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Off-diagonal short time expansion of the heat kernel on a certain nilpotent Lie group

By

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0. Introduction

Let $\mathscr{L}$ be a differential operator of Hörmander type;

$$\mathscr{L} = \frac{1}{2}\sum_{\alpha=1}^{r} V_\alpha^2 + V_0,$$

where V_α, $\alpha = 0, 1, \cdots, r$, are C^∞-vector fields on $\mathbf{R}^d$. Under the condition $(\mathrm{H.1})_\infty$ of these vector fields given in §2 below, the fundamental solution $p(t, x, y)$ of the heat equation $\frac{\partial u}{\partial t} = \mathscr{L}u$ exists. Its short time expansion of the form

$$p(t, x, y) \sim \exp\left(-\frac{d(x, y)^2}{2t}\right) t^{-N/2}(c_0 + c_1 t + \cdots) \text{ as } t \downarrow 0 \tag{0.1}$$

has been studied by many authors in both analytical and probabilistic methods, cf. e.g. J.-M. Bismut [7], T.J.S. Taylor [21], S. Kusuoka [11], S. Watanabe [24], R. Léandre [16], G. Ben Arous [3]. Among others, G. Ben Arous [3] has shown that (0.1) holds with $N = d$ when the pair (x, y) of points x and y is out of the cut-locus, i.e. when

(i) there exists a unique $h_0 \in K_{\min}^{x,y}$,

(ii) the deterministic Malliavin covariance with respect to x and h_0 is non-degenerate,

(iii) x and y are not conjugate along h_0 (i.e. the Hessian of the mapping $h \in K^{x,y} \to \frac{1}{2}\|h\|_H^2$ is non-degenerate at h_0),

cf. §2 for the precise meaning of notions and notations like $K^{x,y}$, $K_{\min}^{x,y}$, the deterministic Malliavin covariance, etc. Also, $d(x, y)$ in (0.1) is the control metric or the Carnot-Caratheodory metric which coincides with the H-norm of elements in $K_{\min}^{x,y}$. Indeed, it was shown by R. Léandre ([13], [14], [15]) that, under the assumption of $(\mathrm{H.1})_\infty$, it holds generally

$$\lim_{t\downarrow 0} 2t \log p(t, x, y) = -d(x, y)^2. \tag{0.2}$$

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When the pair (x, y) is in the cut-locus, we can still expect that (0.1) holds but the exponent N is usually greater than d. In the simplest case of $x = y$, the expansion (0.1) with $d(x, y) = 0$ has been obtained by G. Ben Arous [4], R. Léandre [16] and S. Takanobu [20] under some restriction on the drift vector field V_0. If this restriction is violated, the situation is much more complicated, cf. G. Ben Arous [5], G. Ben Arous-R. Léandre [6].

Consider the case (x, y) is in the cut-locus and $x \neq y$. First we consider the case when (i) is violated but (ii) and (iii) remain valid for every $h_0 \in K^{x,y}_{\min}$. Here, however, the definition of non-conjugacy in (iii) should be modified as:

(iii)' the Hessian of the mapping $h \in K^{x,y} \to \frac{1}{2}\|h\|^2_H$ is non-degenerate at h_0 *in the direction normal to* $K^{x,y}_{\min}$.

Then we can expect that (0.1) holds with $N = d + \dim K^{x,y}_{\min}$ just as in the case of the heat kernel on a sphere with $\mathscr{L}$ = a half of the Laplacian and y is antipodal to x, cf. S.A. Molchanov [17]. Note that $K^{x,y}_{\min}$ is in one-to-one correspondence with the set of minimal geodesics (minimal horizontal curves given in §2) connecting x and y and hence $\dim K^{x,y}_{\min}$ = the dimension of the set of all minimal geodesics connecting x and y. A typical example of this situation is the case of the Heisenberg group realized by $\mathbf{R}^3$ and $x = (0, 0, 0)$, $y = (0, 0, \eta)$, $\eta \neq 0$ (cf. B. Gaveau [9], R. Azencott [2]). In this case, $K^{x,y}_{\min}$ constitutes a one-dimensional submanifold in the Cameron-Martin Hilbert space and $N = 4 = d + \dim K^{x,y}_{\min}$. If, furthermore, the condition (ii) is violated, i.e., the deterministic Malliavin covariance degenerates at $h \in K^{x,y}_{\min}$, we may still expect that (0.1) holds with $N > d + \dim K^{x,y}_{\min}$, however.

Purpose of this paper is to illustrate these situations in a concrete case of the nilpotent Lie group $N_{4,2}$ realized by $\mathbf{R}^{10}$. In this case, an explicit integral representation of the heat kernel was obtained by B. Gaveau [9] (cf. also M. Chaleyat-Maurel [8]) and the short time expansion (0.1) could be obtained directly from it. We follow here, however, a probabilistic approach given by H. Uemura-S. Watanabe [22] which can explain well the role of $\dim K^{x,y}_{\min}$ and the degeneracy of the Malliavin covariance in the determination of N and which may give some insight, we hope, in more general situations.

Finally, we explain briefly our method. First we represent the heat kernel as

$$p(\varepsilon^2, x, y) = \mathbf{E}[\delta_y(X^\varepsilon_1)] \tag{0.3}$$

by a generalized expectation of a generalized Wiener functional in the sense of S. Watanabe [24] where X^ε_t is the solution of the following stochastic differential equation:

$$\begin{cases} dX_t = \varepsilon \sum_{\alpha=1}^{r} V_\alpha(X_t) \circ dw^\alpha_t + \varepsilon^2 V_0(X_t)dt \\ X_0 = x. \end{cases} \tag{0.4}$$

δ_y is, of course, the Dirac's δ-function at $y \in \mathbf{R}^d$. We evaluate the generalized expectation in the right-hand side of (0.3) by appealing to the theory of large deviations and the theory of asymptotic expansions of Wiener functionals as developed in S. Watanabe [24]. Roughly, X_t^ε conditioned by $X_1^\varepsilon = y$ will be concentrated on the set $M^{x,y} = \{c^{x,h}; h \in K_{\min}^{x,y}\}$ of minimal horizontal curves connecting x and y as $\varepsilon \downarrow 0$, actually will be distributed uniformly on this set. It will be shown clearly by our probabilistic method how this limiting behavior of tied-down trajectories X_t^ε is reflected on that of $p(\varepsilon^2, x, y)$ as $\varepsilon \downarrow 0$.

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1. Probabilistic preliminaries

In this section we introduce some notions and results on asymptotic expansions of generalized Wiener functionals as are necessary in the future. The reader is refered to S. Watanabe [23], [24] for details.

Let (W, H, μ) be an abstract Wiener space. $\mathbf{D}_p^s(E)(s \in \mathbf{R}, 1 \leq p < \infty)$ be the completion of $\mathscr{P}(E)(:= \{E\text{-valued polynomial Wiener functionals}\})$ by the norm $\|\cdot\|_{p,s} = \|(I - L)^{s/2} \cdot\|_p$, where L is *the Ornstein-Uhlenbeck operator* (*the number operator*), $\|\cdot\|_p$ is the L^p-norm with respect to the measure μ, and E is a separable Hilbert space. Especially when $E = \mathbf{R}$, we denote $\mathbf{D}_p^s$ instead of $\mathbf{D}_p^s(\mathbf{R})$. Then it holds that $\mathbf{D}_p^0(E) = L^p(E, \mu)$ and $\mathbf{D}_p^s(E)^*$, the dual space of $\mathbf{D}_p^s(E)$, coincides with $\mathbf{D}_q^{-s}(E)$ under the identification of $\mathbf{D}_2^0(E)^*$ $(= L^2(E, \mu)^*)$ with itself, q being the conjugate exponent of p; $1/p + 1/q = 1$.

We define *H-derivative* $D: \mathscr{P}(E) \to \mathscr{P}(H \otimes E)$ by $DF(w)[h] := \lim_{\varepsilon \downarrow 0} \frac{F(w + \varepsilon h) - F(w)}{\varepsilon}$, $h \in H$. Here $H \otimes E$ is a Hilbert space formed of all linear operators from H to E of Hilbert-Schmidt type endowed with the Hilbert-Schmidt inner product. D can be extended to a bounded linear operator $\mathbf{D}_p^s(E) \to \mathbf{D}_p^{s-1}(H \otimes E)$ and we denote this extended linear operator again by D. If D^* is the dual operator of D, then D^* maps from $\mathbf{D}_p^{s+1}(H \otimes E)$ to $\mathbf{D}_p^s(E)$ and $L = -D^*D$. (See also N. Ikeda-S. Watanabe [10] or H. Sugita [19].)

Set $\mathbf{D}^\infty(E) := \bigcap_{s>0} \bigcap_{1<p<\infty} \mathbf{D}_p^s(E)$, $\tilde{\mathbf{D}}^\infty(E) := \bigcap_{s>0} \bigcup_{1<p<\infty} \mathbf{D}_p^s(E)$, $\tilde{\mathbf{D}}^{-\infty}(E) := \bigcup_{s>0} \bigcap_{1<p<\infty} \mathbf{D}_p^{-s}(E)$ and $\mathbf{D}^{-\infty}(E) := \bigcup_{s>0} \bigcup_{1<p<\infty} \mathbf{D}_p^{-s}(E)$. We call an element of $\mathbf{D}^{-\infty}(E)$ *a generalized Wiener functional* in analogy with the Schwartz distribution theory. When $E = \mathbf{R}$ we denote them simply by $\mathbf{D}^\infty, \tilde{\mathbf{D}}^\infty, \tilde{\mathbf{D}}^{-\infty}, \mathbf{D}^{-\infty}$ respectively. For $G \in \mathbf{D}^\infty$ and $\Phi \in \mathbf{D}^{-\infty}$ (or $G \in \tilde{\mathbf{D}}^\infty$ and $\Phi \in \tilde{\mathbf{D}}^{-\infty}$), $G \cdot \Phi (= \Phi \cdot G) \in \mathbf{D}^{-\infty}$ is defined by ${}_{\mathbf{D}^{-\infty}}\langle G \cdot \Phi, F \rangle_{\mathbf{D}^\infty} := {}_{\mathbf{D}^{-\infty}}\langle \Phi, G \cdot F \rangle_{\mathbf{D}^\infty}$ [resp. $:= {}_{\tilde{\mathbf{D}}^{-\infty}}\langle \Phi, G \cdot F \rangle_{\tilde{\mathbf{D}}^\infty}$] for all $F \in \mathbf{D}^\infty$.

For $F(w) = (F^1(w), \cdots, F^d(w)) \in \mathbf{D}^\infty(\mathbf{R}^d)$, i.e. $F^i(w) \in \mathbf{D}^\infty$, $i = 1, \cdots, d$, set $\sigma^{ij}(w) = \langle DF^i(w), DF^j(w) \rangle_H$, $i, j = 1, \cdots, d$. Here $\langle \cdot, * \rangle_H$ means the inner product of H. We call this $d \times d$ matrix valued Wiener functional $\sigma(w) = (\sigma^{ij}(w))_{i,j=1,\cdots,d}$ *the*

Malliavin covariance of F. If $\sigma(w)$ is positive definite for almost all w and furthermore $\{\det\sigma(w)\}^{-1}\in\bigcap_{1<p<\infty}L^p(\mu)$, we say that F is *non-degenerate* (in Malliavin's sense), and in this case, for any $T\in\mathscr{S}'(\mathbf{R}^d)$, a tempered Schwartz distribution on $\mathbf{R}^d$, its *pull-back* $T(F)$ is defined as an element of $\tilde{\mathbf{D}}^{-\infty}$. For $G\in\tilde{\mathbf{D}}^{\infty}$, we denote ${}_{\tilde{\mathbf{D}}^{-\infty}}\langle T(F), G\rangle_{\tilde{\mathbf{D}}^{\infty}}(={}_{\mathbf{D}^{-\infty}}\langle G\cdot T(F), 1\rangle_{\mathbf{D}^{\infty}})$ by $\mathbf{E}[T(F)\cdot G]$ or $\mathbf{E}[G\cdot T(F)]$. Especially when $T=\delta_y$, the Dirac's δ-function at $y\in\mathbf{R}^d$, $\mathbf{E}[G\cdot\delta_y(F)]=\mathbf{E}[G|F=y]\cdot p(y)$, $p(y)$ being the C^∞-density of F.

Let $F(\varepsilon, w)\in\mathbf{D}_p^s(E)$ for all $\varepsilon\in(0, 1]$. If $\|F(\varepsilon, w)\|_{p,s}=o(\varepsilon^n)$ as $\varepsilon\downarrow 0$, we say $F(\varepsilon, w)=o(\varepsilon^n)$ *as* $\varepsilon\downarrow 0$ *in* $\mathbf{D}_p^s(E)$. When $F(\varepsilon, w)\in\mathbf{D}^\infty(E)$ for all $\varepsilon\in(0, 1]$, we say $F(\varepsilon, w)=o(\varepsilon^n)$ *as* $\varepsilon\downarrow 0$ *in* $\mathbf{D}^\infty(E)$ if $F(\varepsilon, w)=o(\varepsilon^n)$ as $\varepsilon\downarrow 0$ in $\mathbf{D}_p^s(E)$ for all $s>0$ and $p\in(1,\infty)$. Similarly we define $F(\varepsilon, w)=o(\varepsilon^n)$ in $\tilde{\mathbf{D}}^\infty(E)$, in $\tilde{\mathbf{D}}^{-\infty}(E)$ and in $\mathbf{D}^{-\infty}(E)$.

