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Indefinite Kähler metrics of constant holomorphic sectional curvature

By

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§ 1. Introduction

Let (M, g) be a connected (indefinite) Riemannian manifold of dimension $n(>1)$. From the well-known fact that a Riemannian manifold (M, g) of constant curvature is conformally flat, making use of a conformal change of a Euclidean metric, the following theorem was obtained:

Theorem A ([3; §27]). *Let (M, g) be an indefinite Riemannian manifold of dimension $n(>1)$ and of index m. If (M, g) is of constant curvature κ, then each point of M has a coordinate neighborhood $\{U; x^1, \cdots, x^n\}$ in which the components g_{pq} $(p, q = 1, \ldots, n)$ of g are given by*

$$g_{pq} = \frac{\varepsilon_p \delta_{pq}}{\left\{1 + \frac{\kappa}{4} \sum_{r=1}^{n} \varepsilon_r (x^r)^2\right\}^2} \quad \text{(not summed for } p\text{)},$$

where $\varepsilon_t = -1$ or 1 according as $t \leqq m$ or $t > m$.

Concerning the fact that a Riemannian manifold of constant curvature is projectively flat (cf. [4; §34]), the following result is known:

Theorem B (cf. [3; §27], [16; §3 *in Chapter V*]). *Let (M, g) be a Riemannian manifold of dimension $n(>1)$. If (M, g) is of constant curvature κ, then each point of M has a coordinate neighborhood $\{U; x^1, \cdots, x^n\}$ in which the components g_{pq} $(p, q = 1, \cdots, n)$ of g are given by*

$$g_{pq} = \frac{\{1 + \kappa \sum_{r=1}^{n} (x^r)^2\} \delta_{pq} - \kappa x^p x^q}{\{1 + \kappa \sum_{r=1}^{n} (x^r)^2\}^2}.$$

Let g' be another indefinite Riemannian metric on (M, g). When the Levi-Civita connection induced from g is projectively related to that induced from g', the metric g is called a projective change of g'. The local expression of g in

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Theorem B has been given without making use of a projective change of a Euclidean metric. On the other hand, T. Levi-Civita [18], L. P. Eisenhart ([3], [4]), B. Kagan [15] and other authors ([17], [25]) investigated projective changes of (indefinite) Riemannian metrics and obtained interesting results. In particular, B. Kagan proved the following theorem:

Theorem C ([15]). *Let $(\boldsymbol{R}^n, g_0)$ be a Euclidean space of dimension $n(>1)$ with the canonical Riemannian metric g_0. If an indefinite Riemannian metric g is a projective change of g_0, then in terms of the natural coordinate system $x^1, \cdots, x^n$ of $\boldsymbol{R}^n$, the components g_{pq} $(p, q = 1, \cdots, n)$ of g are locally expressed by*

(a) $$g_{pq} = \frac{1}{4\kappa\phi^2}\left\{2\phi\frac{\partial^2\phi}{\partial x^p \partial x^q} - \frac{\partial\phi}{\partial x^p}\frac{\partial\phi}{\partial x^q}\right\} \quad \textit{for} \quad \kappa \neq 0,$$

(b) $$g_{pq} = \frac{1}{2\psi}\frac{\partial^2}{\partial x^p \partial x^q}\left(\frac{\phi}{\psi}\right) \quad \textit{for} \quad \kappa = 0,$$

where κ is the constant curvature of g and ϕ (resp. ψ) is a quadratic polynomial (resp. a linear polynomial) of $x^1, \cdots, x^n$.

Theorem C implies that an indefinite Riemannian metric of constant curvature has the local components expressed by (a) or (b) in Theorem C, which contain the local expressions of g in Theorem B as a special case.

By making use of projective changes of a Finsler metric, M. Matsumoto [19] studied a projectively flat Finsler space of constant curvature and obtained some interesting results. And by restricting his consideration to the case of Riemannian manifolds, he showed the local expression (a) in Theorem C.

For the complex case, S. Bochner [2] and other authors ([5], [11], [12]; see [16; Chapter IX]) investigated Kähler metrics of constant holomorphic sectional curvature, T. Ōtsuki and Y. Tashiro [20] studied a holomorphically projective change of a Kähler metric (for the definition, see §2 in this paper), and thereafter several authors ([9], [10], [13], [14], [21], [22], [23], [24], [27]) obtained many interesting results. And in these directions, the following theorems are well known:

Theorem D ([2]). *Let (M, J, g) be a Kähler manifold of real dimension $2n(>2)$ and of constant holomorphic sectional curvature κ. Then each point of M has a real coordinate neighborhood $\{U; x^1, \cdots, x^{2n}\}$ in which the components $g_{ij}(i, j = 1, \cdots, 2n)$ of g are expressed by*

$$g_{ij} = \sum_{a,b=1}^{2n} (\delta_i^a \delta_j^b + J_i^a J_j^b)\frac{\partial^2 f}{\partial x^a \partial x^b},$$

where f is given by

$$f = \frac{1}{\kappa}\log\left\{1 + \frac{\kappa}{2}\sum_{i=1}^{2n}(x^i)^2\right\} \quad \textit{for} \quad \kappa \neq 0,$$

$$f=\frac{1}{4}\sum_{i=1}^{2n}(x^i)^2 \qquad\qquad for \quad \kappa=0,$$

and J_i^j *mean the components of* J *such that* $\delta_{ab}J_i^aJ_j^b=\delta_{ij}$.

Theorem E ([9], [20], [24]). *Let* (M, J, g) *be a Kähler manifold of real dimension* $2n(>2)$. *The following conditions are equivalent*:
(a) (M, J, g) *is of constant holomorphic sectional curvature.*
(b) *The holomorphically projective curvature tensor of* (M, J, g) *vanishes.*
(c) (M, J, g) *is holomorphically projectively flat.*

From Theorems C and E, we can surmise that Theorem D is directly obtained by making use of a holomorphically projective change of a complex Euclidean metric.

Recently, M. Barros and A. Romero [1] investigated an indefinite Kähler manifold of constant holomorphic sectional curvature, and in particular they mentioned the classification of complete and simply-connected indefinite Kähler manifolds of constant holomorphic sectional curvature. But they did not show the local expression of such metrics except an indefinite complex Euclidean metric.

And for the contact case, the present author considered a CHP-change g of the contact Riemannian metric of a K-contact Riemannian manifold and determined such a metric g without a condition for curvature ([6], [7], [8]).

The main purpose of this paper is to write down explicitly all holomorphically projective changes of a complex Euclidean metric in terms of the natural coordinates. Consequently, we can obtain the local expression of an indefinite Kähler metric of constant holomorphic sectional curvature, which is the generalization of Bochner's Theorem D. And by virtue of the similar method, we can show the generalization of Theorem B.

Throughout this paper, we assume that all objects under consideration are differentiable of class C^∞ and all manifolds are connected. And, unless otherwise stated, indices $\{a, b, c, d, h, i, j, k\}$, $\{p, q, r, s\}$ and $\{\alpha, \beta, \gamma\}$ run over the ranges $\{1, \cdots, 2n\}$, $\{1, \cdots, n\}$ and $\{2, \cdots, n\}$ respectively, and we use the summation convention.

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§2. Holomorphically projective changes

Let (M, J, g) be an indefinite Kähler manifold of real dimension $2n(>2)$ and of index $2m$. Then there exists an orthonormal base $\{e_1, \cdots, e_{2n}\}$ with respect to g for the tangent space $T_x(M)$ at each point $x\in M$ such that

$$g(X, Y)=-\sum_{p=1}^{m}X^pY^p+\sum_{p=m+1}^{n}X^pY^p-\sum_{p=n+1}^{n+m}X^pY^p+\sum_{p=n+m+1}^{2n}X^pY^p$$

for any $X=X^ie_i$, $Y=Y^ie_i\in T_x(M)$, and its metric g and its complex structure J

satisfy

$$\nabla g = 0,\ \nabla_X Y - \nabla_Y X - [X, Y] = 0$$

$$J^2 = -I,\ g(JX, JY) = g(X, Y),\ \nabla J = 0,$$

for any $X, Y \in \mathfrak{X}(M)$, where we denote by I, ∇ and $\mathfrak{X}(M)$ the identity tensor, the Levi-Civita connection induced from g and the set of vector fields on M respectively (cf. [1]).

