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Two torsion and homotopy associative H-spaces

By

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§ 0. Introduction

In this note we consider the following question:

If Y is a mod 2 H-space, when does $Y \times S^7$ admit the structure of a homotopy associative mod 2 H-space?

There are several examples that are revealing. First, it is well known that the seven-sphere admits the structure of an H-space, but does not admit a homotopy associative structure. In the case of Lie groups, it is known that at the prime 2, Spin(8) is homotopy equivalent to Spin(7) $\times S^7$ and Spin(7) is homotopy equivalent to $G_2 \times S^7$. Among all the compact simply connected simple Lie groups, only G_2, F_4, Spin(7) and Spin(8) have a subHopf algebra over the Steenrod algebra of the following form

$$A = \frac{\mathbf{Z}_2[x]}{x^4} \otimes \wedge (Sq^2 x) = H^*(G_2; \mathbf{Z}_2), \quad \deg x = 3. \tag{0.1}$$

In this paper we show that this is the key factor in determining if a finite H-space producted with a seven-sphere can admit a homotopy associative H-structure. This can be summarized by the following theorems.

Theorem A. *Let Y be a finite 1-connected complex and suppose $H^*(Y; \mathbf{Z}_2)$ does not contain any subalgebras over the Steenrod algebra of type A. Then $Y \times S^7$ cannot be a homotopy associative H-space.*

Theorem B. *Let Y be a finite 1-connected complex and suppose $H^*(Y; \mathbf{Z}_2)$ has at most one subalgebra of type A over the Steenrod algebra. Then $Y \times (S^7)^k$ cannot be a homotopy associative for $k \geq 3$.*

The first results concerning products with S^7 and homotopy associativity were due to Goncalves, [2], who proved that if Y is any simply-connected compact simple Lie group other than G_2 and Spin(7), then $Y \times S^7$ cannot be a homotopy-associative H-space, even when localized at the prime two. Hubbuck [3] showed that the two-torsion is necessary for their products with S^7 to be the homotopy

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types of topological groups by proving that if Y has no two-torsion in its homology then $Y \times S^7$ cannot be the homotopy type of a topological group. (In fact, he proved the stronger technical result that such a $Y \times S^7$ cannot be the homotopy type of an A_4-space in the sense of Stasheff [10].) Recently, Iwase [4] has strengthened Hubbuck's result by proving that if Y has no two-torsion then the product of Y with S^7 is not a homotopy-associative H-space.

In Hubbuck's and Iwase's work the absence of two-torsion is essential, since it relies in the first case on Iwase's structure theorem [5] for the K-ring of projective n-space and in the second case on Iwase's method of generating complexes. In Theorem A, above, we specify exactly the type of two-torsion that is capable of permitting $Y \times S^7$ to be homotopy-associative. In particular, there is no subalgebra of type A in $H^*(Y)$ if and only if $Sq^1(H^5(Y) \cap \ker Sq^4) = 0$. We follow the method of Goncalves, i.e., we use a certain tertiary cohomology operation defined in [2], and apply it to certain connected covers of $Y \times S^7$. A main ingredient in our work is Lin's description in [7] of the Steenrod connections in finite H-spaces, which we use to compute in the cohomology of these connected covers. Essentially this allows us to compute the fibre of the 3-connective cover of an H-space where we kill off all 4-, 5-, 8- and some 7-dimensional generators. If this fibre does not contain non-primitive 14-dimensional generators or primitive 22-dimensional generators, the original H-space cannot be homotopy associative.

This work generalized the results of Hubbuck and Iwase because it allows for the existence of two torsion and it generalized the results of Goncalves from Lie groups to H-spaces. In the nonfinite case it is interesting to note that there is a splitting $\Omega S^8 \cong \Omega S^{15} \times S^7$. One can trace through our proof to show that the homotopy associativity of ΩS^8 is reflected by the non-primitivity of the 14-dimensional generator of $H^*(\Omega S^8)$.

The above results may be applied to the rational type of an example described by Adams and Wilkerson. In their paper, they cite a rational type of the form

$$\{4, 4, 4, 8, 8, 8, 12, 12, 16, 16, 20, 24, 24, 28\}.$$

This type is not the type of a Lie group, but for every prime $p > 3$ it is shown that it is the type of a loop space. Furthermore, it is the type of a product of a Lie group with S^7. The Lie group is either

$$G = \mathrm{Spin}(15) \times Sp(2) \times F_4 \quad \text{or} \quad G = Sp(7) \times Sp(2) \times F_4.$$

In either case, our results show that $G \times S^7$ cannot be homotopy associative (Theorem 5.1).

The organization of our paper is as follows: In section one we review the proof of Goncalves that the cubes of certain cohomology classes factor through secondary operations. We give an explicit formula for this factorization. In section two we describe the $\mathscr{A}(2)$ subHopf algebra of the cohomology of an H-space generated by its three-dimensional generators. Corollary 2.5 shows that such a subHopf algebra actually splits over the Steenrod algebra into the tensor products of subHopf algebras over $\mathscr{A}(2)$ in a certain range. This allows us to calculate the cohomology of the 3-connective cover of a finite H-space in chapter 3. This is described by chart 3 at the end of chapter 3. In chapter 4, we kill off all 4-, 5-, 8- and some 7-dimensional generators in the 3-connective cover and calculate the cohomology of the fibre. In chapter 5, Theorems A and B are proved.

In a first reading, the reader may want to read the statements of results in chapters 1 through 4 and go on to chapter 5 for the proof of the main theorems.

All spaces are assumed to be one-connected and all coefficients of cohomology are assumed to be $\mathbf{Z}_2$ unless otherwise stated.

§ 1. Factorization of the cube

Given an element u_8 in the cohomology of a space with $u_8 \in \ker Sq^1, Sq^2, Sq^4$, $\deg u_8 = 8$, it was shown in [2] that u_8^3 factors through secondary operations. An explicit factorization is given here.

The following diagram is due to Goncalves and Harper [2]:

$$
\begin{array}{ccc}
K(\mathbf{Z}_2, 9, 11) & \xrightarrow{\Omega A} & K(\mathbf{Z}_2, 16, 17, 23) \\
\downarrow j_1 & & \downarrow j \\
E_1 & \xrightarrow{g} & E \\
\downarrow p_1 & & \downarrow p \\
K(\mathbf{Z}, 8) & \xrightarrow{Sq^8} & K(\mathbf{Z}_2, 16) \\
\left(\begin{smallmatrix} Sq^2 \\ Sq^4 \end{smallmatrix}\right) \Big\downarrow & & \Big\downarrow \left(\begin{smallmatrix} Sq^1 \\ Sq^2 \\ Sq^8 \end{smallmatrix}\right) \\
K(\mathbf{Z}_2, 10, 12) & \xrightarrow{A} & K(\mathbf{Z}_2, 17, 18, 24) \\
B \downarrow & & \downarrow (Sq^8 + Sq^{6,2}, Sq^7 + Sq^{4,2,1}, Sq^1) \\
K(\mathbf{Z}_2, 13, 16) & \xrightarrow[(Sq^{12}, Sq^{6,3})]{} & K(\mathbf{Z}_2, 25).
\end{array} \tag{1.1}
$$

The notation $Sq^{i,j}$ means $Sq^i Sq^i$. The matrices A and B are given by

$$
B = \begin{pmatrix} Sq^{2,1} & Sq^1 \\ Sq^6 & Sq^4 \end{pmatrix} \quad A = \begin{pmatrix} 0 & Sq^5 + Sq^{4,1} \\ Sq^8 & Sq^{4,2} \\ Sq^{14} & Sq^{12} \end{pmatrix}
$$

With the above defining relations, diagram (1.1) is a commutative diagram of infinite loop spaces and infinite loop maps. The composition of successive vertical maps is null homotopic.

