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On T. Petrie's problem concerning homology planes

By

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Introduction

Let X be a smooth algebraic surface defined over the complex number field $\boldsymbol{C}$. We call X a homology plane if the homology groups $H_i(X;\boldsymbol{Z})$ vanish for $1 \leqq i \leqq 4$. If X is topologically contractible, then X is a homology plane.

This paper arose from the following question of tom Dieck and Petrie which evolved from their study of a configuration of curves leading to a homology plane.

Problem. *Let F and G be irreducible curves in $\boldsymbol{P}^2$ such that*

i) $\deg F = m$ *and* $\deg G = m+1$

ii) *F and G are topologically isomorphic to $\boldsymbol{P}^1$.*

iii) *F and G meet in two smooth points.*

Assume that an affine surface which is obtained from $\boldsymbol{P}^2$ by deleting these curves via the construction generalizing the Ramanujam's is a homology plane (*See* § 2 *for the explicit construction*). *Does this imply $m=2$?*

We recall some notations and results concerning homology planes. Embed X into a smooth projective surface, say V, as a Zariski open set so that the boundary divisor $D := V - X$ consists of smooth rational curves, with only normal crossings as its singularity. Define the Kodaira dimension $\kappa(X)$ of X as follows:

$$\kappa(X) = \begin{cases} -\infty & \text{if } |n(D+K_V)| = \emptyset \text{ for any } n>0 \\ \sup\limits_{n>0} \dim \Phi_{|n(D+K_V)|}(V) & \text{otherwise} \end{cases}$$

where K_V is the canonical divisor of V. $\kappa(X)$ takes the value $-\infty$, 0, 1, 2. The above definition does not depend on the choice of an open embedding X into a smooth projective surface V and smooth algebraic surfaces are classified into four classes according to the value $\kappa(X)$.

The affine plane $\boldsymbol{C}^2$ is topologically contractible with $\kappa(\boldsymbol{C}^2) = -\infty$, and this is the unique topologically contractible surface, even the unique homology plane whose Kodaira dimension is $-\infty$.

In [R], Ramanujam exhibited an example of a smooth contractible affine surface R, whose Kodaira dimension is in fact two. Recently Gurjar-Miyanishi [G-M] proved the following:

(1) There are no homology planes of Kodaira dimension 0.

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(2) There exist an infinite number of distinct smooth affine contractible surfaces of Kodaira dimension 1 and any homology plane of Kodaira dimension 1 is one of them.

(3) There exists a smooth affine contractible surface G-M of Kodaira dimension two other than Ramanujam's.

It is natural to ask whether or not there are other topologically contractible surfaces or homology planes of Kodaira dimension two. We remark that a homology plane is a rational affine surface by Gurjar-Shastri [G-S].

In this paper we shall consider a slightly general form of tom Dieck-Petrie's problem. Putting $\deg F=m$ and $\deg G=n$, we treat the problem for an arbitrary pair (m, n). Here we may assume $m\leqq n$.

In §1 we reproduce the proof of a slightly generalized version of the inequality in Gurjar-Shastri. In §2 we shall generalize the example of Ramanujam and using the inequality in §1 we shall prove the following

Theorem. *If the obtained surface is a homology plane whose Kodaira dimension is greater than or equal to zero, then m is less than or equal to three.*

In §3 we shall consider remaining cases $m=3, 2, 1$ separately. Especially when $m=2$, we get two new configulations of curves which lead to contractible surfaces, but obtained surfaces are isomorphic to the example in Gurjar-Miyanishi. Finally in §4 we shall generalize the example of Gurjar-Miyanishi.

Jointly with Prof. M. Miyanishi, we constructed infinitely many topologocally contractible surfaces by looking at the above case $(m, n)=(1, n)$. We include the results as an appendix at the end of the article.

The author would like to thank Professor Ted Petrie and Professor Masayoshi Miyanishi who initiated him to this problem and gave him many useful suggestions.

§1. An auxiliary inequality

Let V be a smooth projective surface and let D be a reduced effective divisor on V. We assume that each irreducible component of D is smooth and D has only normal crossings as its singularity. Let K_V denote the canonical divisor of V. When some multiple of K_V+D is effective, K_V+D has Zariski decomposition $K_V+D=P+N$ in $\mathrm{Pic}(X)\otimes Q$ where P and N satisfy the following properties;

(Z1) P is numerically effective, i.e., $(P\cdot C)$ is non-negative for any curve C in X.

(Z2) N is effective.

(Z3) $PN=0$.

(Z4) The intersection matrix of the curves in the support of N is negative definite.

With the above notations, Miyaoka proved that if K_V+D has Zariski decomposition, the following inequality holds:

$$e(V)-e(D)-\frac{1}{3}(K_V+D)^2+\frac{1}{12}N^2\geqq 0 \tag{1.1}$$

where $e(V)$ denotes the topological euler characterisic. For the Zariski decomposition and Miyaoka's inequality see [F] and [M].

Gurjar and Shastri used some inequality derived from Miyaoka's one to prove that a complex homology 2-cell, which is in our terminology a homology plane, is rational. They proved the inequality assuming that V is not rational. We show that the given argument in Gurjar-Shastri applies for rational surfaces as well. We will reproduce here their proof of the inequality in the case of rational surfaces to fix the notation.

Now we assume that the following four conditions hold.

(i) V is a smooth rational projective surface defined over $\boldsymbol{C}$.

(ii) Each irreducible component of D is isomorphic to $\boldsymbol{P}^1$ and dual graph of D is a tree, i. e., D is simply connected.

(iii) D is minimal, i. e., any (-1) curve in D meets at least three other components.

(iv) $\kappa(V-D)\geqq 0$.

Let $b_i=b_i(D)$ denote the i-th betti number of D and $\beta_i(V)$ denote the i-th betti number of V. Since V is rational, we have the folowing equalities:

$$e(V)=\beta_2(V)+2,$$

$$e(D)=b_0-b_1+b_2,$$

$$K_V{}^2=10-\beta_2(V),$$

which follows from Noether's formula and

$$E_V\cdot D+D^2=2(b_1-b_0),$$

which follows from the adjunction formula. Thus the inequality yields

$$4\beta_2+b_1-b_0-4-3b_2-K_V\cdot D+\frac{1}{4}N^2\geqq 0. \tag{1.2}$$

First we have the following

Lemma 1.1. $N^2<0$.

Proof. Since the dual graph of D is a tree, D contains a component D_0 such that $D_0\cdot(D-D_0)\leqq 1$ and D_0 is isomorphic to $\boldsymbol{P}^1$. Take D_0 to be a terminal component of the tree. Therefore we have

$$(K_V+D\cdot D_0)\leqq -1.$$

By the property of the Zariski decomposition of K_V+D, D_0 is a componet of N. This implies $N^2<0$.

We consider a sequence of contractions of (-1) curves $\pi\colon V\to V''$, where V'' is a relatively minimal rational surface, i. e., either $V''\cong\boldsymbol{P}^2$ or V'' is a minimal ruled surface $\Sigma_a(a\geqq 0)$. Though there may be different models V'' obtained from V as above, we fix V'' once for all.

Write $\pi=\varphi_m\circ\varphi_{m-1}\circ\cdots\circ\varphi_1$ where each φ_j is a contraction of (-1) curve E_j. Put $\psi_0=id_V$ and $\psi_j=\varphi_j\circ\cdots\circ\varphi_1$ for $j\geqq 1$. Now we can rearrange φ_j's and we may assume that if $\pi_1=\varphi_{n_1}\circ\cdots\circ\varphi_1$, then the following holds.

a) Every component of $D'=\pi_1(D)$ is still smooth.

a) For each $j>n_1$, $(E_j\cdot C)\geqq 2$ for at least one component C of $\psi_{j-1}(D)$ i. e., any contraction of $E_j(j>n_1)$ results a singular irreducible component in the image $\phi_j(D)$.

Put $V'=\pi_1(V)$, $\pi_2=\varphi_m\circ\cdots\circ\varphi_{n_1+1}: V'\to V''$. Let ε_1 be the exceptional locus for π_1, i. e., ε_1 is the union of irreducible curves on V contracted to points on V' and let ε_2 be the exceptional locus of π_2 on V'. We denotes the components of ε_1 by $\{L_i\}_{1\leq i\leq n_1}$ and those of ε_2 by $\{M'_i\}_{1\leq i\leq n_2}$, where $m=n_1+n_2$ is the number of contractions in π. The M'_i's are curves on V' and we denote the proper transform of M'_i on V by M_i.

Define for each (-1) curve E_j, the branching number $\beta(E_j)$ by

$$\beta(E_j)=\begin{cases}(\phi_{j-1}(D)-E_j\cdot E_j) & \text{if } E_j\subset\phi_{j-1}(D)\\ (\phi_{j-1}(D)\cdot E_j) & \text{if } E_j\not\subset\phi_{j-1}(D).\end{cases}$$

Note that each component L_i of ε_1 is a proper transform of a (-1) curve E_j on $\phi_{j-1}(V)$, which is uniquely determined. We put $\beta(L_i)=\beta(E_j)$.

Here we assume that the following condition (V) holds.

(V) ε_1 does not contain a curve L_i with $\beta(L_i)\leqq 1$.

