partial z^k_\lambda) f_\lambda G^k_\lambda \wedge G^{a(1)}_{\lambda} \wedge \cdots \wedge G^{a(p)}_{\lambda}$$
$$- p\,{}^0\sum \sum_{k=1}^{n} \phi_{\lambda a(1)\dots a(p)} (\partial f_\lambda/\partial z^k_\lambda) G^k_\lambda \wedge G^{a(1)}_{\lambda} \wedge \cdots \wedge G^{a(p)}_{\lambda}.$$

(3.1) Theorem. ∇_τ *is well-defined by* (3.1.1). (cf. Added in proof)

For the proof of this theorem, first we give the following lemma, which simplifies our calculation.

(3.2) Lemma. *Let $f_1, \dots, f_p$ be functions on a open set U with a system of local coordinates $\{z^0, \dots, z^p, \dots\}$. Then the following equality holds.*

$$\text{(3.2.1)} \qquad \sum_{t=0}^{p} (-1)^t \partial/\partial z^t [D(f_1 \dots f_p)/D(z^0 \dots \hat{t} \dots z^p)] = 0$$

Proof. We show this by induction on p. In case of $p = 1$,

$$\partial/\partial z^0 (\partial f_1/\partial z^1) - \partial/\partial z^1 (\partial f_1/\partial z^0) = 0.$$

Now we assume that the equality (3.2.1) holds for $p-1$. Then:

$$\sum_{t=0}^{p}(-1)^t\partial/\partial z^t[D(f_1\ldots f_p)/D(z^0\ldots\hat{t}\ldots z^p)]$$

$$=\sum_{t=0}^{p}(-1)^t\partial/\partial z^t[\sum_{s=0}^{t-1}(-1)^s\partial f_1/\partial z^s\{D(f_2\ldots f_p)/D(z^0\ldots\hat{s}\ldots\hat{t}\ldots z^p)\}$$

$$+\sum_{s=t+1}^{p}(-1)^{s-1}\partial f_1/\partial z^s\{D(f_2\ldots f_p)/D(z^0\ldots\hat{t}\ldots\hat{s}\ldots z^p)\}],$$

with using Leibniz rule,

$$=\sum_{s=0}^{p}(-1)^{s+1}\partial f_1/\partial z^s\times[\sum_{t=s+1}^{p}(-1)^{t-1}\partial/\partial z^t\{D(f_2\ldots f_p)/D(z^0\ldots\hat{s}\ldots\hat{t}\ldots z^p)\}$$

$$+\sum_{t=0}^{s-1}(-1)^t\partial/\partial z^t\{D(f_2\ldots f_p)/D(z^0\ldots\hat{t}\ldots\hat{s}\ldots z^p)\}].$$

By our induction hypothesis, the inside of the brackets in the last expression is zero. Thus we obtain the equality (3.2.1). □

Proof of (3.1) *Theorem.* By the symbol $\nabla\langle\lambda\rangle(\phi)$, we denote the image of ϕ through the map ∇_τ defined by (3.1.1) with the canonical local frame $(G^0_\lambda,\ldots,G^n_\lambda)$. Using the expression of (3.1.1), we can rewrite the coefficients of $\nabla\langle\lambda\rangle(\phi)$ as follows.

(3.2.2) For $1\leqq a(1)<\cdots<a(p)\leqq n$,

$$\nabla\langle\lambda\rangle(\phi)_{\lambda 0a(1)\ldots a(p)}=\sum_{t=1}^{p}(-1)^t f_\lambda^{p+1}\times\partial/\partial z_\lambda^{a(t)}(\phi_{\lambda 0a(1)\ldots\hat{t}\ldots a(p)}/f_\lambda^p).$$

(3.2.3) For $1\leqq a(0)<\cdots<a(p)\leqq n$,

$$\nabla\langle\lambda\rangle(\phi)_{\lambda a(0)\ldots a(p)}=\sum_{t=0}^{p}(-1)^t f_\lambda^{p+1}\times\partial/\partial z_\lambda^{a(t)}(\phi_{\lambda a(0)\ldots\hat{t}\ldots a(p)}/f_\lambda^p).$$

In the expression above, we must take notice of the difference of the range of the index t. Now we start our proof of well-definedness. As for "the order of poles", it is easy to see that $f_\lambda^q\nabla\langle\lambda\rangle(\phi)_{\lambda c(0)\ldots c(p)}$ is holomorphic for indices $\underline{0}\leqq c(0)<\cdots<c(p)\leqq n$ by using (3.2.2) and (3.2.3). Hence we have only to show:

$$\nabla\langle\lambda\rangle(\phi)_{\lambda a(0)\ldots a(p)}=\nabla\langle\mu\rangle(\phi)_{\lambda a(0)\ldots a(p)},$$

for indices $\underline{0}\leqq a(0)<\cdots<a(p)\leqq n$. First we shall treat the right-hand side of the above. In case of $1\leqq a(0)$, with using (2.4.2), our computation proceeds as follows.

$$\nabla\langle\mu\rangle(\phi)_{\lambda a(0)\ldots a(p)}=\sum\nabla\langle\mu\rangle(\phi)_{\mu 0,b(1)\ldots b(p)}g^p_{\lambda\mu}\{D(g_{\lambda\mu},z_\mu^{b(1)}\cdots z_\mu^{b(p)})/D(z_\lambda^{a(0)}\cdots z_\lambda^{a(p)})\}$$
$$+\sum\nabla\langle\mu\rangle(\phi)_{\mu b(0)\ldots b(p)}g^{p+1}_{\lambda\mu}\{D(z_\mu^{b(0)}\cdots z_\mu^{b(p)})/D(z_\lambda^{a(0)}\cdots z_\lambda^{a(p)})\},$$