A curve $x(t)$ on (M, J, g) is called a *holomorphically planar curve with respect to* ∇ if $x(t)$ satisfies the differential equation

$$\nabla_{\dot{x}(t)}\dot{x}(t) = \alpha(t)\dot{x}(t) + \beta(t) J\dot{x}(t)$$

for certain functions α and β of a real parameter t, where $\dot{x}(t)$ is the tangent vector of $x(t)$. Let g' be another indefinite Kähler metric on (M, J) and ∇' the Levi-Civita connection induced from g'. If ∇ and ∇' have all of their holomorphically planar curves in common, then g is called a *holomorphically projective change*, briefly an *HP-change, of* g'. As is well known (cf. [23], [24]), g is an HP-change of g' if and only if there exists a 1-form P on M satisfying

(2.1) $$\nabla_X Y = \nabla'_X Y + P(X)Y + P(Y)X - P(JX)JY - P(JY)JX$$

for any $X, Y \in \mathfrak{X}(M)$. We can easily see that the 1-form P in (2.1) is equal to the differential df of a certain function f on M and an indefinite Hermitian metric g satisfying (2.1) is Kählerian. In this case, when we denote by K (resp. K') the curvature tensor of g (resp. g'), we have

(2.2) $$K(X, Y)Z = K'(X, Y)Z - \tilde{P}(Y, Z)X + \tilde{P}(X, Z)Y + \tilde{P}(Y, JZ)JX - \tilde{P}(X, JZ)JY - \{\tilde{P}(X, JY) - \tilde{P}(Y, JX)\}JZ,$$

and when we denote by R (resp. R') the Ricci tensor of g (resp. g'), we have

(2.3) $$R(X, Y) = R'(X, Y) - 2n\tilde{P}(X, Y) - 2\tilde{P}(JX, JY),$$

where we put

$$\tilde{P}(X, Y) = \tilde{P}(Y, X) = (\nabla'_X P)Y - P(X)P(Y) + P(JX)P(JY).$$

On the other hand, since g and g' are Kählerian, we have

(2.4) $$R(X, JY) + R(JX, Y) = 0, \quad R'(X, JY) + R'(JX, Y) = 0$$

(for the definite case, see [26; p.71]). From (2.3) and (2.4), we have

(2.5) $$\tilde{P}(X, JY) + \tilde{P}(JX, Y) = 0,$$

(2.6) $$R = R' - 2(n+1)\tilde{P}.$$

Thus, from (2.2), (2.5) and (2.6), we see that the following tensor field H is invariant under an HP-change:

$$H(X, Y)Z = K(X, Y)Z - \frac{1}{2(n+1)}\{R(Y, Z)X - R(X, Z)Y$$
$$- R(Y, JZ)JX + R(X, JZ)JY + 2R(X, JY)JZ\}.$$

We call such a tensor field H the *holomorphically projective curvature tensor.*

An indefinite Kähler manifold (M, J, g) is said to be *holomorphically projectively flat* when, for each point $x \in M$, there exist a neighborhood V of x and an indefinite flat Kähler metric g' on V such that g is an HP-change of g' on V. In this case, the holomorphically projective curvature tensor of g vanishes. Conversely, if the holomorphically projective curvature tensor of an indefinite Kähler manifold (M, J, g) vanishes, then for each point $x \in M$, there exist a neighborhood V' of x and a flat symmetric affine connection ∇' on V' such that $\nabla' J = 0$ and the Levi-Civita connection ∇ induced from g is an HP-change of ∇', that is, ∇ and ∇' satisfy (2.1) (cf. [24]). Thus there exist a neighborhood V of x in V' and an indefinite flat Riemannian metric g' on V whose Levi-Civita connection coincides with ∇' (cf. [4; §29]), and we can choose as g' a Hermitian metric. Therefore g is locally an HP-change of g' and (M, J, g) is holomorphically projectively flat.

A 2-plane π in the tangent space $T_x(M)$ at $x \in M$ is said to be *holomorphic* if $J\pi \subset \pi$, and *non-degenerate with respect to* g if π has a base $\{X_1, X_2\}$ satisfying

$$g(X_1, X_1)g(X_2, X_2) - g(X_1, X_2)^2 \neq 0.$$

For each holomorphic non-degenerate 2-plane $\pi \subset T_x(M)$, its holomorphic sectional curvature $\rho(\pi)$ is defined by

$$\rho(\pi) = \frac{g(K(X, JX)JX, X)}{g(X, X)^2}$$

for a vector $X \in \pi$ such that $g(X, X) \neq 0$. When ρ is constant for all holomorphic non-degenerate 2-planes in $T_x(M)$ at each point $x \in M$, (M, J, g) is said to be *of constant holomorphic sectional curvature.* As is well known (cf. [1]), the curvature tensor K of (M, J, g) of constant holomorphic sectional curvature κ is expressed by

$$K(X, Y)Z = \frac{\kappa}{4}\{g(Y, Z)X - g(X, Z)Y \tag{2.7}$$
$$- g(Y, JZ)JX + g(X, JZ)JY + 2g(X, JY)JZ\}$$

for any $X, Y, Z \in \mathfrak{X}(M)$, and from which, it follows that (M, J, g) is an Einstein space whose Ricci tensor R is given by

$$R = \frac{n+1}{2}\kappa g. \tag{2.8}$$

And it is easily seen that (M, J, g) is of constant holomorphic sectional curvature if and only if (M, J, g) has the vanishing holomorphically projective curvature tensor.

From the argument above, we can obtain the indefinite analogue of Theorem E:

Proposition 1. *Let (M, J, g) be an indefinite Kähler manifold of real dimension $2n(>2)$. Then the following conditions are equivalent:*
(*a*) *(M, J, g) is of constant holomorphic sectional curvature.*
(*b*) *The holomorphically projective curvature tensor of (M, J, g) vanishes.*
(*c*) *(M, J, g) is holomorphically projectively flat.*

§3. The condition for the existence of holomorphically projective changes

Let (M, J, g') be an indefinite Kähler manifold of real dimension $2n(>2)$ and of index $2m$, and assume that (M, J, g') is of constant holomorphic sectional curvature κ'.

When we take a real-valued function f on M, we shall consider an indefinite Kähler metric g which is an HP-change of g' satisfying (2.1) for the 1-form $P = df$. Then it follows from Proposition 1 that g is of constant holomorphic sectional curvature. And from (2.1) we have

$$(3.1)\qquad (\nabla'_X g)(Y, Z) = 2P(X)g(Y, Z) + P(Y)g(X, Z) + P(Z)g(X, Y) - P(JY)g(JX, Z) - P(JZ)g(JX, Y)$$

for any $X, Y, Z \in \mathfrak{X}(M)$, where we denote by ∇' the Levi-Civita connection induced from g'. The integrability condition of (3.1) is given by

$$(3.2)\qquad \begin{aligned} g(K'(X, Y)Z, W) + g(K'(X, Y)W, Z) &= \tilde{P}(Y, Z)g(X, W) - \tilde{P}(X, Z)g(Y, W) - \tilde{P}(Y, JZ)g(JX, W) \\ &\quad + \tilde{P}(X, JZ)g(JY, W) + \tilde{P}(Y, W)g(X, Z) - \tilde{P}(X, W)g(Y, Z) \\ &\quad - \tilde{P}(Y, JW)g(JX, Z) + \tilde{P}(X, JW)g(JY, Z), \end{aligned}$$

where we denote by K' the curvature tensor of g'. From (2.5), (2.7) and (3.2), we have

$$(3.3)\qquad \tilde{\rho} g = \tilde{P} - \frac{\kappa'}{4} g',$$

where we denote by $\{e_1, \cdots, e_{2n}\}$ and $[g^{ij}]$ the local frame and the inverse matrix of the matrix $[g(e_i, e_j)]$ respectively, and we put

$$\tilde{\rho} = \frac{1}{2n} g^{ij} \{\tilde{P}(e_i, e_j) - \frac{\kappa'}{4} g'(e_i, e_j)\}.$$

Covariantly derivating (3.3) and using (2.5), (3.1) and (3.3), we have

$$X\tilde{\rho}\cdot g(Y, Z) = (\nabla'_X\nabla'_Y P - \nabla'_{\nabla'_X Y}P)Z - 2P(X)(\nabla'_Y P)Z - 2P(Y)(\nabla'_Z P)X$$
$$- 2P(Z)(\nabla'_X P)Y + 4P(X)P(Y)P(Z)$$
$$+ \frac{\kappa'}{4}\{2g'(Y, Z)P(X) + g'(X, Z)P(Y) + g'(X, Y)P(Z)$$
$$- g'(JX, Y)P(JZ) - g'(JX, Z)P(JY)\},$$

from which, we have

$$X\tilde{\rho}\cdot g(Y, Z) - Y\tilde{\rho}\cdot g(X, Z) = (K'(X, Y)P)Z + P(K'(X, Y)Z) = 0.$$

Hence we see that $\tilde{\rho}$ is constant. In this case, from (2.6), (2.8) and (3.3), $-4\tilde{\rho}$ is equal to the constant holomorphic sectional curvature κ of g. Therefore, we can obtain

Proposition 2. *Let (M, J, g') be an indefinite Kähler manifold of real dimension $2n(>2)$ and of constant holomorphic sectional curvature κ', and f a real-valued function on M. Assume that there exists a holomorphically projective change g of g' defined by* (2.1) *for the* 1*-form $P = df$.*
(*a*) *If the constant holomorphic sectional curvature κ of g does not vanish, then g and P satisfy*

$$\frac{\kappa}{4}g(X, Y) = \frac{\kappa'}{4}g'(X, Y) - (\nabla'_X P)Y + P(X)P(Y) - P(JX)P(JY), \tag{3.4}$$

$$(\nabla'_X P)JY + (\nabla'_Y P)JX = 2P(X)P(JY) + 2P(JX)P(Y), \tag{3.5}$$

$$(\nabla'_X\nabla'_Y P - \nabla'_{\nabla'_X Y}P)Z \tag{3.6}$$
$$= 2P(X)(\nabla'_Y P)Z + 2P(Y)(\nabla'_Z P)X + 2P(Z)(\nabla'_X P)Y$$
$$- 4P(X)P(Y)P(Z) - \frac{\kappa'}{4}\{2g'(Y, Z)P(X) + g'(X, Z)P(Y)$$
$$+ g'(X, Y)P(Z) - g'(JX, Y)P(JZ) - g'(JX, Z)P(JY)\},$$

(*b*) *if κ vanishes, then g and P satisfy*

$$(\nabla'_X g)(Y, Z) = 2P(X)g(Y, Z) + P(Y)g(X, Z) \tag{3.7}$$
$$+ P(Z)g(X, Y) - P(JY)g(JX, Z) - P(JZ)g(JX, Y),$$

$$(\nabla'_X P)Y - P(X)P(Y) + P(JX)P(JY) - \frac{\kappa'}{4}g'(X, Y) = 0 \tag{3.8}$$

for any $X, Y, Z \in \mathfrak{X}(M)$, where ∇' is the Levi-Civita connection induced from g'.