We shall introduce some more notations:

$$R_2=\cup\{L_i\,|\,\beta(L_i)=2\},\qquad R_3=\cup\{L_i\,|\,\beta(L_i)=3\},$$

$$R_4=\cup\{L_i\,|\,\beta(L_i)\geqq 4\},\qquad S=\varepsilon_2\cap\pi_1(D),$$

$r_i=b_2(R_i)=$the number of the irreducible components in R_i,

$$e_1=n_1-b_2(\varepsilon_1\cap D),$$

$$\sigma=n_2-\sum_{M'_j\in S}(M'^2_j+2).$$

Let $D=\cup D_r$, $\pi_1(D)=\cup D'_s$, and $\pi(D)=\cup D''_t$ be the irreducible decomposition of D, $\pi_1(D)$ and $\pi(D)$ respectively. D_t and D'_t will denote the proper transform of D''_t on V and V' respectively and D_s will denote the proper transform of D'_s on V.

Let $m_{t,i}$ be the multiplicity of a singular point $P_{t,i}$ on D''_t including infinitely near singular points on D''_t. We define

$$\tau=\sum_{t,i}m_{t,i}-2n_2,$$

$$\lambda=\sum_t D''_t\cdot K_{V''}.$$

Here $K_{V'}$, $K_{V''}$ denote the canonical divisors of V', V'' respectively.

For $1\leqq j\leqq n$, let now φ_j contract $\phi_{j-1}(L_j)$ where $L_j\subset\varepsilon_1\cap R_i$ for some $i=2, 3, 4$ i. e., L_i is a proper transform of E_j. Then we have

$$\text{(1.3)}\qquad \sum_r((\phi_{j-1}(D_r))^2+2)\leqq\begin{cases}\sum_r((\phi_j(D_r))^2+2)-i & \text{if } L_j\not\subset D\\ \sum_r((\phi_j(D_r))^2+2)-i+1 & \text{if } L_j\subset D.\end{cases}$$

Here we take $(\phi_{j-1}(D_r))^2+2=0$ (resp. $(\phi_j(D_r))^2+2=0$) if $\phi_{j-1}(D_r)$ (resp. $\phi_j(D_r)$) is a point.

By the adjunction formula, we have

$$-K_V\cdot D=\sum_r(D_r^2+2)=\sum_r(\phi_0(D_r))^2+2)$$

and

$$-K_{V'}\cdot\pi_1(D)=\sum_s(D'^2_s+2)=\sum_s((\pi_1(D_s))^2+2)=\sum_r((\phi_{n_1}(D_r))^2+2).$$

Now applying (1.3), repeatedly for $j=1, 2, \cdots, n_1$ we obtain

$$-K_V\cdot D\leqq -K_{V'}\cdot\pi_1(D)-2r_2-3r_3-4r_4+b_2(\varepsilon_1\cap D)$$

and noting the fact that $n_1=r_2+r_3+r_4$,

$$-K_V\cdot D\leqq -K_{V'}\cdot\pi_1(D)-r_3-2r_4-n_1-e_1.$$

On the other hand, by the genus formula, we have

$$(D_t''^2+2)+D_t''\cdot K_{V'}=\sum_i m_{t,i}(m_{t,i}-1)$$

and since D_t' is the smooth model of D_t'',

$$D_t'^2+2\leqq D_t''^2+2-\sum_i m_{t,i}{}^2=-\sum_t m_{t,i}-D_t''\cdot K_{V'}.$$

Recalling the definition of τ and λ we get

$$\sum_t(D_t'^2+2)\leqq -\tau-2n_2-\lambda.$$

Since

$$-K_{V'}\cdot\pi_1(D)=\sum_s(D_s'^2+2)=\sum_t(D_t'^2+2)+\sum_{M'\in S}(M'^2+2),$$

we have

$$-K_V\cdot D\leqq -\lambda-\tau-\sigma-n_2-n_1-r_3-2r_4-e_1.$$

Now applying (1.2), we obtain

$$0\leqq 4\beta_2-b_0-3b_2-4-\lambda-\tau-\sigma-n_2-n_1-r_3-2r_4-e_1+\frac{1}{4}N^2.$$

All the quantities in this inequality are integers except possibly $(1/4)N^2$. Note that $\beta_2(V'')=\beta_2(V)-n_1-n_2=1$ or 2 because V'' is isomorphic to $\boldsymbol{P}^2$ or Σ_a. We can rewrite the above inequality and obtain the following

Proposition 1.2. *Let V be a smooth rational projective surface and let D be a reduced effective divisor on V. We assume that the conditions* (i) *to* (v) *hold. Then under the above notations we have the following inequality:*

$$3(b_2-\beta_2)+b_0+\lambda+e_1+\sigma+\tau+r_3+2r_4\leqq\begin{cases}-4 & \text{if}\quad V''\cong\boldsymbol{P}^2\\ -3 & \text{if}\quad V''\cong\Sigma_a\end{cases}\tag{1.4}$$

Remark. We can prove the above prposition without the condition (V).

§ 2. A generalization of Ramanujam's example

In this section we shall apply the inequality (1.4) to the special case. Note that quantities σ, τ, r_3, r_4 can be calculated locally on V''.

Let F and G be curves on $\boldsymbol{P}^2$. We assume that the following conditions hold.

A 1) $\deg F=m$, $\deg G=n$ and $m\leqq n$.

A-2) F and G are topologically isomorphic to $\boldsymbol{P}^1$.

A-3) F and G meet each other in two smooth points. Put $F\cap G=\{R, S\}$.

Since F and G are topologically isomorphic to $\boldsymbol{P}^1$, they have only cuspidal singularities. Namely if $\nu: \tilde{F}\to F$ is the normalization, and if $P\in\mathrm{Sing}(F)$, $\nu^{-1}(P)$ consists of a single point.

Let $\pi_2: V' \to \boldsymbol{P}^2$ be the shortest sequences of blowing-ups at singular points of F and G infinitely near singular points such that the proper transforms of F and G on V' become nonsingular. Let $\tilde{\pi}_1: \tilde{V} \to V'$ be the shortest sequences of blowingups such that the total transform $(\pi_2 \circ \tilde{\pi}_1)^{-1}(F \cup G)$ is a divisor with normal croosings. Since $(\pi_2 \circ \tilde{\pi}_1)^{-1}(F \cup G)$ contains a loop, we further blow up $\tilde{V}$ arbitrarily many times at infinitely near points of one of R or S. Let $\pi_0: V \to \tilde{V}$ be the composition of such blowing-ups and put $\pi_1 = \tilde{\pi}_1 \circ \pi_0$. Let E be (-1) curve obtained by the last blowing-up in π_0. Put $\pi = \pi_2 \circ \pi_1$ and $D = \mathrm{Supp}(\pi^{-1}(F \cup G) - E)$. We may assume that D is minimal, i. e., any (-1) curve in D meet at least 3 other components. This assumption restricts the sequence of blowing-ups $\pi_0: V \to \tilde{V}$. Put $X = V - D$. Then our question is the following:

When does X become a contractible surface of $\bar{\kappa}(X) = 2$?

Note that this construction is a generalization of Ramanujam's and we can show that the example of Gurjar-Miyanishi, which was originally constructed from Σ_2, can be obtained also in this way.

Let P be a singular point of F or G, say F. Define d_0 and d_1 by

$$d_0 = \max_A i(F, A; P),$$

$$d_1 = \mathrm{mult}_P F,$$

where A is a curve on $\boldsymbol{P}^2$ passing through P and smooth at P. Since F is singular at P, this quantity d_0 is uniquely determined (and bounded of course). Indeed, we consider the shortest sequence of blowing-ups with centers at P and its infinitely near points such that the proper transform of F is smooth over P. Then as the exceptional curves considerd from the beginning, there appear several (-2) curves and then a curve with self-intersection multiplicity < -2. If there are p_1 curves with self-intersection multiplicity (-2), then $d_0 = p_1 d_1 + d_2$ where d_2 is the multiplicity of the proper transform of F after p_1 blowing-ups. We have $d_1 < d_0$.

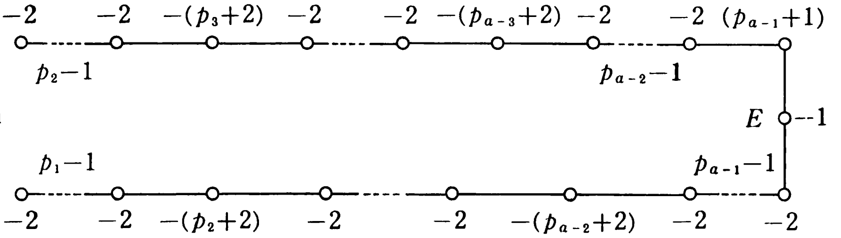
Find integers $d_2, \cdots, d_a$ and $p_1, \cdots, p_a$ by the following Euclidean algorithm;

$$\begin{array}{ll} d_0 = p_1 d_1 + d_2 & 0 < d_2 < d_1 \\ d_1 = p_2 d_2 + d_3 & 0 < d_3 < d_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ d_{a-2} = p_{a-1} d_{a-1} + d_a & 0 < d_a < d_{a-1} \\ d_{a-1} = p_a d_a. & \end{array}$$

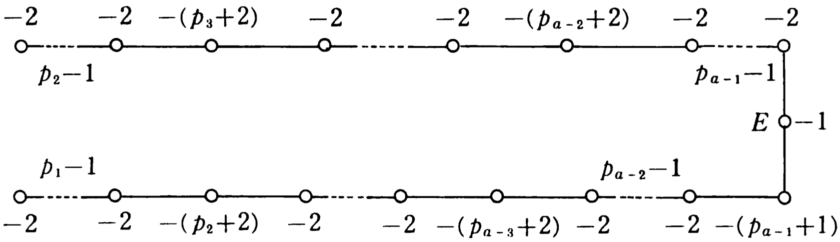
Let $n_2(P)$ denote the local contribution of P to the quantity n_2 and so on. Put $\delta_F(P) = \sigma(P) + \tau(P) + r_3(P) + 2r_4(P)$. We want to calculate these quantities. We will consider four cases separately.