Let $F(\varepsilon, w)\in\mathbf{D}_p^s(E)$ for all $\varepsilon\in(0, 1]$. We say $F(\varepsilon, w)$ *has the asymptotic expansion in* $\mathbf{D}_p^s(E)$:

$$F(\varepsilon, w)\sim f_0(w)+\varepsilon\cdot f_1(w)+\varepsilon^2\cdot f_2(w)+\cdots \textit{as } \varepsilon\downarrow 0 \textit{ in } \mathbf{D}_p^s(E)$$

if $f_i(w)\in\mathbf{D}_p^s(E)$, $i=0, 1, 2,\cdots$, and furthermore for all n,

$$F(\varepsilon, w)-\sum_{i=0}^{n}\varepsilon^i\cdot f_i(w)=o(\varepsilon^n) \text{ as } \varepsilon\downarrow 0 \text{ in } \mathbf{D}_p^s(E).$$

Similarly we define the asymptotic expansion in $\mathbf{D}^\infty(E)$, in $\tilde{\mathbf{D}}^\infty(E)$, in $\tilde{\mathbf{D}}^{-\infty}(E)$ and in $\mathbf{D}^{-\infty}(E)$. For example, we say $F(\varepsilon, w)$ has the asymptotic expansion in $\tilde{\mathbf{D}}^\infty(E)$ when for all n and s, there exists $p=p(s, n)$ such that $f_i(w)\in\mathbf{D}_p^s(E)$, $i=0, 1, 2,\cdots, n$, and $F(\varepsilon, w)-\sum_{i=0}^{n}\varepsilon^i\cdot f_i(w)=o(\varepsilon^n)$ as $\varepsilon\downarrow 0$ in $\mathbf{D}_p^s(E)$.

Let $F(\varepsilon, w)\in\mathbf{D}^\infty(\mathbf{R}^d)$ for all $\varepsilon\in(0, 1]$ and $\sigma(\varepsilon, w)$ be its Malliavin covariance. We say $F(\varepsilon, w)$ is *uniformly non-degenerate* if $F(\varepsilon, w)$ is non-degenerate for all $\varepsilon\in(0, 1]$ and furthermore

$$\varlimsup_{\varepsilon\downarrow 0}\|\{\det\sigma(\varepsilon, w)\}^{-1}\|_p<\infty \qquad \text{for all } p\in(1,\infty).$$

Here we give an important theorem concerning the asymptotic expansion of pull-backs.

Theorem 1.1 (S. Watanabe [24]). *Let a family* $F(\varepsilon, w)\in\mathbf{D}^\infty(\mathbf{R}^d)$, $0<\varepsilon\le 1$, *be uniformly non-degenerate and have the asymptotic expansion in* $\mathbf{D}^\infty(\mathbf{R}^d)$:

$$F(\varepsilon, w)\sim f_0(w)+\varepsilon\cdot f_1(w)+\cdots \textit{ as } \varepsilon\downarrow 0 \textit{ in } \mathbf{D}^\infty(\mathbf{R}^d).$$

Then for all $T\in\mathscr{S}'(\mathbf{R}^d)$, *its pull-back* $T(F(\varepsilon, w))\in\tilde{\mathbf{D}}^{-\infty}$ *and has the asymptotic expansion in* $\tilde{\mathbf{D}}^{-\infty}$:

$$T(F(\varepsilon, w))\sim\Phi_0(w)+\varepsilon\cdot\Phi_1(w)+\cdots \textit{ as } \varepsilon\downarrow 0 \textit{ in } \tilde{\mathbf{D}}^{-\infty}.$$

Furthermore, these coefficients $\Phi_i(w)$, $i=0, 1, 2,\cdots$, *are obtained from the formal Taylor expansion of* T, *i.e. formally from*

$$T(F(\varepsilon, w)) = T(f_0) + \partial T(f_0)(\varepsilon\cdot f_1 + \varepsilon^2\cdot f_2 + \cdots)$$
$$+ \frac{1}{2}\partial^2 T(f_0)(\varepsilon\cdot f_1 + \varepsilon^2\cdot f_2 + \cdots)\otimes(\varepsilon\cdot f_1 + \varepsilon^2\cdot f_2 + \cdots) + \cdots,$$

namely $\Phi_i(w)$ is obtained by picking up all coefficients of ε^i in the right-hand side above. For example, $\Phi_0 = T(f_0)$, $\Phi_1 = \partial T(f_0)f_1$ and $\Phi_2 = \partial T(f_0)f_2 + \frac{1}{2}\partial^2 T(f_0)f_1 \otimes f_1$.

Corollary 1.2. *Under the same assumptions as in* Theorem 1.1,

$$\mathbf{E}[T(F(\varepsilon, w))] \sim \mathbf{E}[\Phi_0(w)] + \varepsilon\cdot\mathbf{E}[\Phi_1(w)] + \cdots \quad \text{as } \varepsilon \downarrow 0.$$

2. Stochastic representation of heat kernels

Here we discuss the stochastic representation of the fundamental solution of heat equations by using the above results. Consider the following differential operator $\mathscr{L}$ of Hörmander type on $\mathbf{R}^d$:

$$\mathscr{L} = \frac{1}{2}\sum_{\alpha=1}^{r} V_\alpha^2,$$

where $V_\alpha(x) = \sum_{i=1}^{d} V_\alpha^i(x)\frac{\partial}{\partial x_i}$, $\alpha = 1,\cdots,r$, and we assume $V_\alpha^i(x) \in C_b^\infty(\mathbf{R}^d) :=$ the totality of C^∞-functions such that all derivatives are bounded. Let $p(t, x, y)$ be the fundamental solution of $\frac{\partial}{\partial t} - \mathscr{L}$, i.e.

$$\begin{cases} \dfrac{\partial}{\partial t}p(t, x, y) = \mathscr{L}_x\, p(t, x, y) \\ \lim\limits_{\varepsilon\downarrow 0} p(t, x, y) = \delta_y(x). \end{cases}$$

$p(t, x, y)$ can be obtained probabilistically by the following way: Let (W_0^r, P) be an r-dimensional Wiener space, i.e. $W_0^r := \{w = (w_t) \in C([0, 1] \to \mathbf{R}^r);\ w_0 = 0\}$ is a Banach space endowed with the norm $\|w\| := \sup_{t\in[0,1]} |w_t|$ and P is the Wiener measure on W_0^r. Let H be *the Cameron-Martin subspace* of W_0^r, i.e. H is a Hilbert space consisted of all absolutely continuous functions on $[0, 1]$ whose Radon-Nikodym derivatives are square integrable with the norm $\|h\|_H := \left(\int_0^1 \left|\frac{dh_t}{dt}\right|^2 dt\right)^{1/2}$. Then (W_0^r, H, P) is an abstract Wiener space. Now consider the following stochastic differential equation (abbr. S.D.E.) on $\mathbf{R}^d$:

$$\text{(2.1)} \qquad \begin{cases} dX_t = \sum_{\alpha=1}^{r} V_\alpha(X_t)\circ dw_t^\alpha \\ X_0 = x. \end{cases}$$

Here $w_t = (w_t^1, \cdots, w_t^r) \in W_0^r$ and $\circ dw_t^\alpha$ denotes the stochastic differential of Stratonovich type. We denote by X_t the solution of S.D.E. (2.1). We assume the following Hörmander-type condition on the vector fields V_α, $\alpha = 1, \cdots, r$:

(H.1)$_\infty$ *If we set*

$$H(n) = \{x \in \mathbf{R}^d;\ \ell.s.\ \{[V_{\alpha_1}, [V_{\alpha_2}, \cdots, [V_{\alpha_{k-1}}, V_{\alpha_k}]] \cdots](x), \quad \alpha_i \in \{1, \cdots, r\},\ k \leq n\} = T_x(\mathbf{R}^d)\}$$

then $$\bigcup_{n=1}^{\infty} H(n) = \mathbf{R}^d.$$

Here $\ell.s.$ means the linear span. In the case $\bigcup_{n=1}^{N} H(n) (= H(N)) = \mathbf{R}^d$, we say the condition (H.1)$_N$ is fulfilled. *From now on we always assume* (H.1)$_\infty$. Then it is known (cf. S. Kusuoka-D.W. Stroock [12]) that the Malliavin covariance $\sigma(t)$ of $X_t \in \mathbf{D}^\infty(\mathbf{R}^d)$ is non-degenerate for each fixed $t \in (0, 1]$, more precisely positive constants $K_1 = K_1(p)$ and K_2 exist such that $\mathbf{E}[|\det \sigma(t)|^{-p}]^{1/p} \leq K_1 t^{-K_2}$, $t \in (0, 1]$, $p \in (1, \infty)$. Hence $\delta_y(X_t) \in \tilde{\mathbf{D}}^{-\infty}$. Moreover we can see that

$$p(t, x, y) = \mathbf{E}[\delta_y(X_t)].$$

Let X_t^ε be a solution of the following S.D.E. (2.2):

$$\text{(2.2)} \qquad \begin{cases} dX_t = \varepsilon \sum_{\alpha=1}^{r} V_\alpha(X_t) \circ dw_t^\alpha \\ X_0 = x. \end{cases}$$

Then it is easy to see that $\{X_t^\varepsilon\} \overset{\mathscr{L}}{\sim} \{X_{\varepsilon^2 t}\}$, so the fundamental solution $p(t, x, y)$ can be expressed also by

$$p(\varepsilon^2, x, y) = \mathbf{E}[\delta_y(X_1^\varepsilon)].$$

In the following we use this representation to study its asymptotic behavior as $\varepsilon \downarrow 0$.

For each $h \in H$, consider the following differential equation:

$$\text{(2.3)} \qquad \begin{cases} \dfrac{dc(t)}{dt} = \sum_{\alpha=1}^{r} V_\alpha(c(t)) \cdot \dfrac{dh_t^\alpha}{dt} \\ c(0) = x. \end{cases}$$

We denote the solution by $c^{x,h}(t)$. Such a curve for some x and h is called *a horizontal curve with respect to* $\{V_\alpha\}$. For all $x, y \in \mathbf{R}^d$, set

$$K^{x,y} = \{h \in H;\ c^{x,h}(1) = y\}.$$

Then under the condition (H.1)$_\infty$, it is well-known that $K^{x,y} \neq \phi$ *for all* $x, y \in \mathbf{R}^d$(cf. J.-M. Bismut [7], Th. 1.14). Thus, for all $x, y \in \mathbf{R}^d$, we set

$$d(x, y) = \min\{\|h\|_H;\ h \in K^{x,y}\}.$$

This defines a metric called *the control metric* of x and y. Let

$$K_{\min}^{x,y} = \{h \in K^{x,y};\ \|h\|_H = d(x, y)\}.$$

Then it is also well-known that $K_{\min}^{x,y} \neq \phi$ (cf. J.-M. Bismut [7], Th. 1.14). We define $M^{x,y}$ by

$$M^{x,y} = \{c^{x,h};\ h \in K_{\min}^{x,y}\}$$

and call its element *the minimal horizontal curve* connecting x and y.

Consider the following differential equation on $d \times d$ matrix:

$$(2.4)\qquad \begin{cases} \dfrac{dY(t)}{dt} = \displaystyle\sum_{\alpha=1}^{r} \partial V_\alpha(c(t))\, Y(t) \cdot \frac{dh_t^\alpha}{dt} \\ Y(0) = I, \end{cases}$$

where $c(t)$ is the solution of (2.3) and $\partial V_\alpha(x)$ is a $d \times d$ matrix whose (i, j)-component is $\partial V_\alpha^i(x)/\partial x_j$. This solution is denoted by $Y^{x,h}(t)$. With this solution we define a $d \times d$ matrix $\sigma^{x,h}$ by

$$\sigma^{x,h} = \sum_{\alpha=1}^{r} \int_0^1 Y^{x,h}(1)\, Y^{x,h}(t)^{-1}\, V_\alpha(c^{x,h}(t)) \otimes Y^{x,h}(1)\, Y^{x,h}(t)^{-1}\, V_\alpha(c^{x,h}(t))\, dt.$$

This $\sigma^{x,h}$ is called *the deterministic Malliavin covariance* with respect to x and h and plays an important role later when we discuss the minimal horizontal curve.

We define *the Hamiltonian function* associated to the vector fields V_α, $\alpha = 1, \cdots, r$, by

$$H(p, x) = \frac{1}{2} \sum_{\alpha=1}^{r} \langle p, V_\alpha(x) \rangle^2, \quad (p, x) \in T^*(\mathbf{R}^d),$$

where $\langle \cdot, * \rangle$ denotes the coupling of elements in $T_x^*(\mathbf{R}^d)$ and $T_x(\mathbf{R}^d)$. Consider the following *Hamilton equation* with respect to $H(p, x)$ above:

$$(2.5)\qquad \begin{cases} \dot{x}_t = \dfrac{\partial H}{\partial p}(p_t, x_t) \\ \dot{p}_t = -\dfrac{\partial H}{\partial x}(p_t, x_t), \end{cases}$$

where $\dot{}$ denotes the time derivative $\dfrac{d}{dt}$. The solution of this equation (2.5) is called *a bicharacteristic*. We denote the bicharacteristic with an initial value (p_0, x_0) by $(p_t(p_0, x_0), x_t(p_0, x_0))$. Now we summarize some results concerning to the bicharacteristic. Refer to J.-M. Bismut [7] for details.