applying (3.2.2) and (3.2.3),

$$=\sum\{D(g_{\lambda\mu}, z_\mu^{b(1)}\cdots z_\mu^{b(p)})/D(z_\lambda^{a(0)}\cdots z_\lambda^{a(p)})\}g_{\lambda\mu}^p\{\sum_{t=1}^{p}(-1)^t f_\mu^{p+1}\partial/\partial z_\mu^{b(t)}(\phi_{\mu 0,b(1)\dots\hat{t}\dots b(p)}/f_\mu^p)\}$$

$$+\sum\{D(z_\mu^{b(0)}\cdots z_\mu^{b(p)})/D(z_\lambda^{a(0)}\cdots z_\lambda^{a(p)})\}g_{\lambda\mu}^{p+1}\{\sum_{t=0}^{p}(-1)^t f_\mu^{p+1}\partial/\partial z_\mu^{b(t)}(\phi_{\mu b(0)\dots\hat{t}\dots b(p)}/f_\mu^p)\}$$

$$=(1/p!)\sum{}^{\#}(\text{the first rerm})+(1/(p+1)!)\sum{}^{\#}(\text{the second term})$$

$$=(1/p!)\sum{}^{\#}\sum_{t=1}^{p}f_\lambda^{p+1}g_{\mu\lambda}\{D(z_\mu^{b(t)}, g_{\lambda\mu}, z_\mu^{b(1)}\dots\hat{t}\dots z_\mu^{b(p)})/D(z_\lambda^{a(0)}\cdots z_\lambda^{a(p)})\}$$

$$\times\partial/\partial z_\mu^{b(t)}(\phi_{\mu 0,b(1)\dots\hat{t}\dots b(p)}/f_\mu^p)+(1/(p+1)!)\sum{}^{\#}\sum_{t=0}^{p}f_\lambda^{p+1}$$

$$\times\{D(z_\mu^{b(t)}, z_\mu^{b(0)}\dots\hat{t}\dots z_\mu^{b(p)})/D(z_\lambda^{a(0)}\cdots z_\lambda^{a(p)})\}\partial/\partial z_\mu^{b(t)}(\phi_{\mu b(0)\dots\hat{t}\dots b(p)}/f_\mu^p)$$

$$=(1/(p-1)!)\sum{}^{\#}f_\lambda^{p+1}g_{\mu\lambda}\{D(z_\mu^{b(1)}, g_{\lambda\mu}, z_\mu^{b(2)}\cdots z_\mu^{b(p)})/D(z_\lambda^{a(0)}\cdots z_\lambda^{a(p)})\}$$

$$\times\partial/\partial z_\mu^{b(1)}(\phi_{\mu 0b(2)\dots b(p)}/f_\mu^p)+(1/p!)\sum{}^{\#}f_\lambda^{p+1}\{D(z_\mu^{b(0)}\cdots z_\mu^{b(p)})/D(z_\lambda^{a(0)}\cdots z_\lambda^{a(p)})\}$$

$$\times\partial/\partial z_\mu^{b(0)}(\phi_{\mu b(1)\dots b(p)}/f_\mu^p).$$

Hence we obtain:

(3.2.4) $\nabla\langle\mu\rangle(\phi)_{\lambda a(0)\dots a(p)}$

$$=\sum f_\lambda^{p+1}g_{\mu\lambda}\{D(\phi_{\mu 0b(1)\dots b(p-1)}/f_\mu^p, g_{\lambda\mu}, z_\mu^{b(1)}\cdots z_\mu^{b(p-1)})/D(z_\lambda^{a(0)}\cdots z_\lambda^{a(p)})\}$$
$$+\sum f_\lambda^{p+1}\{D(\phi_{\mu b(1)\dots b(p)}/f_\mu^p, z_\mu^{b(1)}\cdots z_\mu^{b(p)})/D(z_\lambda^{a(0)}\cdots z_\lambda^{a(p)})\}.$$

Also in the case $a(0)=0$, we can get the following by the similar computation with using (2.4.1) and (3.2.2).

(3.2.5) $\nabla\langle\mu\rangle(\phi)_{\lambda 0,a(1)\dots a(p)}$

$$=-\sum f_\lambda^{p+1}\{D(\phi_{\mu 0,b(1)\dots b(p-1)}/f_\mu^p, z_\mu^{b(1)}\cdots z_\mu^{b(p-1)})/D(z_\lambda^{a(1)}\cdots z_\lambda^{a(p)})\}$$

Next we shall express $\nabla\langle\lambda\rangle(\phi)_{\lambda a(0)\dots a(p)}$ in terms of $\{\phi_{\mu *\dots *}\}$. In case of $1\leqq a(0)$, our computation is carried out as follows. By virtue of (3.2.3) and (2.4.2),

$$\nabla\langle\lambda\rangle(\phi)_{\lambda a(0)\dots a(p)}=\sum_{t=0}^{p}(-1)^t f_\lambda^{p+1}\partial/\partial z_\lambda^{a(t)}(\phi_{\lambda a(0)\dots\hat{t}\dots a(p)}/f_\lambda^p)$$

$$=\sum_{t=0}^{p}(-1)^t f_\lambda^{p+1}\partial/\partial z_\lambda^{a(t)}(f_\lambda^{-p}[\sum g_{\lambda\mu}^{p-1}\phi_{\mu 0b(1)\dots b(p-1)}$$

$$\times\{D(g_{\lambda\mu}, z_\mu^{b(1)}\cdots z_\mu^{b(p-1)})/D(z_\lambda^{a(0)}\dots\hat{t}\dots z_\lambda^{a(p)})\}$$

$$+\sum g_{\lambda\mu}^p\phi_{\mu b(0)\dots b(p-1)}\{D(z_\mu^{b(0)}\dots z_\mu^{b(p-1)})/D(z_\lambda^{a(0)}\dots\hat{t}\dots z_\lambda^{a(p)})\}])$$

$$=g_{\mu\lambda}\sum f_\lambda^{p+1}(\phi_{\mu 0b(1)\dots b(p-1)}/f_\mu^p)[\sum_{t=0}^{p}(-1)^t\partial/\partial z_\lambda^{a(t)}$$