Case I) $d_a = 1\,(a \geqq 3)$ In this case the singular point P can be resolved after blowing up $p_1 + p_2 + \cdots + p_{a-1}$ times successiviely. Tne exceptional locus $\pi_2^{-1}(P)$ is expressed in terms of the dual graph as follows;

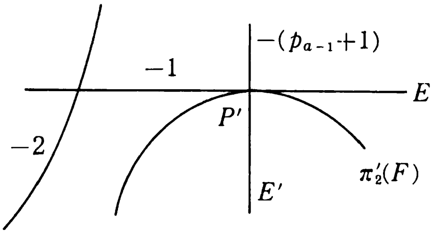
a: even



a: odd



The proper transform $\pi_2'(F)$ and the exceptional locus $\pi_2^{-1}(P)$ intersect as follows;



where E is the (-1) curve obtained by the last blowing-up and E' is the curve with self intersection number $-(p_{a-1}+1)$ in the dual graph. $\pi_2'(F)$ intersects E' transversally and $i(\pi_2'(F), E;\ P')=d_{a-1}$.

Now we can calculate the quantities $n_2(P)$, $\delta(P)$ and so on as follows;

$$n_2(P)=p_1+\cdots+p_{a-1},$$

$$\begin{aligned}\sigma(P)&=n_2(P)-1+(p_2+\cdots+p_{a-1}-1)\\ &=p_1+2(p_2+\cdots+p_{a-1})-2,\end{aligned}$$

$$\begin{aligned}\tau(P)&=p_1d_1+p_2d_2+\cdots+p_{a-1}d_{a-1}-2(p_1+\cdots+p_{a-1})\\ &=d_0+d_1-(p_a+1)-2(p_1+\cdots+p_{a-1}),\end{aligned}$$

$$r_3(P)=p_a,\ r_4(P)=0.$$

We obtain

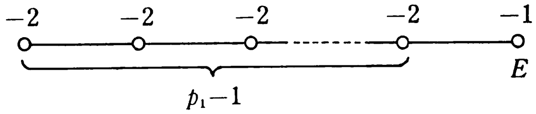
$$\delta_F(P)=d_0+d_1-p_1-3$$

and

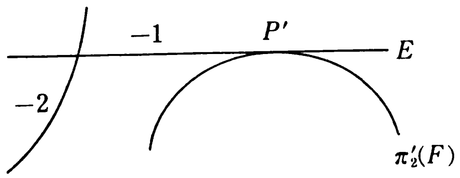
$$\begin{aligned}&\delta_F(P)-n_2(P)\\&\quad=p_1d_1+p_2d_2+\cdots+p_{a-1}d_{a-1}-2p_1-(p_2+\cdots+p_{a-1})-2+p_a\\&\quad=(d_1-2)p_1+(d_2-1)p_2+\cdots+(d_{a-1}-1)p_{a-1}+d_{a-1}-2\\&\quad>0,\end{aligned}$$

since $d_i\geqq 2$ for $2\leqq i\leqq a-1$, $d_1\geqq 3$ and $p_i\geqq 1$.

Case II) $d_a=1$ $(a=2)$ In this case the singular point P can be resolved after blowing up p_1 times successively. The exceptional locus $\pi_2^{-1}(P)$ is expressed in terms of the dual graph as follows;



The proper transform $\pi_2'(F)$ and the exceptional locus $\pi_2^{-1}(P)$ intersect as follows;



where E is the (-1) curve obtained by the last blowing-up and $i(\pi_2'(F), E;\ P')=d_1$. We obtain the quantities $n_2(P)$, $\sigma(P)$ and so on as follows;

$$\begin{aligned}&n_2(P)=p_1,\\&\sigma(P)=n_2(P)-1=p_1-1,\\&\tau(P)=p_1d_1-2p_1,\\&r_3(P)=d_1-1,\qquad r_4(P)=0,\end{aligned}$$

whence

$$\delta_F(P)=p_1d_1+d_1-p_1-2=d_0+d_0-p_1-3$$

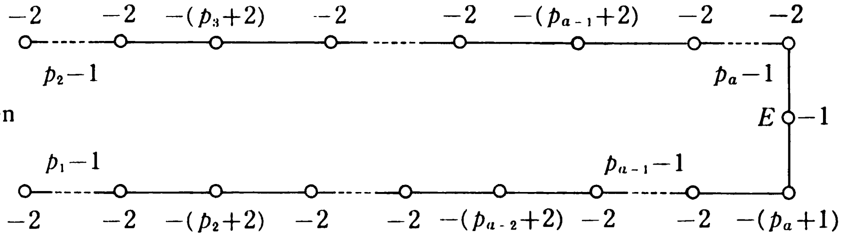
and

$$\delta_F(P)-n_2(P)=(p_1+1)(d_1-2)\geqq 0,$$

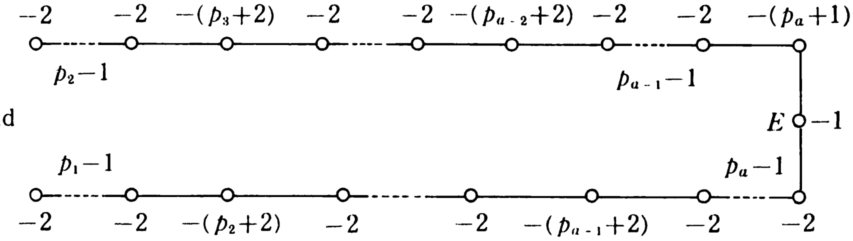
since $d_1\geqq 2$ and $p_1\geqq 1$.

Case III) $d_a>1$ $(a\geqq 2)$ π_2 comprises at least the composition ρ of $p_1+\cdots+p_a$ times blowing-ups and the exceptional locus $\rho^{-1}(P)$ is expressed in terms of the dual graph as follows;

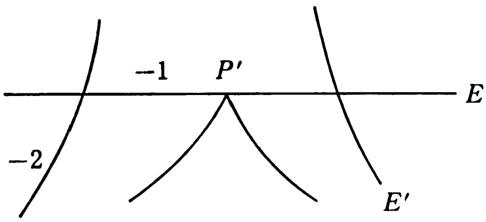
a : even



a : odd



The proper transform $\rho'(F)$ and the exceptional locus $\rho^{-1}(P)$ intersects as follows;



where E is a (-1) curve obtained by the last blowing-up and $i(E, \rho'(F); P')=d_a$. Let $n_2(P)^{(1)}$ denote the local contribution at P to the quantity n_2 which is due to those exceptional curves which are obtained first $p_1+\cdots+p_a$ times blowing-ups. We define $\sigma(P)^{(1)}$, $\tau(P)^{(1)}$ and $\delta_F(P)^{(1)}$ in a similar way. Now we can calculate these quantities as follows;

$$n_2(P)^{(1)}=p_1+\cdots+p_a,$$

$$\begin{aligned}\sigma(P)^{(1)}&\geqq n_2(P)-1+(p_2+\cdots+p_a-1)\\ &\geqq p_1+2(p_2+\cdots+p_a)-2,\end{aligned}$$

$$\tau(P)^{(1)}=p_1d_1+p_2d_2+\cdots+p_ad_a-2(p_1+\cdots+p_a),$$

whence

$$\begin{aligned}\delta_F(P)^{(1)}&=\sigma(P)^{(1)}+\tau(P)^{(1)}\\ &\geqq p_1d_1+p_2d_2+\cdots+p_aP_a-p_1-2\\ &=d_0+d_1-d_a-p_1-2\end{aligned}$$

and

$$\begin{aligned}\delta_F(P)^{(1)}-n_2(P)^{(1)}&\\ &\geqq p_1d_1+p_2d_2+\cdots+p_ad_a-2p_1-(p_2+\cdots+p_a)-2\\ &=(d_1-2)p_1+(d_2-1)p_2+\cdots+(d_a-1)p_a-2\\ &\geqq 0\end{aligned}$$

since $d_a>1$, $d_i\geqq 3$ for $1\leqq i\leqq a-1$ and $p_i\geqq 1$.

Next we put $d_0^{(0)}:=d_a$ and $d_1^{(0)}=\mathrm{multi}_{P'}\rho'(F)$. We consider three cases separately.

Subcase III-1) $d_0^{(0)}>d_1^{(0)}$ We determine integers by the Euclidean algorithm and continue the argument, returning to one of main four cases I) to IV).

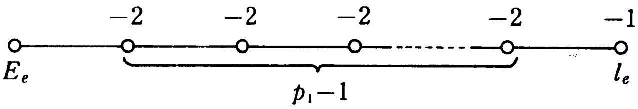
Subcase III-2) $d_0^{(0)}=d_1^{(0)}$ Let W be the surface obtained from P^2 by $p_1+\cdots+p_a$ times blowing-ups with center P and its infinitely near points as above. Let $F^{(0)}=\rho'(F)$, $P_0:=P'=\rho'(F)\cap E$ and let $\mu_1\colon W_1\to W_0:=W$ be the blowing-up with center P_0. Let $F^{(1)}:=\mu_1'(F^{(0)})$, $l_1=\mu_1^{-1}(P_0)$ and let $P_1=F^{(1)}\cap l_1$. Let $d_1^{(1)}:=\mathrm{mult}_{P_1}F^{(1)}$. If $d_0^{(0)}=d_1^{(0)}=d_1^{(1)}$, let $\mu_2\colon W_2\to W_1$ be the blowing-up of W_1 with center P_1. Define $\mu_j\colon W_j\to W_{j-1}$, $F^{(j)}$, l_j and $d_1^{(j)}$ as follows when $d_0^{(0)}=d_1^{(0)}=d_1^{(1)}=\cdots=d_1^{(j-1)}$.

