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# Theorems of Plessner and Riesz types for finely harmonic morphims

Dedicated to Professor Tatsuo Fuji'i'e on his sixtieth birthday

By

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## Introduction

In the previous paper [22], we investigated the behavior of non-negative finely superharmonic functions at the Martin boundary. In this paper, we study the behavior of finely harmonic morphisms at the Martin boundary. Namely, our main results are as follows;

**Theorem 1.** *Let $R$ be a hyperbolic Riemann surface, $R'$ a Riemann surface, $U$ a fine subdomain of $R$ and $\varphi: U \to R'$ a finely harmonic morphism. Then it holds that*

(*i*) *if $R'$ is hyperbolic or $R'-\varphi(U)$ is not polar, then $\varphi$ has a fine limit (see § 5) at almost every point of $\Delta_1(U)$ (see § 4) with respect to the harmonic measure $\omega_x (x \in R)$, and that*

(*ii*) *if $R'$ is parabolic (or compact) and $R'-\varphi(U)$ is polar, then the fine cluster set $\varphi^{\wedge}(\zeta)$ of $\varphi$* (see § 5) consists of a singleton or the Martin compactification *$R'^*_M$ of $R'$ at almost every point $\zeta$ of $\Delta_1(U)$ with respect to $\omega_x$ $(x \in R)$, where we put $R'^*_M = R'$ if $R'$ is compact.*

**Theorem 2.** *Let $R$, $R'$, $U$ and $\varphi$ be as above. If there exists a polar subset $N$ of $R'$ such that $\omega_z(\{\zeta \in \Delta_1(U): \varphi^{\wedge}(\zeta) \subset N\}) > 0$, then $\varphi$ is a constant mapping.*

Theorems 1 and 2 are regarded respectively as the theorems of Plessner and Riesz types for finely harmonic morphisms (cf. [3, Theorems 14.2 and 14.3] or [9, Theorems 7.1p $\sim$ 7.3]). For the proofs we make use of the probabilistic method which is a modification of Doob's one (cf. [7], [8] and [9]). In those proofs Theorem 3. 1 (see § 3) plays an important role.

In § 1 we provide some definitions and a result from fine potenetial theory. We introduce in § 2 a Brownian motion on a Riemann surface and give a stochastic characterization of finely harmonic morphims in § 3. In § 4 we introduce the conditional Brownian motion and state a stochastic characterization

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of fine neighborhoods at a minimal point of the Martin boundary. By using these results, we shall give the proofs of Theorems 1 and 2 in §5.

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## §1. Preliminaries

First we introduce the notations which will be used throughout this paper.

$R$, $R'$: arbitrary Riemann surfaces,

$R_M^*$: the Martin compactification of $R$ (if $R$ is compact, we put $R_M^* = R$),

$\Delta(R)$: the Martin boundary of $R$,

$\Delta_1(R)$: the totality of minimal points in $\Delta(R)$,

$k_\zeta$: the Martin function with pole at $\zeta \in \Delta_1(R)$ (only if $R$ is hyperbolic, this notation is used),

$\omega_z$: the harmonic measure on $\Delta_1(R)$ relative to $z \in R$ and $R$ (only if $R$ is hyperbolic, this notation is used),

$\bar{A}^*$: the closure of a subset $A$ of $R$ in $R_M^*$, and

$CA = R - A$ for a subset $A$ of $R$.

We refer to [3, Ch. 13] for the notion of Martin's compactification. We use the terminology "almost every" or "a.e." to mean "except on a null set with respect to $\omega_z$".

Next we state some notions and a result from fine potential theory. For that purpose we introduce into $R$ the weakest topology which makes all positive superharmonic functions in subdomains of $R$ continuous. Such a topology is called the fine topology (cf. [2, Ch. 1]). Throughout this paper, when "fine" or "finely" is used in a topological context, the topological object under discussion is considered in this fine topology, for example, finely open, fine neighborhood, etc. In addition, for a subset $A$ of $R$, we denote the fine interior and the fine closure of $A$ by $\mathrm{Int}_f A$ and $\mathrm{Cl}_f A$ respectively. For a finely open set $U$, we denote the fine boundary of $U$ by $\partial_f U$.

**Definition 1.1** (B. Fuglede [12, Definition 8.3 and Theorem 14. 1]). Let $U$ be a finely open subset of $R$. A finely continuous mapping $f$: $U \to \mathbf{R}$ is called to be *finely harmonic* in $U$ if for every $x \in U$, there exists a compact fine neighborhood $V$ of $x$ in $U$ such that $f$ is bounded on $V$ and that $f(z) = \int f \, d\varepsilon_z^{CV}$, for every $z \in \mathrm{Int}_f V$, where $\varepsilon_z^{CV}$ is the balayage of the Dirac measure $\varepsilon_z$ at $z$ on $CV$ (cf. [1, Ch. IV]).