(2.6-I) Let $p_t := p_t(p_0, x_0)$, $x_t := x_t(p_0, x_0)$ and $\dot{h}_t := (\langle p_t, V_1(x_t) \rangle, \cdots, \langle p_t, V_r(x_t) \rangle)$. Then

$$c^{x_0,h}(t) = x_t(p_0, x_0).$$

(2.6-II) If the deterministic Malliavin covariance $\sigma^{x_0,h}$, $h \in K_{\min}^{x_0,y}$, is non-degenerate, i.e. $\det \sigma^{x_0,h} > 0$, then there exists a unique p_0 such that

$$c^{x_0,h}(t) = x_t(p_0, x_0).$$

(2.6-III) The following (H.2) is a sufficient condition on vector fields V_α, $\alpha = 1,\cdots,r$, for the non-degeneracy of its deterministic Malliavin covariance:

(H.2) $$\ell.s.\ \{V_1(x_0),\cdots, V_r(x_0), [V_1, Y](x_0),\cdots, [V_r, Y](x_0)\} = T_{x_0}(\mathbf{R}^d)$$

for every fixed $\lambda = (\lambda_1,\cdots,\lambda_r) \in \mathbf{R}^r \backslash \{0\}$ setting $Y = \sum_{\alpha=1}^{r} \lambda_\alpha V_\alpha$.

Namely, if (H.2) is satisfied at $x_0 \in \mathbf{R}^d$, then $\det \sigma^{x_0,h} > 0$ for every $h \in H$ such that $h \neq 0$.

3. Nilpotent Lie groups of order r with n-generators

In this section we introduce a nilpotent Lie group which will be the main subject of this paper (cf. B. Gaveau [9]). Let $V_1,\cdots, V_n$ be C^∞-vector fields. For $I = (i_1,\cdots, i_k) \in \{1,\cdots,n\}^k$ we define $V_{[I]}$ and V_I by

$$V_{[I]} = [V_{i_1}, [V_{i_2},\cdots [V_{i_{k-1}}, V_{i_k}]\cdots],$$
$$V_I = V_{i_1} \cdot V_{i_2} \cdots V_{i_k},$$

and let $|I|$ be the length of I. (In this case $|I| = k$.) It is easy to show that there exist constants A_{IJ} such that

$$V_{[I]} = \sum A_{IJ} \cdot V_J$$

and $A_{IJ} = 0$ if $|I| \neq |J|$.

Definition 3.1. We say that a system of vector fields $\{V_1,\cdots, V_n\}$ is *free of order r* at x if $\sum_{|I| \le r} a_I \cdot V_{[I]}(x) = 0$, $a_I \in \mathbf{R}$, implies $\sum_{|I| \le r} a_I \cdot A_{IJ} = 0$ for all J satisfying $|J| \le r$. Let $\mathbf{V} = \ell.s.\ \{V_1,\cdots, V_n\}$. We say the vector space $\mathbf{V}$ is *free of order r* if $\{V_1,\cdots, V_n\}$ is free of order r for all x.

Definition 3.2. Let $\mathfrak{g}$ be a Lie algebra.
i) $\mathfrak{g}$ is said to be *nilpotent of order r* if $\mathfrak{g} = \mathbf{V}^1 \oplus \cdots \oplus \mathbf{V}^r$ where $\mathbf{V}^i$, $i = 1,\cdots,r$, are vector subspaces of $\mathfrak{g}$ satisfying $\mathbf{V}^2 = [\mathbf{V}^1, \mathbf{V}^1]$, $\mathbf{V}^3 = [\mathbf{V}^1, \mathbf{V}^2],\cdots, \mathbf{V}^r = [\mathbf{V}^1, \mathbf{V}^{r-1}]$, $[\mathbf{V}^1, \mathbf{V}^r] = \{0\}$ and $[\mathbf{V}^i, \mathbf{V}^j] \subset \mathbf{V}^{i+j}$.
ii) Furthermore $\mathfrak{g}$ is said to have n generators if $\dim \mathbf{V}^1 = n$ and moreover $\mathbf{V}^1$ is free of order r.

We say $\mathfrak{g}$ is *a nilpotent Lie algebra of order r with n-generators* if i) and ii) above are satisfied and denote it by $\mathfrak{n}_{n,r}$. Let $N_{n,r}$ be a Lie group corresponding to $\mathfrak{n}_{n,r}$. This $N_{n,r}$ is called *a nilpotent Lie group of order r with n-generators. From now on, we assume* $r = 2$.

Proposition 3.1. *Let* $\mathfrak{n}_{n,2} = \mathbf{V}^1 \oplus \mathbf{V}^2$, $\{V_i, i = 1,\cdots,n\}$ *be a base of* $\mathbf{V}^1$, *and* $V_{jk} := [V_j, V_k]$. *Then a system* $\{V_i, V_{jk};\ 1 \le i \le n,\ 1 \le j < k \le n\}$ *is a base of* $\mathfrak{n}_{n,2}$.

Proof. Set $\sum_{|I| \le 2} a_I \cdot V_{[I]}(x) = 0$ where $a_I = 0$ if $I = (i_1, i_2)$ satisfies $i_1 > i_2$. Since $\mathbf{V}^1$ is free, $\sum_{|I| \le 2} a_I \cdot A_{IJ} = 0$ for all J. Therefore by taking $J = i$, $i = 1,\cdots,n$, or $J = (j, k)$, $1 \le j < k \le n$, we see easily that $a_J = 0$, i.e. $\{V_i, V_{jk};\ 1 \le i \le n,\ 1 \le j < k \le n\}$ is linearly independent. Since $V_{[(i_1,i_2)]} = - V_{[(i_2,i_1)]}$, it is clear that the above system is a base.

With this base we can introduce a canonical coordinate on $N_{n,2}$ as follows:

$$(x_i, x_{(jk)})_{\substack{1 \le i \le n \\ 1 \le j < k \le n}} \longleftrightarrow \exp\Big(\sum_{i=1}^{n} x_i \cdot V_i + \sum_{1 \le j < k \le n} x_{(jk)} \cdot V_{jk} \Big) \in N_{n,2}.$$

Hence $N_{n,2}$ is realized by $\mathbf{R}^{n(n+1)/2}$ under this coordinate and the group action is given as follows by Campbell-Hausdorff's theorem:

$$(x_i, x_{(jk)}) \cdot (y_i, y_{(jk)}) = \left(x_i + y_i,\ x_{(jk)} + y_{(jk)} + \frac{1}{2}(x_j y_k - x_k y_j) \right).$$

Define mappings $L_{(x_i, x_{(jk)})}$ and $R_{(y_i, y_{(jk)})}$ on $\mathbf{R}^{n(n+1)/2}$ by

$$L_{(x_i, x_{(jk)})}(z_i, z_{(jk)}) = (x_i, x_{(jk)}) \cdot (z_i, z_{(jk)}),$$

and

$$R_{(y_i, y_{(jk)})}(z_i, z_{(jk)}) = (z_i, z_{(jk)}) \cdot (y_i, y_{(jk)}).$$

Then both $L_{(x_i, x_{(jk)})}$ and $R_{(y_i, y_{(jk)})}$ are affine mappings with the determinants 1 and so the Haar measure of $N_{n,2}$ is the Lebesgue measure. Under this coordinate V_i is expressed as follows:

$$V_i = \frac{\partial}{\partial x_i} + \frac{1}{2}\left(\sum_{k<i} x_k \frac{\partial}{\partial x_{(ki)}} - \sum_{k>i} x_k \frac{\partial}{\partial x_{(ik)}} \right). \tag{3.1}$$

Set

$$\Delta_{n,2} = \sum_{i=1}^{n} V_i^2.$$

Obviously $\{V_i, 1 \le i \le n\}$ satisfies $(\mathrm{H}.1)_2$. The group $N_{2,2}$ is called *the 3-dimensional Heisenberg group* (cf. B. Gaveau [9], H. Uemura-S. Watanabe [22]), and the group $N_{3,2}$ does not play a different role from $N_{2,2}$ in our future considerations. Thus, in this paper, we assume $n = 4$ and study the group $N_{4,2}$ exclusively.

Notations. (cf. H. Uemura-S. Watanabe [22]) i) $\mathbf{x}\in\mathbf{R}^{10}$ is denoted by $\mathbf{x} = (x_i, x_{(jk)})_{\substack{i=1,\cdots,4\\ 1\le j<k\le 4}}$ or by $[x, X]$ where $x\in\mathbf{R}^4$ and $X\in\mathfrak{o}(4)$:= the totality of 4×4 real skew-symmetric matrices, defined by $x=(x_1,\cdots,x_4)$ and

$$X_{ij}=\begin{cases} x_{(ij)} & \text{if } i<j,\\ -x_{(ji)} & \text{if } i>j,\\ 0 & \text{otherwise.}\end{cases}$$

We also denote such X by $\sum_{i<j} x_{(ij)}\delta_{ij} - \sum_{i>j} x_{(ji)}\delta_{ij}$.

ii) For every $\Omega\in O(4)$ we define a mapping $T(\Omega)$ on $\mathbf{R}^{10}$ by

$$T(\Omega)\mathbf{x}=[\Omega x, \Omega X{}^t\Omega].$$

iii) For $X, Y\in\mathfrak{o}(4)$, define $X\sim Y$ if and only if $X=\Omega Y{}^t\Omega$ for some $\Omega\in O(4)$.

Remark 3.1. Noting that $\Omega X{}^t\Omega\in\mathfrak{o}(4)$ and that $\|X\| = \|\Omega X{}^t\Omega\|$, $\|\cdot\|$ being a 16-dimensional Euclidean norm by regarding X as an element of 16-dimensional Euclidean space, we know $T(\Omega)\in O(10)$. And it is easy to see that ${}^tT(\Omega)=T({}^t\Omega)$.

4. Computation of minimal horizontal curves

In this section we determine all the minimal horizontal curves on $N_{4,2}$ connecting the origin $\mathbf{0}$ and $\mathbf{x}=[0, X]$. For each $h\in K^{\mathbf{0},\mathbf{x}}$, the horizontal curve $c^h(t)=(c^{h,i}(t), c^{h,(jk)}(t))_{\substack{i=1,\cdots,4\\ 1\le j<k\le 4}}$ and the deterministic Malliavin covariance

$$\sigma(=\sigma(h))=\begin{pmatrix}\sigma^{ij} & \sigma^{i(mn)}\\ \sigma^{(kl)j} & \sigma^{(kl)(mn)}\end{pmatrix}_{1\le i,j\le 4,\ 1\le k<l\le 4,\ 1\le m<n\le 4}$$

are given as follows:

$$c^{h,i}(t)=h_t^i,\ i=1,\cdots,4,$$

$$c^{h,(jk)}(t)=\frac{1}{2}\int_0^t\{h_s^j\cdot\dot{h}_s^k - h_s^k\cdot\dot{h}_s^j\}\,ds,\qquad 1\le j<k\le 4,$$

where $\dot{}=\dfrac{d}{dt}$ and $c^h(1)=[0, X]$ and

$$\sigma^{ij}=\delta_{ij},\ 1\le i,j\le 4,$$

$$\sigma^{(kl)j}=\sigma^{j(kl)}=0 \text{ if } k\ne j \text{ and } l\ne j,$$

$$\sigma^{(kl)k}=\sigma^{k(kl)}=-\int_0^1 h_t^l\,dt,$$

$$\sigma^{(kl)l}=\sigma^{l(kl)}=\int_0^1 h_t^k\,dt,$$

$$\sigma^{(kl)(mn)} = 0 \text{ if } \{k, l, m, n\} = \{1, 2, 3, 4\},$$

$$\sigma^{(kl)(kl)} = \int_0^1 \{(h_t^k)^2 + (h_t^l)^2\}\, dt,$$

$$\sigma^{(kl)(kn)} = \int_0^1 h_t^l \cdot h_t^n \, dt,$$

$$\sigma^{(kl)(mk)} = \sigma^{(mk)(kl)} = -\int_0^1 h_t^l \cdot h_t^m \, dt$$

and

$$\sigma^{(kl)(ml)} = \int_0^1 h_t^k \cdot h_t^m \, dt.$$

Proposition 4.1. *If* rank $X = 4$, *the above deterministic Malliavin covariance* σ *is non-degenerate.*

Proof. For all $X \in \mathfrak{o}(4)$, there exists $U \in \mathfrak{o}(4)$ such that $U = u_1(\delta_{12} - \delta_{21}) + u_2(\delta_{34} - \delta_{43})$ and $X \sim U$. If rank $X = 4$, then rank $U = 4$, i.e. $u_1, u_2 \neq 0$. It is enough to prove in the case $X = U$ because

$$\sigma(\Omega h) = T(\Omega)\sigma(h)^t T(\Omega), \qquad \Omega \in O(4),$$

which is easily obtained by that

$$Y^{0,\Omega h}(t) = T(\Omega)\, Y^{0,h}(t)^t\, T(\Omega)$$

and that

$$T(\Omega) \sum_{\alpha=1}^{4} V_\alpha(c^h(t)) \otimes V_\alpha(c^h(t))^t\, T(\Omega)$$
$$= \sum_{\alpha=1}^{4} V_\alpha(c^{\Omega h}(t)) \otimes V_\alpha(c^{\Omega h}(t)),$$

$Y^{x,h}$ and V_α being as in (2.4) and (3.1) respectively.