μ_j is the blowing-up of W_{j-1} with center $P_{j-1}=F^{(j-1)}\cap l_{j-1}$,

$$F^{(j)}=\mu_j'(F^{(j-1)}),\qquad l_j=\mu_j^{-1}(P_{j-1}),$$

$$P_j=F^{(j)}\cap l_j,\qquad d_1=\mathrm{mult}_{P_j}F^{(j)}.$$

Assume $d_0^{(0)}=d_1^{(0)}=d_1^{(1)}=\cdots=d_1^{(e-1)}>d_1^{(e)}$ and put $\mu=\mu_e\circ\cdots\circ\mu_1$. The exceptional locus $\mu^{-1}(P)$ of ρ is expressed in terms of dual graph as follows;



The proper transform $F^{(e)}$ intersect l_e with contact of order $d_1^{(e-1)}$ and E_e is the proper transform of E by μ.

We consider two cases separately.

Subcase III 2-a) $d_1^{(e)}=1$ Let $n_2(P)^{(2)}$ denote the local contribution at P to the quantity n_2 which is due to exceptional curves of blowing-ups $\mu_1, \mu_2, \cdots, \mu_e$ and further $d_1^{(e-1)}$ times blowing-ups with center P_e and its infinitely near points over $F^{(e)}$. Then we obtain

$$n_2(P)^{(2)}=e,$$

$$\sigma(P)^{(2)}=n_2(P)^{(2)}-1=e-1,$$

$$\tau(P)^{(2)}=d_1^{(0)}e-2e,$$

$$r_3(P)^{(2)}=d_1^{(e-1)}-1=d_1^{(0)}-1,$$

whence

$$\delta_F(P)^{(2)} = \sigma(P)^{(2)} + \tau(P)^{(2)} + r_3(P)^{(2)} + 2r_4(P)^{(2)}$$
$$= d_1^{(0)}e - e - 1 + d_1^{(0)} - 1$$

and

$$\delta_F(P)^{(2)} - n_2(p)^{(2)} \geqq (d_1^{(0)} - 2)(e+1) \geqq 0$$

since $d_1^{(0)} \geqq 2$ and $e \geqq 1$.

Subcase III-2ˎb) $d_1^{(e)} > 1$ Let $n_2(P)^{(2)}$ denote the local contribution at P to the quantity n_2 which is due to exceptional curves of blowing-ups $\mu_1, \mu_2, \cdots, \mu_e$. Then we have

$$n_2(P)^{(2)} = e\,,$$
$$\sigma(P)^{(2)} \geqq n_2(P)^{(2)} = e\,,$$
$$\tau(P)^{(2)} = d_1^{(0)}e - 2e\,,$$

whence

$$\delta_F(P)^{(2)} = \sigma(P)^{(2)} + \tau(P)^{(2)} \geqq d_1^{(0)}e - e$$

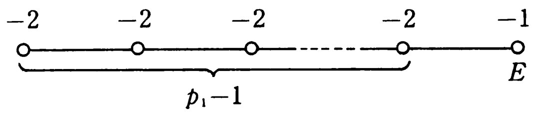
and

$$\delta_F(P)^{(2)} - n_2(p)^{(2)} \geqq (d_1^{(0)} - 2)e \geqq 0$$

since $d_1^{(0)} \geqq 3$ and $e \geqq 1$.

In this case we determine integers for the pair $(d_1^{(e-1)}, d_1^{(e)})$ by the Euclidean algorithm and continue the argument, returning to the one of main four cases I) to IV).

Case IV) $d_a > 1$, $a = 1$ Similarly π_2 comprises at least the composition ρ of p_1 blowing-ups. The exceptional locus $\rho^{-1}(P)$ of ρ is expressed in terms of dual graph as follows;



The proper transform $\rho'(F)$ intersect the last obtained (-1) curve E at P' with contact of order d_1.

If the proper transform $\rho'(F)$ is smooth at P', let $n_2(P)$ denote the local contribution at P to the quantity n_2 which is due to those exceptional curves which are obtained from first p_1 times blowing-ups and further d_1 times blowing-ups with with center P' and its infinitely near points over $\rho'(F)$. We define $\sigma(P)$, $\tau(P)$ and $\delta_F(P)$ in a similar way. Then we have

$$n_2(P) = p_1, \qquad \sigma(P) = p_1 - 1\,,$$
$$\tau(P) = p_1 d_1 - 2p_1, \qquad r_3(P) = d_1 - 1\,,$$

where

$$\delta_F(P) = \sigma(P) + \tau(P) + r_3(P) + 2r_4(P) = p_1 d_1 - p$$

and

$$\delta_F(P) - n_2(P) = (d_1 - 2)(p_1 + 1) \geqq 0$$

since $d_1 \geqq 2$, $p_1 \geqq 2$.

If the proper transform $\rho'(F)$ is still singular at P', let $n_2(P)^{(1)}$ denote the local contribution at P to the quantity n_2 which is due to those exceptional curves which are obtained from first p_1 times blowing-ups. We define $\sigma(P)^{(1)}$, $\tau(P)^{(1)}$ and $\delta_F(P)^{(1)}$ in a similar way. Then we have

$$n_2(P)^{(1)} = p_1, \qquad \sigma(P)^{(1)} \geqq p_1,$$

$$\tau(P)^{(1)} = p_1 d_1 - 2p_1,$$

whence

$$\delta_F(P)^{(1)} = \sigma(P)^{(1)} + \tau(P)^{(1)} \geqq p_1 d_1 - p_1$$

and

$$\delta_F(P) - n_2(P) \geqq p_1(d_1 - 2) \geqq 0$$

since $d_1 \geqq 2$, $p_1 \geqq 2$.

At the next step we proceed as in the subcase III-1) or III-2).

We continue this argument I) to IV) untill we can resolve the singularity at P. Thus we have calculated the local contribution $\delta_F(P)$ as follows;

Proposition 2.1. *We can estimate the local contribution $\delta_F(P)$ as follows;*

$$\delta_F(P) \begin{cases} = d_0 + d_1 - p_1 - 3 & \text{if } d_a = 1,\ a \geqq 2 \\ \geqq d_0 + d_1 - d_a - p_1 - 2 & \text{if } d_a > 1,\ a \geqq 2 \\ \geqq d_0 - p_1 & \text{if } d_a > 1,\ a = 1. \end{cases}$$

In any case we have $\delta_F(P) \geqq n_2(P)$.

From now on, in addition to the conditions A1) to A3), we assume the following conditions hold.

A-4) X is a homology plane, i.e., the homology groups $H_i(X; \boldsymbol{Z})$ vanish for $1 \leqq i \leqq 4$.

A-5) $\pi_1(X) = (e)$.

A-6) $\kappa(X) \geqq 0$.

Note that by a theorem of J. H. C. Whitehead, A-4) and A 5) imply that X is topologically contractible.

We can prove easily the following

Lemma 2.2. *Under the above notations, the assumption* A-4) *implies the following;*

(1) *X is affine.*

(2) *The divisor class group* Pic X *is trivial.*

(3) *The units in the coordinate ring $\Gamma(X)$ of X are just the elements of $\boldsymbol{C}^*$.*

In particular (2) *and* (3) *imply that the irreducible components of D generate freely the divisor class group* Pic V.

For the proof of these, see [F] § 1 and § 2.

This lemma implies the following

Lemma 2.3. *Under the assumption* A-1) *to* A-4) *the integers m and n are mutually prime.*

Proof. From lemma 2.2, (1) and (2) the proper transform $\pi'(l)$ of the line l on $\boldsymbol{P}^2$ has the expression as follows;

$$\pi'(l)=n_F F'(F)+n_G\pi'(G)+\sum n_i D_i,$$

where n_F, n_G and n_i are integers and D_i's are irreducible components of D other than F and G. Taking the direct image by π, we get

$$l\cong n_F F+n_G G$$

in Pic $\boldsymbol{P}^2$. Since Pic $\boldsymbol{P}^2=Zl$, this implies that $m=\deg F$ and n $\deg G$ are mutually prime.

We now apply the inequality obtained in § 1 to our situation.

Theorem 1. *Lel F and G be the irreducible curves on $\boldsymbol{P}^2$. We assume that F, G and the surface X obtained by the above construction satisfy the conditions* A-1) *to* A-4) *and* A-6). *Then have $m=\deg F\leqq 3$.*

Proof. We apply the inequality (1.4). By lemma 2.2, since V is rational, we have $b_2=\beta_2$. We get also

$$b_0=1, \qquad e_1=1, \qquad \lambda=-3(m+n).$$

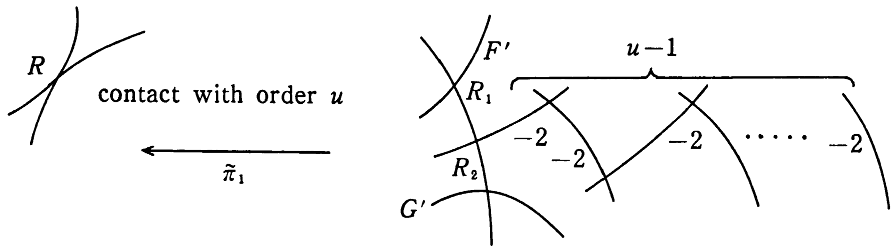
We have already calculated the local contribution $\delta_F(P)$ and $\delta_G(Q)$ for each singular point P of F and Q of G. We put

$$\delta_F=\sum_{P_i}\delta_F(P_i), \qquad \delta_G=\sum_{Q_j}\delta_G(Q_j)$$

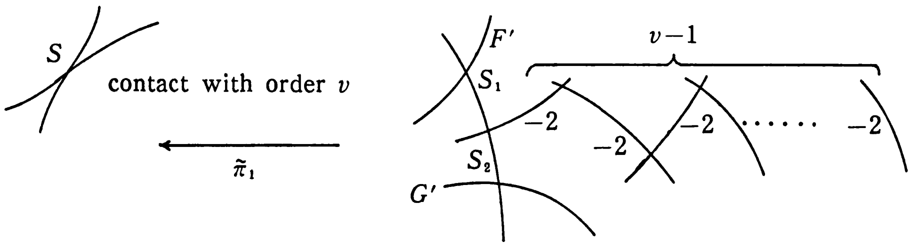
where P_i's and Q_j's are singular poists of F and G respectively. Note that if $m\geqq 3$, F has singilar points and therefore δ_F is positive. Similarly δ_G is positive if G is singular.