$$\{D(g_{\lambda\mu}, z_\mu^{b(1)}\cdots z_\mu^{b(p-1)})/D(z_\lambda^{a(0)}\dots\hat{t}\dots z_\lambda^{a(p)})\}]$$

$$+g_{\mu\lambda}\sum f_\lambda^{p+1}[\sum_{t=0}^{p}(-1)^t\{\partial/\partial z_\lambda^{a(t)}(\phi_{\mu 0b(1)\dots b(p-1)}/f_\mu^p)\}$$

$$\times \{D(g_{\lambda\mu}, z_\mu^{b(1)} \cdots z_\mu^{b(p-1)})/D(z_\lambda^{a(0)} \ldots \hat{t} \ldots z_\lambda^{a(p)})\}]$$

$$+ \sum f_\lambda^{p+1}(\phi_{\mu 0 b(1) \ldots b(p-1)}/f_\mu^p)[\sum_{t=0}^{p}(-1)^t(\partial g_{\mu\lambda}/\partial z_\lambda^{a(t)})$$

$$\times \{D(g_{\lambda\mu}, z_\mu^{b(1)} \cdots z_\mu^{b(p-1)})/D(z_\lambda^{a(0)} \ldots \hat{t} \ldots z_\lambda^{a(p)})\}]$$

$$+ \sum f_\lambda^{p+1}(\phi_{\mu b(0) \ldots b(p-1)}/f_\mu^p)[\sum_{t=0}^{p}(-1)^t \partial/\partial z_\lambda^{a(t)}$$

$$\{D(z_\mu^{b(0)} \cdots z_\mu^{b(p-1)})/D(z_\lambda^{a(0)} \ldots \hat{t} \ldots z_\lambda^{a(p)})\}]$$

$$+ \sum f_\lambda^{p+1}[\sum_{t=0}^{p}(-1)^t\{\partial/\partial z_\lambda^{a(t)}(\phi_{\mu b(0) \ldots b(p-1)}/f_\mu^p)\}$$

$$\times \{D(z_\mu^{b(0)} \cdots z_\mu^{b(p-1)})/D(z_\lambda^{a(0)} \ldots \hat{t} \ldots z_\lambda^{a(p)})\}].$$

Among the five terms above, (3.2) Lemma shows that the first term and the fourth term vanish. Hence we see:

the expression above

$$\begin{aligned} &= g_{\mu\lambda}\sum f_\lambda^{p+1}\{D(\phi_{\mu 0 b(1) \ldots b(p-1)}/f_\mu^p, g_{\lambda\mu}, z_\mu^{b(1)} \cdots z_\mu^{b(p-1)})/D(z_\lambda^{a(0)} \cdots z_\lambda^{a(p)})\} \\ &\quad + \sum f_\lambda^{p+1}(\phi_{\mu 0 b(1) \ldots b(p-1)}/f_\mu^p) \times \{D(g_{\mu\lambda}, g_{\lambda\mu}, z_\mu^{b(1)} \cdots z_\mu^{b(p-1)})/D(z_\lambda^{a(0)} \cdots z_\lambda^{a(p)})\} \\ &\quad + \sum f_\lambda^{p+1}\{D(\phi_{\mu b(0) \ldots b(p-1)}/f_\mu^p, z_\mu^{b(0)} \cdots z_\mu^{b(p-1)})/D(z_\lambda^{a(0)} \cdots z_\lambda^{a(p)})\}. \end{aligned}$$

Since $g_{\mu\lambda} \cdot g_{\lambda\mu} = 1$, the second term above also vaishes. Comparing this with (3.2.4), for $1 \leqq a(0)$, we obtain that:

$$\nabla \langle \mu \rangle (\phi)_{\lambda a(0) \ldots a(p)} = \nabla \langle \lambda \rangle (\phi)_{\lambda a(0) \ldots a(p)}.$$

Also for the case $a(0) = 0$, the similar calculation of $\nabla \langle \lambda \rangle (\phi)_{\lambda 0 a(1) \ldots a(p)}$ with using (3.2) Lemma shows:

$$\nabla \langle \mu \rangle (\phi)_{\lambda 0 a(1) \ldots a(p)} = \nabla \langle \lambda \rangle (\phi)_{\lambda 0 a(1) \ldots a(p)}$$

after comparing that with (3.2.5). Q.E.D.

Let us study an elementary property of τ-derivation ∇_τ. We can obtain an exact sequence:

$$0 \longrightarrow \Omega_{X/S}^p \otimes L^p \longrightarrow \wedge^p J_{X/S}^1(L) \longrightarrow \Omega_{X/S}^{p-1} \otimes L^p \longrightarrow 0$$

from the sequence (1.1.2) with putting $F = L$ and taking the canonical filtration of its p-th wedge. Since $\Omega_{X/S}^p \otimes L^p \simeq \Omega_{X/S}^p(pD_\tau)$, thes sequence gives:

$$(3.3.1) \quad 0 \longrightarrow \Omega_{X/S}^p(mD_\tau) \xrightarrow{\alpha_p} \{\wedge^p J_{X/S}^1(L)\}((m-p)D_\tau) \xrightarrow{\beta_p} \Omega_{X/S}^{p-1}(mD_\tau) \longrightarrow 0.$$

With relation to this sequence (3.3.1), τ-derivation has a fine property as mentioned below.

(3.3) Proposition. *The following diagram is exact commutative.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^p(mD) & \xrightarrow{\alpha_p} & \{\wedge^p J^1(L)\}((m-p)D) & \xrightarrow{\beta_p} & \Omega^{p-1}(mD) & \longrightarrow & 0 \\ & & {\scriptstyle d_{X/S}}\downarrow & & {\scriptstyle \nabla_\tau}\downarrow & & {\scriptstyle -d_{X/S}}\downarrow & & \\ 0 & \longrightarrow & \Omega^{p+1}((m+1)D) & \xrightarrow[\alpha_{p+1}]{} & \{\wedge^{p+1} J^1(L)\}((m-p)D) & \xrightarrow[\beta_{p+1}]{} & \Omega^p((m+1)D) & \longrightarrow & 0, \end{array}$$

where $D = D_\tau$, $\Omega^p = \Omega^p_{X/S}$, $J^1(L) = J^1_{X/S}(L)$ and $d_{X/S} =$ the exterior derivation over S.