Combining Fuglede's theorem [13, Theorem 4.1] with Debiard and Gaveau's theorem [5, Theorem 2], we obtain

**Theorem 1.1.** *Let $U$ be a finely open subset of $\mathbf{R}^2$ and $f$ a finely harmonic function in $U$. Then there exists an $\mathbf{R}^2$-valued function $\mathbf{h}$ in $U$ satisfying the following condition: for every $x \in U$, there exist a compact fine neighborhood $V$ of $x$ and a sequence $\{f_n\}_{n=1}^{+\infty}$ of harmonic functions in neighborhoods of $V$ such that*

*$\{f_n\}_{n=1}^{+\infty}$ converges uniformly to $f$ on $V$ and for every $z \in \mathrm{Int}_f V$, $\{\nabla f_n\}_{n=1}^{+\infty}$ converges strongly to* $\mathbf{h}$ *in $L^2(V, G_z^{\mathrm{Int}_f V}\, dv)$, where $G_z^{\mathrm{Int}_f V}$ is the fine Green's function for $\mathrm{Int}_f V$ with pole at $z \in \mathrm{Int}_f V$ (cf. [14]) and $dv$ is 2-dimensional Lebesgue measure.*

The above function $\mathbf{h}$ is independent of any choice of $V$ and $\{f_n\}_{n=1}^{+\infty}$. We call $\mathbf{h}$ the gradient of $f$ and denote it by $\nabla f = \{\partial f/\partial x_i\}_{i=1}^{2}$. Finally we state the definition of finely harmonic morphisms.

**Definition 1.2** (B. Fuglede [15]). A finely continuous mapping $\varphi$ from a finely open subset $U$ of $R$ into $R'$ is called a *finely harmonic morphism on $U$* if for any finely harmonic function $h$ in a finely open subset $W$ of $R'$, $h \circ \varphi$ is finely harmonic in $\varphi^{-1}(W)$.

## §2. Brownian motion on a Riemann surface

First we state the notion of Brownian motion on $R$ (cf. [25]). Let $\tilde{R}$ be the universal convering surface of $R$ with a natural projection $\pi$. Koebe's theorem (cf. [11, Ch. IV Theorem 4. 1]) states that $\tilde{R}$ is conformally equivalent to one of the following three surfaces, the unit disc $\mathbf{D}$, the complex plane $\mathbf{C}$ and the extended complex plane $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. We introduce into $\tilde{R}$ a Riemannian metric $\tilde{g} = \tilde{g}(z)|dz|^2$ ($z$ is a global coordinate in $\tilde{R} - \{\infty\}$) as follows:

$$\tilde{g}(z) = \frac{4}{(1-|z|^2)^2}, \quad \text{for } \tilde{R} = \mathbf{D},$$

$$\tilde{g}(z) = 1 \qquad , \quad \text{for } \tilde{R} = \mathbf{C},$$

$$\tilde{g}(z) = \frac{4}{(1+|z|^2)^2}, \quad \text{for } \tilde{R} = \hat{\mathbf{C}}.$$

For $\tilde{R} = \hat{\mathbf{C}}$, at the point of infinity, $\tilde{g}(\zeta) = \dfrac{4}{(1+|\zeta|^2)^2}$, in terms of the local coordinate $\zeta = 1/z$. Thus we can introduce into $R$ a Riemannian metric $g$ such that $\tilde{g}$ is the pull-back of $g$. Let $L_{\tilde{g}}$ and $L_g$ be the Laplace-Beltrami operators corresponding to $\tilde{g}$ and $g$ respectively, that is $L_{\tilde{g}} = \dfrac{4}{\tilde{g}(z)} \dfrac{\partial^2}{\partial z\, \partial \bar{z}}$ and $L_g = \dfrac{4}{g(\zeta)} \dfrac{\partial^2}{\partial \zeta\, \partial \bar{\zeta}}$ under the local coordinates $\zeta$ in $R$.

**Definition 2.1.** An $L_g$-diffusion process $\{B(t, x, \omega)\}_{t \geq 0}$ on $R$ starting at $x \in R$ (see [17, Ch. IV Definition 5.3]) is called a *Brownian motion on $R$ strating at $x$.*

Usually we denote $\{B(t, x, \omega)\}_{t \geq 0}$ by $\{B(t)\}_{t \geq 0}$ or $\{B(t, x)\}_{t \geq 0}$ in brief. Now we construct a Brownian motion on $R$. To do this we have only to construct a Brownian motion $\{\tilde{B}(t)\}_{t \geq 0}$ on $\tilde{R}$ starting at $\tilde{x}$ ($\pi(\tilde{x}) = x$), for the projection of $\{\tilde{B}(t)\}_{t \geq 0}$ under $\pi$ gives us the desired one. We first consider the case : $\tilde{x} \neq \infty$. Taking a global coordinate on $\tilde{R} - \{\infty\}$, we obtain a Brownian motion $\{\tilde{B}(t)\}_{t \geq 0}$ on $\tilde{R} - \{\infty\}$ starting at $\tilde{x}$ as a solution of the stochastic differential equation:

$$d\tilde{B}(t)=\frac{1}{(\tilde{g}(\tilde{B}(t)))^{1/2}}\,dW(t)\quad (*),$$

where $\{W(t)\}_{t\geq 0}$ is a complex Brownian motion starting at 0 which is identified with 2-dimensional Bwownian motion in $\mathbf{R}^2$ starting at 0 (cf. [10, p.1]). (*) is solvable by the method of random time change, that is, setting

$$\zeta(t)=x+W(t)\quad\text{and}\quad \Phi(t)=\int_0^t \tilde{g}(\xi(s))\,ds$$

gives us the solution : $\tilde{B}(t)=\xi(\Phi^{-1}(t))$ of (*). Since the life time of almost every Brownian path on $\tilde{R}-\{\infty\}$ is $+\infty$, we find that $\{\tilde{B}(t)\}_{t\geq 0}$ is a Brownian motion on $\tilde{R}$ starting at $\tilde{x}$. For $\tilde{x}=\infty$, using the local coordinate $\zeta=1/z$ leads us to the first case.