We shall calculate the local contributions at R and S, where $\{R, S\}=F\cap G$.

Put $i(F, G; R)=u$, $i(F, G; S)=v$, where $u+v=mn$. First we must blow up at R and its infinitely near points up to order $\leqq u$, all lying on the proper transform of F, to get a normal crossing divisor. This process is $\tilde{\pi}_1$ considered locally at R.



We must also blow up v times at Q.



Moreover, to get a simply connected boundary D, we blow up at one of four points R_1, R_2, S_1, S_2 and its infinitely near points arbitrarily many times and exclude the lastly obtained exceptional curve from D.

Thus the sum $\delta(R, S)$ of the local contributions at R and S is equal to

$$\begin{aligned}\delta(R, S)&=\sigma(R)+\tau(R)+r_3(R)+2r_4(R)+\sigma(S)+\tau(S)+r_3(S)+2r_4(S)\\&=u-1+v-1=mn-2.\end{aligned}$$

Putting these numbers into the inequality in Proposition 1.2, we obtain

$$1-3(m+n)+1+mn-2+\delta_F+\delta_G\leqq -4,$$

or

$$(m-3)(n-3)+\delta_F+\delta_G\leqq 5.$$

By lemma 2.3 and by noting that δ_F and δ_G are positive if $m\geqq 3$ and $n\geqq 3$, we have $m\leqq 4$.

Now we shall exclude the case $m=4$. If so, the arithmetic genus $P_a(F)$ of F is 3. Since F is topologically isomorphic to $\boldsymbol{P}^1$, F must have one singular point with multiplicity three or three singular points with multiplicity two, where some of them might be infinitely near to the other. In any case $\delta_F=3$. Thus the inequality becomes

$$(n-3)+3+\delta_G\leqq 5.$$

Since $n\geqq m=4$, since m and n are mutually prime and since $\delta_G>0$, this is a contradiction.

§ 3. The case $m\leqq 3$

In this section we shall consider three cases $m=1, 2, 3$ separately. In these cases we have obtained so far only incomplete results. We even state some results without details, but include results which are obtained supposing that some conjecture is true.

Case $m=3$ The arithmetic genus $P_a(F)$ is 1 and F has only one singular point of multiplicity 2. We have $\delta_F=1$ and the inequality becomes $\delta_G\leqq 4$.

If $n=5$, G has some singular points, whose multiplicity chains are subsets of the following:

(4), (3, 2, 2, 2), (2, 2, 2, 2, 2, 2)

By a chain consisting of several intergers, we mean that not only the point itself is a singular point with the first number as its multiplicity, but also there are infinitely near singular points whose multiplicities are successively indicated by the integers in the chain. If G has one singular point with multiplicity 4, we have $\delta_G=5$. In the other cases, we have $\delta_G=6$. Thus the case $n=5$ is impossible. The case $n\geqq 7$ is impossible by the same reason.

Subcase $m=3$, $n=4$ In this case arithmetic genus of G is three and G has either one singular point of multiplicity three or three singular points of multiplicity two. First we consider the case when G has unique cuspidal singular point of multiplicity three. Then choosing suitably a system of homogeneous coordinates (X, Y, Z) on $\boldsymbol{P}^2$, we can write the equation of G as

$$Y^3Z=X^2(X^2+aY^2).$$

Write the equation of F as follows.

$$f:=a_1XZ^2+a_2YZ^2+a_3X^2Z+a_4XYZ+a_5ZY^2+a_6X^3+a_7X^2Y+a_8XY^2+a_9Y^3+Z^3$$

Here we may assume the coefficient of Z is one, because if the coefficient of Z is zero, F and G meet at the singular point of G. We may assume also that the points R and S lie outside the line $Y=0$.

The intersection points of F and G are then obtained as common roots of the following two equations:

(1) $\quad z=x^2(x^2+a)$

(2) $\quad f';=z^3+a_1xz^2+a_2z^2+a_3x^2z+a_4xz+a_5z+a_6x^3+a_7x^2+a_8x+a_9=0$

where $x=X/Y$, $z=Z/Y$.

We plug (1) into (2) and let $\tilde{f}$ be the obtained polynomial. The hypothesis that $F\cdot G=uR+vS$ implies that the equation $\tilde{f}=0$ should have two distinct roots. Here $u+v=12$ and it is sufficient to consider u for $1\leqq u\leqq 6$. Thus we get the following equation;

$$\tilde{f}=(x+r)^u\times(x+s)^{12-u}$$

whence, comparing the coefficient of x^{11} we obtain

$$r=-(12-u)s/u.$$

By the comparison of coefficients of terms in x, using a computer, we can determine inductively the coefficients $a, a_1, a_2, \cdots, a_9$ of f as polynomials of s and we also get one relation from the comparison of coefficients of x^7. For $u=1, 2, 4, 5$, we get a contradiction from this relation. For $u=3$ and 6, we can determine the polynomial f, but f splits into the third power of a linear form. Thus in this case there exist no irreducible curves which satisfy the conditions A-1) to A-3).

In the case of a quintic curve with three ordinary cusps, the involved computation is so complicated that we could not obtain any definite answer.

Case $m=2$ To us the following statement seems to be true.

(*) When the degree $G=n$ is fixed, among many possibilities of the combinations of multiplicites of singular points of G, δ_G attains its minimum if the number of singular points is the least.

We assume that this conjecture is true. In the case of a curve in $\boldsymbol{P}^2$, δ_G attains its minimum if G has only one cuspidal singular point with multiplicity $n-1$ and we obtain $\delta_G=2n-5$. Putting these into the inequality (1, 4) we have

$$1-(2+n)\cdot 3+1+2n-5+2n-2\leqq -4.$$

Therefore $n\leqq 7$.

In fact when $n=7$, the combination of multiplicities of the singular points of G is one of the following:

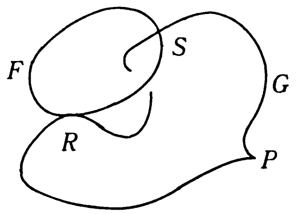
$$(6), (5, 3, 2, 2), (4, 4, 3), (4, 4, 2^3), (3^5), (3^4, 2^3), (3^3, 2^6), (3^2, 2^9), (3, 2^{12}), (2^{15}).$$

where some of these singular points may be infinitely near points to the other. On the other hand we have $\delta_G\leqq 9$ by the inequality (1.4). By calculating δ_G in each case we can see that only the case when G has unique cuspidal singular point is possible. In the case $m=2$, $n=7$ and G has only one cuspidal singular point, we can check, in a similar way as in the case $m=3$, $n=4$, that there exists no curve G which satisfies the condition A-2) and A-3).

In the case when $n=5$ we have nothing to say.

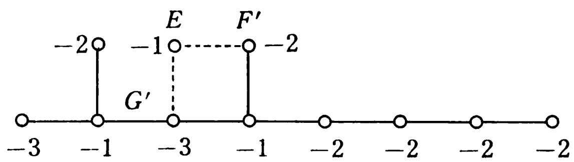
In the case $m=3$, we have some examples of contractible surfaces. We shall describe these examples. First we recall the example of Ramanujam.

Example 1. *Ramanujam's example* Bet F be a conic on $\boldsymbol{P}^2$ and let G be a curve of degree 3 on $\boldsymbol{P}^2$ with a ordinaly cusp P such that $F\cdot G=5R+S$.



After resolving the singularity of G and blowing up successively, we make the set theoretic total transform of $F\cup G$ a divisor with normal crossings. Finally we blow up at S and we denote the obtained surface by V. The dual graph of the total transform of $F\cup G$ on V has the following picture.

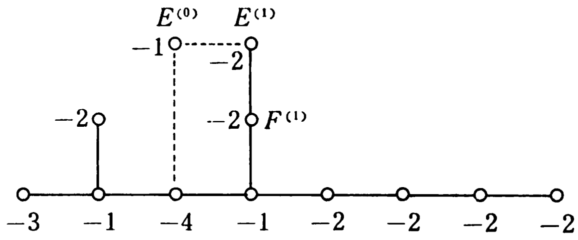
(3.1)



Here we denote the proper transform of F and G by F' and G', and E is the exceptional curve obtained by the blowing up at S. We put $D=$(total transform of $F\cup G$) $-E$ as defined in §2 and put $X=V-D$. This is an example of Ramanujam and X is topologically contractible.