Since $h \in K^{0,[0,U]}$,

$$(4.1) \qquad \begin{cases} \displaystyle\int_0^1 \dot{h}_t^i \, dt = 0, \; i = 1, \cdots, 4, \\ \displaystyle\frac{1}{2}\int_0^1 (h_t^1 \cdot \dot{h}_t^2 - h_t^2 \cdot \dot{h}_t^1)\, dt = u_1 (\neq 0), \\ \displaystyle\frac{1}{2}\int_0^1 (h_t^3 \cdot \dot{h}_t^4 - h_t^4 \cdot \dot{h}_t^3)\, dt = u_2 (\neq 0), \\ \displaystyle\frac{1}{2}\int_0^1 (h_t^i \cdot \dot{h}_t^j - h_t^j \cdot \dot{h}_t^i)\, dt = 0 \text{ if } (i, j) \neq (1, 2), (3, 4). \end{cases}$$

It is easy to show that σ is transformed into the following θ by a general linear mapping: $\theta^{ij} = \delta_{ij}$, $\theta^{(ij)k} = \theta^{k(ij)} = 0$ for all $1 \le i < j \le 4$, $k = 1, \cdots, 4$, and $\theta^{(ij)(kl)}$ are given by replacing h with $\bar{h}$ in $\sigma^{(ij)(kl)}$, where $\bar{h}_t^i := h_t^i - \int_0^1 h_s^i ds$. Clearly (4.1) remains valid under replacing h with $\bar{h}$.

Now it is enough to show that ${}^t\xi\Theta\xi = 0$ implies $\xi = 0$ where we set $\Theta = (\theta^{(ij)(kl)})_{\substack{1 \le i < j \le 4 \\ 1 \le k < l \le 4}}$ and $\xi = {}^t(\xi_{12}, \xi_{13}, \xi_{14}, \xi_{23}, \xi_{24}, \xi_{34})$. Since

$$
\begin{aligned}
{}^t\xi\Theta\xi = \int_0^1 \{&(-\xi_{12}\cdot\bar{h}_t^1 + \xi_{23}\cdot\bar{h}_t^3 + \xi_{24}\cdot\bar{h}_t^4)^2 \\
&+ (\xi_{12}\cdot\bar{h}_t^2 + \xi_{13}\cdot\bar{h}_t^3 + \xi_{14}\cdot\bar{h}_t^4)^2 \\
&+ (\xi_{13}\cdot\bar{h}_t^1 + \xi_{23}\cdot\bar{h}_t^2 - \xi_{34}\cdot\bar{h}_t^4)^2 \\
&+ (\xi_{14}\cdot\bar{h}_t^1 + \xi_{24}\cdot\bar{h}_t^2 + \xi_{34}\cdot\bar{h}_t^3)^2\}\, dt,
\end{aligned}
$$

we see that ${}^t\xi\Theta\xi = 0$ is equivalent to the following (4.2):

$$
\text{(4.2)} \qquad
\begin{cases}
-\xi_{12}\cdot\bar{h}_t^1 + \xi_{23}\cdot\bar{h}_t^3 + \xi_{24}\cdot\bar{h}_t^4 = 0, \\
\xi_{12}\cdot\bar{h}_t^2 + \xi_{13}\cdot\bar{h}_t^3 + \xi_{14}\cdot\bar{h}_t^4 = 0, \\
\xi_{13}\cdot\bar{h}_t^1 + \xi_{23}\cdot\bar{h}_t^2 - \xi_{34}\cdot\bar{h}_t^4 = 0, \\
\xi_{14}\cdot\bar{h}_t^1 + \xi_{24}\cdot\bar{h}_t^2 + \xi_{34}\cdot\bar{h}_t^3 = 0.
\end{cases}
$$

Then substituting (4.2) into (4.1), we can easily show $\xi = 0$. This completes the proof.

Thus, in view of (2.6–II), the minimal horizontal curve in this case is obtained from bicharacteristics. This is also true in the case rank $X = 2$, because we can reduce this case to that of Heisenberg group.

Now we determine the bicharacteristics on $N_{4,2}$. Substituting (3.1), the Hamilton equation (2.5) is given by

$$
\text{(4.3)} \qquad
\begin{cases}
\dot{x}_t^i = p_t^i + \frac{1}{2}\{\sum_{k<i} x_t^k\cdot p_t^{(ki)} - \sum_{k>i} x_t^k\cdot p_t^{(ik)}\}, \\
\dot{x}_t^{(ij)} = \frac{1}{2}(x_t^i\cdot\dot{x}_t^j - x_t^j\cdot\dot{x}_t^i), \\
\dot{p}_t^i = -\frac{1}{2}\{\sum_{i<j} p_t^j\cdot p_t^{(ij)} - \sum_{i>j} p_t^j\cdot p_t^{(ji)}\} \\
\qquad -\frac{1}{4}\{\sum_{\substack{l<j \\ i<j}} x_t^l\cdot p_t^{(ij)}\cdot p_t^{(lj)} + \sum_{\substack{l>j \\ i>j}} x_t^l\cdot p_t^{(ji)}\cdot p_t^{(jl)} \\
\qquad - \sum_{i<j<l} x_t^l\cdot p_t^{(ij)}\cdot p_t^{(jl)} - \sum_{k<j<i} x_t^k\cdot p_t^{(kj)}\cdot p_t^{(ji)}\}, \\
\dot{p}_t^{(ij)} = 0.
\end{cases}
$$

Moreover it is easy to show that

(4.4) $$\dot{h}_t^i (:= \langle \mathbf{p}_t, V_i(\mathbf{x}_t) \rangle) = \dot{x}_t^i.$$

Since $p_t^{(ij)} = p_0^{(ij)}$ and $\{x_t^{(ij)}\}$ are obtained by $\{x_t^i\}$, setting $x_t = {}^t(x_t^1, \cdots, x_t^4)$ and $p_t = {}^t(p_t^1, \cdots, p_t^4)$, we must solve the following equation:

(4.5) $$\frac{d}{dt}\begin{pmatrix} x_t \\ p_t \end{pmatrix} = \begin{pmatrix} -A & I \\ A^2 & -A \end{pmatrix}\begin{pmatrix} x_t \\ p_t \end{pmatrix}.$$

Here $A = (a_{ij})_{i,j=1,\cdots,4} \in \mathfrak{o}(4)$ is given by

$$a_{ij} = \begin{cases} \dfrac{1}{2} \cdot p_0^{(ij)}, & i < j, \\ -\dfrac{1}{2} \cdot p_0^{(ji)}, & i > j, \\ 0, & i = j, \end{cases}$$

and I denotes the 4×4 identity matrix.

Proposition 4.2. *For all $\Omega \in O(4)$,*

$$\begin{cases} \mathbf{p}_t(T(\Omega)\mathbf{p}_0, T(\Omega)\mathbf{x}_0) = T(\Omega)\mathbf{p}_t(\mathbf{p}_0, \mathbf{x}_0) \\ \mathbf{x}_t(T(\Omega)\mathbf{p}_0, T(\Omega)\mathbf{x}_0) = T(\Omega)\mathbf{x}_t(\mathbf{p}_0, \mathbf{x}_0). \end{cases}$$

Proof. It is easy to see that

$$\frac{d}{dt}\begin{pmatrix} \Omega x_t \\ \Omega p_t \end{pmatrix} = \begin{pmatrix} -\Omega A {}^t\Omega & I \\ (\Omega A {}^t\Omega)^2 & -\Omega A {}^t\Omega \end{pmatrix}\begin{pmatrix} \Omega x_t \\ \Omega p_t \end{pmatrix},$$

so the assertion of this proposition is obvious.

Remark 4.1. We know that for all $A \in \mathfrak{o}(4)$, there exist $\Omega \in O(4)$ and $Q \in Q(4) := \{q_1(\delta_{12} - \delta_{21}) + q_2(\delta_{34} - \delta_{43}) \in \mathfrak{o}(4);\ 0 \le q_1 \le q_2\}$ such that

$$A = \Omega Q {}^t\Omega.$$

Thus, by the proposition above, we can conclude that determining all the minimal horizontal curves connecting $\mathbf{0}$ and $\mathbf{x} = [0, X]$ is equivalent to determining all $(\tilde{\mathbf{p}}_0\ (= [\tilde{p}_0, 2Q]), \Omega) \in \mathbf{R}^{10} \times O(4)$, $Q \in Q(4)$, such that the H-norm of h, given by (4.4) from the solution of (4.3) with the initial value $(\tilde{\mathbf{p}}_0, \mathbf{0})$ satisfying $\mathbf{x}_1(\tilde{\mathbf{p}}_0, \mathbf{0}) = T({}^t\Omega)\mathbf{x}$, takes a minimum.

Replacing A by $Q \in Q(4)$ in (4.5), we have

(4.6) $$\begin{cases} \dot{x}_t^{2i-1} = -q_i \cdot x_t^{2i} + p_t^{2i-1} \\ \dot{x}_t^{2i} = q_i \cdot x_t^{2i-1} + p_t^{2i} \\ \dot{p}_t^{2i-1} = -q_i^2 \cdot x_t^{2i-1} - q_i \cdot p_t^{2i} \\ \dot{p}_t^{2i} = -q_i^2 \cdot x_t^{2i} + q_i \cdot p_t^{2i-1}, \qquad i = 1, 2, \end{cases}$$

with initial value $(x_0, p_0) := (0, \tilde{p}_0)$. We denote the solution of (4.6) by $(x_t(\tilde{p}_0), p_t(\tilde{p}_0))$. (*In the folowing we always assume* $\mathbf{x}_0 = \mathbf{0}$, *so we always omit* $\mathbf{x}_0$.) In this case clearly $h_t = x_t$ and the solution of (4.6) is:

a) if $q_i = 0$

$$\begin{cases} x_t^{2i-1}(\tilde{p}_0) = \tilde{p}_0^{2i-1} t \\ x_t^{2i}(\tilde{p}_0) = \tilde{p}_0^{2i} t \\ p_t^{2i-1}(\tilde{p}_0) = \tilde{p}_0^{2i-1} \\ p_t^{2i}(\tilde{p}_0) = \tilde{p}_0^{2i}, \end{cases}$$

b) if $q_i > 0$

$$\begin{cases} x_t^{2i-1}(\tilde{p}_0) = (\tilde{p}_0^{2i-1}/2q_i)\sin 2q_i t + (\tilde{p}_0^{2i}/2q_i)(\cos 2q_i t - 1) \\ x_t^{2i}(\tilde{p}_0) = -(\tilde{p}_0^{2i-1}/2q_i)(\cos 2q_i t - 1) + (\tilde{p}_0^{2i}/2q_i)\sin 2q_i t \\ p_t^{2i-1}(\tilde{p}_0) = (\tilde{p}_0^{2i-1}/2)(\cos 2q_i t + 1) - (\tilde{p}_0^{2i}/2)\sin 2q_i t \\ p_t^{2i}(\tilde{p}_0) = (\tilde{p}_0^{2i-1}/2)\sin 2q_i t + (\tilde{p}_0^{2i}/2)(\cos 2q_i t + 1), \end{cases}$$

thus, always, $\frac{1}{2}\cdot \|h\|_H^2 = \frac{1}{2}\cdot \sum_{i=1}^{4} (\tilde{p}_0^i)^2$.

By the condition $x_1^i(\tilde{p}_0) = 0$, $i = 1, \cdots, 4$, we must have that

$$q_i = r_i \pi,\ r_i \in \mathbf{N} \qquad \text{if } (\tilde{p}_0^{2i-1}, \tilde{p}_0^{2i}) \neq (0, 0),$$

and we set $r_i = 0$ when $\tilde{p}_0^{2i-1} = \tilde{p}_0^{2i} = 0$.

$x_1^{(ij)}(\tilde{\mathbf{p}}_0)$ and $\frac{1}{2}\cdot \|h\|_H^2$ are computed as follows:

i) In the sase $0 = r_1 = r_2$,

$$x_1^{(ij)}(\tilde{\mathbf{p}}_0) = 0 \text{ and } \frac{1}{2}\cdot \|h\|_H^2 = 0.$$

ii) In the case $0 = r_1 < r_2$,

$$x_1^{(ij)}(\tilde{\mathbf{p}}_0) = 0 \qquad \text{if } (ij) \neq (34),$$

$$x_1^{(34)}(\tilde{\mathbf{p}}_0) = \frac{1}{4r_2\pi}\cdot \{(\tilde{p}_0^3)^2 + (\tilde{p}_0^4)^2\}$$

and

$$\frac{1}{2}\cdot \|h\|_H^2 = 2r_2\pi \cdot x_1^{(34)}(\tilde{\mathbf{p}}_0).$$

ii)′ In the case $0 < r_1 = r_2 = r$,

$$x_1^{(12)}(\tilde{\mathbf{p}}_0) = \frac{1}{4r\pi}\cdot \{(\tilde{p}_0^1)^2 + (\tilde{p}_0^2)^2\},$$

$$x_1^{(13)}(\tilde{\mathbf{p}}_0) = \frac{1}{4r\pi}\cdot\{\tilde{p}_0^2\cdot\tilde{p}_0^3 - \tilde{p}_0^1\cdot\tilde{p}_0^4\},$$

$$x_1^{(14)}(\tilde{\mathbf{p}}_0) = \frac{1}{4r\pi}\cdot\{\tilde{p}_0^1\cdot\tilde{p}_0^3 + \tilde{p}_0^2\cdot\tilde{p}_0^4\},$$

$$x_1^{(23)}(\tilde{\mathbf{p}}_0) = \frac{-1}{4r\pi}\cdot\{\tilde{p}_0^1\cdot\tilde{p}_0^3 + \tilde{p}_0^2\cdot\tilde{p}_0^4\},$$

$$x_1^{(24)}(\tilde{\mathbf{p}}_0) = \frac{1}{4r\pi}\cdot\{\tilde{p}_0^2\cdot\tilde{p}_0^3 - \tilde{p}_0^1\cdot\tilde{p}_0^4\},$$

$$x_1^{(34)}(\tilde{\mathbf{p}}_0) = \frac{1}{4r\pi}\cdot\{(\tilde{p}_0^3)^2 + (\tilde{p}_0^4)^2\}$$

and

$$\frac{1}{2}\cdot\|h\|_H^2 = 2r\pi\cdot\{x_1^{(12)}(\tilde{\mathbf{p}}_0) + x_1^{(34)}(\tilde{\mathbf{p}}_0)\}.$$

iii) In the case $0 < r_1 < r_2$,

$$x_1^{(12)}(\tilde{\mathbf{p}}_0) = \frac{1}{4r_1\pi}\cdot\{(\tilde{p}_0^1)^2 + (\tilde{p}_0^2)^2\},$$

$$x_1^{(34)}(\tilde{\mathbf{p}}_0) = \frac{1}{4r_2\pi}\cdot\{(\tilde{p}_0^3)^2 + (\tilde{p}_0^4)^2\},$$

$$x_1^{(ij)}(\tilde{\mathbf{p}}_0) = 0 \qquad \text{otherwise}$$

and

$$\frac{1}{2}\cdot\|h\|_H^2 = 2r_1\pi\cdot x_1^{(12)}(\tilde{\mathbf{p}}_0) + 2r_2\pi\cdot x_1^{(34)}(\tilde{\mathbf{p}}_0).$$

Thus, by setting $\mathbf{x}_1(\tilde{\mathbf{p}}_0) = [0, X(\tilde{\mathbf{p}}_0)]$, we know that:

in the case i), $\operatorname{rank} X(\tilde{\mathbf{p}}_0) = 0$,

in the case ii) or ii)′, $\operatorname{rank} X(\tilde{\mathbf{p}}_0) = 2$

and

in the case iii), $\operatorname{rank} X(\tilde{\mathbf{p}}_0) = 4$.