It follows that there is a primitive element $e \in H^{24}(E)$ such that

$$j^*(e) = (Sq^8 + Sq^{6,2})i_{16} + (Sq^7 + Sq^{4,2,1})i_{17} + Sq^1 i_{23}$$

and e represents the secondary operation $\phi_{0,3}$.

Similarly, there exist stable elements $v_{0,2}, v_{2,2} \in PH^*(E_1)$ defined by

$$j_1^*(v_{0,2}) = Sq^{2,1} i_9 + Sq^1 i_{11}$$

$$j_1^*(v_{2,2}) = Sq^6 i_9 + Sq^4 i_{11}.$$

A calculation using the Adem relations shows

$$j_1^* g^*(e) = Sq^{10,3} i_{11} + Sq^{12,2,1} i_9$$

$$= Sq^{12} j_1^*(v_{0,2}) + Sq^{6,3} j_1^*(v_{2,2}).$$

It follows that (Adams, Goncalves, Harper [1, 2]).

Proposition 1.1. $g^*(e) = Sq^{12} v_{0,2} + Sq^{6,3} v_{2,2}$.

Proof. $j_1^*(g^*(e) + Sq^{12} v_{0,2} + Sq^{6,3} v_{2,2}) = 0$. Therefore, since there is an exact sequence

$$PH^*(K(\mathbf{Z}, 8)) \xrightarrow{p_1^*} PH^*(E_1) \xrightarrow{j_1^*} PH^*(K(\mathbf{Z}_2, 9, 11))$$

it follows that

$$j^*(e) + Sq^{12} v_{0,2} + Sq^{6,3} v_{2,2} = p_1^* \alpha i_8$$

where $\alpha \in \mathscr{A}(2)$ has degree 16. But all such elements αi_8 lie in kernel p_1^*. ■

Recall the following Adem relations which hold on integral classes:

$$Sq^1 Sq^8 = (Sq^5 + Sq^{4,1}) Sq^4$$

$$Sq^2 Sq^8 = Sq^{4,2} Sq^4 + Sq^8 Sq^2.$$

Since $Sq^1 Sq^8$ and $Sq^2 Sq^8$ are both zero on i_8, it follows that there are (unstable) elements $\tilde{v}_{0,3} \in H^{16}(E_1)$, $\tilde{v}_{1,3} \in H^{17}(E_1)$ with

$$j_1^*(\tilde{v}_{0,3}) = (Sq^5 + Sq^{4,1}) i_{11}$$

$$j_1^*(\tilde{v}_{1,3}) = Sq^{4,2} i_{11} + Sq^8 i_9.$$

One checks that $\tilde{v}_{1,3}$ is a suspension.

Since $Sq^1 Sq^8$ is nontrivial on a nine-dimensional class, one can check that [12]

$$\bar{\Delta} \tilde{v}_{0,3} = u_8 \otimes u_8 \text{ where } u_8 = p_1^*(i_8).$$

Hence, $Sq^8 \tilde{v}_{0,3} + u_8^3 \in PH^{24}(E_1)$, because $u_8 \in \ker Sq^1, Sq^2, Sq^4$.

Proposition 1.2. $g^*(e) = Sq^8 \tilde{v}_{0,3} + u_8^3 + Sq^{4,2,1} \tilde{v}_{1,3}$.

Proof. It suffices to check that $j_1^*(Sq^8\tilde{v}_{0,3} + u_8^3 + Sq^{4,2,1}\tilde{v}_{1,3}) = Sq^{10,3}i_{11} + Sq^{12,2,1}i_9$. ■

Corollary 1.3. $u_8^3 = Sq^8\tilde{v}_{0,3} + Sq^{4,2,1}\tilde{v}_{1,3} + Sq^{12}v_{0,2} + Sq^{6,3}v_{2,2}$.

Let E_2 be the fibre of the map that kills the elements $\tilde{v}_{0,3}$, $\tilde{v}_{1,3}$, $v_{0,2}$, $v_{2,2}$. Note that all the elements are uniquely defined with the exception of $\tilde{v}_{0,3}$ because they are the only primitive in their degrees. $\tilde{v}_{0,3}$ can be changed by u_8^2. E_2 is not an H-space, because $\tilde{v}_{0,3}$ is not primitive.

$$(1.2)\qquad \begin{array}{ccc} K(\mathbf{Z}_2, 15, 16, 11, 14) & & \\ \downarrow{\scriptstyle j_2} & & \\ E_2 & & \\ \downarrow{\scriptstyle p_2} & & \\ E_1 & \xrightarrow[Bk_1]{} & K(\mathbf{Z}_2, 16, 17, 12, 15) \\ \downarrow{\scriptstyle j_1} & & \\ K(\mathbf{Z}, 8) & \xrightarrow[Bk_0]{} & K(\mathbf{Z}_2, 10, 12). \end{array}$$

It is easy to check by Corollary 1.3 that

Proposition 1.3. $p_2^*(u_8^2) \neq 0$, $p_2^*(u_8)^3 = 0$.

Looping diagram (1.2) we obtain

$$(1.3)\qquad \begin{array}{ccc} K(\mathbf{Z}_2, 14, 15, 10, 13) & & \\ \downarrow & & \\ \Omega E_2 & & \\ \downarrow & & \\ \Omega E_1 & \xrightarrow[k_1]{} & K(\mathbf{Z}_2, 15, 16, 11, 14) \\ \downarrow & & \\ K(\mathbf{Z}, 7) & \xrightarrow[k_0]{} & K(\mathbf{Z}_2, 9, 11). \end{array}$$

ΩE_2 has two multiplications that give ΩE_2 a loop space structure. We could choose $(Bk_1)^*(i_{16}) = \tilde{v}_{0,3}$ or $(Bk_1)^*(i_{16}) = \tilde{v}_{0,3} + u_8^2$. With respect to these two H-structures the identity map $\Omega E_2 \to \Omega E_2$ has H-deviation $u_7 \otimes u_7$. Following Goncalves [2], for their multiplication

Proposition 1.4. *There is a primitive element* $v \in PH^{22}(\Omega E_2)$ *with* $a_3(v) = u_7 \otimes u_7 \otimes u_7$. *Further, in the Eilenberg-Moore spectral sequence* $\mathrm{Ext}_{H^*(\Omega E_2)}(\mathbf{Z}_2, \mathbf{Z}_2) \Rightarrow H^*(E_2)$

$$d_2[v] = [u_7 | u_7 | u_7].$$

See [11] for a definition of a_3.