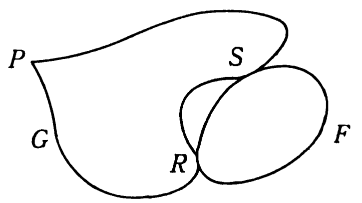
Example 2. In the example 1, we further blow up at the intersection point $E\cap G'$ and let $V^{(1)}$ be the obtained surface. Then the dual graph of the total transform of $F\cup G$ on $V^{(1)}$ has the following picture.

(3.2)



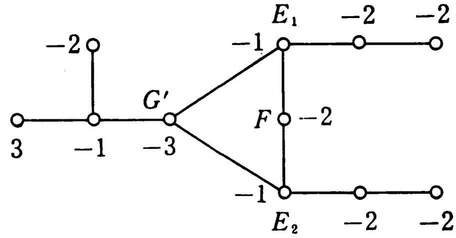
Here $F^{(1)}$, $G^{(1)}$, $E^{(1)}$ are the proper transforms of F', G', E in (3.1) respectively and $E^{(0)}$ is the exceptional curve obtained by blowing up at $E\cap G'$. Put $D^{(1)}=$(total transform of $F\cup G)-E^{(0)}$ and put $X^{(1)}=V^{(1)}-D^{(1)}$. Then we can show that $X^{(1)}$ is isomorphic to the example of Gurjar-Miyanishi and $X^{(1)}$ is topologically contractible.

Example 3. Let F be a conic on $\boldsymbol{P}^2$ and let G be a curve of degree 3 on $\boldsymbol{P}^2$ with an ordinary cusp P such that $F\cdot G=3R+3S$. In fact we can show that such curves exist.



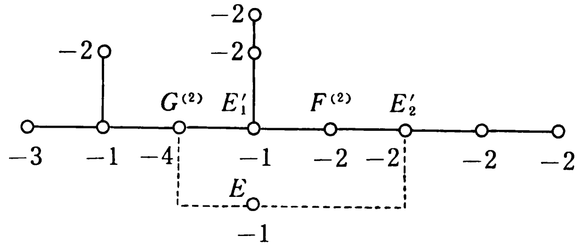
After resolving the singularity of G and blowing up successively we make the set theoretic total transform of $F\cup G$ a divisor with normal crossings. The dual graph of the total transform of $F\cup G$ is as follows.

(3.3)



We further blow up one of $G'\cap E_1$ and $G'\cap E_2$, say $G'\cap E_2$ and let W be the obtained surface. Then the dual graph of the total transform of $F\cap G$ looks as follows.

(3.4)



Here E is the exceptional curve obtained by blowing at $G' \cap E_2$. Let $W^{(2)}$ be the obtained surface. Put $D^{(2)}=$(total transform of $F \cup G$)$-E$ and put $X^{(2)}=W^{(2)}-D^{(2)}$. Then $X^{(2)}$ is isomorphic to the surface in the example 2 and topologically contractible.

These are all example of homology planes we know in the case $m=2$.

The casn $m=1$ We shall treat this case in the appendix of this paper. we recently found new exomples of contractible surfaces in this case.

§ 4. A generalization of Gurjar-Miyanishi's example

In this section we shall generalize the construction of Gurjar-Miyanishi and apply the inequality (1.4).

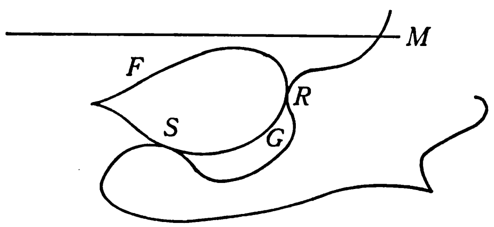
Let Σ_a be a relatively minimal ruled surface with a minimal section M and a fiber l. Let F and G be curves on Σ_a. We assume that the following hold.

B-1) $F \sim mM+mal$, $G \sim nM+(na+1)l$ where m, n are integers.

B-2) F and G are topologically isomorphic to $\boldsymbol{P}^1$.

B-3) F and G meet each other in two smooth points. Put $F \cap G=\{R, S\}$. Put $u=i(F \cdot G\,;\, R)$, $v=i(F \cdot G\,;\, S)$. Then $u+v=m(na+1)$.

B-1) implies that F and M are disjoint and G and M meet transversally at one poit. We have the following figure.



We will consider a ruled surface Σ_a only when $a \geqq 2$, because such a configulation on Σ_1 can be reduced to the configulation on $\boldsymbol{P}^2$ and there do not exist irreducible curves on $\Sigma_0=\boldsymbol{P}^1 \times \boldsymbol{P}^1$ which satisfy B-1) to B-3).

Let $\pi_2: V' \to \Sigma_a$ be the shortest sequences of blowing-ups at singular points of F and G including infinitely near singular points such that the proper transforms of F and G on V' become nonsingular. Let $\tilde{\pi}_1: \tilde{V} \to V'$ be the shortest sequences of blowing-ups such that the total transform $(\pi_2 \circ \tilde{\pi}_1)^{-1}(F \cap G)$ is a divisor with normal crossings.

We further blow up $\tilde{V}$ arbitrarily many times at infinitely near points of one of R or S. Let $\pi_0: V\to\tilde{V}$ be the composition of such blowing-ups and put $\pi_1=\tilde{\pi}_1\circ\pi_0$. let E be (-1) curve obtained by the last blowing-up in π_0. Put $\pi=\pi_2\circ\pi_1$ and $D=\mathrm{Supp}(\pi^{-1}(F\cup G\cup M)-E)$. We may assume that D is minimal. This assumption restricts the sequence of blowing-ups $\pi_0: V\to\tilde{V}$. Put $X=V-D$. Then our question similar to the one in §2 is the following.

When does X become a contractible surface of $\bar{\kappa}(X)=2$?

This construction is a generalization of Gurjar-Miyanishi's. We assume that X is a contractible surface with $\kappa\geqq 0$ and state some known results.

We can calculate the local contribution $\delta_F(P)$ at the singular point P of F by the same way as in §2. Similarly for $\delta_G(Q)$. We define δ_F, δ_G by

$$\delta_F=\sum_{P_i}\delta_F(P_i),\qquad \delta_G=\sum_{Q_i}\delta_G(Q_i),$$

where P_i's and Q_i's are singular points of F and G respectively.

We have

$$b_0=1,\ e_1=1,\ \lambda=-(m+u)(a+2)-2+(a-2),$$

since

$$K_{\Sigma_a}=-2M-(a+2)\bar{l}.$$

The sum $\delta(R, S)$ of the local contribtions at R, S is equal to

$$\delta(R, S)=m(na+1)-2.$$

Putting these into the inequality (1.4) we have

$$1-(m+n)(a+2)+a-4+1+m(na+1)-2+\delta_F+\delta_G\leqq -3,\quad \text{or}$$

$$a(m-1)(n-1)-(m+1)+\delta_F+\delta_G\leqq 2n. \tag{4.1}$$

The arithmetic genus of F and G are as follows;

$$p_a(F)=\frac{1}{2}(m-1)(ma-2),\qquad p_a(G)=\frac{1}{2}na(n-1).$$

Note that the curve F on Σ_a can have a singular point with multiplicity at most m. This implies that if $m\geqq 2$ we must blow up at least $\left\lceil\frac{ma-2}{m}\right\rceil$ times to resolve the singularities of F. Here $\lceil\alpha\rceil$ means the smallest integer which is greater than or equal to α. Similarly if $n\geqq 2$ we must blow up at least a times to resolve the singularities of G. As we proved in Proposition 2.1, this implies

$$\delta_F\geqq\left\lceil\frac{ma-2}{m}\right\rceil \text{ if } m\geqq 2,\qquad \delta_G\geqq a \text{ if } n\geqq 2.$$

Putting these into the inequality (4.1) we obtain the following inequality;

$$(4.2)\quad \begin{aligned}&\{a(n-1)-1\}(m-1)+2a\leqq 2n+2 && \text{if } m\geqq 3 \text{ and } n\geqq 2\\ &a(n+1)\leqq 2n+4 && \text{if } m=2 \text{ and } n\geqq 2\end{aligned}$$

We consider several cases separately.

Case I) $n \geqq 2$

Subcase I-1) $n \geqq 2$, $m \geqq 3$. In this case we have

The left hand side of the inequality (4.2) $\geqq 2a(n-1)-2+2a=2an-2$.

Thus a and a must satisfy

$$2an-2 \leqq 2n+2 \quad \text{or} \quad 2n(a-1) \leqq 4.$$

Since $a \geqq 2$ and $n \geqq 2$, only the case $a=2$ and $n=2$ is possible and if $a=2$ and $n=2$ we have $m=3$. But in this case the original inequality (4.1) becomes $\delta_F+\delta_G \leqq 4$. Since $p_a(F)=4$ and $p_a(G)=2$, $\delta_F=4$ and $\delta_G=2$. Therefore this case does not occur.

Subcase I-2) $n \geqq 2$, $m=2$. The inequality (4.2) becomes

$$(a-2)(n+1) \leqq 2.$$

This implies that only the case $a=2$ is possible and in this case the inequality (4.1) becomes $\delta_F+\delta_G \leqq 5$.

Subcase I-3) $n \geqq 2$, $m=1$. The inequality (4.1) becomes

$$\delta_G \leqq 2n+2.$$

Case II) $n=1$ *or* 0. The inequality (4.1) becomes

$$\delta_F \leqq m+3 \qquad \text{if} \quad n=1,$$

$$\delta_F \leqq (a+1)(m-1)+2 \qquad \text{if} \quad n=0.$$

Gurjar and Miyanishi's example of a contractible surface of Kodaira dimension two belongs one of this case, that is the case $a=2$, $m=2$ and $n=1$.