§4. Holomorphically projective changes of an indefinite complex Euclidean metric

Let $\boldsymbol{R}^{2n}$ be the real vector space of $2n$-tuples of real numbers $[x^i]$ and $\boldsymbol{C}^n$ the complex vector space of n-tuples of complex numbers $[z^p]$. From now on we identify $\boldsymbol{R}^{2n}$ with $\boldsymbol{C}^n$ by the correspondence

$$[x^i] = \begin{bmatrix} x^1 \\ \vdots \\ x^{2n} \end{bmatrix} \longrightarrow [z^p] = \begin{bmatrix} z^1 \\ \vdots \\ z^n \end{bmatrix} = \begin{bmatrix} x^1 + \sqrt{-1}x^{\bar{1}} \\ \vdots \\ x^n + \sqrt{-1}x^{\bar{n}} \end{bmatrix},$$

where $x^{\bar{p}} = x^{n+p}$. We denote by $(\boldsymbol{R}^{2n}, J_0, g_{2s})$ a complex vector space $\boldsymbol{R}^{2n} = \boldsymbol{C}^n$ endowed with the canonical complex structure J_0 and the canonical indefinite complex Euclidean metric g_{2s} of index $2s$, and call it an *indefinite complex Euclidean space of index* $2s$. Then. $(\boldsymbol{R}^{2n}, J_0, g_{2s})$ is of zero holomorphic sectional curvature, and in terms of the natural coordinate system $x^1, \cdots, x^{2n}$ of $\boldsymbol{R}^{2n}$, J_0 and g_{2s} have the following matrix representations:

$$J_0 = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \quad g_{2s} = \begin{bmatrix} -I_s & 0 & 0 & 0 \\ 0 & I_{n-s} & 0 & 0 \\ 0 & 0 & -I_s & 0 \\ 0 & 0 & 0 & I_{n-s} \end{bmatrix},$$

where I_p is an identity matrix of degree p.

We denote by g_{ij} the components of an indefinite Hermitian metric g on $(\boldsymbol{R}^{2n}, J_0, g_{2s})$ in $x^1, \cdots, x^{2n}$ and by $G = [G_{pq}]$ the Hermitian matrix defined by $G_{pq} = g_{pq} + \sqrt{-1}\, g_{\bar{p}q}$.

Assume that the differential $P = df$ of a real-valued function f on $(\boldsymbol{R}^{2n}, J_0, g_{2s})$ and an indefinite Hermitian metric g satisfy (a) or (b) in Proposition 2.

(1) The case where κ is not zero. In terms of $x^1, \cdots, x^{2n}$, (3.6) is rewritten as the system of differential equations

$$\text{(4.1)} \qquad \frac{\partial^3 f}{\partial x^i \partial x^j \partial x^k} = 2\left(\frac{\partial f}{\partial x^i}\frac{\partial^2 f}{\partial x^j \partial x^k} + \frac{\partial f}{\partial x^j}\frac{\partial^2 f}{\partial x^k \partial x^i} + \frac{\partial f}{\partial x^k}\frac{\partial^2 f}{\partial x^i \partial x^j} - 2\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\frac{\partial f}{\partial x^k}\right).$$

When we put $\phi = \pm \exp(-2f)$, from (4.1), we obtain

$$\frac{\partial^3 \phi}{\partial x^i \partial x^j \partial x^k} = 0.$$

Thus, we have

$$\phi = A_{ij}x^i x^j + 2A_i x^i + A,$$

where $A_{ij}(= A_{ji})$, A_i and A are real constants. Therefore, from (3.4), the components g_{ij} of g are given by

$$\text{(4.2)} \qquad g_{ij} = \frac{4}{\kappa\phi^2}\{\phi A_{ij} - (\delta_i^a \delta_j^b + J_i^a J_j^b)(A_{ac}x^c + A_a)(A_{bd}x^d + A_b)\}$$

where $I_{2n} = [\delta_j^i]$, and J_j^i are the components of J_0.

On the other hand, from (3.4) and (3.5), we see that g is Hermitian and

$A_{ab}J_i^aJ_j^b = A_{ij}$. Therefore, from (4.2), the components g_{ij} of g are expressed by

$$g_{ij} = (\delta_i^a\delta_j^b + J_i^aJ_j^b)\frac{\partial^2}{\partial x^a\partial x^b}\left(\frac{1}{\kappa}\log|\phi|\right), \tag{4.3}$$

where, for a scalar f, $|f|$ means its absolute value. In terms of the natural complex coordinate system $z^1, \cdots, z^n$ in $\boldsymbol{C}^n = \boldsymbol{R}^{2n}$, we have

$$\phi = \tilde{A}_{pq}\bar{z}^pz^q + \tilde{A}_p\bar{z}^p + \bar{\tilde{A}}_pz^p + A,$$

$$\begin{aligned} G_{pq} &= \frac{4}{\kappa\phi^2}\{\phi\tilde{A}_{pq} - (\tilde{A}_{pr}z^r + \tilde{A}_p)(\bar{\tilde{A}}_{qs}\bar{z}^s + \bar{\tilde{A}}_q)\} \\ &= \frac{\partial^2}{\partial\bar{z}^p\partial z^q}\left(\frac{4}{\kappa}\log|\phi|\right) \end{aligned} \tag{4.4}$$

where $\tilde{A}_{pq} = A_{pq} + \sqrt{-1}\,A_{\bar{p}q}$, $\tilde{A}_p = A_p + \sqrt{-1}\,A_{\bar{p}}$ and, for a complex-valued quantity Q, $\bar{Q}$ means its conjugate.

(2) The case where κ is zero. In this case, using $x^1, \cdots, x^{2n}$, (3.8) is expressed by the system of differential equations

$$\frac{\partial^2 f}{\partial x^i\partial x^j} = \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} - J_i^aJ_j^b\frac{\partial f}{\partial x^a}\frac{\partial f}{\partial x^b},$$

from which, we have

$$\left\{\begin{aligned} \frac{\partial^2 f}{\partial x^p\partial x^q} &= \frac{\partial f}{\partial x^p}\frac{\partial f}{\partial x^q} - \frac{\partial f}{\partial y^p}\frac{\partial f}{\partial y^q} = -\frac{\partial^2 f}{\partial y^p\partial y^q}, \\ \frac{\partial^2 f}{\partial x^p\partial y^q} &= \frac{\partial f}{\partial x^p}\frac{\partial f}{\partial y^q} + \frac{\partial f}{\partial y^p}\frac{\partial f}{\partial x^q} = \frac{\partial^2 f}{\partial y^p\partial x^q}, \end{aligned}\right. \tag{4.5}$$

where $y^p = x^{\bar{p}}$. When we put $h_p = 2\dfrac{\partial f}{\partial z^p}$, we see from (4.5) that h_p are holomorphic with respect to $z^p = x^p + \sqrt{-1}\,y^p$ and satisfy

$$\frac{\partial h_p}{\partial z^q} = h_ph_q,$$

from which, we have

$$h_p = -\frac{\bar{\tilde{B}}_p}{\bar{\tilde{B}}_qz^q + \bar{\tilde{B}}}, \tag{4.6}$$

where $\tilde{B}_p$ and $\tilde{B}$ are complex constants. From (4.6), we get

$$f = -\log|\bar{\tilde{B}}_pz^p + \bar{\tilde{B}}|.$$

If $[\tilde{B}_p]$ vanishes, then g is affinely related to g_{2s}, and the components g_{ij} of g and G_{pq} are given by

$$(4.7)\qquad g_{ij} = (\delta_i^a \delta_j^b + J_i^a J_j^b)\frac{\partial^2}{\partial x^a \partial x^b}\left(\frac{\phi}{4}\right),$$

$$G_{pq} = \frac{\partial^2}{\partial \bar{z}^p \partial z^q}\phi,$$

where ϕ is the quadratic polynomial used in (1).

