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Level complexes and barycentric subdivisions

By

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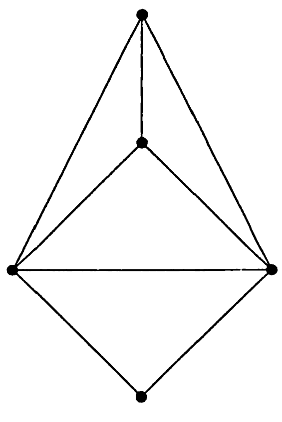
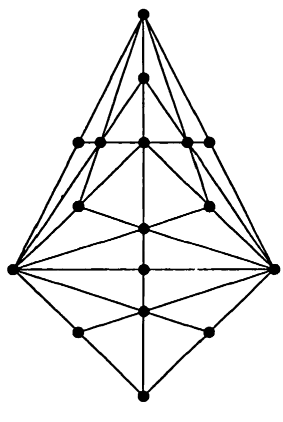
0. Introduction

Let Δ_1 and Δ_2 be finite simplicial complexes (we assume in this paper that all the simplicial complexes are finite) and K a field. Then the Stanley-Reisner rings $K[\Delta_1]$ and $K[\Delta_2]$ of Δ_1 and Δ_2 are defined (see §1 for definition). But in general, the rings $K[\Delta_1]$ and $K[\Delta_2]$ are not isomorphic even though the geometric realizations $|\Delta_1|$ and $|\Delta_2|$ of Δ_1 and Δ_2 are homeomorphic. So from the view point of ring theory (and combinatorial theory), one cannot replace a simplicial complex to a homeomorphic one (e.g. the barycentric subdivision) freely.

But some properties of ring theory (e.g. Cohen-Macaulayness and Buchsbaumness) are topological ones (i.e. if $|\Delta_1|$ and $|\Delta_2|$ are homeomorphic, then $K[\Delta_1]$ has the very property if and only if $K[\Delta_2]$ has it), and some of them are not (e.g. regularity). So it is natural to ask if the given property of ring theory is a topological one.

Consider the level case. If Δ is a pure complex of dimension 2 as in the figure below, then the h-vector of $K[\Delta]$ is (1,2,1) and type $(K[\Delta]) = 2$. So Δ is not level and we see that the property being level is not a topological one, because $|\Delta|$ is homeomorphic to a simplex of dimension 2. If Δ_1 is the barycentric subdivision of Δ, then the h-vector of $K[\Delta_1]$ is (1,14,9) and type $(K[\Delta_1]) = 9$. So we see that Δ_1 is level.

$\Delta =$ 

$\Delta_1 =$ 

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A question arises from the above example. Is there a level complex Δ such that the barycentric subdivision sd(Δ) of Δ is not level ? The purpose of this paper is to deny the existence of such a complex. In fact, we prove a stronger result, i.e. if Δ is a Cohen-Macaulay complex, then sd(Δ) is level except the case that the levelness of Δ is clearly denied by the topological property of $|\Delta|$ (i.e. $\tilde{\chi}(\Delta) \neq 0$ and Δ is not 2-Cohen-Macaulay)(see Theorems 2.3 and 2.5).

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1. Preliminaries

We denote the number of elements of a finite set X by $\sharp X$ and for two sets X and Y we denote by $X - Y$ the set $\{x \in X | x \notin Y\}$.

Let K be a field and fixed throughout this paper. Let $A = K[x_1, \ldots, x_n]$ be a polynomial ring over K. Then A is a $\mathbf{Z}^n$-graded ring in the natural way and if M and N are finitely generated $\mathbf{Z}^n$-graded A-modules then we can define the $\mathbf{Z}^n$-graded structure to $\mathrm{Hom}_A(M, N)$ by $[\mathrm{Hom}_A(M, N)]_\alpha = \{f \in \mathrm{Hom}_A(M, N) | f(M_\beta) \subseteq N_{\alpha+\beta}$ for any $\beta \in \mathbf{Z}^n\}$. So we can also define the $\mathbf{Z}^n$-graded structure of $\mathrm{Ext}_A^i(M, N)$ for any i. Moreover if I is a homogeneous ideal (in this grading) of A then we can also define the $\mathbf{Z}^n$-graded structure to the local cohomology modules $H_I^i(M)$. See [3] and [4] for the details.

