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On a probabilistic properties of Takagi's function

By

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Let

$$\Phi(x)=\begin{cases} 2x & \text{if } 0\leqq x\leqq 1/2, \\ 2-2x & \text{if } 1/2\leqq x\leqq 1, \end{cases}$$

and define

$$f(x)=\sum_{n=1}^{\infty} 2^{-n}\Phi^{(n)}(x),$$

where $\Phi^{(n)}(x)=\Phi(\Phi^{(n-1)}(x))$ such that $\Phi^{(0)}(x)=x$. This function $f(x)$ is known as Takagi's nowhere differentiable continuous function [1]. It should be noted that much later van der Waerden rediscovered the Takagi function (see [2] or [3]). Since then a number of scientists have studied this function from various points of views [4, 5].

The aim of this paper is to investigate another properties of Takagi's function. In particular it is proved that the local modulus of continuity of the function $f(x)$, after appropriate normalization is asymptotically normal (Theorem 1). In a previous paper it had been shown the same for Weierstrass' function [8]. Theorem 2 due to N. Kôno [5] and it seems to me that this another proof is not more difficult than in [5].

Theorem 1. *Let $f(x)$ be the Takagi function then*

$$\lim_{h\downarrow 0} mes\left\{x:\ x\in(0,1)\ \frac{f(x+h)-f(x)}{h\sqrt{\log_2(1/h)}}<y\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-u^2/2}du,$$

where $x+h$ here and below is defined as the sum modulo 1 and $h>0$.

Proof. Let us consider the points x and $x+h$ by binary series

$$x=\sum_{k=1}^{\infty}\varepsilon_k 2^{-k}, \qquad x+h=\sum_{k=1}^{\infty}\varepsilon'_k 2^{-k} \tag{1}$$

and let $\{X_n(x)\}$ be Rademacher system of functions that is $X_n(x)=1-2\varepsilon_n$.

Assume that $h\leqq 1/2$. Then there exist $m=m_h$ such that

$$\frac{1}{2^{m+1}}<h\leqq\frac{1}{2^m} \tag{2}$$

and denote by k_0 random variable $k_0=k_0(x, h)=\{\max k:\ \varepsilon_1=\varepsilon'_1, \cdots, \varepsilon_k=\varepsilon'_k\}$ (it depends on random variable $X_n(x)$). Using now (1) we can write

$$P(k_0=0)=2h, \qquad P(k_0=l)=h2^l \qquad 1\leqq l\leqq m-1,$$

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$$P(k_0<m)=h2^m, \qquad P(k_0=m)=1-h2^m.$$

A simple calculation shows that

$$E(m-k_0)^2<c, \qquad E(m-k_0)^3<c. \tag{3}$$

Here and below c denote positive constants; the same symbol may stand for different constants. Then by Lemma 3 [5]

$$f(x+h)-f(x)=h\sum_{n=1}^{k_0}X_n(x)+\eta_m\zeta_{k_0m}$$
$$+\frac{1}{2}\sum_{n=m+1}^{\infty}\sum_{k=n+1}^{\infty}(X_n(x)X_k(x)-X_n(x+h)X_k(x+h))2^{-k}=\Delta_1+\Delta_2+\Delta_3, \tag{4}$$

where $\eta_m=\sum_{n=m+1}^{\infty}\frac{1-\varepsilon_k'-\varepsilon_k}{2^k}$ and $\zeta_{k_0m}=\sum_{n=k_0+1}^{m}X_n(x)$. We may also write

$$|\zeta_{k_0m}|\leqq(m-k_0), \tag{5}$$

$$|\eta_m|\leqq 2^{-m}. \tag{6}$$

From (2), (5), (6) we deduce that

$$h^{-1}|\Delta_2|\leqq 2(m-k_0). \tag{7}$$

According to (7), (3) and Markov's inequality it follows that

$$\frac{\Delta_2}{h\sqrt{\log_2 h^{-1}}}\xrightarrow{P}0 \quad \text{as} \quad h\longrightarrow 0. \tag{8}$$

Note also that

$$\sum_{n=1}^{k_0}X_n(x)=\sum_{n=1}^{m}X_n(x)-\zeta_{k_0m}. \tag{9}$$

Since $m\leqq\log_2\frac{1}{h}<m+1$, we have

$$\lim_{m\to\infty}P\left(\frac{\sum_{k=1}^{m}X_k(x)}{\sqrt{\log_2 h^{-1}}}<y\right)=G(y) \tag{10}$$

where $G(y)$ is normal (0, 1) distribution function. Taking into consideration (2) and (3) we have

$$\frac{\zeta_{k_0m}}{\sqrt{m}}\xrightarrow{P}0 \quad \text{as} \quad m\longrightarrow\infty. \tag{11}$$

On the other hand by Lemma 4 [5]

$$h^{-1}|\Delta_3|\leqq c. \tag{12}$$

The assertion of the Theorem 1 is now an immediate corollary of (4), (8), (10), (11), and (12).

Theorem 2. *Let $f(x)$ be the Takagi function then*

1) $\overline{\lim_{h\to 0}} \dfrac{f(x+h)-f(x)}{h\sqrt{2\log_2 1/h \log\log\log_2 1/h}} \overset{a.s.}{=} 1,$

2) $\underline{\lim_{h\to 0}} \dfrac{f(x+h)-f(x)}{h\sqrt{2\log_2 1/h \log\log\log_2 1/h}} \overset{a.s.}{=} -1.$

Proof. Denote $a_m=(2m \log\log m)^{1/2}$. Then by (3), (9) we have

$$^{\forall}\varepsilon>0, \qquad \sum_{m=1}^{\infty} P(|m-k_0|>\varepsilon a_m)<\infty.$$

Consequently according to Borel-Cantelli's theorem we may write

$$\frac{m-k_0}{a_m} \overset{a.s.}{\longrightarrow} 0 \quad \text{as} \quad m \longrightarrow \infty \tag{13}$$

and by (7), (2)

$$\frac{|\Delta_2|}{h a_m} \overset{a.s.}{\longrightarrow} 0, \quad \text{as} \quad m \longrightarrow \infty. \tag{14}$$

Furthermore by (13), (5) we have

$$\frac{\zeta_{k_0 m}}{a_m} \overset{a.s.}{\longrightarrow} 0 \quad \text{as} \quad m \longrightarrow \infty. \tag{15}$$

Relations (4), (9), (12), (14) and (15) together with the law of iterated logarithm (see Hartman-Wintner's theorem [6] or [7]) proves the theorem 2.

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