Proof. The problem is local. Hence we shall check the above with taking an arbitrary C.F.-open set U_λ and its canonical local frame as in §2. First we show that $\alpha_{p+1}\cdot d_{X/S} = \nabla_\tau\cdot\alpha_p$. Let us take a local meromorphic p-form:

$$\psi = \sum \psi_{\lambda a(1)\dots a(p)} dz_\lambda^{a(1)} \wedge \cdots \wedge dz_\lambda^{a(p)},$$

namely, a local section of $\Omega^p_{X/S}(mD_\tau)$. Then, it is easy to see:

(3.3.2) $$\alpha_p(\psi) = \sum f_\lambda^p \psi_{\lambda a(1)\dots a(p)} G_\lambda^{a(1)} \wedge \cdots \wedge G_\lambda^{a(p)},$$

which means $$\alpha_p(\psi)_{\lambda a(1)\dots a(p)} = \begin{cases} f_\lambda^p \psi_{\lambda a(1)\dots a(p)} & \text{if } a(1) \geqq 1 \\ 0 & \text{if } a(0) = 0. \end{cases}$$

Hence, by (3.2.2) and (3.2.3), we get:

$$\nabla_\tau\cdot\alpha_p(\psi) = \sum f_\lambda^{p+1} \sum_{t=0}^{n} (-1)^t \partial/\partial z_\lambda^{a(t)}(\psi_{\lambda a(0)\dots\hat{t}\dots a(p)}) \times G_\lambda^{a(0)} \wedge \cdots \wedge G_\lambda^{a(p)}.$$

On the other hand,

$$d_{X/S}(\psi) = \sum\{\sum_{t=0}^{n} (-1)^t \partial/\partial z_\lambda^{a(t)}(\psi_{\lambda a(0)\dots\hat{t}\dots a(p)})\} \times dz_\lambda^{a(0)} \wedge \cdots \wedge dz_\lambda^{a(p)}.$$

Thus we obtain:

$$\begin{aligned} &\alpha_{p+1}\cdot d_{X/S}(\psi) \\ &= \sum f_\lambda^{p+1}\{\sum_{t=0}^{n} (-1)^t \partial/\partial z_\lambda^{a(t)}(\psi_{\lambda a(0)\dots\hat{t}\dots(p)})\} \times G_\lambda^{a(0)} \wedge \cdots \wedge G_\lambda^{a(p)} \\ &= \nabla_\tau\cdot\alpha_p(\psi). \end{aligned}$$

Next we see that $-d_{X/S}\cdot\beta_p = \beta_{p+1}\cdot\nabla_\tau$. We take a local section $\phi = {}^0\sum \phi_{\lambda a(1)\dots a(p)} G_\lambda^{a(1)} \wedge \cdots \wedge G_\lambda^{a(p)}$ of $\{\wedge^p J^1_{X/S}(L)\}((m-p)D_\tau)$. Then,

(3.3.3) $$\beta_p(\phi) = \sum f_\lambda^{-p}\cdot\phi_{\lambda 0, a(2)\dots a(p)} \times dz_\lambda^{a(2)} \wedge \cdots \wedge dz_\lambda^{a(p)},$$

namely, $$\beta_p(\phi)_{\lambda a(1)\dots a(p-1)} = f_\lambda^{-p}\cdot\phi_{\lambda 0 a(1)\dots a(p-1)}$$

for $1 \leqq a(1) < \cdots < a(p-1) \leqq n$,

which implies:

$$(-d_{X/S}\cdot\beta_p(\phi))_{\lambda a(1)\dots a(p)}=\sum_{t=1}^{p}(-1)^t\partial/\partial z_\lambda^{a(t)}(f_\lambda^{-p}\phi_{\lambda 0a(1)\dots\hat{t}\dots a(p)}).$$

On the other hand, using (3.3.3) and (3.2.3),

$$(\beta_{p+1}\cdot\nabla_\tau(\phi))_{\lambda a(1)\dots a(p)}=f_\lambda^{-p-1}\cdot\sum_{t=1}^{p}(-1)^t f_\lambda^{p+1}\partial/\partial z_\lambda^{a(t)}(\phi_{\lambda 0a(1)\dots\hat{t}\dots a(p)}/f_\lambda^p).$$

Hence we get:

$$-d_{X/S}\cdot\beta_p=\beta_{p+1}\cdot\nabla_\tau \qquad \square$$

As the first step for our aim, we constructed τ-derivation ∇_τ in the above. In the second step, we shall define another derivation κ as follows, which will be called "Koszul operator".

(3.4.1) $$\kappa=\alpha_{p-1}\cdot\beta_p\colon\{\wedge^p J^1_{X/S}(L)\}(mD_\tau)\xrightarrow{\beta_p}\Omega^{p-1}_{X/S}((m+p)D_\tau)$$

$$\xrightarrow[\alpha_{p-1}]{}\{\wedge^{p-1}J^1_{X/S}(L)\}((m+1)D_\tau)$$

For a local section $\phi={}^0\sum\phi_{\lambda a(1)\dots a(p)}G_\lambda^{a(1)}\wedge\cdots\wedge G_\lambda^{a(p)}$ of $\{\wedge^p J^1_{X/S}(L)\}(mD_\tau)$, the local expression of $\kappa(\phi)$ is:

(3.4.2) $$\kappa(\phi)=\sum(\phi_{\lambda 0a(1)\dots a(p-1)}/f_\lambda)\times G_\lambda^{a(1)}\wedge\cdots\wedge G_\lambda^{a(p-1)},$$

$$\text{i.e.}\quad \kappa(\phi)_{\lambda a(1)\dots a(p-1)}=\begin{cases}\phi_{\lambda 0a(1)\dots a(p-1)}/f_\lambda & \text{if } a(1)\geqq 1\\ 0 & \text{if } a(1)=0.\end{cases}$$

Then ∇_τ and κ have good properties as mentioned below.

