J. Math. Kyoto Univ. (YMKYAZ)
30-1 (1990) 109-121

On a cyclic covering of a projective manifold

By

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§0. Introduction

The main purpose of this article is to investigate a finite normal cyclic covering of a projective manifold (i.e., the rational function field corresponding to the covering is a cyclic extension). In §1, we consider the structure of a cyclic covering from a field theoritic view point. In §2, we consider the direct image of the structure sheaf by the method of Esnault-Viehweg. And in §3, we applied the result of §1 and §2 to 3 cases. Our main results are as folows.

Proposition 3.3. *Let π $X \to Y$ be a finite cyclic covering of Y where X is normal and Y is non-singular. Let B denote the branch locus of π. Assume:*

(i) *B is an irreducible divisor.*

(ii) *For each $y \in B$, $\pi^{-1}(y)$ consists of one point.*

Then there exists a line bundle F, so that X is embedded in the total space of F.

Proposition 3.4. *Let X be a finite normal cyclic covering of an abelian variety A. Assume that X is of general type, and its covering map is flat. Then,*

$$h^i(\mathcal{O}_X) = h^i(\mathcal{O}_A) \qquad \text{for } 0 \le i < d$$

and

$$h^d(\mathcal{O}_X) \ge n$$

where $d = \dim X = \dim A$ and $n =$ the degree of the covering.

Theorem 3.5. *Let $\pi: S \to \mathbf{P}^2$ be a finite normal covering of $\mathbf{P}^2$ whose covering degree is a prime integer p. Assume that the branch locus of π is $C_1 \cup C_2$, where C_i is a smooth curve whose degree is n_i, and the divisor $C_1 + C_2$ has at most simple normal crossings. Then:*

(i) *There exists a unique integer v with $1 \le v \le p-1$, and singularities of S are all cyclic quotient singularities of type (p, v) or $(p, p-v)$ and they do not appear simutaneously.*

(ii) *The direct image of the structure sheaf of S is isomorphic to*

$$O_{\mathbf{P}^2} \oplus \bigoplus_{i=1}^{p-1} \mathcal{O}_{\mathbf{P}^2}\left(\frac{k}{p}(n_1 + (p-v)n_2) + \left[\frac{k(p-v)}{p}\right]n_2\right)$$

Received, April 4, '88

where [ ] *denotes Gaussian symbol.*

Acknowledgement. I would like to thank Prof. K. Ueno for his valuable suggestions.

Notations and Conventions

The ground field is always a complex number field $\mathbf{C}$.

$h^i(X, \mathcal{O}_X) = h^i(\mathcal{O}_X) = \dim_{\mathbf{C}} H^i(X, \mathcal{O}_X)$

$\mathbf{C}(X)$: the rational function field of X

$\Phi_{|D|}$: the rational map associated to a linear system $|D|$

Let D_1, D_2 be divisors.

$D_1 \sim D_2$ means linear equivalence of two divisors.

§1. Constructions of cyclic coverings

For the first, we remind us of some constructions of cyclic coverings.

Construction 1: Let Y be a projective mainfold and B be a smooth divisor such that $L^{\otimes n} \sim B$ for some $L \in \operatorname{pic}(Y)$ and $n \in \mathbf{N}$. Then, as is well-known, we can construct a finite cyclic covering of Y ramified over B in the total space of L. This construction is a very familiar method, but most of cyclic coverings of Y is not of this type.

Construction 2: Let φ be an element of the retional field of Y, $\mathbf{C}(Y)$, and X be a subvariety in $Y \times \mathbf{P}^1$ which is defined as follows:

Let $\varphi = \varphi_0/\varphi_\infty$ be a local representation, and $[\zeta_0 : \zeta_1]$ be a homogeneous coordinate of $\mathbf{P}^1$. Define:

$$\tilde{X} = \{(y, [\zeta_0 : \zeta_1]) \in Y \times \mathbf{P}^1 | \zeta_0^n \varphi_0 - \zeta_1^n \varphi_\infty = 0\}.$$

Let X be the stein factorization of $p_1|_{\tilde{X}} X^{\sim} \to Y$ where p_1 is a projection to $Y^1 \times \mathbf{P} \to Y$. Then X is a normal finite cyclic covering of Y.

Construction 3: Let B be an effective divisor on Y such that $L^{\otimes n} \sim B$ for some $L \in \operatorname{Pic}(Y)$ and $n \in \mathbf{N}$. Then we can construct a cyclic covering of Y in the total space of L as Constructron 1. Let $n: X' \to X$ be the normalization. Then, X' is a finite normal cyclic covering of Y.

