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The existence of varieties whose hyperplane section is P^r-bundle

By

Eiichi SATO and Heinz SPINDLER

Introduction

In the present paper we consider

Problem 1). For what kind of P^r-bundle ($=Y$) over a projective variety S, does there exist a smooth projective variety X containing Y as an ample divisor? (Remark 2.11)

Problem 2). Let $\{A_i\}$ be a sequence of smooth projective varieties such that A_i is an ample divisor in A_{i+1} for every positive integer i. Assume that A_1 is a P^r-bundle over a non-singular projective variety S. Then, does $\{A_i\}$ terminate or not? (Conjecture of III in [*So*])

When S is a curve, for every P^r-bundle over S ($=Y$), T. Fujita showed the existence of smooth X containing Y in the Problem 1) and gave an example of an infinite sequence in Problem 2). (See 4.21 4.22 in [*Fu*]).

But, when dim $S \geq 2$, Problems 1 and 2 become more complicated rather than the ones in case of curves. In fact, it is proved in [*Fa* + *So*], [*Fa* + *Sa* + *So*], [*Sa* + *Sp*] and [*Sa*] that if S is a smooth surface and there exists a smooth X in Problem 1), Y must be a projective bundle associated with a vector bundle on S except for a special surface S (See ii) in Proposition 3.1). On the other hand, it is known there exist many P^1-bundles, not associated with vector bundles. Moreover, there exists a projective bundle associated with a vector bundle which cannot be ample in any smooth variety. (See Theorem 2 in [*Sa* + *Sp*]).

If a surface S has a suitable good property and F is a vector bundle on S, then we can give a sufficient condition of $P(F)$ to be ample divisor in a smooth variety.

Namely, we have

Theorem II (*in* §2). *Let F be an ample vector bundle over a projective factorial surface S with $H^1(S, F^*) \neq 0$. Assume that*

1) *every curve C on S is numerically effective and* dim $|a_C C| \geq 1$ *with a*

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positive integer a_C,

2) $H^1(S, \mathcal{O}_S) = 0$,

3) *F is generated by its global sections.*

Then, in characteristic zero, there is a linebundle L and an ample vector bundle E on S enjoying the following exact sequence:

$$0 \longrightarrow \mathcal{O}_S \longrightarrow E \longrightarrow F \otimes L \longrightarrow 0.$$

Note here that P^2 and generic surfaces in P^3 with degree ≥ 5 satisfy the above conditions 1), 2).

As for Problem 2), we give a sufficient condition on S under which the sequence $\{A_i\}$ terminates.

In fact, we have

Theorem III (in §. 3) *Let S and A_i be as in the Problem* 2). *Assume that* Pic $S \simeq \mathbf{Z}$ *and* $H^1(S, L) = 0$ *for every linebundle L over S and moreover that the characteristic of the base field is zero. Then, we have the following*

1) *If* $r \geq 2$, *then the sequence terminates.*

2) *If* $r = 1$ *and* dim $S = 2$, *then the sequence terminates except for the case that S is either* P^2 *with* $A_1 = P^1 \times P^2$ *or a suface with* $\kappa(S) = 2$ *and* $q = P_g = 0$.

2′) *if* $S = P^n (n \geq 2)$, *then there is an infinite sequence* $\{A_j\}$ *with* $A_i \simeq P^1 \times P^{n+i-1}$.

Another example of an infinite sequence $\{A_i\}$ satisfying the condition in Problem 2) will be given in 3.7.

Finally, let us state the content in each section briefly. In section 1, given a vector bundle E and an ample vector bundle F enjoying the exact sequence: $0 \to \mathcal{O} \to E \to F \to 0$, we shall consider a necessary and sufficient condition for E to be ample in the sheaf-theoretical language (Theorem II).

In section 2, we shall prove Theorem II. A key for the proof is a vanishing theorem of the first cohomology of a vector bundle (Proposition 2.8).

In section 3, we prove Theorem III.

We work over an algebraically closed field k of any characteristic. Variety means an irreducible, reduced algebraic k-scheme. For a vector bundle E, E^* denotes the dual vector bundle of E.

§.1. Ampleness of a vector bundle

In the present section, let S be an n-dimensional projective variety and F a vector bundle on S of rank $r + 1$.

(1.1) Let us assume that F is ample and let σ be an element in $H^1(S, F^*)$. σ provides us with an extension $E = E_\sigma$ of F by $\mathcal{O}_S$:

$$0 \longrightarrow \mathcal{O}_S \longrightarrow E_\sigma \xrightarrow{\tau} F \longrightarrow 0.$$

Let $p: P(E) \to S$ be the canonical projection.

Throughout this section we shall keep this assumption and the notation.

Our main aim in this section is to show the following.