We define the dimension (Krull dimension) of M written by dim M to be the maximal length of prime ideal chains in the ring $A/\mathrm{ann}(M)$ i.e. $\dim M = \max\{d|$ There exist prime ideals $P_0, \ldots, P_d$ in A such that $\mathrm{ann}(M) \subseteq P_0 \subsetneq \cdots \subsetneq P_d\}$. And the depth of M written by depth M is defined by the following three identical numbers

(i) The length of a maximal M-regular sequence in m.

(ii) $\min\{i | \mathrm{Ext}_A^i(A/m, M) \neq 0\}$

(iii) $\min\{i | H_m^i(M) \neq 0\}$

where $m = (x_1, \ldots, x_n)A$. (All meximal M-regular sequences in m are known to have the same length.) If

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

is the minimal free resolution of M as a graded (in $\mathbf{Z}^n$-grading or in the total degree) A-module, it is known that

$$\operatorname{depth} M = n - \max\{i | F_i \neq 0\}$$

(see [1]). It is also known that depth $M \leq \dim M$ for arbitrary $M \neq 0$ and we say M is a Cohen-Macaulay module if depth $M = \dim M$ or $M = 0$.

A homogeneous system of parameters of a module M is a family $y_1, \ldots, y_d$ of homogeneous (in the total degree) elements of m such that $\dim(M/(y_1, \ldots, y_d)M) = 0$ and $d = \dim M$. It is known that M is Cohen-Macaulay if and only if one of the following equivalent conditions are satisfied.

(i) There exist a homogeneous system of parameters of M such that it is an M-regular sequence.
(ii) Every homogeneous system of parameters of M is an M-regular sequence.

See [7] and [5] for the details.

A standard K-algebra $R = \bigoplus_{i \geq 0} R_i$ is a finitely generated non-negatively graded K-algebra such that $R_0 = K$ and $R = K[R_1]$. If $\dim_K R_1 = n$, then we can construct a degree preserving surjective K-algebra homomorphism $A \to R$, where $A = K[x_1, \dots, x_n]$ is a polynomial ring. Assume R is Cohen-Macaulay and $\dim R = d$. Then if

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow R \longrightarrow 0$$

is the minimal free resolution of R as an A-module, $F_{n-d} \neq 0$ and $F_{n-d+1} = 0$. The rank of F_{n-d} is called the type of R and denoted by type(R). The canonical module of R is defined and denoted by K_R. In our situation $K_R = \mathrm{Ext}_A^{n-d}(R, A)(-n)$ where $M(t)$ is the shift of grading, i.e. $(M(t))_s = M_{s+t}$ (see [3]). So K_R is the cokernel of

$$F^*_{n-d-1}(-n) \longrightarrow F^*_{n-d}(-n)$$

where $* = \mathrm{Hom}_A(\ \ , A)$. The a-invariant of R is also defined by $a(R) = -\min\{i \mid (K_R)_i \neq 0\}$.

For a graded A-module M, we denote the Poicaré series of M by $F(M, \lambda)$, i.e.

$$F(M, \lambda) = \sum_{n \in \mathbf{Z}} (\dim_K M_n)\lambda^n.$$

Then by the Hilbert syzygy theorem,

$$(1 - \lambda)^d F(R, \lambda) = h_0 + h_1\lambda + \cdots + h_s\lambda^s \in \mathbf{Z}[\lambda]$$

$(h_s \neq 0)$. We call the vector $(h_0, h_1, \dots, h_s)$ as the h-vector of R. On the other hand by the result of Stanley [12]

$$F(K_R, \lambda) = (-1)^d F\left(R, \frac{1}{\lambda}\right)$$
$$= \frac{h_s\lambda^{d-s} + h_{s-1}\lambda^{d-s+1} + \cdots + h_0\lambda^d}{(1-\lambda)^d}.$$

Hence $a(R) = s - d$ and

$$h_s = \dim_K((K_R)_{-a(R)}) \leq \dim_K(K_R \otimes_A K) = \dim_K(F^*_{n-d} \otimes_A K) = \mathrm{type}(R).$$

If $h_s = \mathrm{type}(R)$, we say that R is a level ring. (See [13].) So R is level if and only if K_R is generated by $(K_R)_{-a(R)}$ (and if and only if F_{n-d} is generated by $(F_{n-d})_i$ for some i).