It is clear that construction 1 is a special case of construction 3. In this section, we consider the relation between construction 1, 2, and 3.

Let $p: X \to Y$ be a finite normal cyclic covering, and assume that the Galois group $\operatorname{Gal}(\mathbf{C}(X)/\mathbf{C}(Y))$ is isomorphic to $\mathbf{Z}/n\mathbf{Z}$. By field theory, there exist θ in $\mathbf{C}(X)$ whose minimal polynomial is T^n-φ where φ is an element of $\mathbf{C}(Y)$. We introduce following notations:

D_0 : the zero divisor of φ, and $D_0 = \sum_i \nu_i D_i^{(0)}$, its decomposition to irreducible components.

D_∞: the polar divisor of φ, and $D_\infty = \sum_j \mu_j D_j^{(\infty)}$, its decomposition to irreduclble components.

Put $B = D_0 + (n-1)D_\infty$. Then $B \sim nD_0$. So, by construction 3, we can construct a cyclic covering ramified over B in the total space of a line bundle associated to the divisor D_0. Let X_1 be its normalization. Then we have:

Proposition 1.1. Let X, X_1 be as above. Then X and X_1 is isomorphic to each other.

Proof. By the uniquness of the $\mathbf{C}(X)$(resp. $\mathbf{C}(X_1)$)-normalization of Y (see Iitaka [1], Theorem 2.2.4), it is enough to show that $\mathbf{C}(X)=\mathbf{C}(X_1)$. By construction 2, we construct a birational model of X in $Y\times\mathbf{P}^1$. We denote it $\tilde{X}$. We will prove that $\mathbf{C}(\tilde{X})=\mathbf{C}(X_1)$. Let

$$f_0=f_1^{(0)^{\nu_1}}\cdots f_k^{(0)^{\nu_k}} \text{ and } f_\infty=f_1^{(\infty)^{\mu_1}}\cdots f_m^{(\infty)^{\mu_m}}$$

be local equations of D_0 and D_∞ respectively. Then X_1 is constructed as follows:

Put

$$B=D_0+(n-1)D_\infty$$

$$L=\text{ a line bundle associated to divisor } D_0.$$

Then

$$L^{\otimes n}\sim B.$$

Define a subvariety X_1' in the total space of L by the equation $\zeta^n=f_0(f_\infty)^{n-1}$ locally. Then X_1' is a cyclic covering of Y, and its normalization is X_1. Define a raitonal map from X_1 to X as follows:

Locally,

$$X_1 \longrightarrow X_1' \cdots\cdots\cdots\to X$$

$$x \longmapsto (\pi(x),\xi(x)) \mapsto (\pi(x),\xi(x)/f_\infty)$$

(π: the projection of a line bundle).

But, by construction, above rational maps defined over Y. We can easily check that the above maps are birational maps. Therefore $\mathbf{C}(X_1)=\mathbf{C}(\tilde{X})$. Since $\tilde{X}$ is birational to X, so $\mathbf{C}(\tilde{X})=\mathbf{C}(X)$. Therefore $\mathbf{C}(X_1)=\mathbf{C}(X)$. This proves our proposition.

Q.E.D.

§2. The direct image of $\mathcal{O}_X$, $p_*\mathcal{O}_X$

In this section, we assume that the finite morphism p is always flat. Since p is flat and finite, $p_*\mathcal{O}_X$ is locally free sheaf. Moreover, in our case, there is an action of $\mathbf{Z}/n\mathbf{Z}$. Therefore, $p_*\mathcal{O}_X$ is decomposed into the direct sum of line bundles. Next result which are due to H. Esnault-E. Viehweg are important.

Lemma 2.1. *Let D be an effective divisor on Y and $D=B+\sum_j \nu_j E_j$ its decomposition into prime divisors. Suppose that for some invertible sheaf L and* *some integer* $n > 0$, *we have*

$$L^{\otimes n} = \mathcal{O}_Y(D).$$

Then, by Construction 3, *we obtain a finite normal cyclic covering* $p: X \to Y$. *Assume that p is flat. Then*

$$p_*\mathcal{O}_X = \bigoplus_{i=1}^{N-1} L^{(i)^{-1}}$$

$$L^{(i)} = L^{\otimes i} \otimes \mathcal{O}_Y\left(-\sum_j \left[\frac{\nu_j i}{N}\right] E_j\right)$$

where [ ] *is Gaussian symbol.*

For a proof, see Viehweg [6]. By Viehweg [5], if D is an effective divisor with simple normal crossing, X has only rational singularities and p is flat. Therefore, we can calculate numerical invariants of non-singular model of X.