Theorem I. *Let the notations and assumption be as in* (1.1). *Then the following are equivalent to each other.*

1) *E is an ample vector bundle.*

2) *For every curve C on S, $E_{|C}$ is ample.*

3) *For every pair (C, φ) of an irreducible smooth curve C and a finite morphism $\varphi: C \to S$, $\varphi^* E$ is not isomorphic to $\mathcal{O}_C \oplus \varphi^* F$.*

Moreover, assume that the characteristic of the ground field is zero. Then, the above conditions are equivalent to the following:

3′) *For every curve C in S and the normalization ($h: \bar{C} \to C \subset S$) of C $h^* E$ is not isomorphic to $\mathcal{O}_{\bar{C}} \oplus h^* F$.*

Remark 1.1.1. Let M be a vector bundle on a singular curve C, $\varphi: \bar{C} \to C$ the normalization and T the quotient of $\varphi_* \mathcal{O}_{\bar{C}}$ by $\mathcal{O}_C$. Let q be the non-zero element in the image of the natural map $H^0(C. M^* \otimes T) \to H^1(C, M^*)$. Then, the extension defined by q

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E_q \longrightarrow M \longrightarrow 0$$

does not split, but the lifting

$$0 \longrightarrow \mathcal{O}_{\bar{C}} \longrightarrow \varphi^* E_q \longrightarrow \varphi^* M \longrightarrow 0$$

to the normalization does.

For the proof of Theorem I, we shall make several preliminaries.

The following are well-known:

(1.2) 1. The divisor $Y = P(F)$ via τ in $X = P(E)$ defines the tautological line bundle $\mathcal{O}_{P(E)}(1)$ of E which is denoted by L.

2. Since $L_{|Y}$ is the tauotological line bundle of $P(F)$ and F is ample, there is a positive integer m_0 such that for $m \geq m_0$,

1) The complete linear system of $L^{\otimes m}$ yields a birational morphism $\phi_m: X \to \phi_m(X)$.

2) ϕ_m is isomorphic near Y.

See Theorem 4.2 in Chapter III of [*Ha*].

Thus, ϕ_m is a finite morphism if and only if Y is ample, namely, E is ample. Hereafter, let ϕ be such a ϕ_m and let X_ϕ be $\{x \in \phi(X) | \dim \phi^{-1}(x) \geq 1\}$. Then, X_ϕ is at most a finite subset and $A = \phi^{-1}(X_\phi)$ does not intersect with Y.

This notation A will be used very often.

The above result immediately yields.

Proposition 1.3. *Under the notation of* (1.1), *assume that E is not ample. Then, we have*

1) *A is not empty,*

2) $p_{|A}: A \to S$ *is a finite morphism.*

Proof. 1) is obvious. Since $p: X \to S$ is a P^{r+1}-bundle and Y is a member of the tautological line bundle of E, 2) is trivial by virtue of 2) of (1.2). q.e.d.

Next, we obtain

Proposition 1.4. *Let F be an ample vector bundle on a projective variety S. Then, a vector bundle $\mathcal{O}_S \oplus F$ has a unique trivial quotient linebundle.*

Proof. Let L be a trivial quotient linebundle of $\mathcal{O}_S \oplus F$ ($\phi: \mathcal{O}_S \oplus F \to L$). Since F is ample, we infer that $H^0(S, F^*) = 0$ and hence, ϕ is the projection of $\mathcal{O}_S \oplus F$ to $\mathcal{O}_S$. q.e.d.

Applying this to our situation, we obtain

Proposition 1.5. *Under the notation of* (1.1), *let us assume* $E = \mathcal{O}_S \oplus F$. *Then,* A $(= \phi^{-1}(X_\phi))$ *coincides with the unique section induced by the direct summand* $\mathcal{O}_S$ *of* E.

Proof. Since $P(\mathcal{O}_S) \cap P(F)$ is empty, A contains $P(\mathcal{O}_S)$. Now assume that there is an irreducible component B of A which does not coincide with $P(\mathcal{O}_S)$. Pick a closed curve B' in B such that B' is not contained in $P(\mathcal{O}_S)$. Take the normalisation $g; \bar{B} \to B'$ and pull back the vector bundle E by gp on $\bar{B}$. Then, there are at least two sections of $gp^*(P(E))$ which do not intersect with $gp^*P(F)$, namely one section induced by $P(\mathcal{O}_S)$ and another by B'. Hence, it follows that $gp^*(E)$ $(= \mathcal{O}_{\bar{B}} \oplus gp^*(F))$ has another trivial quotient line bundle induced by B'. Noting that gp^*F is ample, Proposition 1.4 leads us to a contradiction. Thus we are done. q.e.d.

Particularly, when S is a curve, we have

Proposition 1.6. *Under the notation of* (1.1), *let S be a curve. Then, the following are equivalent to each other.*

1) *E is not an ample vector bundle.*

2) *A is an irreducible curve in $P(E)$ such that $p_{|A}$ $A \to S$ is a finite surjective morphism and $A \cap P(F) = \phi$.*

More precisely, *for the normalisation of A ($g: \bar{A} \to A$) and its induced bundle map: $\bar{g}: P(gp^*(E)) \to P(E)$, A is equal to $\bar{g}(P(\mathcal{O}_{\bar{A}}))$. (Note that $\mathcal{O}_{\bar{A}}$ is a unique trivial direct summand of $gp^*(E)$)*

Moreover, assume that the characteristic of the base field is zero. Then, the above two are equivalent to

2′) *For the normalization $f: \bar{S} \to S$ of S, f^*E is isomorphic to $\mathcal{O}_{\bar{S}} \oplus f^*F$. (In case that S is smooth, f means the identity)*

Hence, the above $p_{|A}$ *in* 2) *is birational.*

Proof. 2) implies 1) obviously. Now assume 1). Then, we have 2) by virtue of 2) of Proposition 1.3 and the argument in Proposition 1.5. Finally in characteristic zero, it is trivial by virtue of Proposition 4.18. in [*Fu*]. q.e.d.