Therefore the cases that the given matrix X is rank 0 (i.e. $X = O$), rank 2 and rank 4 correspond respectively to the case i), the case ii) or ii)′ and the case iii).

Finally we find the *minimal* horizontal curves x_t connecting $\mathbf{0}$ and $\mathbf{x} = [0, X]$. Equivalently we determine all $h \in K_{\min}^{\mathbf{0},\mathbf{x}}$.

I. The case of $\operatorname{rank} X = 0$, i.e. $X = O$.

In this case clearly $\mathbf{x}_t = \mathbf{0}$ and $h = 0$.

II. The case of rank $X = 2$.

First of all we show that the case ii)′ can be reduced to the case ii). Define $\Theta_\theta \in O(2)$ and $A_{\underline{\theta}}^{(4)} \in O(4)$ by

$$\Theta_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in \mathbf{R},$$

and

$$A_{\underline{\theta}}^{(4)} = \begin{pmatrix} \cos\theta_1 \cdot \Theta_{\theta_2} & -\sin\theta_1 \cdot \Theta_{\theta_3} \\ \sin\theta_1 \cdot \Theta_{\theta_4} & \cos\theta_1 \cdot \Theta_{\theta_5} \end{pmatrix} \tag{4.7}$$

where $\underline{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)$ and $\theta_5 = -\theta_2 + \theta_3 + \theta_4$. Then it is easy to see that for given $Q \in Q(4)$ such that $q_1 = q_2$, $\Omega \in O(4)$ satisfies $\Omega Q {}^t\Omega = Q$ if and only if $\Omega = A_{\underline{\theta}}^{(4)}$, and that for all $z \in \mathbf{R}^4$, there exists $\underline{\theta}$ such that $A_{\underline{\theta}}^{(4)} z = {}^t(0, 0, \tilde{z}_3, \tilde{z}_4)$. Thus if $(\tilde{\mathbf{p}}_0' (= [\tilde{p}_0', \tilde{Q}]), \Omega')$ attains the minimal horizontal curve and furthermore $\tilde{Q}$ is as in ii)′, there exists $\underline{\theta}$ such that

$$T(A_{\underline{\theta}}^{(4)}) \tilde{\mathbf{p}}_0' = [\tilde{\tilde{p}}_0', \tilde{Q}], \quad \tilde{\tilde{p}}_0' = {}^t(0, 0, \tilde{\tilde{p}}_0'^3, \tilde{\tilde{p}}_0'^4).$$

So, by Proposition 4.2 and the invariance of H-norms under the orthogonal mapping, the case ii)′ is reduced to the case ii) (Recall that we set $q_i = 0$ if $\tilde{p}_0^{2i-1} = \tilde{p}_0^{2i} = 0$.). Therefore we only consider the case ii).

Let $U_1 = u(\delta_{34} - \delta_{43})$, $u > 0$, be the matrix satisfying $X \sim U_1$, thus there exists $\Omega \in O(4)$ such that ${}^t\Omega X \Omega = U_1$. All of such Ω are obtained by $\{\Omega_1 A_{\underline{\theta}}^{(2)}; \underline{\theta} = (\theta_1, \theta_2) \in [0, 2\pi)^2\}$, where Ω_1 is an element of $O(4)$ satisfying ${}^t\Omega_1 X \Omega_1 = U_1$ and

$$A_{\underline{\theta}}^{(2)} = \begin{pmatrix} \Theta_{\theta_1} & O \\ O & \Theta_{\theta_2} \end{pmatrix}. \tag{4.8}$$

This is easily seen from the fact that

$${}^t\Omega U_1 \Omega = U_1 \text{ if and only if } \Omega = A_{\underline{\theta}}^{(2)} \text{ for some } \underline{\theta}$$

and that ${}^t\Omega_1' X \Omega_1' = U_1$ implies ${}^t\Omega_1 \Omega_1' U_1 {}^t\Omega_1' \Omega_1 = U_1$.

Since $x_1^{(34)}(\tilde{\mathbf{p}}_0) = u$, $\frac{1}{2} \cdot \|h\|_H^2 = 2r_2 u\pi$ and this takes a minimum when $r_2 = 1$. So $x_1^{(34)}(\tilde{\mathbf{p}}_0) = \frac{1}{4\pi} \cdot \{(\tilde{p}_0^3)^2 + (\tilde{p}_0^4)^2\} = u$, i.e.

$$(\tilde{p}_0^3)^2 + (\tilde{p}_0^4)^2 = 4\pi u.$$

Thus, for some $\alpha \in [0, 2\pi)$, we can write

$$\begin{cases} \tilde{p}_0^3 = \sqrt{4\pi u} \cdot \cos\alpha \\ \tilde{p}_0^4 = \sqrt{4\pi u} \cdot \sin\alpha. \end{cases}$$

Therefore

$$h_t^1(\tilde{\mathbf{p}}_0) = h_t^2(\tilde{\mathbf{p}}_0) = 0$$

and

$$\begin{pmatrix} h_t^3(\tilde{\mathbf{p}}_0) \\ h_t^4(\tilde{\mathbf{p}}_0) \end{pmatrix} = \Theta_\alpha \cdot \begin{pmatrix} \sqrt{u/\pi} \cdot \sin 2\pi t \\ \sqrt{u/\pi} \cdot (1 - \cos 2\pi t) \end{pmatrix}.$$

Noting that $\Theta_\theta \cdot \Theta_\alpha = \Theta_{\theta+\alpha}$, every element of $K_{\min}^{\mathbf{0},\mathbf{x}}$ is obtained by

$$h^{\underline{\theta}} = \Omega_1 A_{\underline{\theta}}^{(2)} \tilde{h},$$

where

$$\tilde{h}_t = {}^t(0, 0, \sqrt{u/\pi} \cdot \sin 2\pi t, \sqrt{u/\pi} \cdot (1 - \cos 2\pi t)). \tag{4.9}$$

Since $\tilde{h}_t^1 = \tilde{h}_t^2 = 0$, we can change $A_{\underline{\theta}}^{(2)}$ to the following $A_\theta^{(1)}$:

$$A_\theta^{(1)} = \begin{pmatrix} I & O \\ O & \Theta_\theta \end{pmatrix}, \quad \theta \in [0, 2\pi). \tag{4.10}$$

Thus every element of $K_{\min}^{\mathbf{0},\mathbf{X}}$ is obtained by

$$h^\theta = \Omega_1 A_\theta^{(1)} \tilde{h}, \quad \theta \in [0, 2\pi). \tag{4.11}$$

III. The case $\operatorname{rank} X = 4$.

III-a) The case $X \sim U_2 = u_1(\delta_{12} - \delta_{21}) + u_2(\delta_{34} - \delta_{43})$, $u_1 > u_2 > 0$.

Similarly to the case II, we know that all $\Omega \in O(4)$ satisfying ${}^t\Omega X \Omega = U_2$ are obtained as in the form $\Omega = \Omega_2 A_{\underline{\theta}}^{(2)}$, $\underline{\theta} = (\theta_1, \theta_2) \in [0, 2\pi)^2$, Ω_2 being any fixed element of $O(4)$ such that ${}^t\Omega_2 X \Omega_2 = U_2$. Also $\frac{1}{2} \cdot \|h\|_H^2 = 2r_1 u_1 \pi + 2r_2 u_2 \pi$, so it takes a minimun when $r_1 = 1$ and $r_2 = 2$. Therefore every element of $K_{\min}^{\mathbf{0},\mathbf{x}}$ is obtained by

$$h^{\underline{\theta}} = \Omega_2 A_{\underline{\theta}}^{(2)} h \tag{4.12}$$

where

$$h_t = {}^t(\sqrt{u_1/\pi} \cdot \sin 2\pi t, \sqrt{u_1/\pi} \cdot (1 - \cos 2\pi t), \sqrt{u_2/2\pi} \cdot \sin 4\pi t, \sqrt{u_2/2\pi} \cdot (1 - \cos 4\pi t)), \tag{4.13}$$

and $A_{\underline{\theta}}^{(2)}$ is as in (4.8).

III-b) The case $X \sim U_3 = u(\delta_{12} - \delta_{21} + \delta_{34} - \delta_{43})$, $u > 0$.

Similarly to the case II or III-a) we know that all Ω satisfying ${}^t\Omega X \Omega = U_3$ are obtained by $\Omega = \Omega_3 A_{\underline{\theta}}^{(4)}$ where Ω_3 is any fixed element of $O(4)$ satisfying ${}^t\Omega_3 X \Omega_3 = U_3$. After all every element of $K_{\min}^{\mathbf{0},\mathbf{x}}$ is obtained by

$$h^{\underline{\theta}} = \Omega_3 A_{\underline{\theta}}^{(4)} h, \quad \underline{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4) \in [0, \pi/2] \times [0, 2\pi)^3, \tag{4.14}$$

where $A_{\underline{\theta}}^{(4)}$ is as in (4.7) and h is given by

$$h_t = {}^t(\sqrt{u/\pi}\cdot\sin 2\pi t,\ \sqrt{u/\pi}\cdot(1-\cos 2\pi t),\ \sqrt{u/2\pi}\cdot\sin 4\pi t,\ \sqrt{u/2\pi}\cdot(1-\cos 4\pi t)). \tag{4.15}$$

5. Asymptotic expansion of the heat kernel on $N_{4,2}$

Here we compute the asymptotic behavior of the heat kernel $p(\varepsilon^2, \mathbf{0}, \mathbf{x})$, $\mathbf{x} = [0, U] \neq \mathbf{0}$. $\mathbf{x}$ is classified into the following three cases (cf. §4).

(Case A) $U \sim u(\delta_{34} - \delta_{43})$, $u > 0$.

(Case B) $U \sim u_1(\delta_{12} - \delta_{21}) + u_2(\delta_{34} - \delta_{43})$, $u_1 > u_2 > 0$.

(Case C) $U \sim u(\delta_{12} - \delta_{21} + \delta_{34} - \delta_{43})$, $u > 0$.

Now consider the following S.D.E. associated to $\mathscr{L}$ on the 4-dimensional Wiener space:

$$\begin{cases} d\mathbf{X}_t = \varepsilon \sum_{\alpha=1}^{4} V_\alpha(\mathbf{X}_t)\circ dw_t^\alpha \\ \mathbf{X}_0 = \mathbf{0} \end{cases} \tag{5.1}$$

where V_α, $\alpha = 1,\cdots,4$, are given in (3.1). We denote the solution by $\mathbf{X}_t^\varepsilon = (X_t^{\varepsilon,i}, X_t^{\varepsilon,(jk)})_{\substack{i=1,\cdots,4 \\ 1\le j<k\le 4}}$. Then $\mathbf{X}_t^\varepsilon$ is obtained in the following concrete form;

$$\begin{cases} X_t^{\varepsilon,i} = \varepsilon w_t^i,\ i = 1,\cdots,4, \\ X_t^{\varepsilon,(jk)} = \varepsilon^2 S^{jk}(t, w),\ 1 \le j < k \le 4, \end{cases}$$

where

$$S^{jk}(t, w) = \frac{1}{2}\int_0^t (w_s^j dw_s^k - w_s^k dw_s^j).$$

Define an $\mathfrak{o}(4)$-valued process $S(t, w)$ by

$$S(t, w) = \sum_{i<j} S^{ij}(t, w)\delta_{ij} - \sum_{i>j} S^{ji}(t, w)\delta_{ij}.$$

Then

$$\begin{aligned} p(\varepsilon^2, \mathbf{0}, \mathbf{x}) &= \mathbf{E}[\delta_{\mathbf{x}}(\mathbf{X}_1^\varepsilon)] \\ &= \mathbf{E}[\delta_{[0,U]}([\varepsilon w_1, \varepsilon^2 S(1, w)])]. \end{aligned}$$

For every $\Omega \in O(4)$, set $U' = {}^t\Omega U \Omega$. Then, recalling Remark 3.1, we see

$$\begin{aligned} &\mathbf{E}[\delta_{[0,U']}([\varepsilon w_1, \varepsilon^2 S(1, w)])] \\ &= \mathbf{E}[\delta_{T({}^t\Omega)[0,U]}([\varepsilon w_1, \varepsilon^2 S(1, w)])] \end{aligned}$$

$$= \mathbf{E}[\delta_{[0,U]}(T(\Omega)[\varepsilon w_1, \varepsilon^2 S(1, w)])]$$
$$= \mathbf{E}[\delta_{[0,U]}([\varepsilon\Omega w_1, \varepsilon^2 S(1, \Omega w)])]$$
$$= \mathbf{E}[\delta_{[0,U]}([\varepsilon w_1, \varepsilon^2 S(1, w)])].$$

Therefore, it is sufficient to treat the following three cases:

(Case A) $U = u(\delta_{34} - \delta_{43})$, $u > 0$.
(Case B) $U = u_1(\delta_{12} - \delta_{21}) + u_2(\delta_{34} - \delta_{43})$, $u_1 > u_2 > 0$.
(Case C) $U = u(\delta_{12} - \delta_{21} + \delta_{34} - \delta_{43})$, $u > 0$.