Let V be a finite set. A simplicial complex Δ with vertex set V is a set of subsets of V such that (i) $\phi \in \Delta$ and (ii) if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$. Note that we do not require that $\{x\} \in \Delta$ for any $x \in V$. An element of Δ is called a face of Δ. For Δ and a face σ of Δ, we define the dimension of σ written dim σ by dim $\sigma = \sharp\sigma - 1$ and dimension of Δ written dim Δ by $\dim \Delta = \max_{\sigma \in \Delta} \dim \sigma$. In particular dim $\{\phi\} = -1$. We define a subcomplex $\Delta \backslash \sigma$ of Δ for $\sigma \in \Delta - \{\phi\}$ by

$$\Delta \backslash \sigma = \{\tau \in \Delta \mid \tau \not\supseteq \sigma\}.$$

And for a subset W of V, we define a subcomplex Δ_W of Δ by

$$\Delta_W = \{\sigma \in \Delta \mid \sigma \subseteq W\}.$$

Note that if $x \in V$ then $\Delta_{V-\{x\}} = \Delta \backslash x$.

Now we define the Stanley-Reisner ring (or face ring) $K[\Delta]$ of Δ (over K) for a simplicial complex Δ with vertex set $V = \{x_1, \dots, x_n\}$. Take a polynomial ring over K whose indeterminates are in one to one correspondence with the elements of V. We denote this polynomial ring by $K[x_1, \dots, x_n] (= A)$ for simplicity. Then

$$K[\Delta] = A/I_\Delta$$

where I_Δ is the ideal generated by $\{x_{j_1} \dots x_{j_t} \mid 1 \le j_1 < \cdots < j_t \le n, \{x_{j_1}, \dots, x_{j_t}\} \notin \Delta\}$. Then it is easily verifield that $\dim K[\Delta] = \dim \Delta + 1$. A complex is said to be Cohen-Macaulay (or level) (over K) if $K[\Delta]$ is Cohen-Macaulay (or level resp.).

Put $f_i =$ (The number of i-dimensional faces in Δ) for each i and we call $(f_{-1}, f_0, \dots, f_{d-1})$ ($d = \dim K[\Delta]$) as the f-vector of Δ. It is easily verified

$$F(K[\Delta], \lambda) = \frac{\sum_{i=0}^{d} f_{i-1} \lambda^i (1-\lambda)^{d-i}}{(1-\lambda)^d}.$$

So if $(h_0, \dots, h_s)$ ($h_s \ne 0$) is the h-vector of $K[\Delta]$, then $s \le d$ and $h_d = (-1)^{d-1}\tilde{\chi}(\Delta)$ ($\tilde{\chi}(\ )$ is the reduced euler characteristic, i.e. $\tilde{\chi}(\Delta) = -f_{-1} + f_0 - \cdots + (-1)^{d-1} f_{d-1}$). Baclawski [2] showed that a Cohen-Macaulay complex Δ is 2-Cohen-Macaulay if and only if $|\tilde{\chi}(\Delta)| = \mathrm{type}(K[\Delta])$. (A complex Δ is called 2-Cohen-Macaulay if Δ is Cohen-Macaulay and for any $x \in V$, $\Delta \backslash x$ is also Cohen-Macaulay and $\dim \Delta \backslash x = \dim \Delta$.) So Δ is 2-Cohen-Macaulay if and only if Δ is level and $\tilde{\chi}(\Delta) \ne 0$.

Finally we make a convention. All the cohomology groups of simplicial complexes and topological spaces are computed with coefficients in K (see, for example, [11] for the difinition). And when considering a simplicial complex Δ with vertex set V we define the support of $\alpha \in \mathbf{Z}^{\sharp V}$ to be a subset of V as follows. Take and fix a bijective map of sets $\varphi: V \to \{1, \dots, \sharp V\}$ and if $\alpha = (\alpha_1, \dots, \alpha_{\sharp V})$ then $\operatorname{supp} \alpha = \{x \in V \mid \alpha_{\varphi(x)} \ne 0\}$. So every time we consider a simplicial complex, we assume that a map φ as above is given and fixed. Especially if we write that Δ is a simplicial complex with vertex set $V = \{x_1, \dots, x_n\}$ then we assume φ is the map such that $\varphi(x_i) = i$ for any $i = 1, \dots, n$, so $\operatorname{supp}\alpha = \{x_j | \alpha_j \neq 0\}$ for $\alpha \in \mathbf{Z}^n$.