Before proving Theorem I, we show

Proposition 1.7. *Under the situation of* (1.1), *assume that* E *is not ample. Then, maintaining the notations in* (1.2), *we have the following.*

1) *For every irreducible curve* C *on* $p(A)$, $p^{-1}(C) \cap A$ *has only one irreducible component which is a curve.*

2) *In characteristic zero,* $p: A \to S$ *is injective outsides at most finitely many points of* A. *When* S *is a normal variety,* $p_{|A}$ $A \to S$ *is surjective if and only if* E *splits.*

Proof. Take a closed irreducible curve C in $p(A)$. Then since $\mathcal{O}_{P(E_{|C})}(1) = \mathcal{O}_{p(E)}(1)_{|p^{-1}(C)}$, we see that $E_{|C}$ is not ample. This and Proposition 1.6 imply 1) and the first part of 2). For the remainder of 2), assume that $p_{|A}: A \to S$ is surjective. Take an irreducible component A' of A such that $p(A') = S$. Then $p: A' \to S$ is a finite, birational morphism by 2) of Proposition 1.6 and therefore, an isomorphism. Consequently it yields a section of p. Noting that $A' \cap Y$ is empty, we see that E splits. q.e.d.

(1.8) *Proof of Theorem* I. 1) obviously implies 2). 3) follows from 2). 3) gives rise to 1) by Proposition 1.3 and Proposition 1.6. Similarly Proposition 1.6 yields the equivalence between 1) and 3'). q.e.d.

Hereafter till § 2, let us consider a vector bundle E on S with the conditions in (1.1) and an additional condition:

(1.9) for every component T_i of A (see 2) of (1.2)), $p|_{T_i}: T_i \to S$ is a closed embedding. (in other words, $E_{|p(T_i)} \simeq \mathcal{O} \oplus F_{|p(T_i)}$.)

Remark 1.10. For the above vector bundle E with (1.9), let $\varphi: C \to S$ be a finite morphism from a curve C to S. Then if $\varphi^* E \simeq \mathcal{O} \oplus \varphi^* F$, there is a compoment T_i of A such that $\varphi(C) \subset T_i$.

Now, let us consider a sufficient condition for E to satisfy the condition 1.9. First, we have

Proposition 1.11. *Let* E *be a vector bundle on an irreducible, reduced curve* S *enjoying* (1.1). *Assume* E *is generated by its global sections and it is not ample. Then* E *splits to* $\mathcal{O}_S \oplus F$. *Consequently,* A *satisfies the condition* 1.9.

Proof. First, by the assumption that E is globally generated:

$$(\sharp) \qquad \bigoplus^{N+1} \mathcal{O}_S \longrightarrow E \longrightarrow 0 \quad (N+1 = \dim H^0(S, E))$$

the rational map ϕ_1 in (1.2) defines a morphism. Let X_1 be a set $\{x \in \phi_1(X) | \dim \phi_1^{-1}(x) \geq 1\}$. Then, we have

Claim. 1. X_1 is not empty.

2. Every fiber of $p: P(E) \to S$ is linearly embedded via ϕ_1 in P^N

3. For each point b in X_1, dim $\phi_1^{-1}(b) = 1$. Moreover, $\phi_1^{-1}(b) \cap p^{-1}(s)$ for each point s in S is one point.
4, There is a morphism $f\colon S \to Gr(N, r+1)$ such that the exact sequence (♯) is the pull back of the following exact sequence via f

$$(\sharp\sharp) \qquad \bigoplus^{N+1} \mathcal{O}_{Gr(N,r+1)} \longrightarrow E(N, r+1)\,(=\mathscr{E}) \longrightarrow 0,$$

where $Gr(N, r+1)$ means the Grassmann manifold parameterizing $(r+1)$-dimensional linear spaces in P^N and $E(N, r+1)$ the universal quotient bundle of rank $r+2$. (We use the notations in § 1 of [*Ta*])

Proof of Claim. Since E is not ample, ϕ_1 is not a finite morphism, which yields 1. Note that a linear system of $\mathcal{O}_{P^n}(1)$ which defines a morphism coindices with the complete linear system. Since $\mathcal{O}_{P(E)}(1)|_{p^{-1}(s)} \simeq \mathcal{O}_{p^{-1}(s)}(1)$ for every point s in S, 2 is obvious. When Y is an m (≥ 2) dimensional subvariety in $P(E)$, there is a point s in S such that dim $p^{-1}(s) \cap Y \geq 1$. Thus we get 3 by virtue of 2. The last assertion is the universality of the Grassmann manifold.

Thus, for every point s in S, we can take $P(E_{|p^{-1}(s)})$ as a linear subspace in P^N by 2 of Claim. To complete our proof, it suffices to show the following.