Next we state characterizations of some potential theoretic notions in terms of Brownian motion.

**Theorem 2.1** (cf. Debiard and Gaveau [4, Corollary of Lemma 1]). *Let $U$ be a compact subset of $R$ such that $\mathrm{Int}_f U$ is not empty, $\varepsilon_x^{CU}$ $(x\in \mathrm{Int}_f U)$ the balayage of the Dirac measure on $CU$, $\{B(t)\}_{t\geq 0}$ a Brownian motion on $R$ starting at $x$, and $\tau$ the first exit time from $U$, namely $\tau=\inf\{t>0;\,B(t)\notin U\}$. Then, $d\varepsilon_x^{CU}(\zeta)=P_x(B(\tau)\in d\zeta)$.*

Kakutani [18] discovered the relation between the type of $R$ and the behavior of almost every Brownian path on $R$.

**Theorem 2.2** (*cf.* [7, *Theorem* 13. 2]). *Let $\{B(t)\}_{t\geq 0}$ be a Brownian motion on $R$ starting at $x$ $(\in R)$ which is defined on $(\Omega,\mathscr{F},\mathscr{F}_t,P_x)$. Then the following two conditions are equivalent:*

(i) *$R$ is parabolic or compact.*

(ii) *for every $y\in R$, every open neighborhood $U$ of $y$ in $R$ and every positive number $M$, there exists a positive number $t(\omega)$ $(\geq M)$ such that $B(t(\omega),\omega)\in U$ a.s. (= almost everywhere with respect to $P_x$).*

For a hyperbolic Riemann surface, Doob showed

**Theorem 2.3** ([8, *Theorem* 10. 2]). *Let $R$ be a hyperbolic Riemann surface and $\{B(t)\}_{t\geq 0}$ a Brownian motion starting at $x$ $(\in R)$. Then there exists $\lim_{t\to+\infty}B(t)\in\Delta_1(R)$ a.s..*

We denote such a limit $\lim_{t\to+\infty}B(t)$ by $B(+\infty)$.

Finally we state the next two well-known lemmas:

**Lemma 2.1.** *Let $R$ be a hyperbolic Riemann surface and $\{B(t)\}_{t\geq 0}$ a Brownian motion on $R$ starting at $x$ $(\in R)$ which is defined on $(\Omega,\mathscr{F},\mathscr{F}_t,P_x)$. Then, $\omega_x(d\zeta)=P_x(B(+\infty)\in d\zeta)$.*

*Proof.* Let $f$ be a bounded continuous function on $\Delta(R)$, $\{R_n\}_{n=1}^{+\infty}$ a canonical exhaustion such that $x \in R_1$, and $\tau_n$ the first exist time of $\{B(t)\}_{t\geq 0}$ from $R_n$. By Uryson's theorem we can extend $f$ to $R$ as a bounded continuous function $F$ on $R_M^*$. By using the same argument as in [21, Theorem 2], we have $H_F^{R_n}(x) = E_x(F(B(\tau_n)))$, where $H_F^{R_n}$ is the Dirichlet solution of $F$ on $R_n$. Letting $n$ be infinity, by [3, Lemma 8.2, and Theorems 8.2 and 13.4] and Theorem 2.3, we have

$$H_f(x) = E_x(f(B(+\infty))),$$

where $H_f$ is the Dirichlet solution of $f$ on $R$. By the definition of $\omega_x$, we have the desired result. q.e.d.

**Lemma 2.2.** *Let $R$ be a parabolic (or compact) Riemann surface, $U$ a finely open subset of $R$, $\{B(t)\}_{t\geq 0}$ a Brownian motion on $R$ starting at $x \in U$ which is defined on $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x)$ and $\tau$ the first exit time of $\{B(t)\}_{t\geq 0}$ from $U$.* If $CU$ *is not polar, $\tau < +\infty$ a.s..*

*Proof.* If $CU$ is not polar, there exists a open subset $D$ of $R$ such that (i) $D \supset U$; (ii) $CD$ is compact in $R$; and (iii) $CD$ is not polar. Let $\{R_n\}_{n=1}^{+\infty}$ be a canonical exhaustion of $R$ such that $x \in R_1$, $\tau_n$ the first exit time of $\{B(t)\}_{t\geq 0}$ from $R_n$, and $\tau'$ the first exit time of $\{B(t)\}_{t\geq 0}$ from $D$. Putting $u_n(x) = P_x(\tau' < \tau_n)$ and $u(x) = P_x(\tau' < +\infty)$, we find that each $u_n$ is a equilibrium potential of $R_n - D$ in $R_n$ (cf. [1, Theorem 3.14]) and that $u$ is a equilibrium potential of $CD$ in $R$, for $u = \lim_{n\to+\infty} u_n$. Since $R$ is parabolic or compact, we find that $\tau' < +\infty$ a.s.. Therefore, we have the desired result because $\tau' \geq \tau$ a.s.. q.e.d.