Assume that $[\tilde{B}_p]$ does not vanish and let us consider the complex coordinate transformation $[z^p] \to [w^p]$ such that $w^p = C_q^p z^q + C^p$, $[C_q^p] \in GL(n, \boldsymbol{C})$, $C_p^1 = \bar{\tilde{B}}_p$ and $C^p = \delta_1^p \bar{\tilde{B}}$. When we denote by $[D_q^p]$ the inverse matrix of $[C_q^p]$ and put $H_{pq} = G_{rs}\bar{D}_p^r D_q^s$ and $\tilde{h}_p = h_q D_p^q$, from $G_{pq} = \bar{G}_{qp}$ and (3.7), we have

$$(4.8)\qquad \begin{cases} H_{pq} = \bar{H}_{qp}, \\ \dfrac{\partial H_{pq}}{\partial u^r} = \bar{\tilde{h}}_p H_{rq} + \tilde{h}_q H_{pr} + 2\dfrac{\partial f}{\partial u^r} H_{pq}, \\ \dfrac{\partial H_{pq}}{\partial v^r} = -\sqrt{-1}\,(\bar{\tilde{h}}_p H_{rq} - \tilde{h}_q H_{pr}) + 2\dfrac{\partial f}{\partial v^r} H_{pq}, \end{cases}$$

where $w^p = u^p + \sqrt{-1}\,v^p$. Since f is dependent only on u^1 and v^1, we see that $\tilde{h}_\alpha (\alpha = 2, \cdots, n)$ vanish and, from (4.8), we have

$$\frac{\partial H_{\alpha\beta}}{\partial u^1} = -\frac{2u^1}{(u^1)^2 + (v^1)^2} H_{\alpha\beta}, \quad \frac{\partial H_{\alpha\beta}}{\partial v^1} = -\frac{2v^1}{(u^1)^2 + (v^1)^2} H_{\alpha\beta},$$

$$\frac{\partial H_{\alpha\beta}}{\partial u^\gamma} = \frac{\partial H_{\alpha\beta}}{\partial v^\gamma} = 0,$$

from which, $H_{\alpha\beta}$ are given by

$$(4.9)\qquad H_{\alpha\beta} = \frac{E_{\alpha\beta}}{|w^1|^2},$$

where $E_{\alpha\beta}$ are complex constants and $E_{\alpha\beta} = \bar{E}_{\beta\alpha}$. Similarly, using (4.8) and (4.9), we have

$$(4.10)\qquad H_{\alpha 1} = -\frac{\bar{w}^1}{|w^1|^4}(E_{\alpha\beta} w^\beta - E_\alpha),$$

$$(4.11)\qquad H_{11} = \frac{1}{|w^1|^4}(E_{\alpha\beta}\bar{w}^\alpha w^\beta - E_\alpha \bar{w}^\alpha - \bar{E}_\alpha w^\alpha + E),$$

where E_α are complex constants and E is a real constant. When we put

$$(4.12)\qquad \begin{cases} \tilde{A}_{pq} = E_{\alpha\beta}\bar{C}_p^\alpha C_q^\beta, \ \tilde{A}_p = -E_\alpha \bar{C}_p^\alpha, \ A = E, \\ \phi = \tilde{A}_{pq}\bar{z}^p z^q + \tilde{A}_p \bar{z}^p + \bar{\tilde{A}}_p z^p + A, \\ \sigma = |w^1|^2 = |\bar{\tilde{B}}_p z^p + \bar{\tilde{B}}|^2 \end{cases}$$

from (4.9), (4.10) and (4.11), we obtain

$$G_{pq} = \frac{1}{\sigma^2}\{\sigma \tilde{A}_{pq} - (\tilde{B}_s \bar{z}^s + \tilde{B})(\tilde{A}_{pr} z^r + \tilde{A}_p)\bar{\tilde{B}}_q - (\bar{\tilde{B}}_s z^s + \bar{\tilde{B}})\tilde{B}_p(\bar{\tilde{A}}_{qr}\bar{z}^r + \bar{\tilde{A}}_q) + \phi \tilde{B}_p \bar{\tilde{B}}_q\} = \frac{\partial^2}{\partial \bar{z}^p \partial z^q}\left(\frac{\phi}{\sigma}\right). \tag{4.13}$$

In terms of $x^1, \cdots, x^{2n}$, from (4.12) and (4.13), σ and the components g_{ij} of g are given by

$$\begin{cases} \sigma = (B_a x^a + B_0)^2 + (B_a J^a_b x^b + B_{\bar{0}})^2, \\ g_{ij} = (\delta^a_i \delta^b_j + J^a_i J^b_j) \dfrac{\partial^2}{\partial x^a \partial x^b}\left(\dfrac{\phi}{4\sigma}\right), \end{cases} \tag{4.14}$$

where $\tilde{B}_p = B_p + \sqrt{-1}\, B_{\bar{p}}$, $\tilde{B} = B_0 + \sqrt{-1}\, B_{\bar{0}}$ and ϕ is the quadratic polynomial of $x^1, \cdots, x^{2n}$ induced from (4.12). Thus, from (4.3), (4.7) and (4.14), we can obtain the complex analogue of Theorem C:

Proposition 3. *In an indefinite complex Euclidean space $(\boldsymbol{R}^{2n}, J_0, g_{2s})$ of index $2s$, if an indefinite Hermitian metric g is an HP-change of g_{2s}, then in terms of the natural coordinate system $x^1, \cdots, x^{2n}$ of $\boldsymbol{R}^{2n}$, the components g_{ij} of g are locally expressed by*

$$g_{ij} = (\delta^a_i \delta^b_j + J^a_i J^b_j) \frac{\partial^2}{\partial x^a \partial x^b} f$$

and the function f is given by one of the followings:

$$(a) \qquad f = \frac{1}{\kappa} \log |\phi| \quad \textit{for} \quad \kappa \neq 0,$$

$$(b) \qquad f = \frac{\phi}{4\sigma} \quad \textit{for} \quad \kappa = 0,$$

$$(c) \qquad f = \frac{\phi}{4} \quad \textit{for} \quad \kappa = 0,$$

where κ is the constant holomorphic sectional curvature of g, $\phi = A_{ij}x^i x^j + 2A_i x^i + A$, $\sigma = (B_i x^i + B_0)^2 + (B_i J^i_j x^j + B_{\bar{0}})^2$, and A_{ij}, A_i, A, B_i, B_0, $B_{\bar{0}}$ are real constants such that $A_{ij} = A_{ji} = J^a_i J^b_j A_{ab}$, $[A_{ij}] \neq 0$, $[B_i] \neq 0$.

§5. The index of g

Let $\tilde{A} = [\tilde{A}_{pq}]$ be the Hermitian matrix taken in §4. Then there exists a unitary matrix U of degree n such that $U^* \tilde{A} U$ is equal to a diagonal matrix

$$D = \begin{bmatrix} -\lambda_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & -\lambda_\eta & & & & & & \\ & & & \lambda_{\eta+1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & \lambda_\zeta & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{bmatrix},$$

and in this case, $\tilde{A}$ is said to be *of singnature* $(\zeta-\eta, \eta)$, where $\lambda_1, \cdots, \lambda_\zeta$ are positive constants and U^* is the adjoint matrix of U. As is well known, in order to determine the index of the indefinite Hermitian metric g, it is sufficient to consider the index of the Hermitian matrix $G=[G_{pq}]$ induced from g in §4. From now on, $r(T)$, $s(T)$, $i(T)$ and $|T|$ mean the rank of a matrix T, the singnature of a symmetric (Hermitian) matrix T, the index of a non-singular symmetric (Hermitian) matrix T and the determinant of a square matrix T respectively. The above-mentioned notations are also used in §6.

(1) The case where κ is not zero and the quadratic hypersurface $\phi=0$ is central. We denote by $[k^i]$ the center of the quadratic hypersurface $\phi=0$ in $\boldsymbol{R}^{2n}$ and put $\tilde{k}^p = k^p + \sqrt{-1}k^{\bar{p}}$. Taking the complex coordinate transformation $[z^p] \to [\theta^p] = U^*([z^p]-[\tilde{k}^p])$, ϕ is expressed by

$$\phi = [\theta^p]^* D[\theta^p] + \phi_0, \tag{5.1}$$

where $\phi_0 = [\tilde{k}^p]^*\tilde{A}[\tilde{k}^p] + [\tilde{A}_p]^*[\tilde{k}^p] + [\tilde{k}^p]^*[\tilde{A}_p] + A$. Furthermore from (4.4), we have

$$U^*GU = \frac{4}{\kappa\phi^2}\{\phi D - D[\theta^p](D[\theta^p])^*\}. \tag{5.2}$$

From (5.1) and (5.2), we get

$$|F| = 0 \qquad \text{for} \quad \zeta < n,$$

$$|F| = (-1)^n \lambda_1 \cdots \lambda_n \phi_0 \phi^{n-1} \qquad \text{for} \quad \zeta = n,$$

where $F = U^*\left(\frac{\kappa\phi^2}{4}G\right)U$. Thus we have

Lemma 4. *G is non-singular if and only if* $\tilde{A}$ *is non-singular and both of* ϕ *and* ϕ_0 *do not vanish.*

Next, we shall consider the index of G. Suppose that G is non-singular and $i(\tilde{A}) = \eta$. When we denote by Λ the diagonal matrix

$$\begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix}$$

and by $\Phi(t)$ the characteristic polynomial of the matrix ${}^t\Lambda^{-1}F\Lambda^{-1}$, from (5.1) and (5.2), we have

$$\Phi(t) = (t-\phi)^{n-1}(t-\phi_0) \qquad \text{for} \quad \eta = 0,$$

$$\Phi(t) = (t+\phi)^{\eta-1}(t-\phi)^{n-\eta-1} \times \left(t^2 + t\sum_{p=1}^{n}\lambda_p\bar{\theta}^p\theta^p - \phi_0\phi\right) \qquad \text{for} \quad 0 < \eta < n,$$

$$\Phi(t) = (t+\phi)^{n-1}(t+\phi_0) \qquad \text{for} \quad \eta = n,$$

from which, by virtue of Sylvester's law of inertia, we obtain

Lemma 5. *Assume that G is non-singular and $i(\tilde{A}) = \eta$.*
(*a*) *If $\phi > 0$, then $i(F) = \eta\ (0 \leqq \eta \leqq n)$ or $\eta + 1\ (0 \leqq \eta < n)$ according as $\phi_0 > 0$ or $\phi_0 < 0$.*
(*b*) *If $\phi < 0$, then $i(F) = n - \eta + 1\ (0 < \eta \leqq n)$ or $n - \eta\ (0 \leqq \eta \leqq n)$ according as $\phi_0 > 0$ or $\phi_0 < 0$.*