Sublemma. *Let Y be a subvariety in $Gr(N, r+1)$ where we have the exact sequence (♯♯) in 4 of the above Claim. Assume that there is a point B in P^N contained in every $(r+1)$-dimensional subspace $P(\mathscr{E}_y)$ $(y \in Y)$. Then, $\mathscr{E}_{|Y}$ has a trivial line bundle as a direct summand.*

Proof Let us consider the Schubert cycle $\Omega = \Omega_{0,N-r,N-r+1,\ldots,N}(B) = \{x \in Gr(N, r+1) | B \in L_x\}$ where L_x is the $(r+1)$-dimensional subspace in P^N corresponding to a point x. Note that Ω is isomorphic to $Gr(N-1, r)$. Now it is well-known that the $(r+2)$-th Chern class Ω' of $\mathscr{E}$ is rationally equivalent to the cycle $\{x \in Gr(N, r+1) \,|L_x$ is contained in some hyperplane on $P^N\}$ $(\simeq Gr(N-1, r+1))$. Since the intersection $\Omega \cdot \Omega'$ is zero modulo the rational equivalence, the $(r+2)$-th Chern class of $\mathscr{E}_{|\Omega}$ is zero (see for example Lemma 1.3 in [*Ta*]). By the assumption that $\mathscr{E}_{|\Omega}$ is globally generated, it has a trivial line bundle as a direct summand. On the other hand, Y is contained in Ω, which yields our desired result. q.e.d. for Sublemma.

Thus by virtue of Proposition 1.4, Proposition 1.5 and 4 of Claim, we complete the proof of our proposition. q.e.d.

The above immediately gives rise to a corollary.
Corollary 1.11.1. Let E be a vector bundle on S enjoying (1.1). Assume that E is generated by its global sections. Then, A has the condition 1.9.

Remark 1.11.2. Under the condition (1.1), assume that $H^1(S, \mathcal{O}_S) = 0$. Then if F is generated by its global sections, so is E.

(1.12) Let us give anothr condition which is equivalent to the conditions in Theorem I. For a curve C on S, let us consider the exact sequence:

$$(\$) \qquad 0 \longrightarrow I_C \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

where I_C is the sheaf of ideals of C in S.

Tensoring F^* to (\$), we have

$$0 \longrightarrow F^* \otimes I_C \longrightarrow F^* \longrightarrow F^*_{|C} \longrightarrow 0,$$

which provides us with an exact sequence

$$(1.12.1) \qquad H^1(S, F^* \otimes I_C) \xrightarrow{h_C} H^1(S, F^*) \xrightarrow{g_C} H^1(C, F^*_{|C})$$

Then we have the following

Remark 1.13. Assume that E enjoys the property (1.9). Then, each condition in Theorem I is equivalent to the following:
4) For every curve C on S, $g_C(\sigma)$ does not vanish.

§.2. Smooth varieties containing a projective bundle as an ample divisor

In the present section, let S be a projective locally factorial surface and let us maintain the notation F, E_σ and the assumption in (1.1), the natation in (1.2) and the additional condition (1.9). Assume furthermore that $\sigma \neq 0$.

Then, our goal in this section is to show Theorem II.

Remark 2.1. In the case where the base space is a curve, Theorem II holds good without any assumption. See Example 4.22 in [*Fu*].

To measure the degree that E is near to an ample vector bundle, we introduce a notation:

(2,2) $\quad B(F, \sigma) = \{$an irreducible, reduced curve C in $S \mid F_{\sigma|C} = \mathcal{O}_C \oplus F_{|C}\}$

Then we obtain

Proposition 2.2.1. *We have*
1) *A in* (1.2) *contains every curve C in $B(F, \sigma)$.*
2) *$B(F, \sigma)$ is a finite set or an empty set.*
3) *E_σ is a ample if and only if $B(F, \sigma)$ is empty.*

Proof. 1) and 3) are obvious by the definition of A and $B(F, \sigma)$ and the condition (1.9). Assume that $B(F, \sigma)$ is an infinite set. Then, $\dim A = 2$ by Proposition 1.3. Thus, there is an irreducible component T_i of A which is a surface. Since it yields a section of p, E splits, which contradicts the assumption $\sigma \neq 0$. q.e.d.

Now, we have the following sequence:

$$\longrightarrow H^1(S, F^*(-C)) \xrightarrow{h_C} H^1(S, F^*) \xrightarrow{g_C} H^1(C, F^*_{|C}) \longrightarrow .$$

(See 1.12.1). Note that if C is in $B(F, \sigma)$, $g_C(\sigma) = 0$.

Hence, we have a non-zero element σ' in $H^1(S, F^*(-C))$ which goes to σ by h_C, which provides us with a non-trivial extension of vector bundle:

$$(2.3) \qquad 0 \longrightarrow \mathcal{O}_S \longrightarrow E_{\sigma'} \longrightarrow F(C) \longrightarrow 0.$$

Then, similarly to $B(F, \sigma)$ above, we investigate $B(F(C), \sigma')$ and, in particular, the relation between them.

Now, let C' be an irreducible smooth projective curve and $\varphi: C' \to S$ a finite morphism with $\dim(\varphi(C') \cap C) \leq 0$. Then, by the exact sequence:

$$0 \longrightarrow F^*(-C) \longrightarrow F^* \longrightarrow F^*_{|C} \longrightarrow 0$$

we have the following exact sequence on C':

$$0 \longrightarrow \varphi^*F^*(-C) \longrightarrow \varphi^*F^* \longrightarrow \varphi^*(F^*_{|C}) \longrightarrow 0.$$

This provides us with the following exact commutative diagram

$$(2.4) \qquad \begin{array}{ccccc} H^1(S, F^*(-C)) & \xrightarrow[i]{} & H^1(S, F^*) & \longrightarrow & H^1(C, F^*_{|C}) \\ \downarrow{\scriptstyle \bar{g}_D} & & \downarrow{\scriptstyle g_D} & & \downarrow \\ H^1(C', \varphi^*F^*(-C)) & \xrightarrow[j]{} & H^1(C', \varphi^*F^*) & \longrightarrow & H^1(C', \varphi^*(F^*_{|C})) . \end{array}$$