Theorem 1.5. *Suppose there is a commutative diagram of a_3-spaces and H-maps*

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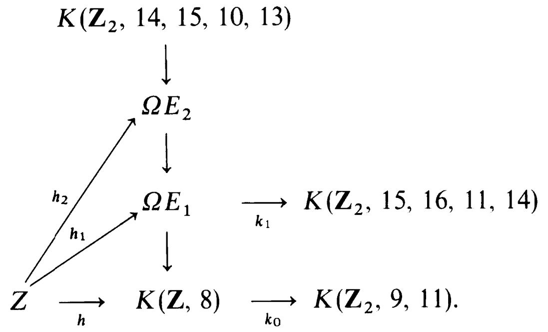


where $h^*(i_7) = z$ and $i_7 \in H^*(K(\mathbf{Z}, 7); \mathbf{Z}_2)$ is the mod 2 *reduction of the fundamental class. If h_1 is an a_3-map, then*

$$a_3(h_2^*(v)) = z \otimes z \otimes z + a_3(h_2)^*(\sigma^* v).$$

Proof. This follows from the composition formula [11]

$$a_3(v \circ h_2) = (\Omega v) a_3(h_2) + a_3(v)(h_2 \wedge h_2 \wedge h_2), \text{ by [11].} \qquad \blacksquare$$

§ 2. The $A(2)$ subHopf algebra generated by $H^3(X)$

In this chapter, we prove that the $\mathscr{A}(2)$ subHopf algebra generated by three-dimensional elements splits over the Steenrod algebra in degrees less than ten into subHopf algebras over $\mathscr{A}(2)$ generated by single three-dimensional generators. This fact will be used in the next chapter to compute the three-connective cover of X in a certain range.

The following theorem is due to Lin [7]:

Theorem 2.1. *Let X be a 1-connected finite H-space with associative* mod 2 *homology ring. Then*

(*a*) $QH^{even}(X) = 0$.
(*b*) $QH^{4k+1}(X) = Sq^{2k} QH^{2k+1}(X)$ *for* $k > 0$.
(*c*) $QH^{2^r + 2^{r+1}k - 1}(X) = Sq^{2^r k} QH^{2^r + 2^r k - 1}(X)$ $k > 0, r > 0$.
(*d*) $Sq^{2^r} QH^{2^r + 2^{r+1}k - 1}(X) = 0$.
(*e*) *X is 2-connected and generators may be chosen to have reduced coproduct in $\xi H^*(X) \otimes H^*(X)$. Hence $H^*(X)$ is primitively generated in degrees less than* 15.

By Theorem 2.1, we conclude

(2.1) $$H^1(X) = H^2(X) = H^4(X) = 0$$

(2.2)
$$PH^6(X) = \xi H^3(X)$$
$$PH^8(X) = 0$$
$$PH^{2^j+1}(X) = Sq^{2^{j-1}} \cdots Sq^2 H^3(X).$$

Let

$$V = \{x \in H^3 | x^2 = 0\}$$
$$V_1 = \{x \in H^3 | x^2 = 0 \text{ or } x^2 \neq 0 \text{ and } Sq^{4,2}x = 0\}.$$

Given a subcoalgebra $W \subset H^*(X)$, let $B(W)$ be the $\mathscr{A}(2)$ subHopf algebra of $H^*(X)$ generated by W.

Lemma 2.2. *There exists a basis $x_1, \ldots, x_l$ for V such that the nonzero $\overline{Sq^{I_j}x_i}$ form a basis for $QB(V)$, where*

$$Sq^{I_j} = \begin{cases} Sq^{2^{j-1}} \cdots Sq^2, & j \geq 2 \\ Sq^0 & j = 1. \end{cases}$$

Proof. Because $PB(V) \cong QB(V)$ in odd degrees, henceforth we will omit the bars from our notation in this proof. Assume by induction that $x_1, \ldots, x_i$ have been chosen so that the nonzero $Sq^{I_{j_1}}x_1, \ldots, Sq^{I_{j_i}}x_i$ are linearly independent.

Let $Sq^{I_k}x$ be a generator of highest degree that does not lie in the $\mathscr{A}(2)$ span of $x_1, \ldots, x_i$, and x lies in V. Then either $Sq^{I_{k+1}}x = 0$ or $Sq^{I_{k+1}}x = \sum a_i Sq^{I_{k+1}}x_i$, $a_i \in \mathbf{Z}_2$. Let $x' = x + \sum a_i x_i$. Then $Sq^{I_{k+1}}x' = 0$ and $Sq^{I_k}x'$ is not in the $\mathscr{A}(2)$ span of $x_1, \ldots, x_i$. Let $x' = x_{i+1}$. By induction, we arrive at a basis for V with the desired properties. ∎

Now consider kernel $Sq^{4,2} \subset H^3(X)$. We have ξ (kernel $Sq^{4,2}$) is a subspace of $H^6(X)$. Pick a basis $y_1^2, \ldots, y_m^2$ for ξ (kernel $Sq^{4,2}$).

Lemma 2.3. *$x_1, \ldots, x_l, y_1, \ldots, y_m$ form a basis for V_1 and the nonzero $Sq^{I_j}x_i$, $Sq^{I_k}y_s$ form a basis for $QB(Y_1)$.*

Proof. By construction $Sq^{4,2}y_s = 0$ so it suffices to show that $x_1, \ldots, x_l$, $y_1, \ldots, y_m$ are linearly independent and $Sq^2x_1, \ldots, Sq^2x_l, Sq^2y_1, \ldots, Sq^2y_m$ are linearly independent.

Suppose $\sum a_i Sq^2 x_i + \sum b_j Sq^2 y_j = 0$. Then applying Sq^1 we get

$$\sum b_j y_j^2 = 0$$

so by construction $b_j = 0$. Hence $a_i = 0$. Similarly the x_i's and y_j's are linearly independent. ∎

Now extend $x_1, \ldots, x_l, y_1, \ldots, y_m$ to a basis for $H^3(X)$ by adding $z_1, \ldots, z_n$.

(2.3) Any nonzero linear combination of the z's has nonzero square.

To see this, suppose

$$\sum c_k z_k^2 = 0.$$

Then $\sum c_k z_k = \sum a_i x_i$. Hence $c_k = 0 = a_i$.

Proposition 2.4. *In degrees less than or equal to nine $PH^*(X)$ has a basis consisting of*

(*a*) *Nonzero* $Sq^{I_j} x_r$ $\quad j \le 3$

(*b*) *Nonzero* $Sq^{I_k} y_s$, y_s^2 $\quad k \le 2$

(*c*) $Sq^{I_l} z_t$, z_t^2 $\quad l \le 3$.

Proof. By (2.1) and (2.2) all primitives of degree less than 9 have the above form so it suffices to show the above elements are linearly independent. By Lemma 2.3 the elements from (a) and (b) are linearly independent.

Consider

$$\sum c_k z_k^2 + \sum b_j y_j^2 = 0.$$

Then

$$\sum c_k z_k + \sum b_j y_j = \sum a_i x_i$$

which implies $c_k = b_j = a_i = 0$. So y_j^2's and z_k^2's form a basis of $PH^6(X)$.