(2) The case where κ is not zero and the quadratic hypersurface $\phi = 0$ is non-central. As is well known, when we take the complex coordinate transformation $[z^p] \to [\theta^p] = U_1^*[z^p] + [\tilde{a}^p]$ for a certain unitary matrix U_1 of degree n and a certain vector $[\tilde{a}^p] \in \boldsymbol{C}^n$, then ϕ is expressed by

$$\phi = [\theta^p]^* D[\theta^p] + [\tilde{b}^p]^*[\theta^p] + [\theta^p]^*[\tilde{b}^p], \tag{5.3}$$

where v is a positive constant and $\tilde{b}^p = v\delta^p_{\zeta+1}$. Furthermore from (4.4) and (5.3), we have

$$U_1^* G U_1 = \frac{4}{\kappa\phi^2}\{\phi D - (D[\theta^p] + [\tilde{b}^p])(D[\theta^p] + [\tilde{b}^p])^*\}. \tag{5.4}$$

From (5.3) and (5.4), we get

$$|F_1| = 0 \qquad \text{for} \quad \zeta < n-1,$$

$$|F_1| = (-1)^{\eta+1}\lambda_1\cdots\lambda_{n-1}v^2\phi^{n-1} \qquad \text{for} \quad \zeta = n-1,$$

where $F_1 = U_1^*\left(\dfrac{\kappa\phi^2}{4}G\right)U_1$. Therefore we have

Lemma 6. *G is non-singular if and only if ϕ does not vanish, $r(\tilde{A}) = n-1$ and*

$$r\left(\begin{bmatrix} A & [\tilde{A}_p]^* \\ [\tilde{A}_p] & \tilde{A} \end{bmatrix}\right) = n+1.$$

Next, we shall consider the index of G. On the same conditions as in Lemma 6, we assume that $s(\tilde{A}) = (n-\eta-1, \eta)$ and denote by Λ the diagonal matrix

$$\begin{bmatrix} \sqrt{\lambda_1} & & & \\ & \ddots & & \\ & & \sqrt{\lambda_{n-1}} & \\ & & & 1 \end{bmatrix}$$

and by $\Phi(t)$ the characteristic polynomial of the matrix ${}^t\Lambda^{-1}F_1\Lambda^{-1}$. Then from (5.3) and (5.4), we have

$$\Phi(t) = (t-\phi)^{n-2}\{t^2 + tv(v - \bar{\theta}^n - \theta^n) - v^2\phi\} \qquad \text{for} \quad \eta = 0,$$

$$\Phi(t) = (t+\phi)^{\eta-1}(t-\phi)^{n-\eta-2}\{t^3 + t^2(\sum_{p=1}^{n-1}\lambda_p\bar{\theta}^p\theta^p + v^2)$$

$$- tv\phi(\bar{\theta}^n + \theta^n) - v^2\phi^2\} \qquad \text{for} \quad 0 < \eta < n-1,$$

$$\Phi(t) = (t+\phi)^{n-2}\{t^2 + tv(v + \bar{\theta}^n + \theta^n) + v^2\phi\} \qquad \text{for} \quad \eta = n-1,$$

from which, we obtain

Lemma 7. *Suppose the same conditions as in Lemma* 6 *and* $s(\tilde{A}) = (n-\eta-1, \eta)$ $(0 \leqq \eta \leqq n-1)$. *Then* $i(F_1) = \eta + 1$ *or* $n - \eta$ *according as* $\phi > 0$ *or* $\phi < 0$.

(3) The case where κ is zero. Suppose that g is given by (b) in Proposition 3 and put

$$H_1 = \begin{bmatrix} A & [\tilde{A}_p]^* \\ [\tilde{A}_p] & \tilde{A} \end{bmatrix}, \quad H_2 = \begin{bmatrix} E & [E_\alpha]^* \\ [E_\alpha] & [E_{\alpha\beta}] \end{bmatrix}.$$

From (4.9), (4.10), (4.11) and (4.12), we have

$$G = \frac{1}{\sigma^2}[C_q^p]^* \begin{bmatrix} 1 & 0 \\ -[w^\alpha] & w^1 I_{n-1} \end{bmatrix}^* H_2 \begin{bmatrix} 1 & 0 \\ -[w^\alpha] & w^1 I_{n-1} \end{bmatrix} [C_q^p], \tag{5.5}$$

$$\tilde{A} = [C_q^p]^* \begin{bmatrix} 0 & 0 \\ 0 & [E_{\alpha\beta}] \end{bmatrix} [C_q^p],$$

$$H_1 = \begin{bmatrix} -1 & 0 \\ 0 & [C_q^p] \end{bmatrix}^* \begin{bmatrix} E & 0 & [E_\alpha]^* \\ 0 & 0 & 0 \\ [E_\alpha] & 0 & [E_{\alpha\beta}] \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & [C_q^p] \end{bmatrix},$$

from which, on the assumption that $[C_q^p]$ is non-singular, it follows that G is non-singular if and only if $\sigma \neq 0$ and H_2 is non-singular and in this case, by virtue of Sylvester's law of inertia,

$$s(G) = s(H_1) = s(H_2), \quad s(\tilde{A}) = s([E_{\alpha\beta}]).$$

On the other hand, since $n - 1 \geqq r([E_{\alpha\beta}]) \geqq r(H_2) - 2$, we only consider two cases where $[E_{\alpha\beta}]$ is of rank $n-2$ or $n-1$.

(3–1) The case where $r([E_{\alpha\beta}]) = n-1$. Then $r(\tilde{A}) = n-1$. If G is non-singular, then $r(H_1) = n$, the quadratic hypersurface $\phi = 0$ is central, and therefore we can use the same method as in (1). Furthermore assume that $s(\tilde{A}) = (n-\eta-1, \eta)$. Then taking the complex coordinate transformation $[z^p] \to [\theta^p] = U^*([z^p] - [\tilde{k}^p])$, we have

$$\phi = -\sum_{p=1}^{\eta} \lambda_p \bar{\theta}^p \theta^p + \sum_{p=\eta+1}^{n-1} \lambda_p \bar{\theta}^p \theta^p + \phi_0, \ \phi_0 \neq 0, \tag{5.6}$$

and $i(G) = \eta$ or $\eta + 1$ according as $\phi_0 > 0$ or $\phi_0 < 0$, where U, $[\tilde{k}^p]$ and ϕ_0 are the same quantities as in (1). On the other hand, from (4.13) and (5.6), we have

$$|G| = |U^*GU| = (-1)^\eta \sigma^{-n-1} \phi_0 \lambda_1 \cdots \lambda_{n-1} \bar{\tilde{B}}'_n \tilde{B}'_n,$$

$$\begin{vmatrix} 1 & 0 & [\tilde{B}_p]^* \\ 0 & A & [\tilde{A}_p]^* \\ [\tilde{B}_p] & [\tilde{A}_p] & \tilde{A} \end{vmatrix} = \left| \begin{bmatrix} I_2 & 0 \\ 0 & U \end{bmatrix}^* \begin{bmatrix} 1 & 0 & [\tilde{B}_p]^* \\ 0 & A & [\tilde{A}_p]^* \\ [\tilde{B}_p] & [\tilde{A}_p] & \tilde{A} \end{bmatrix} \begin{bmatrix} I_2 & 0 \\ 0 & U \end{bmatrix} \right|$$

$$= (-1)^{\eta+1} \phi_0 \lambda_1 \cdots \lambda_{n-1} \bar{\tilde{B}}'_n \tilde{B}'_n,$$

where $[\tilde{B}'_p] = U^*[\tilde{B}_p]$. Thus we get

Lemma 8. *Assume that $s(\tilde{A}) = (n-\eta-1, \eta)$. Then G is non-singular if and only if $\sigma \neq 0$, $r(H_1) = n$ and*

$$r\left(\begin{bmatrix} 1 & 0 & [\tilde{B}_p]^* \\ 0 & A & [\tilde{A}_p]^* \\ [\tilde{B}_p] & [\tilde{A}_p] & \tilde{A} \end{bmatrix}\right) = n+2.$$

In this case, $i(G) = \eta$ or $\eta + 1$ according as $\phi_0 > 0$ or $\phi_0 < 0$.

(3–2) The case where $r([E_{\alpha\beta}]) = n-2$. Then $r(\tilde{A}) = n-2$. If G is non-singular, then $r(H_1) = n$, the quadratic hypersurface $\phi = 0$ is non-central, and therefore we can use the same method as in (2). Furthermore assume that $s(\tilde{A}) = (n-\eta-2, \eta)$. Then taking a certain complex coordinate transformation $[z^p] \to [\theta^p] = U_1^*[z^p] + [\tilde{a}^p]$, we have

$$\phi = -\sum_{p=1}^{\eta} \lambda_p \bar{\theta}^p \theta^p + \sum_{p=\eta+1}^{n-2} \lambda_p \bar{\theta}^p \theta^p + \nu(\bar{\theta}^{n-1} + \theta^{n-1}), \tag{5.7}$$

and $i(G) = \eta + 1$, where U_1, $[\tilde{a}^p]$ and ν are a unitary matrix of degree n, a vector of $\boldsymbol{C}^n$ and a positive constant respectively. And from (4.13) and (5.7), we get

$$|G| = |U_1^* G U_1| = (-1)^{\eta+1}\sigma^{-n-1}\nu^2\lambda_1\cdots\lambda_{n-2}\bar{\bar{B}}'_n\tilde{B}'_n,$$

$$\begin{vmatrix} 1 & 0 & [\tilde{B}_p]^* \\ 0 & A & [\tilde{A}_p]^* \\ [\tilde{B}_p] & [\tilde{A}_p] & \tilde{A} \end{vmatrix} = \left| \begin{bmatrix} I_2 & 0 \\ 0 & U_1 \end{bmatrix}^* \begin{bmatrix} 1 & 0 & [\tilde{B}_p]^* \\ 0 & A & [\tilde{A}_p]^* \\ [\tilde{B}_p] & [\tilde{A}_p] & \tilde{A} \end{bmatrix} \begin{bmatrix} I_2 & 0 \\ 0 & U_1 \end{bmatrix} \right|$$

$$= (-1)^{\eta}\nu^2\lambda_1\cdots\lambda_{n-2}\bar{\bar{B}}'_n\tilde{B}'_n,$$

where $[\tilde{B}'_p] = U_1^*[\tilde{B}_p]$. Thus we get

Lemma 9. *Assume that* $s(\tilde{A}) = (n - \eta - 2, \eta)$. *Then G is non-singular if and only if* $\sigma \neq 0$, $r(H_1) = n$ *and*

$$r\left(\begin{bmatrix} 1 & 0 & [\tilde{B}_p]^* \\ 0 & A & [\tilde{A}_p]^* \\ [\tilde{B}_p] & [\tilde{A}_p] & \tilde{A} \end{bmatrix}\right) = n + 2.$$

In this case, $i(G) = \eta + 1$.