Example. Let l_0, l_∞ be two lines in $\mathbf{P}^2$. Let S be a normal surface corresponding to a field $\mathbf{C}(\mathbf{P}^2)(\theta)$ where $\theta^n = f$, $f \in \mathbf{C}(\mathbf{P}^2)$ and $f = l_0/l_\infty$, and its minimal resolution of S is a rational ruled surface of degree n. Let $p: S \to \mathbf{P}^2$ be a covering map. By the above result, we obtain:

$$p_*\mathcal{O}_S = \mathcal{O}_{\mathbf{P}^2} \oplus \underbrace{\mathcal{O}_{\mathbf{P}^2}(-1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^2}(-1)}_{n-1}.$$

§ 3. Applications

(I) In Wavrik [7], he proved the following;

Theorem 3.1. *Let* $\pi: X \to Y$ *be a cyclic covering of* Y *where* X *and* Y *are complex manifolds. Then we can find a line bundle* F *on* Y *so that* X *is embedded in a total space* F.

For a proof, see Wavrik [7].

Remark. In this above theorem, definition of "a cyclic covering" is slightly different from our definition. His definition is as follows:

Definition of a cyclic covering in the sense of Wavrik [7].

Let $\pi: X \to Y$ be a k-sheeted branched covering of Y, where X and Y are complex manifold. Let C denote the branch locus. We call X a cyclic covering of Y if the following conditions are satisfied:

(i) For each $x \in C$, $\pi^{-1}(x)$ consists of one point.

(ii) The group of covering transformations of $X \backslash \pi^{-1}(C)$ over $Y \backslash C$ is cyclic group of order k.

(iii) Fro each $\in C$ we can find a neighborhood U with coordinates $(z_1, \dots, z_n)$ on U and $(\zeta_1, \dots, \zeta_n)$ on $\pi^{-1}(U)$ such that the map is given by $z_i = \zeta_i$

$(1 \leq i \leq n-1)$, $z_n = \zeta_n^k$.

(iv) If $k \neq 2$, C is connected.

In the above definition, the condition (iii) implies that the branch locus, C is non-singular, and this is essential. Assume that X is normal. Then, of course, C may be singular. In our case, X can not be always embedded in the total space of line bundles. For example, the normal surface S in Example, § 2, has an only singurlity over $l_0 \cap l_\infty$, and this singurality is rational n-ple point. As is well-known, rational surface singularities are hypersurface singularities if and only if they are rational double points. Therefore, if $n \geq 3$, S can not be embedded in any total space of line bundles over $\mathbf{P}^2$.

For a normal finite cyclic covering, we obtain proposition;

Proposition 3.3. *Let $\pi: X \to Y$ be a finite cyclic covering of Y where X is normal and Y is non-singular. Let B denote the branch locus of π. Assume:*

(i) B is an irreducible divisor.

(ii) For each $y \in B$, $\pi^{-1}(y)$ consists of one point.

Then there exists a line bundle F, so that X is embedded in the total space of F.

Proof. Since $\mathbf{C}(X)$ is a cyclic extension of $\mathbf{C}(Y)$, so, by field theory, there exists an element Θ_1 in $\mathbf{C}(X)$ so that its minimal polynomial is $T^n - f$, $f \in \mathbf{C}(Y)$ where $n =$ the degree of the covering. In the following, the notation is the same as § 1. Put:

$$f = \frac{f_1^{(0)^{\nu_1}} \cdots f_k^{(0)^{\nu_k}}}{f_1^{(\infty)^{\mu_1}} \cdots f_l^{(\infty)^{\mu_l}}}$$

where $f_i^{(0)}$ and $f_j^{(\infty)}$ are definining equations of $D_i^{(0)}$ and $D_j^{(\infty)}$, respectively. Put:

$$B = D_0 + (n-1)D_\infty, \qquad L = [D_0].$$

Now, we construct a cyclic covering V' in a total space of L by Construction 3 in § 1. By Proposition 1.1, if V denote a normalization of V', V is isomorphic to X. Therefore, by the assumption (i), we may assume $\nu_2, \dots, \nu_k$, and $\mu_1, \dots, \mu_l$ are all multiple of n and only ν_1 is not. Moreover, by the assumption (ii), g.c.d.$(n, \nu_1) = 1$. Hence there exist integers α, β such that $\alpha\nu_1 = \beta n + 1$. Consider a rational function f^α. This is represented as follows:

$$f^\alpha = \frac{f_1^{(0)^{\nu_1\alpha}} \cdots f_k^{(0)^{\nu_k\alpha}}}{f_1^{(\infty)^{\mu_1\alpha}} \cdots f_l^{(\infty)^{\mu_l\alpha}}} = f_1^{(0)} \left(\frac{f_1^{(0)^\beta} f_2^{(0)^{\nu_2'\alpha}} \cdots f_k^{(0)^{\nu_k'\alpha}}}{f_1^{(\infty)^{\mu_1'\alpha}} \cdots f_l^{(\infty)^{\mu_l'\alpha}}} \right)^n$$

where $\nu_i = n\nu_i'$, $\mu_j = n\mu_j'$ $(i = 2, \dots, k,\ j = 1, \dots, l)$.

Therefore,

$$D_1^{(0)} \sim n\Big(-\beta D_1^{(0)} - \sum_{i=1}^{k} \alpha\nu_i' D_i^{(0)} + \sum_{j=1}^{l} \alpha\mu_j' D_j^{(\infty)}\Big).$$

Put

$$F:=\sum_{j=1}^{l}\alpha\mu'_j D_j^{(\infty)}-\beta D_1^{(0)}-\sum_{i=2}^{k}\alpha\nu'_i D_i^{(0)}.$$

We can construct a cyclic covering of Y in the total space of F by the same method of Construction 1 in § 1. We denote it by X_1. We will prove X_1 is isomorphic to X. (Note that X_1 is a normal variety.) As in the proof of Proposition 1.1, it is enough to show that $\mathbf{C}(X)=\mathbf{C}(X_1)$. By our construction, $\mathbf{C}(X)=\mathbf{C}(Y)(\theta_1)$ and $\mathbf{C}(X_1)=\mathbf{C}(Y)(\theta_2)$ where minimal polynomials of θ_1, θ_2 are T^n-f, T^n-f^α respectively. Since $\alpha\nu_1=\beta n+1$, $\theta_2^{\nu_1}$ satisfies $T^n-f^{\beta n+1}=0$. Therefore the minimal polynomial of $\theta_2^{\nu_1}/f^\beta$ is T^n-f. Therefore $\mathbf{C}(X_1)\supset\mathbf{C}(X)$. But $[\mathbf{C}(X):\mathbf{C}(Y)]=n$. Therefore, $\mathbf{C}(X_1)=\mathbf{C}(X)$.

Q.E.D.

Remark. By the proof of the above proposition, if there exists an element $\theta\in\mathbf{C}(X)$ such that its minimal polynomial T^n-f, $f\in\mathbf{C}(Y)$ and f is of type as follows:

$$f=f_1f_2\cdots f_k\left(\frac{\cdots\cdots}{\cdots\cdots}\right)^n,$$

then, X is always embedded in a total space of a certain line bundle.

(II) Cyclic covering of abelian varieties. Let $p: X\to A$ be a finite cyclic covering of A, where X is a normal variety and A is an abelian variety. Let θ be and element of $\mathbf{C}(X)$ such that $\mathbf{C}(X)=\mathbf{C}(A)(\theta)$ and the minimal polynomial of θ is $T^n-\varphi$ for some $\varphi\in\mathbf{C}(A)$. Let D_0, D_∞ be the zero divisor of φ and the polar divisor of φ respectively, and $D_0=\sum_j \nu_i D_i^{(0)}$ and $D_\infty=\sum_j\mu_j D_j^{(\infty)}$ be their decomposition into irreducible components. We rewrite D_0 and D_∞ as follows:

$$D_0=\sum_i \nu_i D_i^{(0)}=\sum_i(\nu'_i+n\delta_i)D_i^{(0)},$$

$$D_\infty=\sum_j\mu_j D_j^{(\infty)}=\sum_j(\mu'_j+n\lambda_j)D_j^{(\infty)},$$

where δ_i, λ_i, ν_i, μ_j are non-negative integers and $0\le\nu'_i<n$ and $0\le\mu'_j<n$.