We obtain therefore

Proposition 2.5. *Let σ, σ' and C, φ be as above. If φ^*E does not split to $\mathcal{O} \oplus \varphi^*F$, neither does φ^*E_σ.*

Moreover, we have

Corollary 2.6. *Under the same notation as in 2.3, we assume that C is numerically effective. Then, we have*

1) *$F(C)$ is an ample vector bundle.*
2) *$E_{\sigma'}$ has the condition 1.9 and $B(F(C), \sigma') \subset B(F, \sigma)$.*
3) *Particularly if $B(F(C), \sigma')$ is empty, $E_{\sigma'}$ is ample.*

Proof. Since the tensor product of an ample vector bundle and a numerically effective line bundle is ample, we have 1). 2) is obtained by using Proposition 2.5 and 3) by Proposition 2.2.1. q.e.d.

Hereafter we shall discuss a condition for $B(F(C), \sigma')$ to be a proper subset of $B(F, \sigma)$. First, we have a

Proposition 2.7. *Let G be a vector bundle on a complete variety Z and D an irreducible, reduced Cartier divisor on Z. Assume that D is numerically effective and $G_{|D}$ is ample. Then, for a vector bundle $\bar{G}$ on Z, denoting $\bar{G} \otimes \mathcal{O}_Z(mD)$ by $\bar{G}(m)$, we have*

(1) *for every non-negative integer* m, $G(m)_{|D}$ *is an ample vector bundle and* $H^0(D, G^*(-m)_{|D})$ *vanishes.*

(2) $\dim H^1(Z, G^*(-m))$ *is a monotone-decreasing function of* $m(\geq 0)$. *Hence, there is a positive integer* m_0 *such that for every integer* m $(\geq m_0)$, $\dim H^1(Z, G^*(-m))$ *is constant.*

Suppose furthermore that $H^0(Z, G^*(-mD))$ *vanishes for all* $m \geq 0$. *Then, we get*

(3) $H^0(hD, G^*(-m)_{|hD})$ *vanishes for all* $h \geq 1$.

Proof. (1) is trivial. Looking at the following exact sequence

$$(\sharp) \qquad 0 \longrightarrow G^*(-m-1) \longrightarrow G^*(-m) \longrightarrow G^*(-m)_{|D} \longrightarrow 0,$$

we see that (1) implies (2).

Now, let $g_m: H^1(Z, G^*(-m-1)) \to H^1(Z, G^*(-m))$ be the canonical homomorphism induced by ($\sharp$). Note that g_m is injective by (1).

Let us consider an exact sequence

$$0 \longrightarrow G^*(-h) \longrightarrow G^* \longrightarrow G^*_{|hD} \longrightarrow 0,$$

which gives rise to the exact sequence of cohomologies

$$H^0(Z, G^*) \longrightarrow H^0(Z, G^*_{|hD}) \longrightarrow H^1(Z, G^*(-h)) \xrightarrow{i} H^1(Z, G^*).$$

Then, since i can be factored to the product $g_{h-1} g_{h-2}, \ldots, g_0$ i is injective, which yields $H^0(Z, G^*_{|hD}) = 0$ by virtue of the extra assumption. Finally, replacing G^* by $G^*(-m)$, we can prove (3). q.e.d.

Combining the above and 1.7, we get a key result which implies vanishing of the first cohomologies of some vector bundles.

Proposition 2.8. *Let* F *be an ample vector bundle on a normal projective variety* Z *and* D *an irreducible, reduced Cartier divisor on* Z. *Assume that* D *is numerically effective and there is a positive integer* a *such that* $\dim |aD| \geq 1$. *Then, in characteristic zero, there is an integer* m_0 *such that for every integer* $m \geq m_0$, $H^1(Z, F^*(-mD)) = 0$.

Proof. Noting (2) in Proposition 2.7, assume that there is an integer m_0 and a positive constant Q such that for every $m \geq m_0$ $\dim H^1(Z, F^*(-mD)) = Q$. Then, taking a non-zero element α in $H^1(Z, F^*(-mD))(m \geq m_0)$, we consider the exact sequence:

$$0 \longrightarrow F^*((-m-a)D) \longrightarrow F^*(-mD) \longrightarrow F^*(-mD)_{|C_\lambda} \longrightarrow 0$$

where C_λ is a mumber in $|aD|$. It provides us with an exact sequence

$$(2.8.1) \qquad H^0(C_\lambda, F^*(-mD)_{|C_\lambda}) \longrightarrow H^1(Z, F^*((-m-a)D)) \xrightarrow{i} H^1(Z, F^*(-mD)) \xrightarrow{j} H^1(C_\lambda, F^*(-mD)_{|C_\lambda}).$$