Consider

$$\sum c_k Sq^{4,2} z_k + \sum a_i Sq^{4,2} x_i = 0.$$

Then

$$\sum c_k z_k + \sum a_i x_i = z'$$

has the property that if some c_k is nonzero then $(z')^2 \neq 0$ by (2.3). But then

$$\sum c_k z_k + \sum a_i x_i = \sum b_j y_j$$

and $c_k = a_i = b_j = 0$. So $Sq^{4,2} z_k$, $k = 1, \ldots, n$ and the nonzero $Sq^{4,2} x_i$ form a basis for $PH^9(X)$.

Finally, consider

$$\sum a_i Sq^2 x_i + \sum b_j Sq^2 y_j + \sum c_k Sq^2 z_k = 0.$$

Applying Sq^1,

$$\sum b_j y_j^2 + \sum c_k z_k^2 = 0.$$

This implies $b_j = c_k = 0$. We conclude the nonzero $Sq^2 x_i$, $Sq^2 y_j$, $Sq^2 z_k$ form a basis for $PH^5(X)$. ∎

Corollary 2.5. (*a*) $B(V_1) = \wedge (Sq^{I_j} x_r) \otimes \mathbf{Z}_2 \frac{[y_s]}{y_s^4} \otimes \wedge (Sq^2 y_s)$ *as Hopf algebra over* $\mathscr{A}(2)$ *where only the nonzero* $Sq^{I_i} x_r$ *are listed.*
(*b*) $B(H^3(X)) \cong B(V_1) \otimes \mathbf{Z}_2[z_t, Sq^2 z_t, Sq^{4,2} z_t]$ *as Hopf algebras over* $\mathscr{A}(2)$ *in degrees less than or equal to nine.*

§3. Connective covers of finite H-spaces

The strategy for the remainder of the paper is to obtain liftings of a homotopy-associative H-space into ΩE_2 that are a_3-liftings. If X is a homotopy-associative H-space that splits as spaces in the form $X \cong Y \times S^7$, then since S^7 lifts to ΩE_2, we can compose this lifting with the projection map $X \to S^7$ to obtain a lifting of X to ΩE_2. So the only remaining difficulty is to measure the obstructions to lifting by H-maps and a_3-maps.

We will show the following in section 5:

1. If we kill all classes in $H^*(X)$ of degree less than or equal to five then the corresponding cover X_1 of X will lift to ΩE_1, by an H-map.

2. If we kill all classes of degree less than or equal to eight in $H^*(X_1)$ except for certain seven dimensional classes then the lift to ΩE_2 will be an H-map.

For this reason this chapter is devoted to a careful calculation of the cohomology of connective covers of finite H-spaces. We essentially compute all the generators of certain connective covers in degrees less than 9.

The reader may want to skip ahead to section 5 to see how these calculations are used.

We begin by outlining theorems of Moore and Smith [9] for Hopf fibre squares.

Lemma 3.1. *Let $\theta\colon A \to B$ be an epimorphism of mod 2 cohomology Hopf algebras over the Steenrod algebra. Then if A is commutative and associative as a coalgebra, then there exists a Hopf algebra kernel $A \mathbin{\backslash\!\backslash} \theta$ over $\mathscr{A}(2)$.*

Proof. Consider the dual map

$$\mathbf{Z}_2 \longrightarrow B_* \xrightarrow{\theta_*} A_*$$

We have $\theta_*(B_*)$ is a subHopf algebra of a commutative associative Hopf algebra. Therefore $A_* /\!/ \operatorname{im} \theta_*$ is a Hopf algebra over $\mathscr{A}(2)$ and there is an exact sequence of Hopf algebras over $\mathscr{A}(2)$

$$\mathbf{Z}_2 \longrightarrow B_* \longrightarrow A_* \longrightarrow A_* /\!/ \operatorname{im} \theta_* \longrightarrow \mathbf{Z}_2.$$

Dualizing we define $A \mathbin{\backslash\!\backslash} \theta = (A_* /\!/ \operatorname{im} \theta_*)^*$. ■

Now let K be a generalized Eilenberg-MacLane space and let X_1 be the fibre of a map $f\colon X \to K$ between H-spaces, with f an H-map.

$$\begin{array}{ccc} \Omega K & \longrightarrow & \Omega K \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & LK \\ \downarrow{\scriptstyle \pi_1} & & \downarrow \\ X & \underset{f}{\longrightarrow} & K \end{array} \tag{3.1}$$

According to Smith [9], if $H_*(X)$ is associative, there is a filtration of $H^*(X_1)$

$$H^*(X_1) \supset \cdots \supset F^{-2} \supset F^{-1} \supset F^0$$

that is compatible with the Hopf algebra structure of $H^*(X_1)$ and with the Steenrod algebra and a spectral sequence with

$$E_2^{s,t} = \mathrm{Tor}^{s,t}_{H^*(K)}(H^*(X); \mathbf{Z}_2)$$

and $E_\infty = E_0 H^*(X_1)$.

Furthermore by Lemma 2.1, $\Gamma = H^*(K)\backslash\backslash f^*$ is an $\mathscr{A}(2)$ subHopf algebra of $H^*(K)$, hence it is a polynomial algebra. By Kane [6] and Smith [9], the spectral sequence collapses and

$$E_2 \cong H^*(X)/\!/\operatorname{im} f^* \otimes \mathrm{Tor}_\Gamma(\mathbf{Z}_2, \mathbf{Z}_2) \cong E_0 H^*(X_1)$$

as Hopf algebras over the Steenrod algebra, with

$$F^0 = H^*(X)/\!/\operatorname{im} f^*. \tag{3.2}$$

Since Γ is a polynomial algebra,

$$\mathrm{Tor}_\Gamma(\mathbf{Z}_2, \mathbf{Z}_2) = \Lambda(s^{-1,0} Q\Gamma).$$

So

$$E_0 H^*(X_1) = H^*(X)/\!/\operatorname{im} f^* \otimes \Lambda(s^{-1,0} Q\Gamma) \tag{3.3}$$

as Hopf algebras over the Steenrod algebra. The action of the Steenrod algebra on $s^{-1,0} Q\Gamma$ is induced by the map [9, 13.3]

$$\begin{array}{c} Q\Gamma \cong \mathrm{Tor}_\Gamma^{-1}(\mathbf{Z}_2, \mathbf{Z}_2) \\ \downarrow \\ QH^*(K) \cong \mathrm{Tor}^{-1}_{H^*(K)}(\mathbf{Z}_2, \mathbf{Z}_2) \end{array} \tag{3.4}$$

We now use the above data to compute the 3-connective cover of an H-space in degrees less than nine.

By Corollary 2.5 there exists a basis for $H^3(X)$ $x_1, \ldots, x_l$, $y_1, \ldots, y_m$, $z_1, \ldots, z_n$ such that in degrees less than or equal to nine

$$B(H^3(X)) \cong \bigotimes \Lambda(Sq^{I_j} x_r) \otimes \mathbf{Z}_2[y_s] \otimes \Lambda(Sq^2 y_s)$$
$$\otimes\, \mathbf{Z}_2[z_t, Sq^2 z_t, Sq^{4,2} z_t]$$

as Hopf algebras over the Steenrod algebra. For each x_r, y_s, z_t introduce a $K(\mathbf{Z}, 3)$ so that

$$f\colon X \longrightarrow \prod_{r,s,t} K(\mathbf{Z}, 3) = K$$

has the property that each fundamental class hits an x_r, y_s or z_t. Then f is an H-map and the induced map

$$B(H^*(X)) \longleftarrow H^*(K)$$

is an epimorphism of Hopf algebras.