(3.4) Proposition.

(3.4.3) $$\nabla_\tau\cdot\nabla_\tau=0$$

(3.4.4) $$\nabla_\tau\cdot\kappa+\kappa\cdot\nabla_\tau=0$$

(3.4.5) $$\kappa\cdot\kappa=0.$$

Proof. By (3.3) Proposition and the construction (3.4.1) of Koszul operator, (3.4.4) and (3.4.5) obviously hold. Hence we have only to show that (3.4.3) is valid. Let us take a local section ϕ of $\{\wedge^p J^1_{X/S}(L)\}(mD_\tau)$ and prove that the coefficients of the local section $(\nabla_\tau\cdot\nabla_\tau(\phi))$ of $\{\wedge^{p+2}J^1_{X/S}(L)\}(mD_\tau)$ are zero.

$$\begin{aligned}&(\nabla_\tau\cdot\nabla_\tau(\phi))_{\lambda 0a(1)\dots a(p+1)}\\ &=\sum_{t=1}^{p+1}(-1)^t f_\lambda^{p+2}\partial/\partial z_\lambda^{a(t)}(f_\lambda^{-p-1}\nabla_\tau(\phi)_{\lambda 0a(1)\dots\hat{t}\dots a(p+1)})\\ &=\sum_{t=1}^{p+1}(-1)^t f_\lambda^{p+2}\partial/\partial z_\lambda^{a(t)}\{f_\lambda^{-p-1}\sum_{s=1}^{t-1}(-1)^s f_\lambda^{p+1}\partial/\partial z_\lambda^{a(s)}(\phi_{\lambda 0a(1)\dots\hat{s}\dots\hat{t}\dots a(p+1)}/f_\lambda^p)\\ &\qquad+f_\lambda^{-p-1}\sum_{s=t+1}^{p+1}(-1)^{s-1}f_\lambda^{p+1}\partial/\partial z_\lambda^{a(s)}(\phi_{\lambda 0a(1)\dots\hat{t}\dots\hat{s}\dots a(p+1)}/f_\lambda^p)\}\end{aligned}$$

$$= \sum_{s<t} (-1)^{t+s} f_\lambda^{p+2} (\partial^2/\partial z_\lambda^{a(t)} \partial z_\lambda^{a(s)}) (\phi_{\lambda 0 a(1)\dots\hat{s}\dots\hat{t}\dots a(p+1)}/f_\lambda^p)$$

$$- \sum_{s>t} (-1)^{s+t} f_\lambda^{p+2} (\partial^2/\partial z_\lambda^{a(t)} \partial z_\lambda^{a(s)}) (\phi_{\lambda 0 a(1)\dots\hat{t}\dots\hat{s}\dots a(p+1)}/f_\lambda^p) = 0.$$

Also for $a(0) \geqq 1$, the similar calculation shows that

$$(\nabla_\tau \cdot \nabla_\tau(\phi))_{\lambda a(0)\dots a(p+1)} = 0. \qquad \square$$

Based on the result of (3.4) Proposition, we can make sheaves of meromorphic L-valued 1-jet forms into a sheaf of complex as follows, which will be called "L-valued 1-jet de Rham complex".

(3.5) Theorem. *Let $f: X \to S$ be a smooth morphism of complex manifolds with relative dimension n, L a line bundle over X, and m a non-negative integer. Take a global section $\tau \in \Gamma(X, L)$. Then, there exists a sheaf of double complex $\Xi_{X/S}^{**}(m, \tau)$ as follows.*

$$\Xi_{X/S}^{p,q}(m, \tau) = \begin{cases} \{\wedge^{p-q} J_{X/S}^1(L)\}((m+q)D_\tau) & \text{if } n \geqq p \geqq q \geqq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\nabla_\tau : \Xi_{X/S}^{p,q}(m, \tau) \longrightarrow \Xi_{X/S}^{p+1,q}(m, \tau) \qquad (\tau\text{-derivation cf. (3.1.1)})$$

$$\kappa : \Xi_{X/S}^{p,q}(m, \tau) \longrightarrow \Xi_{X/S}^{p,q+1}(m, \tau) \qquad (\text{Koszul operator cf. (3.4.2)})$$

For the index m, $\{\Xi_{X/S}^{**}(m, \tau)\}$ forms an inductive system by canonical inclusions depending on "the order of poles" along the divisor D_τ. Put $\Xi_{X/S}^{**}(*, \tau)$ to be the inductive limit of the inductive system. Then $\Xi_{X/S}^{**}(*, \tau)$ induces a sheaf of simple complex $\Xi_{X/S}^{*}(\tau)$ (L-valued 1-jet de Rham complex) by giving total degree and total derivation $\triangle_\tau = \nabla_\tau + \kappa$. Moreover, this sheaf of complex $\Xi_{X/S}^{*}(\tau)$ is naturally quasi-isomorphic to the usual meromorphic de Rham complex with poles only along D_τ.

$$\Omega_{X/S}^{*}(*D_\tau) \xrightarrow{Qis.} \Xi_{X/S}^{*}(\tau)$$

Proof. It is obvious by its construction and (3.3) Proposition. $\square$

§4. Spectral sequences induced by L-valued 1-jet de Rham complexes

Let M be a compact complex manifold, E a holomorphic vector bundle of rank r over M and $L = O_{P(E)}(1)$ the tautological line bundle of the projective bundle $f: P = P(E) = (E^\vee - \{0\}/\mathbb{C}^*) \to M$. Then we have the following lemma, which gives a key for construcing spectral sequences from "E-valued" cohomologies to topolgical cohomologies.