(Case A) $U = u(\delta_{34} - \delta_{43})$, $u > 0$.

In this case every element h^θ of $K_{\min}^{\mathbf{0},\mathbf{x}}$ is obtained as in (4.11):

$$h^\theta = A_\theta^{(1)}\tilde{h}, \ \theta \in [0, 2\pi),$$

where $A_\theta^{(1)}$ and $\tilde{h}$ are given in (4.10) and (4.9), respectively.

We want to obtain the asymptotic behavior of the heat kernel $p(\varepsilon^2, \mathbf{0}, \mathbf{x})$ as $\varepsilon \downarrow 0$ through the expression $p(\varepsilon^2, \mathbf{0}, \mathbf{x}) = \mathbf{E}[\delta_{\mathbf{x}}(\mathbf{X}_1^\varepsilon)]$ by evaluating the generalized expectation of the right-hand side. Roughly, the family of diffusions $\{\mathbf{X}_t^\varepsilon\}$ conditioned by $\mathbf{X}_1^\varepsilon = \mathbf{x}$ will be concentrated on the family $M^{\mathbf{0},\mathbf{x}}$, actually, will be distributed uniformly on $M^{\mathbf{0},\mathbf{x}}$ as $\varepsilon \downarrow 0$. To see how this fact will be reflected on the asymptotic behavior of $p(\varepsilon^2, \mathbf{0}, \mathbf{x})$, we will proceed as in H.Uemura-S.Watanabe [22].

First, we need the following lemma.

Lemma 5.1.A (cf. H.Uemura-S.Watanabe [22]). *For every fixed $\theta_0 \in [0, 2\pi)$, there exists $\eta_0 > 0$, such that for each η, $0 < \eta < \eta_0$, there exists $\gamma = \gamma(\eta) > 0$ satisfying*

$$\int_{|\theta-\theta_0|<\eta} \delta_0\left(\frac{d}{d\theta}\langle A_\theta^{(1)}\tilde{h}, w\rangle_H\right)\cdot\left(-\frac{d^2}{d\theta^2}\langle A_\theta^{(1)}\tilde{h}, w\rangle_H\right)d\theta = 1$$
$$\text{on } \{w; \|w - A_{\theta_0}^{(1)}\tilde{h}\|_2 < \gamma\}.$$

and

$$\{\theta; \|A_\theta^{(1)}\tilde{h} - A_{\theta_0}^{(1)}\tilde{h}\|_2 < \gamma\} \subset \{\theta; |\theta - \theta_0| < \eta\}. \tag{5.2}$$

Here $\langle h, w\rangle_H$ is the extended H-inner product of $h \in H$ and $w \in W_0^4$ defined by

$$\langle h, w\rangle_H = \sum_{i=1}^4 \int_0^1 \dot{h}_t^i dw_t^i,$$

and $\|\cdot\|_2$ is defined by

$$\|w\|_2^2 = |w_1|^2 + \int_0^1 |w_t|^2 dt, \ w \in W_0^4.$$

Proof. Let $F(\theta, w) = \dfrac{d}{d\theta}\langle A_\theta^{(1)}\tilde{h}, w\rangle_H$ and its Jacobian $\dfrac{d^2}{d\theta^2}\langle A_\theta^{(1)}\tilde{h}, w\rangle_H$ be denoted by $J(\theta, w)$. Clearly $J(\theta, w)$ is continuous with respect to the norm $|\theta| + \|w\|_2$ and it is easy to check that

$$J(\theta_0, A_{\theta_0}^{(1)}\tilde{h}) = -4\pi u(\neq 0).$$

So we can find η_0 and γ_0 such that $J(\theta, w) < 0$ for all $(\theta, w) \in \{\theta; |\theta - \theta_0| < \eta_0\} \times \{w; \|w - A_{\theta_0}^{(1)}\tilde{h}\|_2 < \gamma_0\}$. Furthermore for any $\eta < \eta_0$, we can choose $\gamma = \gamma(\eta) < \gamma_0$ such that for every $w \in \{w; \|w - A_{\theta_0}^{(1)}\tilde{h}\|_2 < \gamma\}$ there exists some $\theta_w \in \{\theta; |\theta - \theta_0| < \eta\}$ satisfying $F(\theta_w, w) = 0$. The reason is as follows:

Let $W_\eta = \{\theta; |\theta - \theta_0| < \eta\}$ and $F_w^\eta = \{F(\theta, w); \theta \in W_\eta\}$. That $0 \in F_{\tilde{h}\theta_0}^\eta$ is easily seen from that $F(\theta_0, A_{\theta_0}^{(1)}\tilde{h}) = 0$. On the other hand it is easy to show that if $x \in F_{w_0}^\eta \cap (\bigcup_{w_n \neq w_0, w_n \to w_0} F_{w_n}^\eta)^c$, then $x \in \partial F_{w_0}^\eta$. But F_w^η is open and hence if $x \in F_{w_0}^\eta$, there exists $\gamma(\eta) > 0$ such that $x \in F_w^\eta$ for all w satisfying $\|w - w_0\|_2 < \gamma(\eta)$. Setting $w_0 = A_{\theta_0}^{(1)}\tilde{h}$ and $x = 0$, we conclude the above statement.

Let $G(w) \in \tilde{\mathbf{D}}^\infty$ be a Wiener functional whose support is contained in $\{w; \|w - A_{\theta_0}^{(1)}\tilde{h}\|_2 < \gamma\}$. Then

$$\begin{aligned}
&\mathbf{E}\left[\int_{|\theta - \theta_0| < \eta} \delta_0(F(\theta, w)) \cdot (-J(\theta, w))\, d\theta\, G(w)\right] \\
&= \int_{|\theta - \theta_0| < \eta} \mathbf{E}[\delta_0(F(\theta, w)) \cdot (-J(\theta, w)) \cdot G(w)]\, d\theta \\
&= \lim_{n \uparrow \infty} \int_{|\theta - \theta_0| < \eta} \mathbf{E}[\varphi_n(F(\theta, w)) \cdot (-J(\theta, w)) \cdot G(w)]\, d\theta.
\end{aligned}$$

Here $\{\varphi_n\}$ is a sequence in $\mathscr{S}(\mathbf{R}^d)$(:= the Schwartz space of rapidly decreasing C^∞-functions on $\mathbf{R}^d$) which converges to δ_0 in the distribution sense. Now clearly (5.2) is satisfied for all γ small enough. Note that the support of $G(w)$ is contained in $\{w; \|w - A_{\theta_0}^{(1)}\tilde{h}\|_2 < \gamma\}$. Thus, by the change of variable $x = F(\theta, w)$, the above is equal to

$$\begin{aligned}
&\lim_{n \uparrow \infty} \mathbf{E}\left[\int_{F_w^\eta} \varphi_n(x)\, dx\, G(w)\right] \\
&= \mathbf{E}[G(w)],
\end{aligned}$$

and this completes the proof.

Remark 5.1. We can easily show that

$$-\frac{d^2}{d\theta^2}\langle A_\theta^{(1)}\tilde{h}, w\rangle_H = \langle A_\theta^{(1)}\tilde{h}, w\rangle_H$$

and

$$\frac{d}{d\theta}\langle A_\theta^{(1)}\tilde{h}, w\rangle_H = \langle A_{\theta+(\pi/2)}^{(1)}\tilde{h}, w\rangle_H,$$

so the equality in Lemma 5.1.A is equivalent to

$$\int_{|\theta-\theta_0|<\eta} \delta_0(\langle A^{(1)}_{\theta+(\pi/2)}\tilde{h}, w\rangle_H)\cdot\langle A^{(1)}_\theta \tilde{h}, w\rangle_H \, d\theta = 1.$$

Since $K_1 := K^{\mathbf{0},\mathbf{x}}_{\min}$ is compact, for all $\gamma > 0$, there exist $\{h^{\tilde{\theta}_1}, \cdots, h^{\tilde{\theta}_n}\} \subset K_1$ such that $K_1 \subset \bigcup_{i=1}^n V_i$ where

$$V_i = \{w \in W_0^4;\ \| w - h^{\tilde{\theta}_i}\|_2^2 < \gamma^2/2\}.$$

Set

$$U_i = \{w \in W_0^4;\ \| w - h^{\tilde{\theta}_i}\|_2^2 < \gamma^2\} \supset V_i.$$

Let $\psi(\xi) \in C^\infty(\mathbf{R})$ satisfy $0 \le \psi \le 1$, $\psi(\xi) = 1$ on $|\xi| \le \gamma^2/2$ and $\psi(\xi) = 0$ on $|\xi| \ge \gamma^2$. Set $\Psi_i(w) = \psi(\| w - h^{\tilde{\theta}_i}\|_2^2)$. Then it is easy to see that $\Psi_i \in \mathbf{D}^\infty$ and

$$I_{U_i}(w) \ge \Psi_i(w) \ge I_{V_i}(w).$$

Setting $\Phi(w) = 1 - \prod_{i=1}^n (1 - \Psi_i(w))$, we see clearly

$$1 - \Phi(w) \le I_{\bigcap_{i=1}^n V_i^c}$$

and $\bigcap_{i=1}^n V_i^c$ is a closed set which is disjoint from K_1. Now

$$\begin{aligned} p(\varepsilon^2, \mathbf{0}, \mathbf{x}) &= \mathbf{E}[\delta_{\mathbf{x}}(\mathbf{X}_1^\varepsilon)] \\ &= \mathbf{E}[\delta_{\mathbf{x}}(\mathbf{X}_1^\varepsilon)(1 - \Phi(\varepsilon w))] + \mathbf{E}[\delta_{\mathbf{x}}(\mathbf{X}_1^\varepsilon)\Phi(\varepsilon w)] \\ &= J_1^{(1)} + J_2^{(1)}. \end{aligned}$$

Here γ which appears in the definition of Φ is the constant $\gamma(\eta)$ in Lemma 5.1.A associated with η which will be decided in Lemma 5.4 below.

Lemma 5.2 (cf. S. Watanabe [24] Lemma 3.3). *There exists a constant $c > 0$ such that*

$$J_1^{(1)}(= \mathbf{E}[\delta_{\mathbf{x}}(\mathbf{X}_1^\varepsilon)(1 - \Phi(\varepsilon w))]) = O(\exp\{-(\|\tilde{h}\|_H^2 + c)/2\varepsilon^2\}).$$

Proof. Clearly for every $\delta > 0$,

$$\begin{aligned} &\mathbf{E}[\delta_{\mathbf{x}}(\mathbf{X}_1^\varepsilon)(1 - \Phi(\varepsilon w))] \\ &= \mathbf{E}[\delta_{\mathbf{x}}(\mathbf{X}_1^\varepsilon)\cdot\psi(|X_1^\varepsilon - \mathbf{x}|^2/\gamma^2)\cdot(1 - \Phi(\varepsilon w))]. \end{aligned}$$

By an integration by parts, the above integral can be given in the form

$$\sum \mathbf{E}[P_k(\varepsilon, w)\psi^{(l)}(|\mathbf{X}_1^\varepsilon - \mathbf{x}|^2/\delta^2)\prod_{i=1}^n (1 - \psi)^{(m_i)}(\| \varepsilon w - h^{\tilde{\theta}_i}\|_2^2)\varphi(\mathbf{X}_1^\varepsilon)],$$

where $P_k(\varepsilon, w)$ is a polinomial of $\mathbf{X}_1^\varepsilon$, $|\mathbf{X}_1^\varepsilon - \mathbf{x}|^2$, $\|\varepsilon w - h^{\tilde{\theta}_i}\|_2^2$, $\gamma(\varepsilon)$(:= the inverse of the Malliavin covariance of $\mathbf{X}_1^\varepsilon$) and their derivatives, and φ is a bounded continuous function on $\mathbf{R}^{10}$. Appealing to S.Kusuoka-D.W.Stroock [12], we know

$$\mathbf{E}[|P_k(\varepsilon, w)|^p]^{1/p} = O(\varepsilon^{-k}) \qquad \text{for some } k \in \mathbf{N}.$$

Thus there exists a constant M such that

$$J_1^{(1)} \leq \varepsilon^{-l} M \cdot P[|\mathbf{X}_1^\varepsilon - \mathbf{x}| \leq \delta\gamma,\ \varepsilon w \in \bigcap_{i=1}^n V_i^c]^{1/q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. By R.Azencott [1], we have

$$\begin{aligned}
&\overline{\lim_{\varepsilon \downarrow 0}}\ \varepsilon^2 \log P[|\mathbf{X}_1^\varepsilon - \mathbf{x}| \leq \delta\gamma,\ \varepsilon w \in \bigcap_{i=1}^n V_i^c] \\
&\leq -\inf\left\{\frac{1}{2} \cdot \|h\|_H^2;\ |c^{\mathbf{0},h}(1) - \mathbf{x}| \leq \delta\gamma,\ h \in \bigcap_{i=1}^n V_i^c\right\}.
\end{aligned}$$

Now the right-hand side of the above inequality is strictly less than $-\frac{1}{2} \cdot \|\tilde{h}\|_H^2$ by taking δ small enough, because, otherwise, by taking $\delta = 1/m$, there exist $h_m \in H$ satisfying $|c^{\mathbf{0},h_m}(1) - \mathbf{x}| \leq \gamma/m$, $h_m \in \bigcap_{i=1}^n V_i^c$ and $\overline{\lim}\ \|h_m\|_H^2 \leq \|\tilde{h}\|_H^2$. Then taking a subsequence $\{h_{m'}\}$ of $\{h_m\}$, there exists $\bar{h}$ such that $h_{m'} \to \bar{h}$ weakly. Such $\bar{h}$ satisfies $\|\bar{h}\|_H^2 \leq \|\tilde{h}\|_H^2$, $c^{\mathbf{0},\bar{h}}(1) = \mathbf{x}$ and $\bar{h} \in \bigcap_{i=1}^n V_i^c$. Therefore $\bar{h} \in K_1$ and this is a contradiction because $\bigcap_{i=1}^n V_i^c$ and K_1 are disjoint.