(4) If g is given by (c) in Proposition 3, then its components g_{ij} are equal to constants A_{ij}, and therefore $i(g) = i([A_{ij}])$.

§6. Theorems

From the argument in §5, we can obtain

Theorem 10. *Let* $(\boldsymbol{R}^{2n}, J_0, g_{2s})$ *be an indefinite complex Euclidean space of real dimention* $2n(>2)$ *with the canonical complex structure* J_0 *and the canonical indefinite complex Euclidean metric* g_{2s} *of index* $2s$, *and let* $x^1, \cdots, x^{2n}$ *be the natural coordinate system of* $\boldsymbol{R}^{2n}$. *If an indefinite Hermitian metric g is a holomorphically projective change of* g_{2s}, *then in terms of* $x^1, \cdots, x^{2n}$, *the components* g_{ij} *of g are locally expressed by*

$$g_{ij} = (\delta_i^a\delta_j^b + J_i^a J_j^b)\frac{\partial^2}{\partial x^a \partial x^b}f,$$

and satisfy one in Table 1, *where* J_j^i *are the components of* J_0.

Corollary 11. *On the same conditions as in Theorem* 10, *if a holomorphically projective change g of* g_{2s} *is positive-definite, then in terms of* $x^1, \cdots x^{2n}$, *the components* g_{ij} *of g are locally expressed by*

$$g_{ij} = (\delta_i^a\delta_j^b + J_i^a J_j^b)\frac{\partial^2}{\partial x^a \partial x^b}f$$

and satisfy one in Table 2. *In addition, if this metric g is globally defined on* $\boldsymbol{R}^{2n}$, *then g is of type* I_1 *or VI in Table* 2, *and its constant holomorphic sectional curvature is non-negative.*

Table 1

<table>
<tr><th>type</th><th>f</th><th>$\phi=0$</th><th>signature of $[A_{ij}]$</th><th>ϕ</th><th>ϕ_0</th><th>κ</th><th>index of g</th></tr>
<tr><td rowspan="4">I</td><td rowspan="12">$\frac{1}{\kappa}\log\vert\phi\vert$</td><td rowspan="8">central</td><td rowspan="8">$(2n-2\eta, 2\eta)$</td><td rowspan="2">$\phi>0$</td><td rowspan="2">$\phi_0>0$</td><td>$\kappa>0$</td><td>2η</td></tr>
<tr><td>$\kappa<0$</td><td>$2n-2\eta$</td></tr>
<tr><td rowspan="2">$\phi<0$</td><td rowspan="2">$\phi_0<0$</td><td>$\kappa>0$</td><td>$2n-2\eta$</td></tr>
<tr><td>$\kappa<0$</td><td>2η</td></tr>
<tr><td rowspan="2">II_1</td><td rowspan="2">$\phi>0$</td><td rowspan="2">$\phi_0<0$ $(\eta<n)$</td><td>$\kappa>0$</td><td>$2\eta+2$</td></tr>
<tr><td>$\kappa<0$</td><td>$2n-2\eta-2$</td></tr>
<tr><td rowspan="2">II_2</td><td rowspan="2">$\phi<0$</td><td rowspan="2">$\phi_0>0$ $(\eta>0)$</td><td>$\kappa>0$</td><td>$2n-2\eta+2$</td></tr>
<tr><td>$\kappa<0$</td><td>$2\eta-2$</td></tr>
<tr><td rowspan="2">III_1</td><td rowspan="4">non-central</td><td rowspan="4">$(2n-2\eta-2, 2\eta)$ $(\eta<n)$</td><td rowspan="2">$\phi>0$</td><td rowspan="2"></td><td>$\kappa>0$</td><td>$2\eta+2$</td></tr>
<tr><td>$\kappa<0$</td><td>$2n-2\eta-2$</td></tr>
<tr><td rowspan="2">III_2</td><td rowspan="2">$\phi<0$</td><td rowspan="2"></td><td>$\kappa>0$</td><td>$2n-2\eta$</td></tr>
<tr><td>$\kappa<0$</td><td>2η</td></tr>
<tr><td rowspan="2">IV</td><td rowspan="3">$\frac{\phi}{4\sigma}$</td><td rowspan="2">central</td><td rowspan="2">$(2n-2\eta-2, 2\eta)$ $(\eta<n)$</td><td rowspan="2"></td><td>$\phi_0>0$</td><td rowspan="4">$\kappa=0$</td><td>2η</td></tr>
<tr><td>$\phi_0<0$</td><td>$2\eta+2$</td></tr>
<tr><td>V</td><td>non-central</td><td>$(2n-2\eta-4, 2\eta)$ $(\eta<n-1)$</td><td></td><td></td><td>$2\eta+2$</td></tr>
<tr><td>VI</td><td>$\frac{\phi}{4}$</td><td>central</td><td>$(2n-2\eta, 2\eta)$</td><td></td><td></td><td>2η</td></tr>
</table>

κ is the constant holomorphic sectional curvature of g, $\phi=\phi(x^i)=A_{ij}x^ix^j + 2A_ix^i + A$, $\sigma=(B_ix^i+B_0)^2+(B_iJ^i_jx^j+B_{\bar{0}})^2 \neq 0$, $[k^i]$ is the center of a central quadratic hypersurface $\phi(x^i)=0$ and $\phi_0=\phi(k^i)$, where A_{ij}, A_i, A, B_i, B_0 and $B_{\bar{0}}$ are real constants such that $A_{ij}=A_{ji}=J^a_iJ^b_jA_{ab}$ and for the cases of types IV and V, the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & {}^t[B_i] \\ 0 & 1 & 0 & 0 & {}^t(J_0[B_i]) \\ 0 & 0 & A & 0 & {}^t[A_i] \\ 0 & 0 & 0 & A & {}^t(J_0[A_i]) \\ [B_i] & J_0[B_i] & [A_i] & J_0[A_i] & [A_{ij}] \end{bmatrix}$$

is non-singular.

Let g be the indefinite Kähler metric of type I, II or III mentioned in Theorem 10, which has nonzero constant holomorphic sectional curvature κ. We put $G_{pq} = g_{pq} + \sqrt{-1}g_{\bar{p}q}$ for the components g_{ij} of g and we shall consider $G=[G_{pq}]$.

Table 2

type	f	$\phi=0$	signature of $[A_{ij}]$	ϕ	ϕ_0	$\varkappa$
I_1	$\frac{1}{\varkappa}\log\lvert\phi\rvert$	central	$(2n, 0)$	$\phi>0$	$\phi_0>0$	$\varkappa>0$
			$(0, 2n)$	$\phi<0$	$\phi_0<0$	
I_2			$(0, 2n)$	$\phi>0$	$\phi_0>0$	$\varkappa<0$
			$(2n, 0)$	$\phi<0$	$\phi_0<0$	
II_1			$(2, 2n-2)$	$\phi>0$	$\phi_0<0$	
II_2			$(2n-2, 2)$	$\phi<0$	$\phi_0>0$	
III_1		non-central	$(0, 2n-2)$	$\phi>0$		
III_2			$(2n-2, 0)$	$\phi<0$		
IV	$\frac{\phi}{4\sigma}$	central	$(2n-2, 0)$		$\phi_0>0$	$\varkappa=0$
VI	$\frac{\phi}{4}$	central	(2n, 0)			