(4.1) Lemma. *In the situation above, there is a canonical O_P-linear isomorphism*:

$$\Phi : f^*E \xrightarrow{\sim} J_{P/M}^1(L). \tag{4.1.1}$$

Hence we see also that

$$(4.1.2)\qquad f_*((\wedge^p J^1_{P/M}(L))\otimes L^m)=\begin{cases}\wedge^p E\otimes S^m(E) & \text{if } m\geqq 0\\ 0 & \text{if } m<0,\end{cases}$$

(4.1.3)

$$R^q f_*((\wedge^p J^1_{P/M}(L))\otimes L^m)=\begin{cases}0 & \text{if } 0<q<r-1\\ 0 & \text{if } q=r-1,\ m>-r\\ S^{-m-r}(E^\vee)\otimes \det E^\vee\otimes \wedge^p E & \text{if } q=r-1,\ m\leqq -r.\end{cases}$$

Proof. If we can prove (4.1.1), then (4.1.2) and (4.1.3) are obvious by projection formula. Hence we have only to show (4.1.1). Let us take an open set U of M and consider:

$$\Gamma(U,E)\simeq\Gamma(U,f_*L)=\Gamma(f^{-1}(U),L)\xrightarrow{j^1_{P/M}}\Gamma(f^{-1}(U),J^1_{P/M}(L)),$$

where $j^1_{P/M}$ denotes 1-jet map introduced in § 1. Since $j^1_{P/M}$ has O_M-linearity, we can get an O_P-linear homomorphism $\Phi:=f^*(j^1_{P/M}):f^*E\to J^1_{P/M}(L)$. We shall show that this Φ gives a desired isomorphism. Because our problem is local, we may assume that $E=\bigoplus_{k=1}^{r} O_M Z^k$ and M has a system of coordinates $(v^1,\dots,v^m)$. Then, $\operatorname{rank} J^1_{P/M}(L)=1+\operatorname{rel\cdot dim} P/M=\operatorname{rank} f^*E$. Hence we have only to show the surjectivity of Φ. Let us choose an arbitrary point x of P. We can regard the free basis $\{Z^1,\dots,Z^r\}$ of E as a system of homogeneous fibre coordinates, suppose that the point x is included by the open set U_a which is defined by $Z^a\neq 0$. Then we have a system of local fibre coordinates $((Z^1/Z^a),\dots\hat{a}\dots(Z^r/Z^a))$ on U_a of f: $P\to M$. Put $(G^0_a, G^1_a,\dots\hat{a}\dots,G^r_a)$ to be the canonical local frame of $J^1_{P/M}(L)$ on the C.F.-open set $(U_a, ((Z^1/Z^a),\dots\hat{a}\dots,(Z^r/Z^a); v^1,\dots,v^m), Z^a)$ of the line bundle L over M, and $\{f^*Z^1,\dots,f^*Z^r\}$ the free basis of f^*E induced by $\{Z^1\dots Z^r\}$. Then:

$$\Phi(f^*Z^k)=(Z^k/Z^a)G^0_a+G^k_a \qquad \text{if } k\neq a,$$
$$\Phi(f^*Z^a)=G^0_a.$$

By these equalities, we can easily see the surjectivity of Φ. □

(4.2) Remark. (i) Let us recall the sequence (1.1.2) with putting F to be $L=O_P(1)$ and f: $X\to S$ to be f: $P=P(E)\to M$. Then Lemma (4.1) shows us that this sequence is nothing but Euler sequence of the sheaf of relative 1-forms of the projective bundle:

$$(4.3.1)\qquad 0\longrightarrow \Omega^1_{P/M}\xrightarrow{u} f^*E\otimes L^\vee\xrightarrow{v} O_P\longrightarrow 0.$$

(ii) The map Φ of Lemma (4.1) does not coincide with the composition of the two maps $j^1_{P/M}$: $L\to J^1_{P/M}(L)$ and $v: f^*E\to L$ in the above.

With placing $f\colon X \to S$ to be $f\colon P = P(E) \to M$ and F to be $L = O_P(1)$, let us consider the sequence (1.1.3) tensored by $L^\vee$:

$$0 \longrightarrow f^*\Omega^1_M \longrightarrow J^1_P(L) \otimes L^\vee \longrightarrow J^1_{P/M}(L) \otimes L^\vee \longrightarrow 0. \tag{4.3.2}$$

After taking k-th wedge of the above, we find that $\wedge^k\{J^1_P(L) \otimes L^\vee\}$ has a natural filtration $\{F^{(k)}_a\}$ such that

$$\begin{aligned} &\wedge^k(J^1_P(L) \otimes L^\vee) = F^{(k)}_0 \supseteq \ldots \supseteq F^{(k)}_{k+1} = 0, \\ &F^{(k)}_a / F^{(k)}_{a+1} \simeq f^*\Omega^a_M \otimes \{\wedge^{k-a}(J^1_{P/M}(L) \otimes L^\vee)\}. \end{aligned} \tag{4.3.3}$$

Now we sppose that a global section τ of E is given. Then this section can be canonically regarded as a global section of $L = O_{P(E)}(1)$ and defines a divisor D_τ of $P = P(E)$. Hence the filtration (4.3.3) induces a filtration $\{F^{(p,q)}_a(m)\}$ of $\Xi^{p,q}_P(m, \tau) \simeq \{\wedge^{p-q}(J^1_P(L) \otimes L^\vee)\}((m+p)D_\tau)$ such that

$$\begin{aligned} &\Xi^{p,q}_P(m, \tau) = F^{(p,q)}_0(m) \supseteq \ldots \supseteq F^{(p,q)}_{p-q+1}(m) = 0, \\ &F^{(p,q)}_a(m)/F^{(p,q)}_{a+1}(m) \simeq f^*\Omega^a_M \otimes \{\wedge^{p-q-a}(J^1_{P/M}(L) \otimes L^\vee)\}((m+p)D_\tau). \end{aligned} \tag{4.3.4}$$

These filtrations behave well towards the action of τ-derivation and Koszul operator as follows.

