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Compactness of the space of incompressible stable minimal surfaces without boundary

Dedicated to Professor Shingo Murakami on his 60th birthday

By

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Plateau problem, which asks an area-minimizing (or stable minimal) surface with a given fixed boundary, has been attacked from various aspects. Its solution is known to be non-unique in general. Several investigations have been devoted to this problem, uniqueness and non-uniqueness (Tomi [7], Nitsche [4], etc.).

Recently Morgan [3] proved, using techniques in geometric measure theory, that for surfaces of higher dimensions, there are only finitely many solutions in the real analytic category. He gave possibility that geometric measure theory works the study of the solution spaces.

In this paper, using the convergence in geometric measure theory following Morgan, we prove the compactness of the space of incompressible stable minimal surface of a fixed topological type *without boundary*, whose areas are bounded by a given constant, in any 3-dim. compact Riemannian manifold.

1. Theorem.

The following notations will be used throughout this paper:

M: 2-dim. compact smooth connected surface without boundary.

N: 3-dim. compact smooth Riemannian manifold without boundary (or with convex boundary).

Definition (cf. Schoen-Yau [6]). For $f \in C^0(M, N)$, f is said to be *incompressible* if and only if

(i) $f_\# : \pi_1(M) \to \pi_1(N)$ is injective in case $g>0$, or

(ii) f is homotopically non-trivial in case $g=0$,

where g denotes the genus of M.

The notion of incompressibility is utilized originally in 3-manifold topology, but it is an appropriate non-degeneracy condition for area-minimizing problems.

We consider the space of incompressible smooth maps of M into N up to parametrization:

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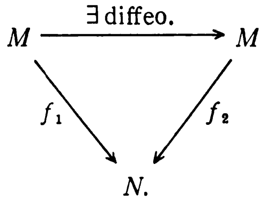
$$\mathcal{I}:=\{f\in C^{\infty}(M, N);\ \text{incompressible}\}/\sim,$$

where $f_1\sim f_2$ ($f_1, f_2\in C^{\infty}(M, N)$) if and only if the following diagram is valid:



$\mathcal{I}$ has the quotient topology induced from $C^{\infty}(M, N)$. Let $[f]$ be the equivalence class of $f\in C^{\infty}(M, N)$, and let $area[f]$ denote the area of $[f]$, which is well-defined. $[f]$ is well-defined to be *minimal* if f is minimal, i.e. the first variation of the area at f is zero. $[f]$ is well-defined to be *stable* if f is stable, i.e. the second variation of the area at f is non-negative. Note also that the incompressibility is well-defined for $[f]$. Now we are in a position to consider the space of stable minimal incompressible maps:

$$\mathcal{M}:=\{[f]\in\mathcal{I};\ \text{stable minimal in } \mathcal{I}\}\subset\mathcal{I}.$$

By the well-known arguments [2], we have

$$\mathcal{M}\subset\{\text{immersions}\}/\sim.$$

The purpose of this paper is to show:

Theorem. *For any constant $C>0$, the space of incompressible stable minimal maps of the area $\leqq C$*

$$\mathcal{M}\cap\{[f];\ area[f]\leqq C\}$$

is compact.

We are grateful with Joel Hass for pointing out a mistake in our original version.

2. Proof of Theorem.

Let $\{f\}_{i=1}^{\infty}\in\mathcal{M}$ be any sequence. For any open ball $B_r(x)$ of radius r in N, let $C_j^{(1)}, \cdots, C_j^{(p)}$ denote the component of $f_j^{-1}B_r(x)$. In general, p depends on j, but by taking a subsequence and by adopting a sufficiently small r, we may suppose that p is independent of j because $p=p(j)$ is bounded. Indeed the boundedness follows from the following two facts:

With respect to i and j,

(i) $\boldsymbol{M}((f_j)_{\#}v(M))\llcorner B_r(x))$ is bounded ($\leqq C$)

(ii) $\boldsymbol{M}((f_j)_{\#}v(C_j^{(i)}))$ is bounded below (cf. Schoen's curvature estimate [5]),

where $\boldsymbol{M}(\ )$ denotes the mass of varifold.

Then for the integral varifold $(f_j)_{\#}(v(C_j^{(i)})$ ($i=1, \cdots, p$; $j=1, 2, \cdots,$), we have

(1) (*stationariness condition*)
The varifold $(f_j)_{\#}v(C_j^{(i)})$ is stationary.

(2) (*volume condition*)
There exists $r>0$ (independent of i, j) such that $\boldsymbol{M}((f_j)_{\#}v(C_j^{(i)}))\leqq(1+\eta)\pi r^2$, where η is the constant in Allard's regularity theorem.

(3) (*density condition*)
$\Theta^2((f_j)_{\#}v(C_j^{(i)}), x)\geqq 1$ for $\|(f_j)_{\#}v(C_j^{(i)})\|$-a.e. $x\in M$.

The condition (1) and (3) is obvious. By Schoen's curvature estimate [5], we can verify also the condition (2) for $j=1, 2, \cdots$. On the other hand there exists an integral varifold V such that taking a subsequence if necessary, $(f_j)_{\#}(v(C_j^{(i)}))\to V^{(i)}$ in the sense of varifold convergence. Obviously the integral varifold $V^{(i)}$ satisfied the condition (1) and (3) instead of $(f_j)_{\#}v(M)$. Then we can verify the following two facts (cf. Morgan [3]):

(i) $\|(f_j)_{\#}v(C_j^{(i)})\|\to\|V^{(i)}\|$ as measure, where $\|(f_j)_{\#}v(C_j^{(i)})\|$, $\|V^{(i)}\|$ denote the total variation measures of the varifolds $(f_j)_{\#}v(C_j^{(i)})$, $V^{(i)}$ respectively.

(ii) *Haus. dist.* $(\mathrm{spt}(f_j)_{\#}v(C_j^{(i)}), \mathrm{spt}\, V^{(i)})\to 0$ as $j\to\infty$,

where *Haus. dist.* (A, B) denotes the Hausdorff distance between A and B.

Then it follows from the above two facts that the condition (2) is satisfied also for $V^{(i)}$ in place of $(f_j)_{\#}v(C_j^{(i)})$. Hence by Allard' regularity theorem, $(f_j)_{\#}v(C_j^{(i)})$ $(j=1, 2, \cdots,)$ and $V^{(i)}$ are $C^{1,\alpha}$-graph over a plane. Note that they all satisfy the minimal surface equation. By a priori estimates, $f_j \llcorner C_j^{(i)}$ converges to a smooth map representing $V^{(i)}$ in C^∞-topology. Thus we have the local (hence global) C^∞-convergence of (a subsequence of) $\{f_j\}_{j=1}^{\infty}$. Note that the incompressibility is preserved by C^∞-convergence. The proof is completed.

Remark 1. Note that in general, $\mathcal{M}\cap\{[f]\,;\ area[f]\leqq C\}$ is not of a finite number. Indeed, let $N=M\times S^1$ endowed with the canonical product metric. Then $\mathcal{M}=\{M\times\{t\}\,;\ t\in S^1\}$, for any sufficiently large C.

Remark 2. Note that even if all f_i's are embeddings, its limit map may be a covering immersion onto an embedded non-orientable surface.

The following remark is due to Joel Hass:

Remark 3. $\mathcal{M}$ per se is not necessarily compact. For instance, let $N=S^1\times S^1\times S^1$. Then $\mathcal{M}$ is not compact, since 2-dim. minimal torus embeddings $T_{p,q}$ of slope (p, q) does not converge as p/q tends to an irrational number.

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