If we take the complex coordinate transformations $[z^p] \to [\theta^p]$ used in § 5 and $[\theta^p] \to [\omega^p]$ such that
for the case of type I,

$$\theta^p = \frac{\sqrt{|\kappa\phi_0|}}{\sqrt{2\lambda_p}}\,\omega^p,$$

for the case of type II_1,

$$\begin{cases} \theta^{\eta+1} = \dfrac{\sqrt{-2\phi_0}}{\sqrt{|\kappa|\lambda_{\eta+1}}\,\omega^{\eta+1}}, \\ \theta^p = \dfrac{\sqrt{-\phi_0}\,\omega^p}{\sqrt{\lambda_p}\,\omega^{\eta+1}} \quad (p \neq \eta+1), \end{cases}$$

for the case of type II_2,

$$\theta^\eta = \frac{\sqrt{2\phi_0}}{\sqrt{|\kappa|\lambda_\eta}\,\omega^\eta}, \quad \theta^p = \frac{\sqrt{\phi_0}\,\omega^p}{\sqrt{\lambda_p}\,\omega^\eta} \quad (p \neq \eta),$$

for the case of type III_1,

$$\begin{cases} \theta^p = \dfrac{\sqrt{2\nu|\kappa|}\,\omega^p}{\sqrt{\lambda_p}(\sqrt{2} - \sqrt{|\kappa|}\,\omega^{\eta+1})} \quad (p \leqq \eta), \\ \theta^p = \dfrac{\sqrt{2\nu|\kappa|}\,\omega^{p+1}}{\sqrt{\lambda_p}(\sqrt{2} - \sqrt{|\kappa|}\,\omega^{\eta+1})} \quad (\eta < p < n), \end{cases}$$

$$\theta^n = \frac{\sqrt{2} + \sqrt{|\kappa|}\omega^{n+1}}{\sqrt{2} - \sqrt{|\kappa|}\omega^{n+1}},$$

for the case of type III_2,

$$\begin{cases} \theta^p = \dfrac{\sqrt{2\nu|\kappa|}\,\omega^p}{\sqrt{\lambda_p}(\sqrt{2} - \sqrt{|\kappa|}\,\omega^n)} & (p < n), \\ \theta^n = -\dfrac{\sqrt{2} + \sqrt{|\kappa|}\omega^n}{\sqrt{2} - \sqrt{|\kappa|}\omega^n}, \end{cases}$$

then, for G of each type, we get the inequality

$$\phi_1 = 1 + \frac{\kappa}{2}\sum_{p=1}^{n} \varepsilon_p \bar{\omega}^p \omega^p > 0$$

after a change of ordering if necessary, and G_{pq} are transformed into an indefinite Fubini-Study metric

$$\begin{aligned} G_{rs}\frac{\overline{\partial z^r}}{\partial \omega^p}\frac{\partial z^s}{\partial \omega^q} &= \frac{2}{(\phi_1)^2}\{\phi_1\varepsilon_p\delta_{pq} - \frac{\kappa}{2}\varepsilon_p\omega^p\varepsilon_q\bar{\omega}^q\} \quad \text{(not summed for } p \text{ and } q) \\ &= \frac{\partial^2}{\partial\bar{\omega}^p\partial\omega^q}\left(\frac{4}{\kappa}\log\phi_1\right), \end{aligned}$$

where we put $i(g) = 2m$ and $\varepsilon_t = -1$ or 1 according as $t \leqq m$ or $t > m$.

Next, let g be an indefinite Kähler metric of type IV or V mentioned in Theorem 10, which has zero holomorphic sectional curvature. When we put $i(g) = 2m$, the Hermitian matrix H_2 in (5.5) is of index m and H_2 is induced from the canonical indefinite complex Euclidean metric g_{2m} of index $2m$ on $\boldsymbol{R}^{2n} = \boldsymbol{C}^n$ by a linear transformation of complex coordinates. When we consider the system of differential equations.

$$\left[\frac{\partial\omega^p}{\partial w^q}\right] = \frac{1}{(w^1)^2}\begin{bmatrix} 1 & 0 \\ -[w^\alpha] & w^1 I_{n-1} \end{bmatrix}$$

for holomorphic functions ω^p, we have the solutions

$$\omega^1 = -\frac{1}{w^1} + D^1, \ \omega^\alpha = \frac{w^\alpha}{w^1} + D^\alpha,$$

where $D^1, \cdots, D^n$ are complex constants. Thus, from (5.5), we see that the components of g are induced from those of g_{2m} by the linear fractional transformation of complex coordinates.

And if g is of type VI mentioned in Theorem 10 and $i(\tilde{A})=m$, then g is affinely related to g_{2s}, and thus the components of g are induced from those of g_{2m} by a linear transformation of complex coordinates.

Therefore from the argument above and Proposition 1, we can obtain the generalization of Bochner's Theorem D as follows:

Theorem 12. *Let (M, J, g) be an indefinite Kähler manifold of real dimension $2n(>2)$ and of index $2m$. If (M, J, g) is of constant holomorphic sectional curvature κ, then each point of M has a real coordinate neighborhood $\{U; x^1, \cdots, x^{2n}\}$ in which the components g_{ij} of g are expressed by*

$$g_{ij}=(\delta_i^a\delta_j^b+J_i^a J_j^b)\frac{\partial^2}{\partial x^a\partial x^b}f,$$

where f is given by

$$f=\frac{1}{\kappa}\log\left[1+\frac{\kappa}{2}\sum_{p=1}^{n}\varepsilon_p\{(x^p)^2+(x^{n+p})^2\}\right] \quad \textit{for} \quad \kappa\neq 0,$$

$$f=\frac{1}{4}\sum_{p=1}^{n}\varepsilon_p\{(x^p)^2+(x^{n+p})^2\} \quad \textit{for} \quad \kappa=0,$$

and $\varepsilon_t=-1$ or 1 according as $t\leqq m$ or $t>m$.

§7. Remarks

7–1. Projective changes of a Euclidean metric. Applying our method to a projective change of an indefinite real Euclidean metric, from Kagan's theorem, we can obtain the following result: Let $(\boldsymbol{R}^n, g_s)$ be an indefinite real Euclidean space of real dimension $n(>1)$ with the canonical indefinite Euclidean metric g_s of index s. If an indefinite Riemannian metric g is a projective change of g_s, then in terms of the natural coordinate system $x^1, \cdots, x^n$ of $\boldsymbol{R}^n$, the components g_{pq} of g are locally expressed by

(a) $$g_{pq}=\frac{1}{4\kappa\phi^2}\left\{2\phi\frac{\partial^2\phi}{\partial x^p\partial x^q}-\frac{\partial\phi}{\partial x^p}\frac{\partial\phi}{\partial x^q}\right\} \quad \textit{for} \quad \kappa\neq 0,$$

(b) $$g_{pq}=\frac{1}{2\psi}\frac{\partial^2}{\partial x^p\partial x^q}\left(\frac{\phi}{\psi}\right) \quad \textit{for} \quad \kappa=0,$$

(c) $$g_{pq}=\frac{\partial^2}{\partial x^p\partial x^q}\left(\frac{\phi}{2}\right) \quad \textit{for} \quad \kappa=0,$$

and satisfy one in Table 3, where κ is the constant curvature of g, $\phi=\phi(x^p)=A_{pq}x^px^q+2A_px^p+A$, $\psi=B_px^p+B$, $[k^p]$ is the center of a central quadratic hypersurface $\phi(x^p)=0$, $\phi_0=\phi(k^p)$, and $A_{pq}(=A_{qp})$, A_p, A, B_p, B are real constants such that $[A_{pq}]\neq 0$, $[B_p]\neq 0$ and for the case of (b) in Table 3, the matrix

Table 3

<table>
<tr><th>g_{pq}</th><th>$\phi=0$</th><th>signature of $[A_{pq}]$</th><th>ϕ</th><th>ϕ_0</th><th>κ</th><th>index of g</th></tr>
<tr><td rowspan="12">(a)</td><td rowspan="8">central</td><td rowspan="8">$(n-\eta,\ \eta)$</td><td rowspan="4">$\phi>0$</td><td rowspan="2">$\phi_0>0$</td><td>$\kappa>0$</td><td>η</td></tr>
<tr><td>$\kappa<0$</td><td>$n-\eta$</td></tr>
<tr><td rowspan="2">$\phi_0<0$ $(\eta<n)$</td><td>$\kappa>0$</td><td>$\eta+1$</td></tr>
<tr><td>$\kappa<0$</td><td>$n-\eta-1$</td></tr>
<tr><td rowspan="4">$\phi<0$</td><td rowspan="2">$\phi_0>0$ $(\eta>0)$</td><td>$\kappa>0$</td><td>$n-\eta+1$</td></tr>
<tr><td>$\kappa<0$</td><td>$\eta-1$</td></tr>
<tr><td rowspan="2">$\phi_0<0$</td><td>$\kappa>0$</td><td>$n-\eta$</td></tr>
<tr><td>$\kappa<0$</td><td>η</td></tr>
<tr><td rowspan="4">non-central</td><td rowspan="4">$(n-\eta-1,\ \eta)$ $(\eta<n)$</td><td rowspan="2">$\phi>0$</td><td rowspan="4"></td><td>$\kappa>0$</td><td>$\eta+1$</td></tr>
<tr><td>$\kappa<0$</td><td>$n-\eta-1$</td></tr>
<tr><td rowspan="2">$\phi<0$</td><td>$\kappa>0$</td><td>$n-\eta$</td></tr>
<tr><td>$\kappa<0$</td><td>η</td></tr>
<tr><td rowspan="3">(b)</td><td rowspan="2">central</td><td rowspan="2">$(n-\eta-1,\ \eta)$ $(\eta<n)$</td><td rowspan="2"></td><td>$\phi_0>0$</td><td rowspan="4">$\kappa=0$</td><td>η</td></tr>
<tr><td>$\phi_0<0$</td><td>$\eta+1$</td></tr>
<tr><td>non-central</td><td>$(n-\eta-2,\ \eta)$ $(\eta<n-1)$</td><td></td><td></td><td>$\eta+1$</td></tr>
<tr><td>(c)</td><td>central</td><td>$(n-\eta,\ \eta)$</td><td></td><td></td><td>η</td></tr>
</table>

$$\begin{bmatrix} 1 & 0 & {}^t[B_p] \\ 0 & A & {}^t[A_p] \\ [B_p] & [A_p] & [A_{pq}] \end{bmatrix}$$

is non-singular.