## §3. A characterization of finely harmonic morphisms

In this section we suppose that $R$ is an arbitrary Riemann surface. First we give a characterization of finely harmonic functions. For this purpose we need the following notion:

**Definition 3.1** (cf. [10, §2.3]). Let $(\Omega, \mathscr{F}, P)$ be a probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$, $\tau$ a stopping time with respect to $\{\mathscr{F}_t\}_{t\geq 0}$ and $\{X_t\}_{0\leq t<\tau}$ a stochastic process on $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$. Then $\{X_t\}_{0\leq t<\tau}$ is called to be a *local martingale* with respect to $\{\mathscr{F}_t\}_{t\geq 0}$ if there exists a sequence $\{\tau_n\}_{n=1}^{+\infty}$ of stopping times satisfying the conditions:

( i ) for each $n$, $\tau_n < \tau$ a.s. ( = almost everywhere on $\Omega$ with respect to $P$);

(ii) $\{\tau_n\}_{n=1}^{+\infty}$ converges increasingly to $\tau$ a.s.;

(iii) each $\{X_{t\wedge\tau_n}\}_{t\geq 0}$ is a martingale with respect to $\{\mathscr{F}_{t\wedge\tau_n}\}_{t\geq 0}$, where $t \wedge \tau_n = \min\{t, \tau_n\}$ and $\mathscr{F}_{t\wedge\tau_n} = \{A \in \mathscr{F} : \{t \wedge \tau_n \leq s\} \cap A \in \mathscr{F}_s,$ for every $s \geq 0\}$.

By Theorems 1.1 and 2.1 and Itô's formula we have the following:

**Proposition 3.1.** *Let $U$ be a finely open subset of $R$ and a mapping $f\colon U \to \mathbf{R}$* finely continuous. Then the following two conditions are equivalent:

(*i*) *$f$ is finely harmonic in $U$;*

(ii) *for all* $x \in U$, *let* $\{B(t)\}_{t\geq 0}$ *be a Bwownian motion on* $R$ *starting at* $x$ *which is defined on a probability space* $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x)$, *and* $\tau$ *the first exit time of* $\{B(t)\}_{t\geq 0}$ *from* $U$. *Then* $\{f(B(t))\}_{0\leq t<\tau}$ *is a local martingale with respect to* $\{\mathscr{F}_t\}_{t\geq 0}$.

We refer to [21, Lemma 1] for the proof of this proposition. Next we obtain a stochastic characterization of finely harmonic morphisms after introducing a stochastic notion.

**Definition 3.2** (B. Øksendal [24]). Let $U$ be a finely open subset of $R$ and $\varphi: U \to R'$ a finely continuous mapping. Then we say that $\varphi$ *preserves the paths of Brownian motion* if, for every $x$ ($\in U$) and a Brownian motion $\{B(t)\}_{t\geq 0}$ on $R$ starting at $x$ which is defined on a probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x)$, the following conditions are fulfiled:

( i ) there exists a mapping $\sigma(t, \omega)$ ($= \sigma(t)$): $[0, +\infty] \times \Omega \to [0, +\infty]$ such that, for every $\omega \in \Omega$, $\sigma(*, \omega)$: $[0, +\infty] \to [0, +\infty]$ is continuous and strictly increasing and such that, for every $t$ ($\geq 0$), $\sigma(t, *)$: $\Omega \to [0, +\infty]$ is measurable with respect to $\mathscr{F}_{t\wedge\tau}$, where $\tau$ is he first exit time of $\{B(t)\}_{t\geq 0}$ from $U$;

( ii ) $\varphi^*(\omega) = \lim_{t\to\tau(\omega)-0} \varphi(B(t, \omega))$ exists a.s. on $\{\omega \in \Omega: \sigma(\tau(\omega), \omega) < +\infty\}$;

(iii) there exist a probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathscr{F}}_t, \hat{P})$ and a Brownian motion $\{A(t, \varphi(x), \omega, \hat{\omega})\ (= A(t))\}_{t\geq 0}$ on $R$ starting at $\varphi(x)$ which is defined on $(\Omega \times \hat{\Omega}, \mathscr{F} \times \hat{\mathscr{F}}, \mathscr{F}_{\sigma^{-1}(t)} \times \hat{\mathscr{F}}_t, P_x \times \hat{P})$ such that $A(t) = \varphi(B(\sigma^{-1}(t)))$ a.s. on $\{\sigma(\tau) > t\} \times \hat{\mathscr{F}}_t$, and such that $A(\sigma(\tau)) = \varphi^*$ a.s..