Remark. We can obtain more restrictions by considering each case in detail. Moreover if we assume that the conjecture (*) in §3 is true, we can exclude some possibilities of values a, m and n, For example in the case II) $n=1$, if we assume $m \geqq 3$, we can show that only the following four cases are possible.

$$(a, m, n)=(2, 4, 1), (2, 3, 1), (3, 3, 1), (3, 2, 1).$$

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Appendix: Examples of homology planes of general type

By

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In the present article, we shall construct infinitely many examples of homology planes $X_{n,m}^{(1)}$ and $X_{n,m}^{(2)}$ with Kodaira dimesnion 2, i. e., of general type. Among them, $X_{n,1}^{(1)}$ is topologically contractible, though we do not know yet whether or not so are the others.

1. Construction. Let C and D be irreducible curves of respective degree 1 and n on the projective plane $\boldsymbol{P}^2$. We assume that $C \cdot D=(n-m)P+mQ$, where $1 \leqq m \leqq \left[\frac{n}{2}\right]$ and P, Q are distinct smooth points on D and that D has a unique cuspidal singular point R of multiplicity $n-1$. Then, choosing suitably a system of homogeneous coordinates (X, Y, Z) on $\boldsymbol{P}^2$, we can write the equation of D as

$$Y^{n-1}Z=X^2(X^{n-2}+a_2X^{n-4}Y^2+\cdots+a_{n-2}Y^{n-2}).$$

Write the equation of C as $Z=bX+cY$. We may assume that the points P and Q lie outside the line $Y=0$. The intersection points of C and D are then obtained as roots of the following equation

$$x^2(x^{n-2}+a_2x^{n-4}+\cdots+a_{n-2})=bx+c,$$

where $x=X/Y$. The hypothsis that $C \cdot D=(n-m)P+mQ$ implies that the above equation should have two distinct roots. Therefore we have the following relation

$$x^2(x^{n-2}+a_2x^{n-4}+\cdots+a_{n-2})-bx-c=(x+u)^{n-m}(x+v)^m$$

whence, by the comparison of coefficients of terms in x, we obtain

$$v=-\left(\frac{n-m}{m}\right)u.$$

Since $a_2, \cdots, a_{n-2}, b$ and c are expressed as polynomials in u and v, they are expressed in a unique way as homogeneous polynomials in u; $\deg_u a_i=i$ for $2 \leqq i \leqq n-2$, $\deg_u b=n-1$ and $\deg_u c=n$. Therefore, replacing homogeneous coordinates X, Y, Z by $u^{n-1}X$, u^nY and Z respectively, we can determine the curves C and D uniquely upto projective transformations as well as the intersection points P and Q. Here we note that $u \neq 0$ because $P \neq Q$. If $m=1$, the curves D and C are respectively given by

$$Y^{n-1}Z=X^2\left\{X^{n-2}-\sum_{i=2}^{n-2}(i-1)\binom{n}{i}X^{n-i-2}Y^i\right\}$$

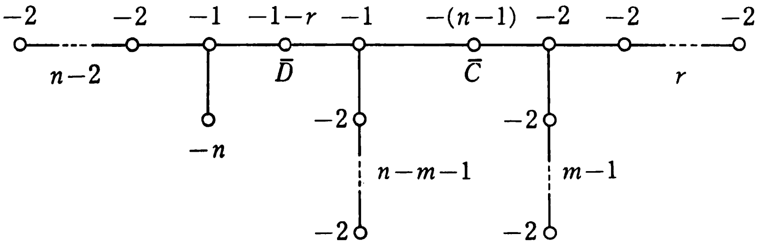
and

$$Z=(n^2-2n)X+(n-1)Y\,.$$

The intersection points P and Q are then given by

$$P=(-1,\ -1,\ -n^2+3n-1)\quad\text{and}\quad Q=(n-1,\ 1,\ (n-1)^3)\,.$$

We perform the following blowing-ups with centers P, Q, R and their infinitely near points: Blow up the point P and its infinitely near points $n-m$ times to separate the proper transforms of C and D, blow up the point Q and its infinitely near points m times first to separate the proper transforms of C and D, then continue the blowing-ups $r+1$ times with centers infinitely near points of Q lying on the proper transform of D, and finally blow up the point R and its infinitely near points minimally to make the proper transform of D smooth. Let $\sigma: V\to \boldsymbol{P}^2$ be the above sequence of blowing-ups, and let $\bar{C}$ and $\bar{D}$ be the proper transforms of C and D, repectively. Let E ce the (-1) curve appearing in the last stage of the $(m+r+1)$ blowing-ups with centers Q and its infinitely near points, and let $X:=V-(\sigma^{-1}(C+D)-E)$. We obtain the following dual graph of $\sigma^{-1}(C+D)-E$:



Note that $(\bar{C}^2)=1-n$ and $(\bar{D}^2)=-1-r$. Next we consider the intersection matrix associated with this dual graph and require that the discriminant have absolute value -1, i. e., the intersection matrix is unimodular. In order to compute the discriminant, we can make avail of the result in Gurjar-Shastri [2; Lemma 11.3]. Denote the vertices $\bar{D}$ and the (-1) curve between $\bar{D}$ and $\bar{C}$ by Δ and Γ respectively. Deleting these two adjacent vertices Δ and Γ from the above dual graph, it is a union of three connected components, each of which has negative-definite intersection form. Call these connected components G_1, G_2, G_3 counted from the left. The intersection matrix of $G_1+\Delta$ is diagonalized so that the entry of Δ is $-1-r+n(n-1)$. Similarly, the intersection matrix of $G_2+\Gamma+G_3$ is diagonalized so that the entry of Γ is

$$\frac{r-(m^2-m-1)}{(n-m)\{(n-m-1)r+(m+1)n-2m-1\}}\,.$$

Now the intersection matrix of the graph G is unimodular if and only if

$$\{n(n-1)-r-1\}\{r-(m^2-m-1)\}-(n-m)\{(n-m-1)r+(m+1)n-2m-1\}=\pm 1$$

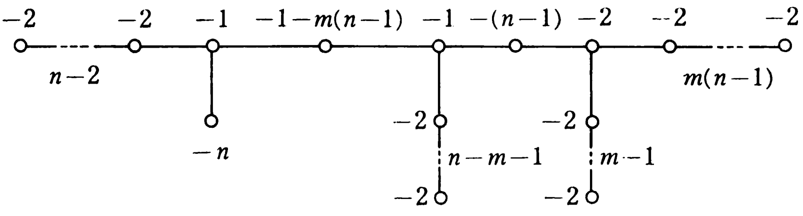
provided $n^2-n-1\geqq r\geqq m^2-m-1$. Taking the value -1 forcibly, this equation is equal to

$$(r-mn+m)(r-mn+m+2)=0.$$

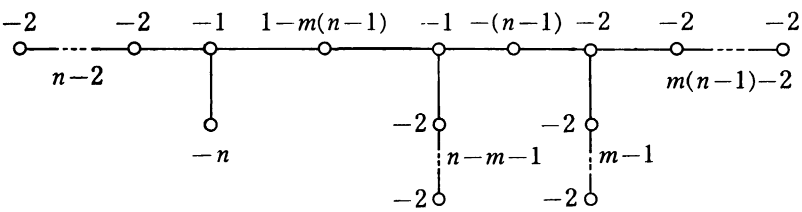
So, the intersection matrix of G is unimodular either if $r=mn-m$ provided $\dfrac{n^2-n-1}{n-1}\geqq m$ and $n\geqq\dfrac{m^2-1}{m}$ or if $r=mn-m-2$ provided $\dfrac{n^2-n+1}{n-1}\geqq m$ and $n\geqq\dfrac{m^2+1}{m}$. The condition $\dfrac{n^2-n-1}{n-1}\geqq m$ or $\dfrac{n^2-n+1}{n-1}\geqq m$ follows automatically from the hypothesis that $m\leqq\left[\dfrac{n}{2}\right]$ and $3\leqq n$. Similarly, the condition $n\geqq\dfrac{m^2-1}{m}$ or $n\geqq\dfrac{m^2+1}{m}$ follows automatically from the same hypothesis. On the other hand, the Picard group $\mathrm{Pic}(V)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ is generated by irreducible components of $\sigma^{-1}(C+D)-E$ if $r\neq m(n-1)-1$, for $(m+r+1-mn)E$ is an integral combination of these components. Moreover, these components are independent over $\boldsymbol{Q}$ because rank $\mathrm{Pic}(V)$ equals to the number of irreducible components of $\sigma^{-1}(C+D)-E$. Hence if the intersection matrix of G is unimodular, $\mathrm{Pic}(V)$ is a free abelian group generated by irreducible components of $\sigma^{-1}(C+D)-E$. Note that $H^2(V;\boldsymbol{Z})\cong\mathrm{Pic}(V)$ because V is rational. Looking at the long exact sequence of integral cohomology groups associated with a pair $(V, V-X)$ and making use of the Lefschetz duality, we easily conclude that X is a homology plane. Thus we have proved the following:

Theorem 1. *For $n\geqq 3$ and $\left[\dfrac{n}{2}\right]\geqq m\geqq 1$, we have homology planes of general type $X_{n,m}^{(i)}$ $(i=1, 2)$ which are embedded into a smooth projective rational surface V so that the boundary divisor $V-X_{n,m}^{(i)}$ is a divisor with simple normal crossings and with the following dual graph:*

$X_{n,m}^{(1)}$



$X_{n,m}^{(2)}$

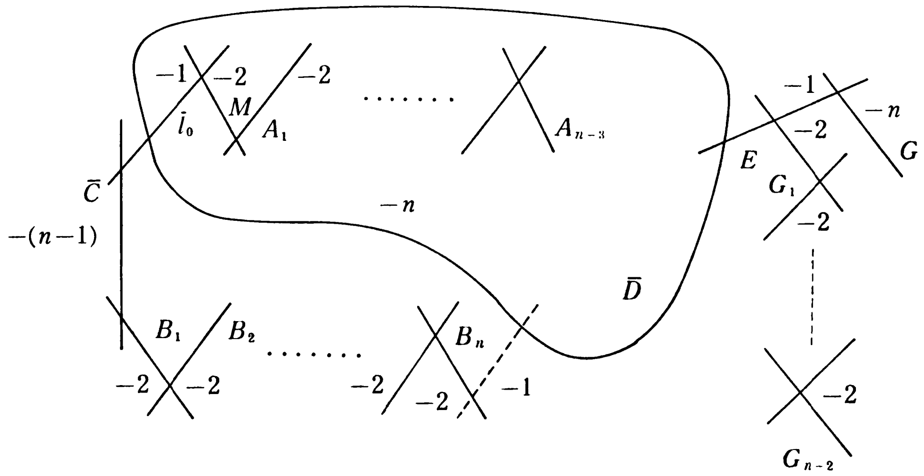


If $(n, m)\neq(n', m')$ then $X_{n,m}^{(i)}$ $(i=1, 2)$ is not isomorphic to $X_{n',m'}^{(1)}$ or $X_{n',m'}^{(2)}$.