By Lemma 3.1, this map has a Hopf algebra kernel which we will denote by $\ker f^*$. If X_1 is the fibre of f, then we have

$$E_0 H^*(X_1) \cong H^*(X) /\!/ \operatorname{im} f^* \otimes \Lambda(s^{-1,0} Q \ker f^*)$$

as Hopf algebras over the Steenrod algebra.

The following is a chart that describes a portion of Q $(\ker f^*)$:

Chart 1

		$Q \ker f^*$
$A.\ \wedge(x_r, Sq^2 x_r, \ldots, Sq^{I_k} x_r)$ $Sq^{I_{k+1}} x_r = 0$	$f^*(i_r) = x_r$	$i_r^2, (Sq^2 i_r)^2, \ldots, (Sq^{I_k} i_r)^2,$ $Sq^{I_l} i_r,\ l > k.$
$B.\ \mathbf{Z}_2 \dfrac{[y_s]}{y_s^4} \otimes \wedge(Sq^2 y_s)$	$f^*(i_s) = y_s$	$i_s^4, (Sq^2 i_s)^2, Sq^{I_l} i_s l > 2$
$C.\ \mathbf{Z}_2[z_t, Sq^2 z_t, Sq^{4,2} z_t]$	$f^*(i_t) = z_t$	$Sq^{I_l} i_t$ if $Sq^{I_l} z_t = 0, l > 3$ $(Sq^2 i_t)^2$ and $(i_t)^4$ are possible but lie in degree greater than or equal to 10.

Lemma 3.2. *In degrees less than or equal to nine,* $\ker f^*$ *is a polynomial algebra on the generators of the types A, B, C listed in Chart* 1.

We now list the generators in low degrees that occur in $\Lambda(s^{-1,0} Q \ker f^*)$

Chart 2

Hopf algebras of Type A	$\wedge(s^{-1,0} Q \ker f^*)$	$\mathscr{A}(2)$ action in $E_0 H^*(X_1)$
$\wedge(x_r)$	$w_5 = s^{-1,0}(i_r^2)$, $w_4 = s^{-1,0}(Sq^2 i_r)$	$Sq^1 w_4 = w_5$
	$w_8 = s^{-1,0}(Sq^{4,2} i_r)$	$Sq^4 w_4 = w_8$
$\wedge(x_r, Sq^2 x_r)$	$w_5 = s^{-1,0}(i_r^2)$, $w_8 = s^{-1,0}(Sq^{4,2} i_r)$	$Sq^4 w_5 = w_9$
	$w_9 = s^{-1,0}(Sq^2 i_r)^2$	$Sq^1 w_8 = w_9$
$\wedge(x_r, Sq^2 x_r, Sq^{4,2} x_r)$	$w_5 = s^{-1,0}(i_r^2)$, $w_9 = s^{-1,0}(Sq^2 i_r)^2$	$Sq^4 w_5 = w_9$
	$w_{16} = s^{-1,0}(Sq^{8,4,2} i_r)$	
$\wedge(x_r, \ldots, Sq^{I_j} x_r)$	$w_5 = s^{-1,0}(i_r^2)$, $w_9 = s^{-1,0}(Sq^2 i_r)^2$	$Sq^4 w_5 = w_9$
$j > 2$	$w_{2^{j+1}} = s^{-1,0}(Sq^{I_{j+1}} i_r)$	
Hopf Algebras of Type B		
$\mathbf{Z}_2 \dfrac{[y_s]}{y_s^4} \otimes \wedge(Sq^2 y_s)$	$w_9 = s^{-1,0}(Sq^2 i_s)^2$	$Sq^1 w_8 = w_9$
	$w_8 = s^{-1,0}(Sq^{4,2} i_5)$	$Sq^2 w_9 = w_{11}$
	$w_{11} = s^{-1,0}(i_s^4)$	
Hopf Algebras of Type C		
$\mathbf{Z}_2[z_t, Sq^2 z_t, Sq^{4,2} z_t]$	No generators of degree less than 10.	

Proposition 3.3. *In degrees less than or equal to nine*

(*a*) *All even generators not in* im π_1^* *occur in degrees 4 or 8, can be chosen to be primitive, and have infinite height.*

(*b*) *All odd generators not in* im π_1^* *occur in degrees* 5 *or* 9, *can be chosen to be primitive and have height two.*

Proof. Let w correspond to an even generator of $\wedge(s^{-1,0}Q\ker f^*)$. Then Chart 2 implies $\{w\}\in E_0H^*(X_1)$ has the form

$$s^{-1,0}(Sq^{I_j}i) \quad \text{for some} \quad i\in H^3(K).$$

Hence $Sq^{I_j}i\in\ker f^*$ and has nonzero projection in $QH^*(K)$. Hence,

$$Sq^{I_{l+1}}i = Sq^{2^l}\cdots Sq^{2^j}Sq^{I_j}i\in\ker f^* \quad l\geq j$$

and therefore

$$Sq^{2^l}\cdots Sq^{2^j}\{w\}\neq 0 \quad \text{in} \quad \wedge(s^{-1,0}Q\ker f^*).$$

But this corresponds to $w^{2^{l-j+1}}\neq 0$. So w has infinite height.

By Chart 2 all generators of $\wedge(s^{-1,0}Q\ker f^*)$ in degrees less than ten occur in degrees 4, 5, 8, 9. Given a generator w, it must belong to F^{-1}.

Since the filtration is compatible with the Hopf algebra structure of $H^*(X_1)$, we must have

$$\bar{\Delta}w\in F^{-1}\otimes F^0 + F^0\otimes F^{-1}.$$

But $F^0 = H^*(X)/\!/\mathrm{im}\, f^*$. By Theorem 2.1, F^0 begins in degree 7. Therefore, all the w's may be chosen to be primitive.

Now if deg w is odd, $\{w\}^2=0$ in $E_0H^*(X_1)$ since $\Lambda(s^{-1,0}Q\ker f^*)$ is exterior. We have

$$\{w_9\} = s^{-1,0}(Sq^2i)^2$$

$$\{w_5\} = s^{-1,0}(i^2)$$

By (3.4)

$$Sq^9\{w_9\} = s^{-1,0}Sq^9(Sq^2i)^2 = 0$$

$$Sq^5\{w_5\} = s^{-1,0}Sq^5(i^2) = 0.$$

So $w_9^2\in F^0$ and $w_5^2\in F^0$.