This completes the proof.

In the following, therefore, we consider $J_2^{(1)}$. Let

$$\Phi = 1 - \prod_{i=1}^n (1 - \Psi_i) = \sum_{i=1}^n \Phi_i,$$

where $\Phi_1 = \Psi_1$, $\Phi_2 = \Psi_2(1 - \Psi_1)$, $\Phi_3 = \Psi_3(1 - \Psi_1)(1 - \Psi_2), \cdots$. Then clearly $\Phi_i \cdot I_{U_i} = \Phi_i$, $i = 1, \cdots, n$. By Lemma 5.1.A and Remark 5.1,

$$\int_{|\theta - \tilde{\theta}_i| < \eta} \delta_0(\langle A_{\theta+(\pi/2)}^{(1)} \tilde{h}, w\rangle_H) \langle A_\theta^{(1)} \tilde{h}, w\rangle_H\, d\theta \cdot \Phi_i(w) = \Phi_i(w), \qquad i = 1, \cdots, n.$$

So

$$\begin{aligned}
J_2^{(1)} &= \mathbf{E}[\delta_{\mathbf{x}}(\mathbf{X}_1^\varepsilon)\Phi(\varepsilon w)] \\
&= \sum_{i=1}^n \mathbf{E}[\delta_{\mathbf{x}}(\mathbf{X}_1^\varepsilon)\Phi_i(\varepsilon w)]
\end{aligned}$$

$$= \sum_{i=1}^{n} \mathbf{E}[\delta_{[0,U]}([\varepsilon w_1, \varepsilon^2 S(1, w)])\Phi_i(\varepsilon w)]$$

$$= \sum_{i=1}^{n} \int_{|\theta - \tilde{\theta}_i| < \eta} \mathbf{E}[\delta_{[0,U]}([\varepsilon w_1, \varepsilon^2 S(1, w)])\delta_0(\langle h^{\theta + (\pi/2)}, \varepsilon w\rangle_H) \cdot \langle h^\theta, \varepsilon w\rangle_H \cdot \Phi_i(\varepsilon w)]\, d\theta$$

$$= \sum_{i=1}^{n} \int_{|\theta - \tilde{\theta}_i| < \eta} \exp(-\|h^\theta\|_H^2/2\varepsilon^2) \cdot \mathbf{E}[\exp(-\langle h^\theta, w\rangle_H/\varepsilon) \times \delta_{[0,O]}((\varepsilon w_1, \varepsilon \int_0^1 (h_s^{\theta,i}\, dw_s^j - h_s^{\theta,j}\, dw_s^i) + \varepsilon^2 S^{ij}(1, w))) \times \delta_0(\langle h^{\theta + (\pi/2)}, h^\theta + \varepsilon w\rangle_H) \cdot \langle h^\theta, h^\theta + \varepsilon w\rangle_H \cdot \Phi_i(h^\theta + \varepsilon w)]\, d\theta,$$

Where the last equality is due to the Cameron-Martin transformation (abbr. C-M transformation) $w \to w + (h^\theta/\varepsilon)$. Now we give some notations.

Notations. For $w, \tilde{w} \in W_0^4$, we define 4×4 matrices $w \otimes \tilde{w}$, $\dot{w} \otimes \tilde{w}$ and $w \otimes \dot{\tilde{w}}$ as follows:

$$(w \otimes \tilde{w})_{ij} = \int_0^1 w_t^i \cdot \tilde{w}_t^j\, dt,$$

$$(\dot{w} \otimes \tilde{w})_{ij} = \int_0^1 \tilde{w}_t^j\, dw_t^i$$

and

$$(w \otimes \dot{\tilde{w}})_{ij} = \int_0^1 w_t^i\, d\tilde{w}_t^j.$$

Of course, we define them only when the right-hand sides have meaning as ordinary or stochastic integrals.

Remark 5.2. It is easy to see that

$$S(1, w) = \frac{1}{2}(w \otimes \dot{w} - \dot{w} \otimes w)$$

and that, for every 4×4 matrix A,

$$(Aw) \otimes \tilde{w} = A(w \otimes \tilde{w}),\ w \otimes (A\tilde{w}) = (w \otimes \tilde{w})^t A,$$

$$(A\dot{w}) \otimes \tilde{w} = A(\dot{w} \otimes \tilde{w}),\ \dot{w} \otimes (A\tilde{w}) = (\dot{w} \otimes \tilde{w})^t A,$$

$$(Aw) \otimes \dot{\tilde{w}} = A(w \otimes \dot{w}) \text{ and } (w \otimes A\dot{\tilde{w}}) = (w \otimes \dot{\tilde{w}})^t A.$$

Then

$$J_2^{(1)} = \sum_{i=1}^{n} \int_{|\theta - \tilde{\theta}_i| < \eta} \exp(-\|A_\theta^{(1)}\tilde{h}\|_H^2/2\varepsilon^2) \cdot \mathbf{E}[\exp(-\langle A_\theta^{(1)}\tilde{h}, w\rangle_H/\varepsilon)$$

$$\times \delta_0(\varepsilon w_1)\delta_O(\varepsilon(A_\theta^{(1)}\tilde{h} \otimes \dot{w} - \dot{w} \otimes A_\theta^{(1)}\tilde{h}) + \frac{\varepsilon^2}{2}(w \otimes \dot{w} - \dot{w} \otimes w))$$
$$\times \delta_0(\langle A_{\theta+(\pi/2)}^{(1)}\tilde{h}, A_\theta^{(1)}\tilde{h} + \varepsilon w\rangle_H)$$
$$\times \langle A_\theta^{(1)}\tilde{h}, A_\theta^{(1)}\tilde{h} + \varepsilon w\rangle_H \cdot \Phi_i(A_\theta^{(1)}\tilde{h} + \varepsilon w)]\, d\theta$$

and noting that $A_\theta^{(1)} \in O(4)$ and Remark 5.2, this is equal to

$$\exp(-\|\tilde{h}\|_H^2/2\varepsilon^2) \cdot \sum_{i=1}^n \int_{|\theta-\tilde{\theta}_i|<\eta} \mathbf{E}[\exp(-\langle \tilde{h}, {}^tA_\theta^{(1)}w\rangle_H/\varepsilon)$$
$$\times \delta_0(\varepsilon A_\theta^{(1)} \cdot {}^tA_\theta^{(1)}w_1)$$
$$\times \delta_O(A_\theta^{(1)}\{\varepsilon(\tilde{h} \otimes {}^tA_\theta^{(1)}\dot{w} - {}^tA_\theta^{(1)}\dot{w} \otimes \tilde{h})$$
$$+ \frac{\varepsilon^2}{2}({}^tA_\theta^{(1)}w \otimes {}^tA_\theta^{(1)}\dot{w} - {}^tA_\theta^{(1)}\dot{w} \otimes {}^tA_\theta^{(1)}w)\}\, {}^tA_\theta^{(1)})$$
$$\times \delta_0(\langle {}^tA_\theta^{(1)} \cdot A_{\theta+(\pi/2)}^{(1)}\tilde{h}, \tilde{h} + \varepsilon \cdot {}^tA_\theta^{(1)}w\rangle_H)$$
$$\times \langle \tilde{h}, \tilde{h} + \varepsilon \cdot {}^tA_\theta^{(1)}w\rangle_H \cdot \Phi_i(A_\theta^{(1)}(\tilde{h} + \varepsilon \cdot {}^tA_\theta^{(1)}w))]\, d\theta.$$

By the invariance of Wiener measure under an orthogonal transformation, we see, noting also that ${}^tA_\theta^{(1)} \cdot A_{\theta+(\pi/2)}^{(1)} = A_{\pi/2}^{(1)}$,

$$J_2^{(1)} = \exp(-\|\tilde{h}\|_H^2/2\varepsilon^2) \cdot \sum_{i=1}^n \int_{|\theta-\tilde{\theta}_i|<\eta} \mathbf{E}[\exp(-\langle \tilde{h}, w\rangle_H/\varepsilon)$$
$$\times \delta_{[0,O]}(T(A_\theta^{(1)})[\varepsilon w_1, \varepsilon(\tilde{h} \otimes \dot{w} - \dot{w} \otimes \tilde{h}) + \frac{\varepsilon^2}{2}(w \otimes \dot{w} - \dot{w} \otimes w)])$$
$$\times \delta_0(\langle A_{\pi/2}^{(1)}\tilde{h}, \tilde{h} + \varepsilon w\rangle_H)$$
$$\times \langle \tilde{h}, \tilde{h} + \varepsilon w\rangle_H \cdot \Phi_i(A_\theta^{(1)}\tilde{h} + \varepsilon A_\theta^{(1)}w)]\, d\theta.$$

Since $\langle A_{\pi/2}^{(1)}\tilde{h}, \tilde{h}\rangle_H = 0$, $-\langle \tilde{h}, w\rangle_H/\varepsilon = 2\pi \cdot S^{34}(1, w)$ under the condition that $(\tilde{h} \otimes \dot{w} - \dot{w} \otimes \tilde{h}) + \frac{\varepsilon}{2}(w \otimes \dot{w} - \dot{w} \otimes w) = 0$ and that $A_\theta^{(1)}w_1 = 0$ (note that $\tilde{h}^1 = \tilde{h}^2 \equiv 0$) and $T(A_\theta^{(1)}) \in O(10)$, we have finally,

$$J_2^{(1)} = \exp(-\|\tilde{h}\|_H^2/2\varepsilon^2) \cdot \sum_{i=1}^n \int_{|\theta-\tilde{\theta}_i|<\eta} \mathbf{E}[\exp\{2\pi \cdot S^{34}(1, w)\}$$
$$\times \delta_0(\varepsilon w_1)\delta_O(\varepsilon(\tilde{h} \otimes \dot{w} - \dot{w} \otimes \tilde{h}) + \frac{\varepsilon^2}{2}(w \otimes \dot{w} - \dot{w} \otimes w))$$
$$\times \delta_0(\langle A_{\pi/2}^{(1)}\tilde{h}, \varepsilon w\rangle_H)$$
$$\times \langle \tilde{h}, \tilde{h} + \varepsilon w\rangle_H \cdot \Phi_i(A_\theta^{(1)}\tilde{h} + \varepsilon A_\theta^{(1)}w)]\, d\theta.$$

Define $\mathbf{R}^{11}$-valued Wiener functional $g_0^{(1)}(w)$ by

$$(5.3) \qquad g_0^{(1)}(w) = (w_1, S^{12}(1, w), (\tilde{h} \otimes \dot{w} - \dot{w} \otimes \tilde{h})_{ij,\, 1 \le i < j \le 4,\, (i,j) \ne (1,2)}, \langle A_{\pi/2}^{(1)}\tilde{h}, w\rangle_H),$$

then by Lemma 5.4 and Lemma 5.5, given below, we can conclude that

$$J_2^{(1)} \sim \exp(-\|\tilde{h}\|_H^2/2\varepsilon^2)\cdot\varepsilon^{-12}\sum_{i=1}^{n}\int_{|\theta-\tilde{\theta}_i|<\eta}\Phi_i(A_\theta^{(1)}\tilde{h})\,d\theta \tag{5.4}$$

$$\mathbf{E}[\exp\{2\pi\cdot S^{34}(1,w)\}\,\delta_0(g_0^{(1)}(w))]\cdot\|\tilde{h}\|_H^2 \qquad \text{as } \varepsilon\downarrow 0.$$

Lemma 5.3.A. $\mathbf{E}[\exp\{2\pi S^{34}(1,w)\}\,\delta_0(g_0^{(1)}(w))] = \dfrac{3}{2^7\pi^3u^3}$.