2. Main theorem

In this section we state the main result of this paper. First we note the following

Lemma 2.1. *Let A be a finitely generated non-negatively graded K-algebra such that $A_0 = K$ and M a finitely generated graded Cohen-Macaulay A-module of dimension d. Then the canonical map*

$$\operatorname{Ext}^d_A(K, M) \longrightarrow H^d_m(M)$$

is injective where $m = \bigoplus_{i>0} A_i$.

Proof. Take a homogeneous system of parameters $\theta_1, \theta_2, \ldots, \theta_d$ of M and put $M_i = M/(\theta_1, \ldots, \theta_i)M$ for $i = 0, 1, \ldots, d$. Then by taking the long exact sequence of the following short exact sequence,

$$0 \longrightarrow M_i \xrightarrow{\theta_{i+1}} M_i \longrightarrow M_{i+1} \longrightarrow 0$$

we get the following commutative diagram for any $i = 0, 1, \ldots, d-1$,

$$\begin{array}{ccc} \operatorname{Ext}^{d-i-1}_A(K, M_i) & \longrightarrow & \operatorname{Ext}^{d-i-1}_A(K, M_{i+1}) \\ \downarrow & & \downarrow \\ H^{d-i-1}_m(M_i) & \longrightarrow & H^{d-i-1}_m(M_{i+1}) \end{array}$$

$$\begin{array}{ccccc} \longrightarrow & \operatorname{Ext}^{d-i}_A(K, M_i) & \xrightarrow{\theta_{i+1}} & \operatorname{Ext}^{d-i}_A(K, M_i) \\ & \downarrow & & \downarrow \\ \longrightarrow & H^{d-i}_m(M_i) & \longrightarrow & H^{d-i}_m(M_i) \end{array}$$

where the vertical maps are the canonical ones. Since M_i is Cohen-Macaulay of dimension $(d-i)$, we see that

$$\operatorname{Ext}^{d-i-1}_A(K, M_i) = H^{d-i-1}_m(M_i) = 0.$$

On the other hand, the map $\operatorname{Ext}^{d-i}_A(K, M_i) \xrightarrow{\theta_{i+1}} \operatorname{Ext}^{d-i}_A(K, M_i)$ is the zero map since θ_{i+1} annihilates K. So we have the following commutative diagram for $i = 0, 1, \ldots, d-1$.

$$\begin{array}{ccc} \operatorname{Ext}^{d-i-1}_A(K, M_{i+1}) & \simeq & \operatorname{Ext}^{d-i}_A(K, M_i) \\ \downarrow & & \downarrow \\ H^{d-i-1}_m(M_{i+1}) & \hookrightarrow & H^{d-i}_m(M_i) \end{array}$$

And we see that the canonical map $\mathrm{Ext}_A^{d-i}(K, M_i) \to H_m^{d-i}(M_i)$ is injective if and only if the canonical map $\mathrm{Ext}_A^{d-i-1}(K, M_{i+1}) \to H_m^{d-i-1}(M_{i+1})$ is injective.

So we only have to show that the canonical map $\mathrm{Hom}_A(K, M_d) \to H_m^0(M_d)$ is injective. But this easily follows from the facts that

$$\mathrm{Hom}_A(K, M_d) \cong 0 \underset{M_d}{:} m$$

$$H_m^0(M_d) \cong \bigcup_{j\geq 0} 0 \underset{M_d}{:} m^j$$

and the canonical map corresponds to the inclusion map $0 \underset{M_d}{:} m \hookrightarrow \bigcup_{j\geq 0} 0 \underset{M_d}{:} m^j$.

Q.E.D.