Again by Theorem 2.1, $P(H^{18}(X)/\!/\mathrm{im}\, f^*)=0$ and $P(H^{10}(X)/\!/\mathrm{im}\, f^*) = 0$. Hence $w_5^2 = w_9^2 = 0$. ∎

Proposition 3.4. $QH^{14}(X_1)=0$, $QH^{22}(X_1)=0$,

$$PH^{22}(X_1) = PH^{22}(X)/\!/\mathrm{im}\, f^*$$

$$= \xi H^{11}(X)/\!/\mathrm{im}\, f^*$$

Proof. It is easy to check that all the even generators either belong to $H^*(X)/\!/\mathrm{im}\, f^*$ or come from $\wedge(s^{-1,0}Q\ker f^*)$. All even generators of $\wedge(s^{-1,0}Q\ker f^*)$ occur in degrees 2^l for some $l \geq 2$. Hence since $QH^{14}(X)/\!/\mathrm{im}\, f^* = 0$ by Theorem 2.1 it follows that $QH^{14}(X_1) = 0$. By an argument similar to Proposition 3.3 any w_{11} with $\{w_{11}\} \in s^{-1,0}Q\ker f^*$ can be chosen to have $w_{11}^2 = 0$. Hence, $QH^{22}(X_1) = 0$, and

(3.5) $$PH^{22}(X_1) = PH^{22}(X)/\!/\mathrm{im}\, f^* = \xi(H^{11}(X)/\!/\mathrm{im}\, f^*)$$

by Theorem 2.1 and the fact that $H^{11}(X)/\!/ f^*$ is primitive. ■

The following chart describes the structure of the Hopf algebras produced in $H^*(X_1)$:

Chart 3

Hopf algebra factors of $H^*(X)$	Hopf algebra factors of $H^*(X_1)$ in degrees less than 12
$\wedge(x_3)$	$\mathbf{Z}_2[w_4] \otimes \wedge(w_5)$, $Sq^1 w_4 = w_5$
$\wedge(x_3, Sq^2x_3)$	$\mathbf{Z}_2[w_8] \otimes \wedge(w_5, w_9)$, $Sq^4 w_5 = w_9$, $Sq^1 w_8 = w_9$
$\wedge(x_3, Sq^2x_3, Sq^{4,2}x_3, \ldots, Sq^{I_j}x_3)$	$\wedge(w_5, w_9)$, $Sq^4 w_5 = w_9$
$\mathbf{Z}_2\dfrac{[y_s]}{y_s^4} \otimes \wedge(Sq^2 y_s)$	$\mathbf{Z}_2[w_8] \otimes \wedge(w_9, w_{11})$, $Sq^1 w_8 = w_9$, $Sq^2 w_9 = w_{11}$
$\mathbf{Z}_2[z_t, Sq^2 z_t, Sq^{4,2} z_t]$	No generators of degree less than 10.

§4. $H^*(X_2)$

In this chapter, we consider the cohomology of the fibre of a map $f_1 : X_1 \to K_1$ where f_1 is an H-map, K_1 a generalized Eilenberg-MacLane space in degrees 4, 5, 7 and 8. If X_2 is the fibre of f_1, then X_2 is a homotopy associative H-space. The main result of this chapter will be to show that in degrees 14 and 22, all primitives and generators are in the image of lower-degree primitives as long as there is no factor of $H^*(X)$ of type B. The method of computation is the same as that used to compute $H^*(X_1)$.

It follows that

$$E_0H^*(X_2) = H^*(X_1)/\!/\mathrm{im}\, f_1^* \otimes \Lambda(s^{-1,0}Q\ker f_1^*).$$

Since $QH^{14}(X_1) = 0$ and $PH^{22}(X_1) \subset \xi(PH^*(X)/\!/\mathrm{im}\, f^*)$ by Proposition 3.4, it follows that any new 14- or 22-dimensional primitives or generators must come from $\wedge(s^{-1,0}Q\ker f_1^*)$. We proceed to calculate all such elements. Our recipe for defining f_1 comes from Chart 3, and the following chart.

Chart 4

Hopf algebra factor of $H^*(X_1)$	K_1 factor
1. $\mathbf{Z}_2[w_4] \otimes \wedge(w_5)$ $Sq^1 w_4 = w_5$	$K(\mathbf{Z}_2, 4), f_1^*(i_4) = w_4$
2. $\mathbf{Z}_2[w_8] \otimes \wedge(w_5, w_9)$ $Sq^1 w_8 = w_9 = Sq^4 w_5$	$K(\mathbf{Z}, 5) \times K(\mathbf{Z}_2, 8)$ $f_1^*(i_5) = w_5, f_1^*(i_8) = w_8$
3. $\Lambda(w_5, w_9)$ $Sq^4 w_5 = w_9$	$K(\mathbf{Z}, 5), f_1^*(i_5) = w_5$
4. $\mathbf{Z}_2[w_8] \otimes \wedge(w_9, w_{11})$ $Sq^1 w_8 = w_9, Sq^2 w_9 = w_{11}$	$K(\mathbf{Z}_2, 8), f_1^*(i_8) = w_8$

Occasionally we will also introduce $K(\mathbf{Z}, 7)$ factors to kill off elements of $H^*(X)/\!/\operatorname{im} f^*$.

Lemma 4.1. *$H^*(X_2)$ is four-connected and all elements of degree less than twelve are primitive.*

Proof. By construction and Theorem 2.1, $F^0 = H(X_1)/\!/\operatorname{im} f_1^*$ is six-connected and by the recipe for f_1, $\wedge(s^{-1,0} Q \ker f_1^*)$ will be four-connected. It follows that $E_0 H^*(X_2)$ is four-conected, so $H^*(X_2)$ is four-connected. It also follows that since F^0 is a subHopf algebra, all elements of degree less than 12 of F^0 are primitive. Further

$$\bar{\Delta} F^{-1} \subset F^{-1} \otimes F^0 + F^0 \otimes F^{-1}.$$

Hence all generators of $\wedge(s^{-1,0} Q \ker f_1^*)$ are primitive in degrees less than twelve. ■

Proposition 4.2. *If $H^*(X)$ does not contain subHopf algebras of type B, then all fourteen-dimensional generators of $H^*(X_2)$* are primitive and in the *$\mathscr{A}(2)$ image of primitive classes of degrees less than twelve.*

Proof. By Proposition 3.4 $QH^{14}(X_1) = 0$, so $Q(H^{14}(X_1)/\!/\operatorname{im} f_1 = 0$. Hence all fourteen-dimensional generators of $H^*(X_2)$ come from $\wedge(s^{-1,0} Q \ker f_1^*)$. These elements in degree 14 come from 15-dimensional elements of $H^*(K_1)$ in $\ker f_1^*$. Since f_1^* is a map of Hopf algebras, it is easy to check that $Q \ker f_1^*$ is spanned by a submodule of $PH^{15}(K_1)$. We show, in fact, that every 15-dimensional admissible of $H^*(K_1)$ lies in kernel f_1^* and in the $\mathscr{A}(2)$ image of admissibles of degree less than 12.