(4.3) Proposition.

$$\nabla_\tau(F^{(p,q)}_a(m)) \subseteq F^{(p+1,q)}_a(m). \tag{4.3.5}$$

$$\kappa(F^{(p,q)}_a(m)) \subseteq F^{(p,q+1)}_a(m). \tag{4.3.6}$$

Proof. Let us take an arbitrary C.F-open set $U = (U, (w^1, \ldots, w^u; v^1, \ldots, v^b), e)$ of $L = O_{P(E)}(1)$ over M (cf. (2.1.1)) and the canonical local frame $(G^0, G^1, \ldots, G^u, G^{u+1}, \ldots, G^{u+b})$ of $\underline{J^1_P(L)}$ with regarding the C.F.-open set U as a C.F-open set over the point Spec($\mathbb{C}$), where $u = \operatorname{rank} E - 1$, $b = \dim M$, and $\{G^{u+1}, \ldots, G^{u+b}\}$ corresponds to $\{dv^1 \otimes e, \ldots dv^1 \otimes e\}$. Then, the subsheaf $F^{(p,q)}_a(m)$ is "generated" by the local sections $\{G^{a(1)} \wedge \cdots \wedge G^{a(p-q)}\}$ which contain some of $\{G^{u+1}, \ldots, G^{u+b}\}$ more than or equal to $\underline{a}$ as factors. By local expressions (3.1.1) and (3.4.2), we can see that ∇_τ and κ never decrease the number of $G^{u+1}, \ldots, G^{u+b}$. Hence $\{F^{(*,*)}_a(m)\}$ is closed under the action of τ-derivation and Koszul operator. □

Based on the result above, we define subcomplexes of $\Xi^*_P(m, \tau)$ and those of L-valued 1-jet de Rham complex $\Xi^*_P(\tau) = \operatorname{Ind.lim}_m \Xi^*_P(m, \tau)$ as follows:

$$F_a(m) = \bigoplus_{p,q} F^{(p,q)}_a(m) \subseteq \Xi^*_P(m, \tau), \tag{4.3.7}$$

$$F_a = \operatorname{Ind.lim}_m F_a(m) \subseteq \Xi^*_P(\tau), \tag{4.3.8}$$

where $\Xi^*_P(m, \tau) = (\Xi^n_P(m, \tau), \nabla_\tau + \kappa)$ and $\Xi^n_P(m, \tau)$ denotes $\bigoplus_{p+q=n} \Xi^{p,q}_P(m, \tau)$.

To make a spectral sequence from "E-valued" cohomologies to topological cohomologies, we need a preparation of homological algebra, namely Leray spectral sequence for hypercohomologies as mentioned in the sequel.

(4.4) Proposition. *Let $f\colon X \to Y$ a continuous map of topological spaces and $C^* = \{C^t\}_{t=0,1,\dots}$ a sheaf of complex bounded below on the topological space X. Then there is a spectral sequence*:

(4.4.1) $$E_2^{p,q} = \mathbb{H}^p(Y, R^q f_*(C^*)) \Rightarrow \mathbb{H}^{p+q}(X, C^*).$$

Proof. To simplify our terminology, we shall use "complex" and "bicomplex" in the sequel instead of "sheaf of complex" and "sheaf of bicomplex", respectively. Let us take an injective Cartan-Eilenberg resolution $\{I^{**}\}$ of the complex C^* such that the sequence: $0 \to C^a \to I^{a,*}$ is exact for all integers $\underline{a}$. Then we can make an injective Cartan-Eilenberg resolution $\{M^{***}\}$ of a double complex $\{f_*(I^{**})\}$ on Y. Hence the sequence: $0 \to f_*(I^{a,b}) \to M^{a,b,*}$ is exact. By $\{s(f_*(I^{**}))\}$, we denote the total simple complex of the double complex $\{f_*(I^{**})\}$. Through a suitable canonical bigrading, $\{M^{***}\}$ can be regarded as a Cartan-Eilenberg resolution $\{{}_{12}s(M^{***})\}$ of the complex $\{s(f_*(I^{**}))\}$. Next we shall consider the functor $T := \Gamma(Y, -)$ from the category of abelian sheaves on Y to the category of abelian groups. Then there is a right hyperderived functor $\mathbb{R}^*T(-)$, which satisfies:

$$\mathbb{R}^a T(s(f_*(I^{**}))) = H^a(\Gamma(Y, s({}_{12}s(M^{***})))) = H^a(s(\Gamma(Y, M^{***}))),$$

where $s(\Gamma(Y, M^{***}))$ denotes a simple complex of abelian groups induced by the tricomplex of abelian groups $\{\Gamma(Y, M^{***})\}$ with total grading. Since the sheaf $f_*(I^{a,b})$ is flasque, the sequence:

$$0 \longrightarrow \Gamma(Y, f^*(I^{a,b})) \longrightarrow \Gamma(Y, M^{a,b,*})$$

is exact for all integers a and b. Hence $\mathbb{R}^a T(s(f_*(I^{**}))) \simeq H^a(s(\Gamma(Y, f_*(I^{**}))))$ $\simeq \mathbb{H}^a(X, C^*)$. On the other hand, we take the sheaf of cohomologies $\underline{H}^{\,*}_{II}(f_*(I^{**}))$ in the second index. Then $\{\underline{H}^{\,q}_{II}(f_*(I^{**}))\}$ forms a complex by the first index. In this situation, the result of E.G.A.III (cf. [G-1] Proposition (11.7.2)) gives us a spectral sequence:

$$\begin{aligned} {}^{(c)}E_2^{p,q} &= \mathbb{R}^p T(\underline{H}^{\,q}_{II}(f_*(I^{**}))) = \mathbb{H}^p(Y, R^q f_*(C^*)) \\ &\Rightarrow \mathbb{R}^{p+q} T(s(f_*(I^{**}))) \simeq \mathbb{H}^{p+q}(X, C^*). \end{aligned}$$ □