Next we state the following

Lemma 2.2. *Let Δ be a Cohen-Macaulay complex of dimension $(d-1)$ with vertex set V. If W is a subset of V such that $W \notin \Delta$, then $\tilde{H}^{d-\sharp W-1}(\Delta_{V-W}) = 0$.*

Proof. Let α be the element of $\{-1, 0\}^n$ such that $\mathrm{supp}\,\alpha = W$. Then by the results of Hochster (see [6] Theorem (5.1) see also [8] Theorem 1 and Corollary 1, [14] Theorem II.4.1) we see that

$$\mathrm{Ext}_A^d(K, K[\Delta])_\alpha \cong \tilde{H}^{d-\sharp W-1}(\Delta_{V-W})$$

and

$$H_m^d(K[\Delta])_\alpha = 0$$

because $W \notin \Delta$. Since the canonical map $\mathrm{Ext}_A^i(K, K[\Delta]) \to H_m^i(K[\Delta])$ is a $\mathbf{Z}^n$-graded map for any i, we see the conclusion by Lemma 2.1. *Q.E.D.*

The main theorem of this paper is the following

Theorem 2.3. *Let Δ be a Cohen-Macaulay complex of dimension $(d-1)$ and $\Gamma = \mathrm{sd}(\Delta)$ be its barycentric subdivision. If $\tilde{\chi}(\Delta) = 0$, then Γ is a level complex and $a(K[\Gamma]) = -1$.*

Proof. Let $V = \Delta - \{\phi\}$ be the vertex set of Γ, $n = \sharp V$ and

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow K[\Gamma] \longrightarrow 0$$

by the minimal free resolution of $K[\Gamma]$ as a $\mathbf{Z}^n$-graded A-module, where $A = K[x \mid x \in V]$. Then by Hochster's formula ([6] Theorem (5.1). See also [14] Theorem II.4.8.), we see

$$(F_j \underset{A}{\otimes} K)_\alpha \cong \begin{cases} \tilde{H}^{\sharp\mathrm{supp}\alpha - j - 1}(\Gamma_{\mathrm{supp}\alpha}) & \text{if } \alpha \in \{0, 1\}^n \\ 0 & \text{otherwise} \end{cases}$$

for any $\alpha \in \mathbf{Z}^n$ and any $j \in \mathbf{Z}$. So in the total degree

$$(F_{n-d} \underset{A}{\otimes} K)_{n-i} \cong \bigoplus_{W \subseteq V, \sharp W = i} \tilde{H}^{d-i-1}(\Gamma_{V-W})$$

for any $i \in \mathbf{Z}$.

Now we

Claim 2.4. *If* $\sharp W \neq 1$, *then* $\tilde{H}^{d-\sharp W-1}(\Gamma_{V-W}) = 0$.

Proof of the claim. If $W = \phi$, then

$$\dim_K \tilde{H}^{d-1}(\Gamma_{V-W}) = \dim_K \tilde{H}^{d-1}(\Gamma) = |\tilde{\chi}(\Gamma)| = |\tilde{\chi}(\Delta)| = 0$$

since Γ is Cohen-Macaulay (cf. the main theorem of [10]). Assume $\sharp W \geq 2$. By Lemma 2.2 we may assume $W = \{\sigma_1, \dots, \sigma_t\}$ and $\sigma_1 \subsetneq \sigma_2 \subsetneq \cdots \subsetneq \sigma_t \in \Delta$. Let $V_0 = \{x_1, \dots, x_m\}$ be the vertex set of Δ and assume that $x_1, \dots, x_m$ are affinely independent elements of m-dimensional euclidean space. The geometric realization X of Δ is

$$\{\lambda_1 x_1 + \cdots + \lambda_m x_m | \lambda_1 + \cdots + \lambda_m = 1, \lambda_1 \geq 0, \dots, \lambda_m \geq 0, \{x_i | \lambda_i > 0\} \in \Delta\}$$

and we consider X as the geometric realization of Γ. Then for any $\tau \in V$, $\lambda_1 x_1 + \cdots + \lambda_m x_m \notin |\Gamma_{V-\{\tau\}}|$ if and only if $\lambda_i > \lambda_j$ whenever $x_i \in \tau$ and $x_j \notin \tau$. So if $q = \mu_1 x_1 + \cdots + \mu_m x_m \in |\Gamma_{V-W}|$, then we can take $i_1, \dots, i_t, j_1, \dots, j_t$ such that

$$x_{i_l} \in \sigma_l,\ x_{j_l} \notin \sigma_l,\ \mu_{i_l} \leq \mu_{j_l}$$

for any $l = 1, \dots, t$.