Thus, since $H^0(aD, F^*(-mD)_{|aD}) = 0$ by (3) in Proposition 2.7, we infer that for almost all elements C_λ in $|aD|$, $H^0(C_\lambda, F^*(-mD)_{|C_\lambda}) = 0$. Since i in (2.8.1) is an isomorphism, $j(\alpha)$ vanishes, which shows the fact that for such C_λ, $E_{\alpha|C_\lambda} = \mathcal{O} \oplus F(mD)_{|C_\lambda}$, where E_α is a non-trivial extension of vector bundles by α. Now, note that $F(mD)$ is ample and let X be $P(E_\alpha)$ and Y be $P(F)$. Applying (1.2) to X, Y and $P(E_{\alpha|C_\lambda})$, $P(F(mD)_{|C_\lambda})$, we see that $A \cap p^{-1}(C_\lambda)$ contains the section $P(\mathcal{O}_{C_\lambda})$ corresponding to trivial quotient line bundle $\mathcal{O}_{C_\lambda}$ of $E_{\alpha|C_\lambda} = \mathcal{O} \oplus F(mD)_{|C_\lambda}$ by Proposition 1.5 and therefore, A (1.2) dominates Z with respect to the canonical projection: $X \to Z$. Hence, from 2) in Proposition 1.7, it follows that E_α splits to $\mathcal{O} \oplus F(mD)$, which contradicts the assumption that Q is positive. q.e.d.

(2.9) Now, let us maintain the notation in (1.1) and assume the condition 1.9. Moreover suppose that C is in $B(F, \sigma)$ such that there exists a positive integer a with $\dim |aC| \geq 1$. Then note that C is numerically effective.

Consider the exact sequence of cohomologies:

$$H^1(S, F^*(-m-1)C) \xrightarrow{h_m} H^1(S, F^*(-mC)) \xrightarrow{g_m} H^1(C, F^*(-mC)_{|C})$$

induced by the exact sequence:

$$0 \longrightarrow F^*((-m-1)C) \longrightarrow F^*(-mC) \longrightarrow F^*(-mC)_{|C} \longrightarrow 0.$$

Then, we have

Proposition 2.9. *Under the above situation* 2.9, *assume that the characteristic of the base field is zero. Then there is a positive integer t and a non-zero element σ_t in $H^1(S, F^*(-tC))$ satisfying the following*:

i) $\sigma = (h_{t-1} \cdots h_1 h_0)(\sigma_t)$

ii) $E_{\sigma_t|C} \neq \mathcal{O} \oplus F(tC)$, *where E_{σ_t} is a vector bundle defined by σ_t*.

Proof. Take an element $\sigma_1 \in H^1(S, F^*(-C))$ such that $h_0(\sigma_1) = \sigma$ and $g_0(\sigma) = 0$ by the condition. Now consider two cases

(2.9.1) $$g_1(\sigma_1) \neq 0$$

(2.9.2) $$g_1(\sigma_1) = 0.$$

In the first case, σ_1 is what we want.

In the latter case taking the same procedure as above, we can take a non-zero element $\sigma_2 \in H^1(S, F^*(-2C))$ such that $h_1(\sigma_2) = \sigma_1$. On the other hand, Proposition 2.8 says that such procedure must terminate, which implies that there are elements σ_j $(1 \leq j \leq t-1)$ in $H^1(S, F^*(-jC))$ such that $h_j(\sigma_{j+1}) = \sigma_j$ and $g_t(\sigma_t) \neq 0$. Thus we are done. q.e.d.

Proof of Theorem II. By the assumption (2), (3) and Remark 1.11.2, E_σ is generated by its global sections. Thus, the vector bundle satisfies the condition 1.9 by Corollary 1.11.1 Thus, the descending induction on the order of $B(F, \sigma)$ proves Theorem II by Corollary 2.6 and Proposition 2.9.1. q.e.d.

Example 2.10. Let us consider a vector bundle F on a smooth projective surface S enjoying conditions and assumptions in Theorem II. Then $P(F)$ can be contained in a smooth projective variety as an ample divisor. Compare Example 4.21 in [*Fu*].

Remark 2.11. Given a projective variety Y, we can always construct a projective variety X containing Y as an ample divisor if we allow X to have singularities. For example, we embed Y in a projective space P^N by a very ample line bundle of Y and make a cone X ($\subset P^{N+1}$) by Y and a point outsides P^N. Then, X is a desired one. But there are several projective varieties that cannot be ample in any smooth projective variety (See [*So*], [*Fu*])

As a corollary to Theorem II we have the following

Corollary 2.12. *Under the same conditions as in Theorem* II, *let us assume that for every curve C on S,* dim $|C| \geq 1$ *(e.g. P^2) and moreover, that* dim $H^1(S, F^*) = 1$. *Then, a vector bundle E_σ defined by a non-zero element σ in $H^1(S, F^*)$ is an ample vector bundle. (e.g. $S = P^2$ and $F = T_{P^2}$)*

Proof. Assume that E_σ is not ample. Then, by the assumption and Remark 2.2.1, $B(F, \sigma)$ is not empty. Take an irreducible, reduced curve C in $B(F, \sigma)$. As shown in (2.2), there is an non-zero element σ' in $H^1(S, F^*(-C))$ such that $h_C(\sigma') = \sigma$ where $h_C\colon H^1(S, F^*(-C)) \xrightarrow{h_C} H^1(S, F^*)$ is a canonical homomorphism (2.2). Since E_σ splits on almost all C_λ in $|C|$, E_σ splits to $\mathcal{O} \oplus F$ in the same way as in Proposition 2.8. Namely, σ is the zero element, which gives a contradiction. q.e.d.