**Theorem 3.1.** *Let* $U$ *be a fine subdomain of* $R$ *and* $\varphi: U \to R'$ *a finely continuous mapping. Then* $\varphi$ *is a non-constant finely harmonic morphism on* $U$ *if and only if* $\varphi$ *preserves the paths of Brownian motion.*

*Proof.* Let $\tilde{R}$ and $\tilde{R}'$ be universal covering surfaces of $R$ and $R'$ with natural projections $\pi$ and $\pi'$ respectively, and $\tilde{U} = \pi^{-1}(U)$. Since any fine domain is arcwise connected (cf. [19]), we can consider a lift of $\varphi$ and denote it by $\tilde{\varphi}$. Since $\pi$ is analytic, we find that $\tilde{U}$ is a fine subdomain of $\tilde{R}$ and that $\tilde{\varphi}$ is a non-constant finely harmonic morphism on $U$ if and only if $\tilde{\varphi}$ is a non-constant finely harmonic morphism on $\tilde{U}$. We see from the construction of a Brownian motion on $R$ that $\varphi$ preserves the paths of Brownian motion if and only if $\tilde{\varphi}$ preserves the paths of Brownian motion. Hence, we have only to prove this theorem in replacing $R$, $R'$, $U$ and $\varphi$ by $\tilde{R}$, $\tilde{R}'$, $\tilde{U}$ and $\tilde{\varphi}$ respectively. Suppose that $\tilde{\varphi}$ is a non-constant finely harmonic morphism on $\tilde{U}$. If $\tilde{U} \subset \mathbf{C}$, $\tilde{\varphi}(U) \subset \mathbf{C}$, $\tilde{g} = 1$ and $\tilde{g}' = 1$, we see from B. Øksendal [24, Theorem 1] or Masaoka [20, Main Theorem] that $\tilde{\varphi}$ preserves the paths of Brownian motion. Hence, by Proposition 3.1 and the construction of a Brownian motion of $R$ in §2, we obtain the desired result.

Next we suppose that $\tilde{\varphi}$ preserves the paths of Brownian motion. Let $u$ be a finely harmonic function in a finely open subset $W$ of $\tilde{R}'$. Since $\tilde{\varphi}$ preserves the paths of Brownian motion, $\tilde{\varphi}$ is finely continuous in $\tilde{\varphi}^{-1}(W)$, for every $x$ ($\in \tilde{\varphi}^{-1}(W)$), there exists a compact fine neighborhood $U(x)$ ($\subset \tilde{\varphi}^{-1}(W)$) of $x$ such that $u \circ \tilde{\varphi}$ is bounded on $U(x)$. To check the integral equation in Definition 1.1

for $u \circ \tilde{\varphi}$, let $(\tilde{B}(t))_{t \geq 0}$ be a Brownian motion on $\tilde{R}$ starting at $z$ $(\in \mathrm{Int}_f U(x))$ which is defined on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathscr{F}}_t, \tilde{P}_z)$ and $\tilde{\tau}$ the first exit time of $\{\tilde{B}(t)\}_{t \geq 0}$ from $U(x)$. Then we see from Proposition 3.1 and the optional sampling theorem [6, Ch. VI Theorem 15] that $\{(u \circ \tilde{\varphi})(\tilde{B}(\sigma^{-1}(t) \wedge \tilde{\tau}))\}_{t \geq 0}$ is a martingale with respect to $\{\mathscr{F}_{\sigma(t)^{-1} \wedge \tilde{\tau}}\}_{t \geq 0}$, where $\sigma(t)$ is the same function as in Definition 2.1. By Lebesgue's bounded convergence theorem and Theorem 2.1, we have

$$\begin{aligned}(u \circ \tilde{\varphi})(z) &= \lim_{t \to +\infty} E_z((u \circ \tilde{\varphi})(\tilde{B}(\sigma^{-1}(t) \wedge \tilde{\tau}))) \\ &= E_z((u \circ \tilde{\varphi})(\tilde{B}(\tilde{\tau}))) \\ &= \int u \circ \tilde{\varphi}\, d\varepsilon_z^{CU(x)}, \quad \text{for all} \quad z \in \mathrm{Int}_f U(x). \qquad \text{q.e.d.}\end{aligned}$$

## § 4. A stochastic characterization of finely open neighborhoods at a minimal point

In this section we suppose that $R$ is hyperbolic. First we state several definitions.

**Definition 4.1** (cf. [23, Theorem 5]). For a point $\zeta \in \Delta_1(R)$ and a subset $A$ of $R$, $A$ is called to be *thin* at $\zeta$ if $\hat{R}_{k_\zeta}^A \neq k_\zeta$, where $\hat{R}_{k_\zeta}^A$ is te balayage of $k_\zeta$ on $A$, that is

$$\hat{R}_{k_\zeta}^A(z) = \liminf_{x \to z} \inf\{s(x)\colon s \text{ is non-negative superharmonic in } R \text{ and } s \geq k_\zeta \text{ on } A\}.$$

**Definition 4.2** (cf. [3, p. 145]). For a point $\zeta \in \Delta_1(R)$ and a finely open subset $U$ of $R$, $U \cup \{\zeta\}$ is called a *finely open neighborhood of* $\zeta$ if $CU$ is thin at $\zeta$. We denote by $\mathscr{G}_\zeta$ the totality of finely open subsets $U$ of $R$ such that $U \cup \{\zeta\}$ is a finely open neighborhood of $\zeta$.

**Definition 4.3.** For a finely open subset $U$ of $R$, we define $\Delta_1(U)\colon = \{\zeta \in \Delta_1(R)\colon U \in \mathscr{G}_\zeta\}$.