Chart 5

All degree 15 admissibles of $H^*(K_1)$	kernel f_1^*
$Sq^{7,3,1} i_4$	$Sq^2 i_4; Sq^{3,1} i_4 = Sq^{2,2} i_4$ and $H^6(X_1) = 0$
$Sq^{7,3} i_5$	$Sq^3 i_5$; if $Sq^2 w_5 = x_7$, then $Sq^1 x_7 = 0$ by Theorem 2.1.
$Sq^{6,1} i_8$	$Sq^{4,1} i_8 = Sq^1(Sq^4 i_8) + Sq^2(Sq^3 i_8)$; $Sq^4 i_8 \in \ker f_1^*$ because $PH^{12}(X_1) = 0$ by Theorem 2.1 since $PH^{12}(X_1) = PH^{12}(X)/\!/\operatorname{im} f^*$ $Sq^2 i_8 \in \ker f_1^*$ since $Sq^2 w_8 \in PH^{10}(X)/\!/\operatorname{im} f^* = 0$. $Sq^{6,1} i_8 = Sq^2(Sq^{4,1} i_8)$

$Sq^{4,2,1}i_8$	$Sq^{4,2,1}i_8 = Sq^{4,2}(Sq^1 i_8 + Sq^4 i_5)$ This is for a Hopf algebra of type 2 in Chart 4. Hence $Sq^1 i_8 + Sq^4 i_5 \in \ker f_1^*$
$Sq^{5,2}i_8$	$Sq^2 i_8 \in \ker f_1^*$, see argument for $Sq^{6,1}i_8$
$Sq^7 i_8$	$Sq^4 i_8 \in \ker f_1^*$, see argument for $Sq^{6,1}i_8$
$Sq^{6,2}i_7$	$Sq^2 i_7 \in \ker f_1^*$ since $(H^*(X)/\!/\operatorname{im} f^*)^9 = 0$ by Theorem 2.1.

Remark 4.1. Note that in the case of a type B subHopf algebra $\mathbf{Z}_2\frac{[y_s]}{y_s^4} \otimes \wedge(Sq^2 y_s)$, $H^*(X_1)$ would contain a factor of the form $\mathbf{Z}_2[w_8] \otimes \wedge(w_9, w_{11})$ with $Sq^{2,1}w_8 = w_{11}$ and hence, if $Sq^4 w_{11} = 0$, $H^*(X_2)$ would contain an element corresponding to $s^{-1,0}Sq^{4,2,1}w_8$ in degree 14 that may not be primitive. ■

Proposition 4.3. *If $H^*(X)$ does not contain a subHopf algebra of type B, then all elements of $PH^{22}(X_2)$ lie in the $\mathscr{A}(2)$ image of lower-dimensional primitives.*

Proof. By Proposition 3.4 $PH^{22}(X_1) = \xi P(H^{11}(X)/\!/\operatorname{im} f^*)$, $QH^{22}(X_1) = 0$.

All 22-dimensional primitives of $H^*(X_2)$ are either generators or squares of eleven-dimensional elements.

Consider $F^0 = H^*(X_1)/\!/\operatorname{im} f_1^* \subset H^*(X_2)$. We have the following commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
P(\xi H^{11}(X_1)) & \longrightarrow & P(\xi H^{11}(X_1)/\!/\operatorname{im} f_1^*) \\
\downarrow & & \downarrow \\
PH^{22}(X_1) & \longrightarrow & P(H^{22}(X_1)/\!/\operatorname{im} f_1^*) \\
\downarrow & & \downarrow \\
QH^{22}(X_1) & \longrightarrow & Q(H^{22}(X_1)/\!/\operatorname{im} f_1^*) \longrightarrow 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array} \tag{4.1}$$

with exact columns, and exact bottom row.

Since $QH^{22}(X_1) = 0$ by Proposition 3.4, it follows that $PH^{22}(X_1)/\!/\operatorname{im} f_1^* = P(\xi H^{11}(X_1)/\!/\operatorname{im} f_1^*)$. But $H^{11}(X_1)/\!/\operatorname{im} f_1^* \subset F^0$ is primitive by Lemma 4.1; hence all elements of $F^0 \subset H^*(X_2)$ are in the $\mathscr{A}(2)$ image of eleven-dimensional primitives of $H^*(X_2)$. Now the following chart shows that all 22-dimensional primitives that arise from $\wedge(s^{-1,0}Q\ker f_1^*)$ also lie in the $\mathscr{A}(2)$ image of elements of degree less than 12 of $H^*(X_2)$. By Lemma 4.1, these elements are all primitive.

As in Chart 5, we show *every* admissible of $H^*(K_1)$ in degree 23 lies in the $\mathscr{A}(2)$ image of some lower dimensional element of kernel f_1^*.

Chart 6

All 23 dimensional admissibles of $H^*(K_1)$	Kernel f_1^*
$Sq^{11}(Sq^{5,2,1}i_4)$	$Sq^{5,2,1}w_4$ lies in $PH^{12}(X_1) = PH^{12}(X)/\!/\operatorname{im} f^* = 0$ by Theorem 2.1.
$Sq^{11}(Sq^{5,2}i_5)$	$Sq^{5,2}w_5 \in PH^{12}(X_1) = 0$
$Sq^{11}(Sq^{3,1}i_8)$	$Sq^{3,1}w_8 = Sq^2Sq^2w_8$ and $Sq^2w_8 \in PH^{10}(X_1) = 0$ by Theorem 2.1
$Sq^{10}(Sq^{4,1}i_8)$	$Sq^{4,1}w_8 = Sq^1(Sq^4w_8) + Sq^2(Sq^1Sq^2w_8)$ Same argument as above.
$Sq^{9,4,2}i_8$	$Sq^2w_8 = 0$
$Sq^{10,5}i_8$	$Sq^4w_8 = 0$
$Sq^{11,4}i_8$	$Sq^4w_8 = 0$
$Sq^{8,4,2,1}i_8$	$Sq^{2,1}w_8 = Sq^2(Sq^1w_8 + Sq^4w_5)$. This is for type 2 Hopf algebra of chart 4. $Sq^1w_8 = Sq^4w_5$
$Sq^{10,4,2}i_7$	$Sq^2 f_1^*(i_7) = 0$ since by Theorem 2.1 $PH^9(X) = Sq^{4,2}PH^3(X)$
$Sq^{11,5}i_7$	$Sq^5 f_1^*(i_7) = Sq^4Sq^1 f_1^*(i_7) + Sq^{2,1}(Sq^2 f_1^*(i_7))$ $Sq^1 f_1^*(i_7) = 0 = Sq^2 f_1^*(i_7)$ by Theorem 2.1.

Remark 4.2. If $H^*(X)$ contained a Hopf algebra of type B and $Sq^{8,4}w_{11} = 0$, then $PH^{22}(X_2)$ would contain an element corresponding to $s^{-1,0}Sq^{8,4,2,1}i_8$ in degree 22. If $Sq^4w_{11} = 0$ and if $s^{-1,0}Sq^{4,2,1}i_8$ is not primitive in $H^{14}(X_2)$ then this 22-dimensional primitive would not be in the image of Steenrod operations applied to lower dimenional primitives.

§5. Application

We now can prove Theorems A and B of the Introduction.