Put

$$\begin{aligned}B&=D_0+(n-1)D_\infty\\&=\sum_i(\nu'_i+n\delta_i)D_i^{(0)}+\sum_j(n-1)(\mu'_j+n\lambda_j)D_j^{(\infty)}.\end{aligned}$$

Then,

$$\begin{aligned}&\sum_i\nu'_iD_i^{(0)}+(n-1)\sum_j\mu'_jD_j^{(\infty)}\\&\quad=B-n(\sum_i\delta_iD_i^{(0)}+(n-1)\sum_j\lambda_jD_j^{(\infty)})\end{aligned}$$

$$\sim nD_0 - n(\sum_i \delta_i D_i^{(0)} + (n-1)\sum_j \lambda_j D_j^{(\infty)})$$

$$= n(D_0 - \sum_i \delta_i D_i^{(0)} - (n-1)\sum_j \lambda_j D_j^{(\infty)})$$

Put

$$L = [D_0 - \sum_i \delta_i D_i^{(0)} - (n-1)\sum_j \lambda_j D_j^{(\infty)}]$$

We construct a cyclic covering V' in the total space of a line bundle L. Let V denote its normalization. We will show that V is isomorphic to X. By the same argument as before, it is enough to show that $\mathbf{C}(V) = \mathbf{C}(X)$. By $f_i^{(0)}$ and $f_i^{(\infty)}$, we denote their defining equations of $D_i^{(0)}$ and $D_j^{(\infty)}$ respectively. Then, by using a local representation, the field $\mathbf{C}(X)$ is equal to

$$\mathbf{C}(X) = \mathbf{C}(A)(\theta),$$

where

$$\theta^n = \frac{f_1^{(0)^{\nu_1}} \cdots f_k^{(0)^{\nu_k}}}{f_1^{(\infty)^{\mu_1}} \cdots f_l^{(\infty)^{\mu_l}}}$$

Let X' be a cyclic covering of A obtained by Construction 2 in § 1 with respect to

$$\varphi = \frac{f_1^{(0)^{\nu_1}} \cdots f_k^{(0)^{\nu_k}}}{f_1^{(\infty)^{\mu_1}} \cdots f_l^{(\infty)^{\mu_l}}}.$$

Clearly, X' is birational to X. We define a rational map from V to X' as follows:

$$\Psi : V \longrightarrow V' \cdots\cdots\cdots \to X'$$

$$v \longmapsto (\pi(v), \zeta(v)) \mapsto \cdots\cdots\cdots \to$$

$$\left(\pi(v), \frac{\zeta(v) f_1^{(0)^{\delta_1}} \cdots f_k^{(0)^{\delta_k}} f_1^{(\infty)^{(n-1)\lambda_1}} \cdots f_l^{(\infty)^{(n-1)\lambda_l}}}{f_1^{(\infty)^{\mu_1}} \cdots f_l^{(\infty)^{\mu_l}}}\right)$$

where π is the projection from a total space of L to Y, and ζ is its fibre coordinate. By our construction,

$$\left(\frac{\zeta(v) f_1^{(0)^{\delta_1}} \cdots f_k^{(0)^{\delta_k}} f_1^{(\infty)^{(n-1)\lambda_1}} \cdots f_l^{(\infty)^{(n-1)\lambda_l}}}{f_1^{(\infty)^{\mu_1}} \cdots f_l^{(\infty)^{\mu_l}}}\right)^n$$

$$= \frac{\zeta(v)^n f_1^{(0)^{n\delta_1}} \cdots f_k^{(0)^{n\delta_k}} f_1^{(\infty)^{n(n-1)\lambda_1}} \cdots f_l^{(\infty)^{n(n-1)\lambda_l}}}{f_1^{(\infty)^{n\mu_1}} \cdots f_l^{(\infty)^{n\mu_l}}}$$

$$= \frac{f_1^{(0)^{\nu'_1 + n\delta_1}} \cdots f_k^{(0)^{\nu'_k + n\delta_k}} f_1^{(\infty)^{(n-1)(\mu'_1 + n\lambda_1)}} \cdots f_l^{(\infty)^{(n-1)(\mu'_1 + n\lambda_l)}}}{f_1^{(\infty)^{n\mu_1}} \cdots f_l^{(\infty)^{n\mu_l}}}$$

$$= \frac{f_1^{(0)^{\nu_1}} \cdots f_k^{(0)^{\nu_k}} f_1^{(\infty)^{(n-1)\mu_1}} \cdots f_l^{(\infty)^{(n-1)\mu_l}}}{f_1^{(\infty)^{n\mu_1}} \cdots f_l^{(\infty)^{n\mu_l}}}$$

$$= \frac{f_1^{(0)^{\nu_1}} \cdots f_k^{(0)^{\nu_k}}}{f_1^{(\infty)^{\mu_1}} \cdots f_l^{(\infty)^{\mu_l}}}.$$

By the above calculation, it is easy to see that Ψ is birational map. Hence $\mathbf{C}(X) = \mathbf{C}(V') = \mathbf{C}(V)$. Therefore, V is isomorphic to X.