**Definition 4.4** (cf. [8] and [10, Ch. 3]). Let $\{p(t, z, dy)\}$ be the transition probability of a Brownian motion $\{B(t)\}_{t \geq 0}$ on $R$ starting at $x$ $(\in R)$ which is defined on a probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x)$. If a diffusion process $\{B^\zeta(t)\}_{t \geq 0}(\zeta \in \Delta_1(R))$ has the transition probability $\left\{p^\zeta(t, z, dy)\colon = \frac{k_\zeta(y)}{k_\zeta(z)} p(t, z, dy)\right\}$, $\{B^\zeta(t)\}_{t \geq 0}$ is called a *Brownian motion on R starting at x conditioned to exit R at* $\zeta$.

In details we refer to [8] or [10, Ch. 3] for a Brownian motion on $R$ starting at $x$ conditioned to exit $R$ at $\zeta$. By Doob [8, Theorem 14.2], we have the following characterization of finely open neighborhoods of a minimal point:

**Theorem 4.1.** *Let $U$ be a finely open subset of $R$, $\{B^\zeta(t)\}_{t \geq 0}(\zeta \in \Delta_1(U))$ a Brownian motion on $R$ starting at $x$ $(\in U)$ conditioned to exit $R$ at $\zeta$ and $\tau^\zeta$ the first exit time of $(B^\zeta(t))_{t \geq 0}$ from $U$. Then, if we take an arbitrary finely open set $V$ $(\in \mathscr{G}_\zeta)$, there exists a positive number $\delta(\omega)$ such that $B^\zeta(t, \omega) \in V \cap U$ for $t \geq \delta(\omega)$ a.s. on $\{\tau^\zeta(\omega) = +\infty\}$.*

## § 5. Proofs of Theorems 1 and 2

In this section we suppose that $R$ is hyperbolic. First we introduce the notion of fine cluster sets.

**Definition 5.1.** Let $U$ be a finely open subset of $R$, and $\varphi: U \to R'$ a finely continuous mapping. Then we define *the fine cluster set* $\varphi^{\wedge}(\zeta)$ of $\varphi$ at $\zeta\,(\in \Delta_1(U))$ as follows: $\varphi^{\wedge}(\zeta) = \bigcap_{V \in \mathscr{G}_\zeta} \overline{\varphi(V \cap U)^*}$. In particular, if $\varphi^{\wedge}(\zeta)$ consists of a singleton, we say that $\varphi$ has a fine limit at $\zeta$.

**Lemma 5.1.** *Let $U$ be a fine subdomain of $R$, $\{B(t, x)\,(= B(t))\}_{t \geq 0}\,(x \in U)$ a Brownian motion on $R$ starting at $x$ which is defined on a probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x)$, $\tau(x)\,(= \tau)$ the first exit time of $\{B(t, x)\}_{t \geq 0}$ from $U$, $\mu_x$ the measure defined on $\Delta(R)$ by $\mu_x(E) = \omega_x(E \cap \Delta_1(U))$ for every Borel subset $E$ of $\Delta(R)$, and $\nu_x$ the measure defined on $\Delta(R)$ by $\nu_x(E) = P_x(B(+\infty) \in E \cap \Delta_1(U), \tau = +\infty)$ for every Borel subset $E$ of $\Delta(R)$. Then $\mu_x$ is absolutely continuous with respect to $\nu_x$.*

*Proof.* We may suppose that $\partial_f U$ consists of only regular points since the totality of irregular points in $\partial_f U$ is a polar set. By [22, Lemma 5.3], we can take a finely open subset $U_1$ of $U$ such that (i) $\mathrm{Cl}_f U_1 \subset U$; (ii) $\partial_f U_1$ consists of only regular points; and (iii) $\omega_x(\Delta_1(U) - \Delta_1(U_1)) = 0$. Let $\tau'(z)\,(z \in \partial_f U)$ be the first exit time of a Brownian motion $\{B(t, z)\}_{t \geq 0}$ on $R$ starting at $z$ from $C(\mathrm{Cl}_f U_1)$. Since $U$ and $C(\mathrm{Cl}_f U_1)$ are nearly Borel sets with respect to a Brownian motion on $R$ (cf. [2, Proposition VII, 8] and [17, Theorem 4.2.2 and 4.3.1]), $\tau(x)$ and $\tau'(z)$ are stopping times with respect to $\{\mathscr{F}_t\}_{t \geq 0}$. We define inductively sequences $\{\sigma_n\}_{n=1}^{+\infty}$ and $\{\delta_n\}_{n=1}^{+\infty}$ of stopping times with respect to $\{\mathscr{F}_t\}_{t \geq 0}$ as follows:

$$\sigma_1 = \tau,$$

$$\delta_1 = \begin{cases} \sigma_1 + \tau'(B(\sigma_1)) \circ \theta_{\sigma_1} & \text{on } \{B(\sigma_1) \in \partial_f U\} \\ +\infty & \text{on } \Omega - \{B(\sigma_1) \in \partial_f U\}, \end{cases}$$

$$\sigma_{n+1} = \begin{cases} \delta_n + \tau(B(\delta_n)) \circ \theta_{\delta_n} & \text{on } \{B(\delta_n) \in \mathrm{Cl}_f U_1\} \\ +\infty & \text{on } \Omega - \{B(\delta_n) \in \mathrm{Cl}_f U_1\}, \end{cases}$$