Furthermore, by the same method as in Theorem 12, we can obtain the generalization of Theorem B as follows: Let (M, g) be an indefinite Riemannian manifold of dimension $n(>1)$ and of index m. If (M, g) is of constant curvature κ, then each point of M has a coordinate neighborhood $\{U; y^1, \cdots, y^n\}$ in which the components g_{pq} of g are expressed by

$$g_{pq} = \frac{\{1 + \kappa \sum_{r=1}^{n} \varepsilon_r (y^r)^2\} \varepsilon_p \delta_{pq} - \kappa \varepsilon_p y^p \varepsilon_q y^q}{\{1 + \kappa \sum_{r=1}^{n} \varepsilon_r (y^r)^2\}^2} \quad \text{(not summed for } p \text{ and } q\text{)},$$

where $1 + \kappa \sum_{r=1}^{n} \varepsilon_r (y^r)^2 > 0$ and $\varepsilon_t = -1$ or 1 according as $t \leqq m$ or $t > m$.

7–2. Generalized changes of a Euclidean metric. Generalizing some changes of (indefinite) Riemannian metrics mentioned in § 1, we can consider the following problem: Let (M, g') be an indefinite Riemannian manifold of real dimension $n(>1)$ and ∇' the Levi-Civita connection induced from g'. When a symmetric tensor field T of type (1, 2) is given on (M, g'), does there exist an indefinite Riemannian metric g on (M, g') whose Levi-Civita connection ∇ satisfies

$$\nabla_X Y = \nabla'_X Y + T(X, Y) \tag{7.1}$$

for any $X, Y \in \mathfrak{X}(M)$?

It is well-known that, if g is another indefinite Riemannian metric on (M, g'), then there exists a unique symmetric tensor field T of type (1, 2) satisfying (7.1).

The changes of (indefinite) Riemannian metrics mentioned in § 1 are those which g' and T satisfy some conditions, and we have been able to solve (7.1) for such special g' and T. But for an arbitrary symmetric tensor field T of type (1, 2) given on (M, g'), there does not necessarily exist an indefinite Riemannian metric whose Levi-Civita connection satisfies (7.1).

It is easily seen that (7.1) is equivalent to

$$(\nabla'_X g)(Y, Z) = g(T(X, Y), Z) + g(Y, T(X, Z)) \tag{7.2}$$

and the integrability condition of (7.2) is

$$\begin{aligned} g(K'(X, Y)Z, W) + g(K'(X, Y)W, Z) &= g((\nabla'_Y T)(X, Z), W) - g((\nabla'_X T)(Y, Z), W) \\ &\quad + g(T(Y, T(X, Z)), W) - g(T(X, T(Y, Z)), W) \\ &\quad + g(Z, (\nabla'_Y T)(X, W)) - g(Z, (\nabla'_X T)(Y, W)) \\ &\quad + g(Z, T(Y, T(X, W))) - g(Z, T(X, T(Y, W))) \end{aligned} \tag{7.3}$$

where K' means the curvature tensor of g'. Our problem is to determine a non-degenerate symmetric solution g of (7.2).

Now we shall show two examples as follows: Let $(\boldsymbol{R}^n, g_0)$ be a real Euclidean space of real dimension $n(>1)$ with the canonical Euclidean metric g_0 and the natural coordinate system $x^1, \cdots, x^n$. We denote by $\left\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\right\}$ and $\{dx^1, \cdots, dx^n\}$ the natural frame and its dual frame respectively.

(1) Let T be a symmetric tensor field on $\boldsymbol{R}^n$ given by

$$T = \sum_{p=1}^{n} \frac{\partial}{\partial x^p} \otimes dx^p \otimes dx^p.$$

If a symmetric tensor field g of type (0, 2) on $\boldsymbol{R}^n$ satisfies (7.2) for this T, then the components g_{pq} of g satisfy

$$\frac{\partial g_{pq}}{\partial x^r} = (\delta_{rp} + \delta_{rq}) g_{pq} \quad \text{(not summed for } p \text{ and } q).$$

Solving the above system of differential equations, we get

$$g_{pq} = A_{pq}\exp(x^p + x^q) \quad \text{(not summed for } p \text{ and } q\text{)},$$

from which, it follows that g is non-degenerate if and only if $[A_{pq}]$ is non-singular, where $A_{pq} = A_{qp}$ are constants.

(2) Let T be a symmetric tensor field on $\boldsymbol{R}^n$ given by

$$T = \frac{\partial}{\partial x^1} \otimes (dx^1 \otimes dx^1 + dx^2 \otimes dx^2).$$

If a symmetric tensor field g of type (0, 2) on $\boldsymbol{R}^n$ satisfies (7.2) for this T, then from (7.3), we have

$$(X^1Y^2 - X^2Y^1)\left\{g\left(W, Z^2\frac{\partial}{\partial x^1}\right) + g\left(Z, W^2\frac{\partial}{\partial x^1}\right)\right\} = 0$$

for any $X, Y, Z, W \in \mathfrak{X}(\boldsymbol{R}^n)$, where $X = X^p\dfrac{\partial}{\partial x^p}$ and so on. Taking $X, Y, Z, W \in \mathfrak{X}(\boldsymbol{R}^n)$ such that

$$X^1Y^2 - X^2Y^1 \neq 0, \quad Z = \frac{\partial}{\partial x^2}, \quad W = \frac{\partial}{\partial x^r},$$

we obtain

$$g_{1r} = 0 \quad (r = 1, \cdots, n),$$

from which, it follows that g is degenerate and there exist no indefinite Riemannian metrics on $\boldsymbol{R}^n$ whose Levi-Civita connections satisfy (7.1) for this T.

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References

[1] M. Barros and A. Romero, Indefinite Kähler manifolds, Math. Ann., **261** (1982), 55–62.
[2] S. Bochner, Curvature in Hermitian metric, Bull. Amer. Math. Soc., **53** (1947), 179–195.
[3] L. P. Eisenhart, Riemannian Geometry, Princeton Univ. Press, 1964.
[4] L. P. Eisenhart, Non-Riemannian Geometry, Amer. Math. Soc., Colloq. Publ., 8, 1968.
[5] S. -S. Eum, Notes on Kaehlerian metric, Kyungpook Math. J., **1** (1958), 13–21.
[6] S. Fujimura, On changes of affine connections in an almost contact manifold, Tensor, New Ser., **38** (1982), 142–146.
[7] S. Fujimura, On changes of affine connections in an almost contact manifold, II, Tensor, New Ser., **41** (1984), 116–118.
[8] S. Fujimura, Some changes of metrics on a K-contact Riemannian manifold, Mem. Res. Inst. Sci. Eng., Ritsumeikan Univ., **45** (1986), 1–5.
[9] S. I. Goldberg, Note on projectively Euclidean Hermitian manifolds, Proc. Nat. Acad. Sci.

U.S.A., **42** (1956), 128–130.

[10] I. Hasegawa and K. Yamauchi, On infinitesimal holomorphically projective transformations in compact Kaehlerian manifolds, Hokkaido Math. J., **8** (1979), 214–219.

[11] N. S. Hawley, Constant holomorphic curvature, Canad. J. Math., **5** (1953), 53–56.

[12] J. Igusa, On the structure of a certain class of Kaehler varieties, Amer. J. Math., **76** (1954), 669–678.

[13] S. Ishihara, Holomorphically projective changes and their groups in an almost complex manifold, Tôhoku Math. J., **9** (1957), 273–297.

[14] S. Ishihara and S. Tachibana, A note on holomorphically projective transformations of a Kählerian space with parallel Ricci tensor, Tôhoku Math. J., **13** (1961), 193–200.

[15] B. Kagan, Über eine Erweiterung des Begriffes vom projektiven Räume und dem zugehörigen Absolut, Trudy Sem. Vektor. Tenzor. Anal., **1** (1933), 12–101.

[16] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, I, II, Interscience Publishers, New York, 1963, 1969.

[17] M. Kurita, Geodesic correspondence of Riemann spaces, J. Math. Soc. Japan, **8** (1956), 22–39.

[18] T. Levi-Civita, Sulle transformazioni delle equazioni dinamiche, Ann. Mat. Pura Appl., **24** (1896), 255–300.

[19] M. Matsumoto, Projectively flat Finsler spaces of constant curvature, J. Nat. Acad. Math. India, **1** (1983), 142–164.

[20] T. Ōtsuki and Y. Tashiro, On curves in Kaehlerian spaces, Math. J. Okayama Univ., **4** (1954), 57–78.

[21] T. Sakaguchi, On the holomorphically projective correspondence between Kählerian spaces preserving complex structure, Hokkaido Math. J., **3** (1974), 203–212.

[22] S. Tachibana, On an application of the stereographic projection to CP^m, Kyungpook Math. J., **12** (1972), 183–197.

[23] S. Tachibana and S. Ishihara, On infinitesimal holomorphically projective transformations in Kählerian manifolds, Tôhoku Math. J., **12** (1960), 77–101.

[24] Y. Tashiro, On a holomorphically projective correspondence in an almost complex space, Math. J. Okayama Univ., **6** (1957), 147–152.

[25] P. Venzi, Geodätische Abbildungen in riemannscher Mannigfaltigkeiten, Tensor, New Ser., **33** (1979), 313–321.

[26] K. Yano, Differential Geometry on Complex and Almost Complex Spaces, Pergamon Press, Oxford, 1964,

[27] Y. Yoshimatsu, H-projective connections and H-projective transformations, Osaka J. Math., **15** (1978), 435–459.