In the following, we assume that $p: X \to A$ is flat. Then, by Lemma 2.1,

$$p_*\mathcal{O}_X \simeq \mathcal{O}_A \oplus \bigoplus_{m=1}^{n-1} L^{(m)^{-1}}$$

where

$$L^{(m)} = L^m \otimes \mathcal{O}_A\left(- \sum_i \left[\frac{v'_i m}{n}\right] D_i^{(0)} - \sum_j \left[\frac{(n-1)\mu'_j m}{n}\right] D_j^{(\infty)}\right)$$

$$L^m \otimes \mathcal{O}_A\left(- \sum_i \left[\frac{v'_i m}{n}\right] D_i^{(0)} - \sum_j \left((\mu'_j - 1)m + \left[\frac{m(n-\mu'_j}{n}\right]\right) D_j^{(\infty)}\right).$$

By our construction,

$$L^n \sim {\sum_i}' v'_i D_i^{(0)} + (n-1){\sum_j}' \mu'_j D_j^{(\infty)}$$

where ${\sum_i}'$, ${\sum_j}'$ mean that the sum are taken for non-zero v'_i, v'_j. Therefore,

$$\begin{aligned} L^{(m)^n} = L^{mn} \otimes \mathcal{O}_A\Bigg(& - \sum_i n\left[\frac{v'_i m}{n}\right] D_i^{(0)} \\ & - \sum_j \left(n(\mu'_j - 1)m + n\left[\frac{m(n-\mu'_j}{n}\right]\right) D_j^{(\infty)}\Bigg) \\ \sim \mathcal{O}_A\Bigg(& \sum_i m v'_i D_i^{(0)} + m(n-)\sum_j \mu'_j D^{(\infty)} - \sum_i n\left[\frac{v'_i m}{n}\right] D_i^{(0)} \\ & - \sum_j \left(mn(\mu'_j - 1) + n\left[\frac{m(n-\mu'_j}{n}\right]\right) D_j^{(\infty)}\Bigg) \\ = \mathcal{O}_A\Bigg(& \sum_i \left(m v'_i - n\left[\frac{v'_i m}{v}\right]\right) D_i^{(0)} \\ & + \sum_j \left(m(n-\mu'_j) - n\left[\frac{m(n-\mu'_j}{n}\right]\right) D_j^{(\infty)}\Bigg). \end{aligned}$$

By our consruction, $mv'_i > n\left[\dfrac{v'_i m}{v}\right]$, $m(n-\mu'_j) > n\left[\dfrac{m(n-\mu'_j}{n}\right]$.

Now we obtain the following.

Proposition 3.4. *Let X be a finite normal cyclic covering of an abelian variety A. Assume that X is of general type, and its covering map is flat. Then,*

$$h^i(\mathcal{O}_X) = h^i(\mathcal{O}_A) \qquad \text{for } 0 \le i < d$$

and

$$h^d(\mathcal{O}_X) \ge n$$

where $d = \dim X = \dim A$ *and* $n =$ *the degree of the covering.*

Proof. By the above calculation,

$$p_* \mathcal{O}_X \simeq \mathcal{O}_A \oplus \bigoplus_{m-1}^{n-1} L^{(m)^{-1}},$$

where p is the covering map, and $L^{(m)}$ is as above.

Claim. The divisor;

$$\sum_i \left(m\nu_i' - n \left[\frac{\nu_i' m}{n} \right] \right) D_i^{(0)} + \sum_j \left(m(n - \mu_j') - n \left[\frac{m(n - \mu_j')}{n} \right] \right) D_j^{(\infty)}$$

is an ample divisor.

Assume the above claim. We obtain that $L^{(m)}$ is ample. Then, by Riemann-Roch Theorem for an abelian varieties (see Mumford [7]), we obtain

$$h^d(A, L^{(m)}) > 0.$$

Therefore,

$$h^d(X, \mathcal{O}_X) \geq n.$$

Proof of Claim. Since an effective divisor on an abelian is always numerically effective, it is enough to show that the divisor

$$D = \sum_i{}' D_i^{(0)} + \sum_j{}' D_j^{(\infty)}$$

is ample, where $\sum_i{}', \sum_j{}'$ denotes that the sums are taken for non-zero ν_i' and μ_j'. Assume that D is not ample. Then, by Iitaka [1], Proposition 10.6, there exists an abelian variety A_1 such that

(i) $\Phi_{|mD|}: A \to A_1$ gives a structure of an abelian fibre space and $\dim A_1 = \kappa(D, A)$.