Now our preparations are finished. Let us go back to our first situation of §4. Since D_τ is f-ample and M is compact, for a sufficiently large m, we get:

$$R^q f_*(\Xi^*_P(m, \tau)) = 0 \qquad (q > 0).$$

Then, with applying (4.4) Proposition to our case, the spectral sequence (4.4.1) degenerates and shows us that

$$\mathbb{H}^n(M, f_* \Xi_P^*(m, \tau)) \simeq \mathbb{H}^n(P(E), \Xi_P^*(m, \tau)) \tag{4.4.2}$$

for a sufficiently large m. Moreover we may assume that the sheaf of complex $f_* \Xi_P^*(m, \tau)$ has a filtration $\{f_*(F_a(m))\}$ which satisfies:

$$\begin{aligned} &f_*(F_a(m)) = \bigoplus f_*(F_a^{(p,q)}(m)), \\ &f_* \Xi_P^{p,q}(m, \tau) \supseteq f_*(F_0^{(p,q)}(m)) \supseteq \dots \supseteq f_*(F_{p-q+1}^{(p,q)}(m)) = 0, \\ &f_* F_a^{(p,q)}(m)/f_* F_{a+1}^{(p,q)}(m) \simeq \Omega_M^a \otimes \wedge^{p-q-a} E \otimes S^{m+q+a}(E), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{for } m \gg 0). \end{aligned} \tag{4.4.3}$$

After taking the inductive limit of the right hand side of (4.4.2) for the index m, (3.5) Theorem shows us that

$$\begin{aligned} \operatorname*{Ind.lim}_m \mathbb{H}^n(M, f_* \Xi_P^*(m, \tau)) &\simeq \mathbb{H}^n(P(E), \Xi_P^*(\tau)) \\ &\simeq \mathbb{H}^n(P(E), \Omega_p^*(*D_\tau)) \simeq H^n(P(E) - D_\tau, \mathbb{C}), \end{aligned} \tag{4.4.4}$$

where we used Hironaka's theorem on resolution of singularities to show the last isomorphism for D_τ with general singularities. As for admitting nilpotent structure for D_τ, the cofinality of inductive systems for D_τ and for $(D_\tau)_{red}$ brings us the same results.

Next we shall show that $H^n(P(E) - D_\tau, \mathbb{C}) \simeq H^n(M - Z_\tau, \mathbb{C})$, where Z_τ denotes the zero locus of τ on M as a section of E. In fact, for a point x of M, the point x is contained by Z_τ if and only if the fibre $f^{-1}(x) \subseteq P(E)$ is included by the divisor D_τ. Hence we have a commutative diagram:

$$\begin{array}{ccc} P(E) & \hookleftarrow & P(E) - D_\tau = : V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_0 = f|_V} \\ M & \hookleftarrow & M - Z_\tau \end{array} .$$

Since $f_0 \colon V \to M - Z_\tau$ is an $\mathbb{A}^{r-1}$-fibre bundle, Leray-Hirsch theorem shows the desired isomorphism. Thus, using (4.4.4), we obtain:

$$\operatorname*{Ind.lim}_m \mathbb{H}^n(M, f_* \Xi_P^*(m, \tau)) \simeq H^n(M - Z_\tau, \mathbb{C}). \tag{4.4.5}$$

On the other hand, for a sufficiently large m, $f_* \Xi_P^*(m, \tau)$ has the filtration (4.4.3). Then the filtration $\{f_* F_a(m)\}$ canonically induces a spectral sequence:

$$E_1^{p,q} = \operatorname*{Ind.lim}_m \mathbb{H}^q(M, \Omega_M^p \otimes f_* \Xi_{P/M}^*(m, \tau)) \Rightarrow H^{p+q}(M - Z_\tau, \mathbb{C}). \tag{4.4.6}$$

Moreover, hypercohomologies have a canonical spectral sequence. Hence we obtain a double spectral sequence as follows.

(4.5) Theorem. *Let M be a compact complex manifold, E a holomorphic vector bundle of rank r over M and $L = \mathcal{O}_{P(E)}(1)$ the tautological line bundle of the projective bundle $f \colon P = P(E) = (E^\vee - \{0\}/\mathbb{C}^*) \to M$. Take a global section τ of E. Then there exists a double spectral sequence:*

$$E_1^{c,(a,b)} = \operatorname*{Ind.lim}_{m} \bigoplus_{a(1),a(2),b} H^b(M, \Omega_M^c \otimes \wedge^{a(1)-a(2)} E \otimes S^{m+a(2)}(E))$$

$$(a(1) + a(2) = a,\ r - 1 \geqq a(1) \geqq a(2) \geqq 0)$$

$$\Rightarrow E_1^{c,a+b} = \operatorname*{Ind.lim}_{m} \mathbb{H}^{a+b}(M, \Omega_M^c \otimes f_* \Xi_{P/M}^*(m, \tau))$$

$$\Rightarrow H^{a+b+c}(M - Z_\tau, \mathbb{C}).$$

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Added in proof. One started from the fact (4.1.1) in the case $M = \{pt\}$ and came to a suitable definition for the τ-derivation as a generalization of the Koszul derivation over the affine space. Recently, one noticed that a long time ago, Ogus also introduced a nice derivation $d_{OG} = d\colon P^a(m) \cong \{\wedge^a J_{X/S}^1(L)\}((m-a)D_\tau) \to P^{a+1}(m) \cong \{\wedge^{a+1} J_{X/S}^1(L)\}((m-a-1)D_\tau)$ in [O] for another purpose. His derivation d_{OG} is the same as ours except a slight difference in handling the order of poles (cf. (3.3) Proposition). Moreover, without any violent calculation, his derivation d_{OG} is naturally derived from the exterior derivation of a logarithmic de Rham complex depending only on the line bundle L, and not on the section τ. Nevertheless, by two reasons, one still think that it may be worth presenting this topic in such a rough style. One reason is that the order of poles must be adjusted as ours with using the section τ for the compatibility with the meromorphic de Rham complex. The other one is that up to the present, one is not sure to accomplich forthcoming applications of this work without using the explicit formulae such as (3.1.1).