To verify the last assertion, note that if $X_{n,m}^{(i)}$ is isomorphic to $X_{n',m'}^{(1)}$, for example, then the graphs of the boundary divisors attached to $X_{n,m}^{(i)}$ and $X_{n',m'}^{(1)}$ must be transformed from one to the other by the blowing-up-and-downs with centers lying on the boundary divisors. This is clearly impossible. The same reasoning relying on the difference between the boundary graphs of $X_{n,m}^{(i)}$ and a homology plane of Kodaira dimension 1 (cf. [1]), it can be shown that $X_{n,m}^{(i)}$ is a homology plane of general type.

Note also that $X_{3,1}^{(1)}$ has the same boundary graph as Ramanujam's surface [4]. Indeed, consider the smooth projective surface V containing $X_{3,1}^{(1)}$ which is constructed in the arguments. Let A be an irreducible conic on $\boldsymbol{P}^2$ meeting the cubic D with $A\cdot D=P+4Q+S$, where S is a smooth point on D other than P and Q. The proper transform $\sigma'(A)$ on V is a (-1) curve meeting the (-1) curve E (cf. the previous notations). Now, contracting the components in the subgraphs G_1, G_3, the component Γ and $\sigma'(A)$, we have a birational morphism $\tau: V\to\boldsymbol{P}^2$ which maps $\sigma'(D)$ and the unique component in the subgraph G_2 to a cuspidal cubic and a smooth conic meeting each other with $\tau(\sigma'(D))\cdot\tau(\sigma'(A))=5P'+Q'$ as in Ramanujam's example for smooth points P' and Q' on $\tau(\sigma'(D))$.

2. Topological contractability. We shall consider the surface $X:=X_{n,1}^{(1)}$. With the previous notations, X is embedded into a smooth projective surface V with the boundary divisor illustrated below:



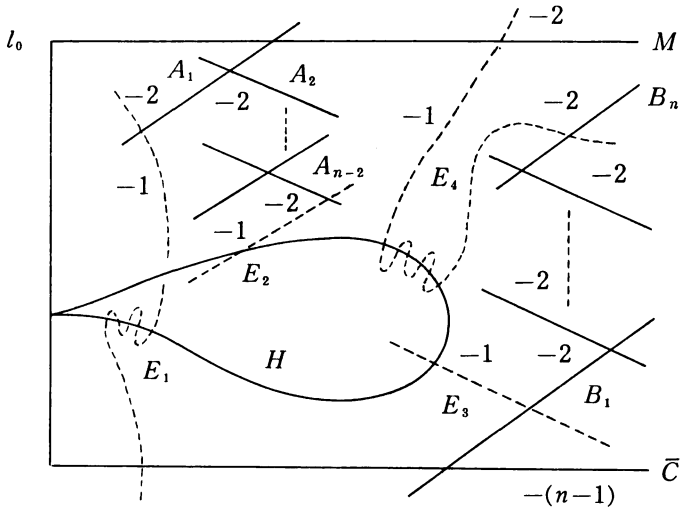
Contracting $E, G_1, \cdots, G_{n-2}$ and $\bar{D}$, we have a morphism $\rho: V\to W$ such that $l_0:=\rho(\bar{l})$ defines a $\boldsymbol{P}^1$-fibration $p: W\to\boldsymbol{P}^1$ on W. The fibration p has two cross-sections M and $\bar{C}$, and the components $A_1, \cdots, A_{n-3}$ and $B_1, B_2, \cdots, B_n$ are contained in singular fibers of the $\boldsymbol{P}^1$-fibration p. Moreover, the image $H:=\rho(G)$ meets a general fiber l of p with $(H\cdot l)=n-1$, i.e., H is an $(n-1)$-section of p. Moreover, H has a cuspidal singular point R of multiplicity $n-1$ lying on the fiber l_0.

Contracting further the components of singular fibers of p not meeting the section M, we obtain a birational morphism $\sigma: W\to\Sigma_2$ from W to the Hirzebruch surface Σ_2 of degree 2 so that M is the minimal section, $S:=\sigma\cdot\rho(\bar{C})$ is a cross-section with $S\sim M+2l$ and $T:=\sigma(H)$ is an $(n-1)$-section with $T\sim(n-1)(M+2l)$. Note that W has Picard number $2n+1$ and there are $2n-1$ curves on W to be contracted under σ, including the components $B_1, \cdots, B_n$ and $A_2, \cdots, A_{n-3}$. So, there are four (-1) curves $E_i (1 \leqq i \leqq 4)$ such that

(i) $E_1+A_1+\cdots+A_{n-3}+E_2$ and $E_3+B_1+\cdots+B_n+E_4$ are singular fibers of p, and there are no other singular fibers of p; if $n=3$, E_1+E_2 is a singular fiber;

(ii) $(E_1\cdot A_1)=(E_2\cdot A_{n-3})=(E_1\cdot \bar{C})=(E_2\cdot H)=1$ and $(E_1\cdot H)=n-2$; if $n=3$, $(E_1\cdot E_2)=(E_2\cdot M)=(E_1\cdot \bar{C})=1$;

(iii) $(E_3\cdot B_1)=(E_4\cdot B_n)=(E_3\cdot H)=(E_4\cdot M)=1$ and $(E_4\cdot H)=n-2$;



The morphism $p: W \to \boldsymbol{P}^1$ restricted onto X is a fibration onto A^1. So, we have an exact sequence of the fundamental groups,

$$\pi_1(l-\{(n+1)\text{points}\}) \longrightarrow \pi_1(X) \longrightarrow \pi_1(A^1) \longrightarrow 1,$$

where l is a general fiber of p and $l\cap X=l-\{P_0, \cdots, P_n\}$ with $P_0=l\cap M$, $\{P_1, \cdots, P_{n-1}\}=l\cap H$ and $P_{n+1}=l\cap \bar{C}$. Take loops $\gamma_0, \cdots, \gamma_{n+1}$ on l around $P_0, \cdots, P_{n+1}$, oriented by the natural complex orientation on X. Then $\pi_1(X)$ is generated by the images of $\gamma_0, \cdots, \gamma_{n+1}$. Write the image of γ_i in $\pi_1(X)$ by $[\gamma_i]$. As loops around the curve H, we have $[\gamma_1]=\cdots=[\gamma_{n-1}]$, which we denote by y. Consider the curve E_2. Take loops γ and δ on E_2 around the points $E_2\cap H$ and $E_2\cap A_{n-3}$, respectively, oriented by the natural orientation on X. Then $[\gamma]=[\gamma_1]$ and $[\gamma][\delta]=1$. Hence $[\delta]=[\gamma^{-1}]$. Let δ' be an oriented loop on the curve A_1 around the point $A_1\cap M$. Since the branch $A_1+\cdots+A_{n-3}$ is contracted to a rational double point, we know that $[\delta']=[\delta]^{n-2}$(cf. Mumford [3]), and $[\delta']=[\gamma_0]$ which we call x. Hence $x=[\delta]^{n-2}=y^{2-n}$. Next, look at the curve E_3. Let β and β' be oriented loops on E_3 around the points $H\cap E_3$ and $E_3\cap B_1$, respectively. Then $[\beta][\beta']=1$ and $[\beta]=y$. Bring the loops γ_{n+1} and β' along the curves $\bar{C}$ and B_1, respectively, near the point $\bar{C}\cap E_3$. Since we can take $D_* \times D^*$ as an open neighbourhood of $\bar{C}\cap E_3$, where D^* is a punctured disk and since $\pi_1(D^*\times D^*)\cong \boldsymbol{Z}\times \boldsymbol{Z}$, we know that $[\gamma_{n+1}]$ and $[\beta']$ commute with each other. Denote $[\gamma_{n+1}]$ by z. Then $\pi_1(X)$ is generated by x, y and z with relations $x=y^{2-n}$ and $yz=zy$. Hence $\pi_1(X)$ is abelian. Since $H_1(X; \boldsymbol{Z})=(0)$ by Theorem 1, we have $\pi_1(X)=(1)$. By a theorem of J. H. C. Whitehead, X is topologically contractible. We have thus proved:

Theorem 2. *With the notations in Theorem* 1, *the surface* $X_{n,1}^{(1)}$ *is topologically contractible for every* $n \geqq 3$.

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