§.3. A sequence of ample divisors

First we need the following

Proposition 3.1. *Let B be a smooth ample divisor in a smooth projective variety X and $\pi\colon B \to T$ a P^r-bundle over a smooth variety. Then,*

i) *When $r \geq 2$, π is extended to a P^{r+1}-bundle $\phi\colon X \to T$ with* dim $T \leq r+1$ *and B is the tautological line bundle in X with respect to ϕ. Namely, there is an exact sequence of vector bundles on T:*

$$0 \longrightarrow \mathcal{O} \longrightarrow M \longrightarrow N \longrightarrow 0,$$

such that $P(M) = X$, $P(N) = B$ and M is ample.

ii) *Let $r = 1$. Assume that T is a curve but B is not isomorphic to $P^1 \times P^1$ or that T is a surface which is neither P^2 nor a surface of general type with $p_g = q = 0$. Then, the same conclusion as in i) above holds. If $T = P^2$, then X is a P^2-bundle over P^2 unless $B = P^1 \times P^2$.*

ii)'. *In case of $T = P^n (n \geq 3)$, $B = P^1 \times P^n$ and X is a P^{n+1}-bundle over P^1.*

See [*So*], [*Fa + Sa + So*], [*Sa + Sp*] and [*Sa*].

Thus we can restate Problem 2) in terms of vector bundles.

Proposition 3.2. *Under the condition in Problem* 2, *we have*
1) *When* $r \geq 2$, A_i *is a* P^{r+i-1}*-bundle over* S. *For each* i, *there is an ample vector bundle* E_i *of* rank $(r+i)$, *a line bundle* L_i *on* S *and an exact sequence*

$$0 \longrightarrow \mathcal{O} \longrightarrow E_i \longrightarrow E_{i-1} \otimes L_{i-1} \longrightarrow 0$$

such that $P(E_i) = A_i$.
2) *When* $r = 1$, *we assume that* S *is a surface which is neither* P^2 *nor one of general type with* $p_g = q = 0$ *or that in case of* $S = P^2$, B *is not isomorphic to* $P^1 \times P^2$. *Then, the same conclusion as above* 1) *holds.*
2)′ *If* $r = 1$ *and* $S = P^n (n \geq 3)$, *then* $A_i \simeq P^1 \times P^{n+i-1}$.

Taking Proposition 3.2 into account, let us consider Problem 2).
(3.3) First, let us study a sequence of couples of vector bundles E_i and line bundles L_i on a smooth projective variety S ($\dim S \geq 2$) such that for each integer i (≥ 1), they enjoy the following:
0) rank $E_i = r + i$.
1) Pic $S \simeq \mathbf{Z}L$, where $L$is ample. Hence, L_i can be written in a form $L^{\otimes a_i}$ with a_i an integer.
2) There is an exact sequence:

$$0 \longrightarrow \mathcal{O}_S \longrightarrow E_i \longrightarrow E_{i-1} \otimes L_{i-1} (= E_{i-1}(a_{i-1})) \longrightarrow 0.$$

3) E_i is ample for $i \geq 1$ and $L_1 = \mathcal{O}_S$ ($a_1 = 0$).

Then, we show

Theorem 3.4. *Under the above condition* (3.3), *let us assume that* r *is positive and* $H^1(S, M)$ *vanishes for all line bundles* M *on* S. *Then, the above sequence terminates.*

Proof. The sequence 2) in (3.3) can be described as follows:

$$0 \longrightarrow \bigoplus_{i=2}^{m} \mathcal{O}_S(\sum_{j=i}^{m} a_j) \longrightarrow E_m(a_m) \longrightarrow E_1(\sum_{i=1}^{m} a_i) \longrightarrow 0.$$

Letting $b_i = \sum_{j=1}^{i} a_j$ with $b_0 = 0$, we reformulate the above in the following form:

$$*_m \qquad 0 \longrightarrow \bigoplus_{i=1}^{m-1} \mathcal{O}_S(b_m - b_i) \longrightarrow E_m(a_m) \longrightarrow E_1(b_m) \longrightarrow 0.$$

Here, recall that $E_1(b_m)$ is ample.
Now assume the existence of an infinite sequence $\{E_m\}$.
Then we obtain

Claim. $B = \{b_n | n \in \mathbf{N}\}$ is bounded.

Proof. Dualizing the above sequence $*_m$, we have

$$0 \longrightarrow E_1^*(-b_m) \longrightarrow E_m^*(-a_m) \longrightarrow \bigoplus_{i=1}^{m-1} \mathcal{O}_S(b_i - b_m)\ (= N_m) \longrightarrow 0.$$

Note that the first cohomology of the above quotient vector bundle vanishes by our assumption. On the other hand, since E_{m+1} is ample, the exact sequence in 2) of (3.3) does not split and therefore, $H^1(S, E_m^*(-a_m))$ does not vanish. Thus, B is bound by Serre's vanishing Theorem and Serre's duality.

From now on, we show that the above claim yields two results which contradict each other.

First, the Claim implies dim $H^1(S, E_1^*(-b_m))$ is bounded when m runs over every positive integer m. On the other hand since $E_m(a_m)$ is ample $H^0(S, E_m^*(-a_m)) = 0$. Thus dim $H^0(S, N_m)$ must be bounded as a function of m. Secondarily, we can take an integer $\bar{b}$ such that the cardinarity of $U = \{m | b_m = \bar{b}\}$ is not finite by the above argument. It follows that dim $H^0(S, N_m)$ is not bounded when m ranges over U, which is a contradiction. Thus we are done.

q.e.d.