Proof of Theorem A. Let X be a homotopy associative H-space and suppose X is mod 2 equivalent to $Y \times S^7$. Suppose $H^*(X)$ does not contain any $\mathscr{A}(2)$ subHopf algebras of the form

$$\mathbf{Z}_2\frac{[y]}{y^4} \otimes \wedge(Sq^2y), \quad \deg y = 3.$$

If $h'\colon S^7 \to K(\mathbf{Z}, 7)$ is the integral class of $H^7(S^7)$ then there is a commutative diagram

$$(5.1)\qquad \begin{array}{ccccccc}
X_2 & & & & \Omega E_2 & & \\
\downarrow{\scriptstyle p_2} & & & \nearrow{\scriptstyle h_2'} & \downarrow & & \\
X_1 & & & & \Omega E_1 & \xrightarrow[k_1]{} & K(\mathbf{Z}_2, 15, 16, 11, 14) \\
\downarrow{\scriptstyle p_1} & & & \nearrow{\scriptstyle h_1'} & \downarrow & & \\
X & \xrightarrow{\pi} & S^7 & \xrightarrow[h']{} & K(\mathbf{Z}, 7) & \xrightarrow[k_0]{} & K(\mathbf{Z}_2, 9, 11)
\end{array}$$

By Theorem 2.1, X is 2-connected. Therefore $h'\pi$ is an a_3-map, and $D_{h_1'\pi}$ factors through $K(\mathbf{Z}_2, 8, 10)$.

Again by Theorem 2.1, any element of $H^l(X \wedge X)$ for $l = 8, 10$ has factors that lie in the $\mathscr{A}(2)$ subHopf algebra generated by $H^3(X)$. Therefore, if X_1 is the 3-connective cover of X, then

$$h_1'\pi p_1 \quad \text{is an} \quad H\text{-map}.$$

Similarly, $a_3(h_1'\pi p_1)$ factors through $K(\mathbf{Z}_2, 7, 9)$. Since X_1 is 3-connected it follows that

$$(5.2)\qquad h_1'\pi p_1 \quad \text{is an} \quad a_3\text{-map}.$$

This implies if $h_2'' = h_2'\pi p_1$, then $D_{h_2''}$ factors through $K(\mathbf{Z}_2, 14, 15, 10, 13)$. Examining Chart 2 of chapter 3, one checked that $H^l(X_1 \wedge X_1)$, $l = 14, 15, 10, 13$, involves the elements w_4, w_5, w_9 except possibly in the case of $H^{14}(X_1 \wedge X_1)$ which could involve $(h_1'\pi p_1)^*(i_7)$ or a seven-dimensional class in $H^*(X)/\!/\operatorname{im} f^*$. Note $H^*(X)/\!/\operatorname{im} f^*$ is six-connected by Theorem 2.1.

Case 1. If $[D_{h_2''}] \in H^l(X_1 \wedge X_1)$ does not contain $(h_1'\pi p_1)^*(i_7) \otimes (h_1'\pi p_1)^*(i_7)$, then by killing off w_4, w_5, w_9 and possibly elements of $H^7(X)/\!/\operatorname{im} f^*$ we obtain an H-map $h_2 = h_2'\pi p_1 p_2$

$$X_2 \xrightarrow{h_2} \Omega E_1 .$$

Case 2. If $[D_{h_2''}]$ has $(h_1'\pi p_1)^*(i_7) \otimes (h_1'\pi p_1)^*(i_7)$ as a summand, then by changing the k-invariant $(Bk_1)^*(i_{16}) = \tilde{v}_{0,3} + i_8^2$ where i_8 is the lifting of the fundamental class, then this changes the H-structure of ΩE_2 so that $[D_{h_2''}]$ does not have $(h_1'\pi p_1)^*(i_7) \otimes (h_1'\pi p_1)^*(i_7)$ as a summand. (See the remark before Proposition 1.4.)

In either case the map

$$(5.3)\qquad X_2 \xrightarrow{h_2} \Omega E_1 \quad \text{is an} \quad H\text{-map}.$$

Now $a_3(h_2)$ factors through $K(\mathbf{Z}_2, 13, 14, 9, 12)$. By Lemma 4.1, $H^*(X_2)$ is four-connected so

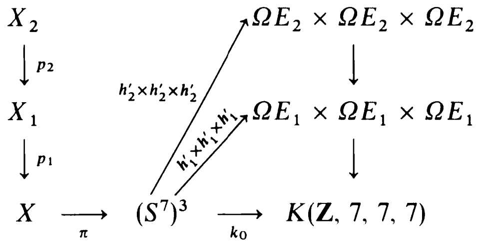
$$(5.4)\qquad h_2 \quad \text{is an} \quad a_3\text{-map}.$$

By Theorem 1.5, there is an element $h_2^*(v) \in PH^{22}(X_2)$ with $a_3(h_2^*(v)) = z \otimes z \otimes z \neq 0$ where $z = h^*(i_7)$, $h = h'\pi p_1 p_2$. But by Proposition 4.3, $h_2^*(v) = \sum \alpha_i z_i$ where $z_i \in PH^*(X_2)$ have degree less than 22 and $\alpha_i \in \mathscr{A}(2)$. This implies

$$z \otimes z \otimes z \in \sum \alpha_i a_3(z_i).$$

But X_2 has the homotopy type of $Y_2 \times S^7$ for some space Y_2, so $z \otimes z \otimes z$ is not in the image of Steenrod operations. Further $z \otimes z \otimes z \notin \text{image } \bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta}$. We conclude X could not have been a homotopy associative H-space. This proves Theorem A.

Proof of Theorem B. We construct X_2 in the same manner as before. There are maps



If $h_1 = (h'_1 \times h'_1 \times h'_1)\pi p_1$ and D_{h_1} involves $s_i \otimes s_j$ where s_i, s_j are 7-dimensional spherical classes, then by changing the H-structure of $\Omega E_2 \times \Omega E_2 \times \Omega E_2$, we still can make $h_2 = (h'_2 \times h'_2 \times h'_2)\pi p_1 p_2$ an H-map, and therefore an a_3-map.

Now there exist elements $h_2^*(v_i)$, $i = 1, 2, 3$ with

$$a_3(h_2^*(v_i)) = s_i \otimes s_i \otimes s_i \quad \text{mod} \quad \text{im}\bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta}.$$

Since $\dim Sq^1[H^5(Y) \cap \ker Sq^4] \leq 1$, there exists at most one subHopf algebra of type B in $H^*(X)$.

Therefore $H^*(X_2)$ contains at most one nonprimitive generator w_{14}, and one primitive 22-dimensional generator not in the $\mathscr{A}(2)$ image of lower dimensional primitives. The analysis due to Goncalves [2, p. 19] shows that if V is the 3-dimensional vector space spanned by $s_i \otimes s_i \otimes s_i$, $i = 1, 2, 3$, then

$$\frac{V}{\text{im}\,\bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta}} \quad \text{has dimension at least two.}$$

It follows that there must be at least two linearly independent 22-dimensional primitives with nonzero a_3-invariant. This is a contradiction, and completes the proof of Theorem B. ∎

Theorem 5.1. *Let X be a finite H-space with rational generators in degrees*

$$\{3, 3, 3, 7, 7, 7, 11, 11, 15, 15, 19, 23, 23, 27\}.$$

Then X has the same rational type as $G \times S^7$ where G is either Spin(15) $\times Sp(2) \times F_4$ *or* $Sp(7) \times Sp(2) \times F_4$. *In either case, if X is* mod 2 *equivalent to* $G \times S^7$, *then X cannot be homotopy associative*

Proof. In the process of taking connective covers, the only possibility of creating 14- and 22- dimensional generators occurs in the connective cover of F_4. But Goncalves [2] shows that in the 3-connective cover of F_4, $Sq^4 w_{11} = x_{15}$, so no 14-dimensional or 22-dimensional generators are created.

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