Now we assume that $\sigma_1 = \{x_1, \dots, x_s\}$ and let $p = \frac{1}{s}(x_1 + \cdots + x_s)$ be the barycenter of σ_1 and $Y = |\Delta \backslash \sigma_1|$. Then by the proof of [9] Lemma 2.2, we see that

$$F: (X - p) \times I \longrightarrow X - p$$

$$(q, a) \longmapsto (1-a)q + ar(q)$$

is a strong deformation retraction of $X - p$ to Y, where $r(q) = \frac{1}{\varepsilon(q)}(q - (1 - \varepsilon(q))p)$,

$\varepsilon(q) = 1 - \min_{1 \leq i \leq s} \{s\mu_i\}$, $q = \mu_1 x_1 + \cdots + \mu_m x_m$ and $I = \{a \in \mathbf{R} | 0 \leq a \leq 1\}$.

If we write

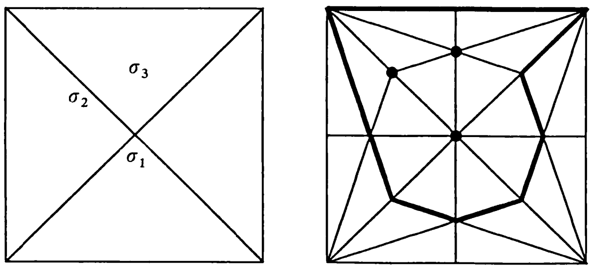
$$r(q) = \nu_1 x_1 + \cdots + \nu_m x_m$$

we see that

$$x_i \in \sigma_l,\ x_j \notin \sigma_l,\ \mu_i \leq \mu_j \Longrightarrow \nu_i \leq \nu_j.$$

So by the above argument,

$$F(Z \times I) \subseteq Z$$



where $Z = |\Gamma_{V-W}|$. Therefore we see that Y is a strong deformation retract of Z. We also see that Y is a strong deformation retract of $|\Gamma_{V-\{\sigma_1\}}|$ by the same way. So

$$\begin{aligned} &\tilde{H}^{d-\sharp W-1}(\Gamma_{V-W}) \\ &\cong \tilde{H}^{d-\sharp W-1}(Y) \\ &\cong \tilde{H}^{\sharp(V-\{\sigma_1\})-(n-d+\sharp W-1)-1}(\Gamma_{V-\{\sigma_1\}}) \\ &\mathrel{<\!\!\!\oplus} F_{n-d+\sharp W-1} \bigotimes_A K \\ &= 0 \end{aligned}$$

since Γ is Cohen-Macaulay of dimension $(d-1)$ (cf. Hochster's formula). And the proof of the claim is complete.

Now we return to the proof of Theorem 2.3. By Claim 2.4 and Hochster's formula, we see that

$$F_{n-d} \bigotimes_A K = (F_{n-d} \bigotimes_A K)_{n-1}$$

in the total degree. So F_{n-d} is generated by $(F_{n-d})_{n-1}$ and F^*_{n-d} is generated by $(F^*_{n-d})_{-n+1}$. Since the canonical module $K_{K[\Gamma]}$ of $K[\Gamma]$ is a homomorphic image of $F^*_{n-d}(-n)$, we see that $K_{K[\Gamma]}$ is generated by $(K_{K[\Gamma]})_1$. This means that $K[\Gamma]$ is a level ring and $a(K[\Gamma]) = -1$. *Q.E.D.*

As a corollary of the above theorem we have the following

Theorem 2.5. *Let Δ be a level complex and $\Gamma = \mathrm{sd}(\Delta)$ the barycentric subdivision of Δ. Then Γ is also level.*

Proof. The case $\tilde{\chi}(\Delta) = 0$ is proved by Theorem 2.3. If $\tilde{\chi}(\Delta) \neq 0$, then Δ is 2-Cohen-Macaulay and by [2] Theorem 2.5 (see also [15] and [9] Theorem 3.3) Γ is also 2-Cohen-Macaulay. So Γ is level. *Q.E.D.*

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