The above gives rise to a corollary.

Corollary 3.5. *Let our condition be as in* (3.3). *Assume that there is a smooth subvariety T in S such that T satisfies the same assumption as S in Theorem* 3.4. *Then, the sequence A_i terminates.*

Note that a restriction of an ample vector bundle to a closed subvariety is ample.

Remark 3.6. If S is a generic smooth surface of the degree r (≥ 4) in P^3 or an n (≥ 3)-dimensional complete intersection, then the condition 1) of (3.3) and the assumption in Theorem 3.4 are satisfied.

Proof of Theorem III. 1) and 2) are obvious by virtue of i) and ii) of Proposition 3.1 and Theorem 3.4. Next, let us consider 2)′.

Taking 2) and 2)′ in Proposition 3.2 into account, it suffices to consider the case where $A_1 = P^n \times P^1$ and to construct an exact sequence of vector bundles of P^1:

$$0 \longrightarrow \mathcal{O} \xrightarrow{\ i\ } \bigoplus^{r+1} \mathcal{O}(r) \longrightarrow \bigoplus^{r} \mathcal{O}(r+1) \longrightarrow 0.$$

For this purpose, take $(r+1)$ points $A_1, \ldots, A_{r+1}$ in P^1 and let s_i be a section of $H^0(P^1, \mathcal{O}(r))$ such that $s_i(A_j) = 0$ for every $j \neq i$. Then $s_1, \ldots, s_{r+1}$ are a basis of $H^0(P^1, \mathcal{O}(r))$. Now let a non-zero section of $\mathcal{O}$ correspond to $(s_1, \ldots, s_{r+1})$ in $\bigoplus^{r+1} \mathcal{O}(r)$. Then we see easily that i is an injection as an vector bundle. Hence, we have only to show

Claim. Coker (i) $= \bigoplus^{r} \mathcal{O}(r+1)$.

Proof. Letting E' the left-hand side of the above, we see $E' = \bigoplus_{i=1}^{r} \mathcal{O}(b_i)$ with $b_1 \leq b_2 \leq \ldots \leq b_r$ and $r \leq b_1$. Now, assume that $b_1 = r$.

Then, the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \bigoplus^{r+1} \mathcal{O}(r) \longrightarrow E' \longrightarrow 0$$

provides us with another exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \bigoplus^{r} \mathcal{O}(r) \longrightarrow \bigoplus_{i=2}^{r} \mathcal{O}(b_i) \longrightarrow 0,$$

which contradicts the property that $s_1, \ldots, s_{r+1}$ form a basis of $H^0(P^1, \bigoplus \mathcal{O}(r))$. Therefore, we infer that $b_i = r + 1$, since $c_1(E') = r(r + 1)$. This completes the proof of our claim and then Theorem III is proved. q.e.d.

(3.7) We notice that the infinite sequence $\{A_i\}$ in the final case of Theorem III is essentially the one in case of dim $S = 1$. Therefore, let us give another example in an n (≥ 2)-dimensional case.

Let C be a smooth complete curve of genus g. Then, fixing a point p_0 in C, the correspondance $C^m \ni (p_1, \ldots, p_m) \to (\ldots, \sum_{i=1}^{m} \int_{p_0}^{p_j} \omega_i, \ldots)$ gives rise to a canonical morphism $\alpha_m: S^m(C) \to J(C)$ from m-fold symmetric product of C to the jacobian variety of C, where $\omega_i (1 \leq i \leq g)$ is a basis of $H^1(C, \mathcal{O}_C)$. It is well-known that if $m \geq 2g - 1$, α_m is a P^{m+1-g}-bundle over $J(C)$.

Moreover we have the following diagram:

$$\begin{array}{ccccc} C^m & \xrightarrow{s_m} & S^m(C) & \xrightarrow{\alpha_m} & J(C) \\ \beta_m \downarrow & & \gamma_m \downarrow & & \\ C^{m+1} & \xrightarrow{s_{m+1}} & S^{m+1}(C) & & \end{array}$$

where β_m is a morphism defined by $(p_1, \ldots, p_m) \to (p_1, \ldots, p_m, p_{m+1})$ and it naturally yields γ_m.

Then we have

Proposition 3.8. *Letting $A_m = S^{m+r}(C)$, A_m is an ample divisor in A_{m+1}. Thus, there is an infinite sequence $\{A_i\}$ satisfying condition in Problem 2).*

Proof. When we consider the morphism $q_j: C^m \to C^{m+1}$ defined by $q_{j+1}(p_1, \ldots, p_m) = (p_1, \ldots, p_{j-1}, p_0, p_j, \ldots, p_m)$, we see that $s_{m+1}(q_j(C^m)) = \gamma_m(S^m(C))$ for every j. On the other hand, it is easy to check that the union of $q_j(C^m) (0 \leq j \leq m)$ is ample in C^{m+1} by Nakai's criterion. Thus, since s_m is a finite morphism, we get the desired result. q.e.d.

DEPARTMENT OF MATHEMATICS
COLLEGE OF GENERAL EDUCATION
KYUSHU UNIVERSITY

MATHEMATISCHE INSTITUT
DER UNIVERSITÄT
BUNSENSTRASSE 3–5
3400 GÖTTINGEN
FEDERAL REPUBLIC OF GERMANY

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