$$\delta_{n+1} = \begin{cases} \sigma_{n+1} + \tau'(B(\sigma_{n+1})) \circ \theta_{\sigma_{n+1}} & \text{on } \{B(\sigma_{n+1}) \in \partial_f U\} \\ +\infty & \text{on } \Omega - \{B(\sigma_{n+1}) \in \partial_f U\}, \end{cases}$$

where, for a stopping time $\sigma$ we denote the sift operator by $\theta_\sigma$ (cf. [1, pp. 136, 137 and 155]). By Lemma 2.1 and Theorem 4.1, and the strong Markov property, we have

$$(*)\quad \mu_x(E) = P_x(B(+\infty)\in E\cap\Delta_1(U))$$
$$= \sum_{n=1}^{+\infty} P_x(B(+\infty)\in E\cap\Delta_1(U),\ \sigma_n = +\infty)$$
$$= \nu_x(E) + \sum_{n=2}^{+\infty} E_x(\nu_{B(\delta_{n-1})}(E) \colon \delta_{n-1} < +\infty)$$

Suppose that there exists $x$ $(\in U)$ such that $\nu_x(E) = 0$. Putting $f(z) = \nu_z(E)$, we see from the strong Markov property and [12, Theorem 14. 6] that $f$ is a non-negative finely harmonic in $U$. Hence, by the minimun principle (cf. [12, Theorem 12.6]), $f$ is identically zero in $U$. Therefore, by $(*)$, $\mu_x(E) = 0$. q.e.d.

Next we prove Theorems 1 and 2.

The proof of Theorem 1.

*The proof of* (*i*). Let $\{B(t)\}_{t\geq 0}$ be a Brownian motion on $R$ starting at $x$ $(\in U)$ which is defined on $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x)$ and $\tau$ the first exit time of $\{B(t)\}_{t\geq 0}$ from $U$. First we show that there exists $\lim_{t\to+\infty} \varphi(B(t))$ $(\in R'\cup\Delta_1(R'))$ a.s. on $\{\tau = +\infty\}$. By Theorem 3.1 we can define $\sigma(t)$ and $\{A(t, \varphi(x), \omega, \hat{\omega})\}_{t\geq 0}$ as in Definition 3.2. Let $\tau'(\varphi(x), \omega, \hat{\omega})$ $(= \tau'(\omega, \hat{\omega}))$ be the first exit time of $\{A(t)\}_{t\geq 0}$ from $\varphi(U)$. Since $\sigma(\tau(\omega), \omega) \leq \tau'(\omega, \hat{\omega})$ a.s. on $\{\tau = +\infty\} \times \hat{\mathscr{F}}$, we find that there exists $\lim_{t\to+\infty} \varphi(B(t))$ $(\in R'\cup\Delta_1(R'))$ a.s. on $\{\tau = +\infty\}$. In fact, if $R'$ is parabolic (or compact) and $C(\varphi(U))$ is not polar, this fact follows from Theorem 3.1 and Lemma 2.2. If $R'$ is hyperbolic, this fact follows from Theorems 2.3 and 3.1. Let $\{B^{\zeta}(t, x, \omega)\}_{t\geq 0}$ be a Brownian motion on $R$ starting at $x$ conditioned to exit $R$ at $\zeta$ $(\in\Delta_1(U))$ and $\tau^{\zeta}$ the first exit time of $\{B^{\zeta}(t)\}_{t\geq 0}$ from $U$. Then, by Lemma 2.1 and [10, p.96 (4)], we have

$$\int_{\Delta_1(U)} P_x^{\zeta}(\tau^{\zeta} = +\infty)\ \omega_x(d\zeta)$$
$$= P_x(B(+\infty)\in\Delta_1(U),\ \tau = +\infty)$$
$$= P_x(\text{There exists } \lim_{t\to+\infty} \varphi(B(t))\in R'\cup\Delta_1(R'),\ B(+\infty)\in\Delta_1(U) \text{ and } \tau = +\infty)$$
$$= \int_{\Delta_1(U)} P_x^{\zeta}(\text{There exists } \lim_{t\to+\infty} \varphi(B^{\zeta}(t))\in R'\cup\Delta_1(R') \text{ and } \tau^{\zeta} = +\infty)\,\omega_x(d\zeta).$$

Thus, we find that, at a.e. $\zeta\in\Delta_1(U)$, there exists $\lim_{t\to+\infty} \varphi(B^{\zeta}(t))$ $(\in R'\cup\Delta_1(R'))$ a.s. on $\{\tau^{\zeta} = +\infty\}$. We consider such a point $\zeta\in\Delta_1(U)$. To prove (i), we have only to prove that $\varphi\hat{\ }(\zeta)$ is a singleton. We assume that $\varphi\hat{\ }(\zeta)\cap R' \neq \phi$. For the remaining case, using the same argument as in the following proof, we have the desired result. Let $\zeta'$ be a point of $\varphi\hat{\ }(\zeta)\cap R'$. For an arbitrary finely open set $V$ $(\in\mathscr{G}_{\zeta})$ and an arbitrary open neighborhood $D$ of $\zeta'$, $D\cap\varphi(V) \neq \phi$, that is