(ii) There exists an ample divisor Δ on A_1 such that $D = \Phi_{|nD|}^*(\Delta)$.

Let B be an abelian subvariety of A which is a fibre of $\Phi_{|mD|}$. By Poincaré reducibility, we obtain the commutative diagram:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{p}} & B \times A_1 \\ {\scriptstyle \tilde{\alpha}} \downarrow & & \downarrow {\scriptstyle \alpha} \\ X & \xrightarrow{p} & A \end{array}$$

where α and $\tilde{\alpha}$ are étale morphism, and $\kappa(X) = \kappa(\tilde{X}) = d$. By (i), (ii) as above, we obtain $(\alpha^* L^{(m)})^n|_{B \times \{a\}}$ is a trivial line bundle for $a \in A_1$, and $\alpha^* L^{(m)}|_{B \times \{a\}}$ is the same for all $a \in A_1$. Therefore, we obtain a n-fold étale cyclic covering of $B \times A_1$, say $\tilde{B} \times A_1$. Eventually, we obtain the commutative diagram:

$$\begin{array}{ccc} \tilde{\tilde{X}} & \xrightarrow{\tilde{\tilde{p}}} & \tilde{B} \times A_1 \\ \tilde{\beta} \downarrow & & \downarrow \beta \\ \tilde{X} & \xrightarrow{\tilde{p}} & B \times A_1 \end{array}$$

where β and $\tilde{\beta}$ are ètale morphisms and $\kappa(\tilde{\tilde{X}}) = \kappa(\tilde{X}) = d$. By our construction, $\beta^* \alpha^* L^{(m)}$ is considered a pullback of some line bundle over A_1. Therefore X has a structure of an abelian fibre space. This implies $\kappa(X) < \dim X$. This is contradiction.

Q.E.D.

Remark. By the proof of Propsition 3.3, we know the structure of a cyclic covering of an abelian variety. And if its covering map is flat, we can compute cohomology of its structure sheaf by using Kempf's Theorem (see Kempf [2]). Note that we can obtain many examples of a normal cyclic covering over an abelian variety which have the same cohomology as an abelian variety. Note that they are not of general type by Proposition 3.3.

(III) Remark on S. Yamamoto's paper. In [8], S. Yamamoto proved the following.

Theorem 3.5(Yamamoto [8]). *A 3-sheeted covering space of* $\mathbf{P}^2$ *branched along* $C_1 \cup C_2$, *which are two smooth curves with at most simple normal crossings, is either*

a normal surface whose singularities are all rational double points

or

a normal surface whose singularities are all rational triple points.

Moreover, for the first case, we obtain

$$p_g(S) = g(C_1) + g(C_2) - \frac{1}{9}(C_1 - 2C_2)(2C_1 - C_2)$$

and for the second,

$$p_g(S) = g(C_1) + g(C_2) - \frac{2}{9}(C_1 - C_2)^2$$

where $g(C_1)$ *is a genus of* C_1.

We will extend the above theorem to p-sheeted covering where p is a prime integer. By the result of M. Oka (see Oka [4]), $\pi_1(\mathbf{P}^2 \setminus (C_1 \cup C_2))$ is an abelian group. Therefore, for a normal p-sheeted covering S of $\mathbf{P}^2$ branched along $C_1 \cup C_2$ which satisfies the above conditions, $\mathbf{C}(S)$ is a cyclic extension of $\mathbf{C}(\mathbf{P}^2)$ with degree p. Hence, we can apply the resluts in §§1 and 2 to this case, and we obtain the following:

Theorem 3.5. *Let* $\pi: S \to \mathbf{P}^2$ *be a finite normal cover over* $\mathbf{P}^2$ *whose covering* *degree is a prime integer p. Assume that the branch locus of* π *is* $C_1 \cup C_2$, *where* C_1 *is a smooth curve whose degree is* n_i, *and the divisor* $C_1 + C_2$ *has at most simple normal crossings. Then*

(i) *There exists a unique integer* ν *with* $1 \leq \nu \leq p-1$, *and singularities of S are all cyclic quotient singularities of type* (p, ν) *or* $(p, p-\nu)$ *and they do not appear simultaneously.*

(ii) *The direct image of a structure sheaf of S is isomorphic to*

$$\mathcal{O}_{\mathbf{P}^2} \oplus \bigoplus_{k=1}^{p-1} \mathcal{O}_{\mathbf{P}^2}\left(\frac{k}{p}(n_1 + (p-\nu)n_2) + \left[\frac{k(p-\nu)}{p}\right] n_2\right)$$

where [ ] *denotes Gaussian symbol.*

Remark. Since qoutient singularities are rational, we can compute numerical invariants of a minimal resolution of S by using the above results. For $p = 3$, $\nu = 1$ and $p = p$, $\nu = p - 1$, we obtain Yamamoto's results.

Proof. (i) Under the above assumption, π is a cyclic covering of order p. Therefore, the rational function field $\mathbf{C}(S)$ is obtained as follows:

$$\mathbf{C}(S) = \mathbf{C}(\mathbf{P}^2)(\theta)$$

where

$$\theta^p = \varphi, \text{ for some } \varphi \in \mathbf{C}(\mathbf{P}^2).$$

Let

$$(\varphi)_0 = \sum_i \nu_i D_i^{(0)}$$

and

$$(\varphi)_\infty = \sum_j \mu_j D_j^{(\infty)}$$

be irreducible decompositions into prime divisors with respect to the zero divisor of φ and the polar divisor of φ respectively. By the assumption, we may assume that either

(a) $D_1^{(0)} = C_1$, $D_1^{(\infty)} = C_2$ and all $\nu_i (i \geq 2)$, $\mu_j (j \geq 2)$ are divisible by p,

or

(b) $D_1^{(0)} = C_1$, $D_2^{(0)} = C_2$ and all $\nu_i (i \geq 3)$, $\mu_j (j \geq 1)$ are divisible by p.

Case (a) Let f_1, f_2 be local equations for $D_1^{(0)}$ and $D_1^{(\infty)}$ respectively. Then, locally φ is represented as follows:

$$\varphi = \frac{f_1^{\nu_1} g_1^p}{f_2^{\mu_1} g_2^p}$$

where $(g_1^p) = \sum_{i \geq 2} \nu_i D_i^{(0)}$ and $(g_2^{(p)} = \sum_{j \geq 2} \mu_j D_j^{(\infty)}$. By the assumption, $g.c.d.(\nu_1, p) = 1$. Therefore there exists a pair of integers (k_1, l_1) such that $k_1 \nu_1 + p l_1 = 1$. Hence,

$$\varphi^{k_1} = \frac{f_1^{k_1\nu_1} g_1^{k_1 p}}{f_2^{k_1\mu_1} g_2^{k_1 p}} = \frac{f_1}{f_2^{k_1\mu_1}}\left(\frac{g_1^{k_1}}{g_2^{k_1} f_1^{l_1}}\right)^p.$$

Let ν, l_2 be a unique integer such that

$$k_1\mu_1 = pl_2 + \nu \qquad 0 < \nu < p.$$

Then

$$\varphi^{k_1} = \frac{f_1}{f_2^{\nu}}\left(\frac{g_1^{k_1}}{f_2^{l_2} f_2^{l_2} g_2^{k_1}}\right)^p = f_1 f_2^{(p-\nu)}\left(\frac{g_1^{k_1}}{f_1^{l_1+1} f_2^{l_2} g_2^{k_1}}\right)^p$$

Let L be the line bundle which is linear equivalent to

$$\left(\sum_{j\geq 2} k_1\mu_j D_j^{(\infty)} + (l_1 + 1)D_1^{(0)} + l_2 D_1^{(\infty)} - \sum_{i\geq 2} k_1\nu_i D_i^{(0)}\right)$$

Then

$$L^{\otimes p} \sim D_1^{(0)} + (p-\nu)D_1^{(\infty)}$$

and we can construct a normal cyclic covering $\tilde{S}$ which ramified over $C_1 \cup C_2$. Obiously, $\mathbf{C}(S) = \mathbf{C}(\mathbf{P}^2) \supset \mathbf{C}(\mathbf{P}^2)(\theta^{k_1}) = \mathbf{C}(\tilde{S})$ and $[\mathbf{C}(\mathbf{P}^2)(\theta)\colon \mathbf{C}(\mathbf{P}^2)] = [\mathbf{C}(\mathbf{P}^2)(\theta^{k_1})\colon \mathbf{C}(\mathbf{P}^2)] = p$. Therefore $\mathbf{C}(S) = \mathbf{C}(\tilde{S})$, and $S \simeq \tilde{S}$. Moreover, by the local equation in the total space of L, singularities of S are all cyclic quotient singularities of type (p, ν).

A proof for case (b) is similar to case (a), so we omitt it.

(ii) By the results of Esnault-Viehweg (see Viehweg [5]), π is flat morphism. Therefore, we can apply Lemma 2.1, and obtain the desired result.

Q.E.D.

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