$\varphi^{-1}(D)\cap V\neq\phi$. Here, suppose that $\varphi^{-1}(D)$ is thin at $\zeta$. Since $\hat{R}_{k_\zeta}^{\mathrm{Cl}_f(\varphi^{-1}(D))} = \hat{R}_{k_\zeta}^{\varphi^{-1}(D)}$ (cf. [1, Ch. VI Lemma 4.3]), we find that $\mathrm{Cl}_f(\varphi^{-1}(D))$ is thin at $\zeta$, that is $C(\mathrm{Cl}_f(\varphi^{-1}(D)))\in\mathscr{G}_\zeta$. This is a contradiction. Thus $\varphi^{-1}(D)$ is not thin at $\zeta$. Hence [8, Theorem 14. 2] states that, for any positive number $M$, there exists $t(\omega)$ $(\geq M)$ such that $\varphi(B^\zeta(t(\omega),\omega))\in D$ a.s. on $\{\tau^\zeta=+\infty\}$. Since $D$ is an arbitrary open neighborhood of $\zeta'$ and there exists $\lim_{t\to+\infty}\varphi(B^\zeta(t))$ $(\in R'\cup\Delta_1(R'))$ a.s. on $\{\tau^\zeta=+\infty\}$, we find that there exists $\lim_{t\to+\infty}\varphi(B^\zeta(t))=\zeta'$ a.s. on $\{\tau^\zeta=+\infty\}$. Therefore, $\varphi\hat{}(\zeta)=\{\zeta'\}$.

*The proof of* (*ii*). Suppose that $R'$ is parabolic (or compact) and $C(\varphi(U))$ is polar. Let $\{B(t)\}_{t\geq 0}$ be a Brownian motion on $R$ starting at $x$ $(\in U)$ which is defined on $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x)$ and $\tau$ the first exit time of $\{B(t)\}_{t\geq 0}$ from $U$. By Theorem 3.1, we can define $\sigma(t)$ as in Definition 3.2. Since $R'$ is parabolic or compact, by Theorems 2.2 and 3.1 we find that (i) $\bigcap_{t\geq 0}\overline{\bigcup_{s\geq t}\{\varphi(B(s))\}^*}=R'^*_M$ a.s. on $\{\tau=+\infty,\ \sigma(\tau)=+\infty\}$; and (ii) there exists $\lim_{t\to+\infty}\varphi(B(t))$ $(\in R')$ a.s. on $\{\tau=+\infty,\ \sigma(\tau)<+\infty\}$. Let $\{B^\zeta(t)\}_{t\geq 0}$ $(\zeta\in\Delta_1(U))$ be a Brownian motion on $R$ starting at $x$ conditioned to exit $R$ at $\zeta$ which is defined on $(\Omega, \mathscr{F}, \mathscr{F}_t, P^\zeta_x)$ and $\tau^\zeta$ the first exit time of $\{B^\zeta(t)\}_{t\geq 0}$ from $U$. By using the same argument as in the proof of Theorem 1 (i), we find that, at a.e. $\zeta\in\Delta_1(U)$, (i) $\bigcap_{t\geq 0}\overline{\bigcup_{s\geq t}\{\varphi(B^\zeta(s))\}^*}=R'^*_M$ a.s. on $\{\tau^\zeta=+\infty,\ \sigma(\tau^\zeta)=+\infty\}$; and (ii) there exists $\lim_{t\to+\infty}\varphi(B^\zeta(t))$ $(\in R')$ a.s. on $\{\tau^\zeta=+\infty,\ \sigma(\tau^\zeta)<+\infty\}$. We take such a point $\zeta\in\Delta_1(U)$. Then, if $P^\zeta_x(\tau^\zeta=+\infty,\ \sigma(\tau^\zeta)=+\infty)>0$, by Theorem 4.1, we find that $\varphi\hat{}(\zeta)=R'^*_M$. If $P^\zeta_x(\tau^\zeta=+\infty,\ \sigma(x^\zeta)=+\infty)=0$, by using the same argument as in the proof of Theorem 1 (i), we find that $\varphi\hat{}(\zeta)$ consists of a singleton. q.e.d.

*The proof of Theorem* 2. Suppose that $\varphi$ is not a constant mapping on $U$. Let $\{B(t)\}_{t\geq 0}$ be a Brownian motion on $R$ starting at $x$ $(\in U)$ which is defined on $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x)$ and $\tau$ the first exit time of $\{B(t)\}_{t\geq 0}$ from $U$. By Theorem 3.1, we can define $\sigma(t)$ as in Definition 3.2. If $\varphi\hat{}(\zeta)\subset N$, we see from Theorem 1 that $\varphi$ has a fine limit at $\zeta$. Hence, by Lemma 5.1, the argument in the proof of Theorem 1 and the assumption of this theorem, we find that $P_x(\lim_{t\to+\infty}\varphi(B(t))\in N,\ \tau=+\infty,\ \sigma(\tau)<+\infty)>0$. On the other hand, by Theorem 3.1 we find that $P_x(\lim_{t\to+\infty}\varphi(B(t))\in N,\ \tau=+\infty,\ \sigma(\tau)<+\infty)=0$, since $N$ is polar. This is a contradiction. q.e.d.

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