Proof. Define $\xi_k^{(i)}$, $\eta_k^{(i)}$, $k = 1, 2, \cdots$, and $\eta_0^{(i)}$, $i = 1, \cdots, 4$, by

$$\xi_k^{(i)} = \sqrt{2}\int_0^1 \sin 2\pi kt\, dw_t^i,\quad k = 1, 2, \cdots,\ i = 1, \cdots, 4, \tag{5.5}$$

$$\eta_k^{(i)} = \sqrt{2}\int_0^1 \cos 2\pi kt\, dw_t^i,\ k = 1, 2, \cdots,\ i = 1, \cdots, 4,$$

and

$$\eta_0^{(i)} = w_1^i, \qquad i = 1, \cdots, 4.$$

Then we can easily show that

$$S^{ij}(1,w) = \frac{1}{2\pi}\left[\sum_{k=1}^{\infty}\frac{1}{k}\{\xi_k^{(j)}(\eta_k^{(i)} - \sqrt{2}\cdot\eta_0^{(i)}) - \xi_k^{(i)}(\eta_k^{(j)} - \sqrt{2}\cdot\eta_0^{(j)})\}\right],$$

$$1 \le i < j \le 4,$$

$$(\tilde{h}\otimes\dot{w} - \dot{w}\otimes\tilde{h})_{13} = -\sqrt{u/2\pi}\,\xi_1^{(1)},$$

$$(\tilde{h}\otimes\dot{w} - \dot{w}\otimes\tilde{h})_{14} = \sqrt{u/2\pi}\,(\eta_1^{(1)} - \sqrt{2}\cdot\eta_0^{(1)}),$$

$$(\tilde{h}\otimes\dot{w} - \dot{w}\otimes\tilde{h})_{23} = -\sqrt{u/2\pi}\,\xi_1^{(2)},$$

$$(\tilde{h}\otimes\dot{w} - \dot{w}\otimes\tilde{h})_{24} = \sqrt{u/2\pi}\,(\eta_1^{(2)} - \sqrt{2}\cdot\eta_0^{(2)}),$$

$$(\tilde{h}\otimes\dot{w} - \dot{w}\otimes\tilde{h})_{34} = \sqrt{u/2\pi}\,\{\xi_1^{(4)} + (\eta_1^{(3)} - \sqrt{2}\cdot\eta_0^{(3)})\}$$

and

$$\langle A_{\pi/2}^{(1)}\tilde{h}, w\rangle_H = \sqrt{2\pi u}\,(\xi_1^{(3)} - \eta_1^{(4)}).$$

Thus

$$\mathbf{E}[\exp\{2\pi S^{34}(1,w)\}\,\delta_0(g_0^{(1)}(w))]$$

$$= \mathbf{E}[\delta_0(\eta_0^{(1)}, \eta_0^{(2)}, S^{12}(1,w), -\sqrt{u/2\pi}\cdot\xi_1^{(1)}, \sqrt{u/2\pi}(\eta_1^{(1)} - \sqrt{2}\cdot\eta_0^{(1)}),$$

$$-\sqrt{u/2\pi}\cdot\xi_1^{(2)}, \sqrt{u/2\pi}(\eta_1^{(2)} - \sqrt{2}\cdot\eta_0^{(2)}))]$$

$$\times\ \mathbf{E}[\exp\{\sum_{k=1}^{\infty}\frac{1}{k}\{\xi_k^{(4)}(\eta_k^{(3)} - \sqrt{2}\cdot\eta_0^{(3)}) - \xi_k^{(3)}(\eta_k^{(4)} - \sqrt{2}\cdot\eta_0^{(4)})\}\}$$

$$\delta_0(\eta_0^{(3)}, \eta_0^{(4)}, \sqrt{u/2\pi}\{\xi_1^{(4)} + (\eta_1^{(3)} - \sqrt{2}\cdot\eta_0^{(3)})\}, \sqrt{2\pi u}(\xi_1^{(3)} - \eta_1^{(4)}))]$$

$$= J_3^{(1)} \times J_4^{(1)}.$$

By Proposition 5.1 below, we see that $J_3^{(1)} = \frac{3}{16\pi u^2}$. On the other hand,

$$J_4^{(1)} = \mathbf{E}\left[\exp\left\{\sum_{k=2}^{\infty} \frac{1}{k}(\xi_k^{(4)}\eta_k^{(3)} - \xi_k^{(3)}\eta_k^{(4)})\right\}\right]$$
$$\times \mathbf{E}[\exp(\xi_1^{(4)}\eta_1^{(3)} - \xi_1^{(3)}\eta_1^{(4)}) | \eta_0^{(3)} = \eta_0^{(4)} = \xi_1^{(4)} + \eta_1^{(3)} = \xi_1^{(3)} - \eta_1^{(4)} = 0]$$
$$\times \{(2\pi)^2 \cdot \sqrt{\det C}\}^{-1},$$

where C is the covariant matrix of $(\eta_0^{(3)}, \eta_0^{(4)}, \sqrt{u/2\pi}\,(\xi_1^{(4)} + \eta_1^{(3)}), \sqrt{2\pi u}\,(\xi_1^{(3)} - \eta_1^{(4)}))$ and it is easy to see that $\det C = 4u^2$. So, by a slight computation, we have

$$J_4^{(1)} = \prod_{k=2}^{\infty}\left(1 - \frac{1}{k^2}\right)^{-1}$$
$$\times \mathbf{E}\left[\exp\left(-\frac{1}{2}(\{(\xi_1^{(4)} - \eta_1^{(3)})/\sqrt{2}\}^2 + \{(\xi_1^{(3)} + \eta_1^{(4)})/\sqrt{2}\}^2)\right)\right] \times \frac{1}{8\pi^2 u}$$
$$= \prod_{k=2}^{\infty}\left(1 - \frac{1}{k^2}\right)^{-1} \times \left((1/\sqrt{2\pi})\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 \times \frac{1}{8\pi^2 u}$$
$$= \frac{1}{8\pi^2 u}.$$

Thus the assertion of this lemma is concluded.

Proposition 5.1. *Let $J_3^{(1)}$ be as in the proof of above lemma. Then*

$$J_3^{(1)} = \frac{3}{16\pi u^2}.$$

Proof. Noting that

$$\int_{-\infty}^{\infty} e^{-2\pi i t x} dt = \delta_x,$$

it is easy to see that

$$J_3^{(1)} = \int_{-\infty}^{\infty} \mathbf{E}\left[\exp\left\{-2\pi i t \cdot \sum_{k=1}^{\infty} \frac{1}{k}(\xi_k^{(1)}\eta_k^{(2)} - \xi_k^{(2)}\eta_k^{(1)})\right\}\right.$$
$$\left. | \eta_0^{(1)} = \eta_0^{(2)} = \xi_1^{(1)} = \xi_1^{(2)} = \eta_1^{(1)} = \eta_1^{(2)} = 0\right] \cdot p_1(0)\, dt,$$

where $p_1(x)$ is the density of the law of $(\eta_0^{(1)}, \eta_0^{(2)}, -\sqrt{u/2\pi} \cdot \xi_1^{(1)}, \sqrt{u/2\pi} \cdot (\eta_1^{(1)} - \sqrt{2} \cdot \eta_0^{(1)}), -\sqrt{u/2\pi} \cdot \xi_1^{(2)}, \sqrt{u/2\pi} \cdot (\eta_1^{(2)} - \sqrt{2} \cdot \eta_0^{(2)}))$ and $p_1(0) = \frac{1}{2\pi u^2}$. By a slight computation, the above conditional expectation is equal to

$$\prod_{k=2}^{\infty} \mathbf{E}\left[\exp\left\{-2\pi i \cdot \frac{t}{k}(\xi_k^{(1)}\eta_k^{(2)} - \xi_k^{(2)}\eta_k^{(1)})\right\}\right]$$

$$= \prod_{k=2}^{\infty} \frac{k^2}{4\pi^2 t^2 + k^2}$$

$$= (1 + 4\pi^2 t^2)\frac{2\pi^2 t}{\sinh 2\pi^2 t}.$$

Thus

$$J_3^{(1)} = \frac{1}{2\pi u^2}\int_{-\infty}^{\infty}(1 + 4\pi^2 t^2)\frac{2\pi^2 t}{\sinh 2\pi^2 t}dt$$

$$= \frac{3}{16\pi u^2}.$$

It is easy to see that $\|\tilde{h}\|_H^2 = 4\pi u$ and hence,

$$J_2^{(1)} \sim \exp\left(\frac{2\pi u}{\varepsilon^2}\right)\varepsilon^{-12}\frac{3}{2^5\pi^2 u^2}\sum_{i=1}^{n}\int_{|\theta - \tilde{\theta}_i| < \eta} \Phi_i(A_\theta^{(1)}\tilde{h})\,d\theta.$$

Now

$$\sum_{i=1}^{n}\int_{|\theta - \tilde{\theta}_i| < \eta} \Phi_i(A_\theta^{(1)}\tilde{h})\,d\theta$$

$$= \sum_{i=1}^{n}\int_{|\theta - \tilde{\theta}_i| < \theta} I_{U_i}(A_\theta^{(1)}\tilde{h})\cdot\Phi_i(A_\theta^{(1)}\tilde{h})\,d\theta$$

$$= \sum_{i=1}^{n}\int_{0}^{2\pi} I_{U_i}(A_\theta^{(1)}\tilde{h})\cdot\Phi_i(A_\theta^{(1)}\tilde{h})\,d\theta$$

$$= \sum_{i=1}^{n}\int_{0}^{2\pi} \Phi_i(A_\theta^{(1)}\tilde{h})\,d\theta = 2\pi.$$

We have, therefore,

$$J_2^{(1)} \sim \exp\left(-\frac{2\pi u}{\varepsilon^2}\right)\varepsilon^{-12}\frac{3}{16\pi u^2} \qquad \text{as } \varepsilon \downarrow 0.$$

Therefore, we can now conclude the following.

Theorem 5.1.A. *In* Case A, *i.e.*, $\mathbf{x} = [0, U]$, $U \sim u(\delta_{34} - \delta_{43})$, $u > 0$,

$$p(\varepsilon^2, \mathbf{0}, \mathbf{x}) \sim \exp\left(-\frac{2\pi u}{\varepsilon^2}\right)\varepsilon^{-12}\frac{3}{16\pi u^2} \qquad \textit{as } \varepsilon \downarrow 0.$$

(Case B) $U = u_1(\delta_{12} - \delta_{21}) + u_2(\delta_{34} - \delta_{43})$, $u_1 > u_2 > 0$.

In this case every element $h^{\underline{\theta}}$ of $K_{\min}^{\mathbf{0},\mathbf{x}}$ is obtained as in (4.12):

$$h^{\underline{\theta}} = A_{\underline{\theta}}^{(2)}h, \ \underline{\theta} = (\theta_1, \theta_2) \in [0, 2\pi)^2,$$

where $A_{\underline{\theta}}^{(2)}$ and h are as in (4.8) and (4.13), respectively. We set $h^{[1]}$ and $h^{[2]}$ by

$$h_t^{[1]} = {}^t(\sqrt{u_1/\pi}\sin 2\pi t,\ \sqrt{u_1/\pi}(1-\cos 2\pi t),\ 0,\ 0)$$

and

$$h_t^{[2]} = {}^t(0,\ 0,\ \sqrt{u_2/2\pi}\sin 4\pi t,\ \sqrt{u_2/2\pi}(1-\cos 4\pi t)).$$

Similarly as in Case A, we need only to evaluate

$$J_2^{(2)} := \mathbf{E}[\delta_{\mathbf{x}}(\mathbf{X}_1^\varepsilon)\,\Phi(\varepsilon w)]$$

where $\mathbf{X}_t^\varepsilon$ is a solution of S.D.E. (5.1) and Φ is defined as in Case A associated with $K_2 := K_{\min}^{\mathbf{0},\mathbf{x}}$, $\mathbf{x} = [0, U]$. Again $\gamma(\eta)$ used in the definition of Φ is given by the following lemma with η determined by Lemma 5.4 below. In the following we use the same notations as in Case A.

Lemma 5.1.B. *For every $\underline{\theta}_0 \in [0, 2\pi)^2$, there exists $\eta_0 > 0$ such that for each $\eta \in (0, \eta_0)$ there exists $\gamma(\eta) > 0$ satisfying*

$$\int_{|\underline{\theta}-\underline{\theta}_0|<\eta} \delta_0\left(\left(\frac{\partial}{\partial\theta_i}\langle A_{\underline{\theta}}^{(2)}h, w\rangle_H\right)_{i=1,2}\right) \det\left\{\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\langle A_{\underline{\theta}}^{(2)}h, w\rangle_H\right)_{i,j=1,2}\right\} d\theta_1 d\theta_2 = 1$$
$$on\ \{w;\ \|w - A_{\underline{\theta}_0}^{(2)}h\|_2 < \gamma\}$$

and

$$\{\underline{\theta};\ \|A_{\underline{\theta}}^{(2)}h - A_{\underline{\theta}_0}^{(2)}h\|_2 < \gamma\} \subset \{\underline{\theta};\ |\underline{\theta} - \underline{\theta}_0| < \eta\}.$$

Proof is similar to Lemma 5.1.A and omitted.

Remark 5.3. It is easy to see that

$$\frac{\partial}{\partial\theta_1}\langle A_{\underline{\theta}}^{(2)}h, w\rangle_H = \langle A_{(\theta_1+(\pi/2),\theta_2)}^{(2)}h^{[1]}, w\rangle_H,$$

$$\frac{\partial}{\partial\theta_2}\langle A_{\underline{\theta}}^{(2)}h, w\rangle_H = \langle A_{(\theta_1,\theta_2+(\pi/2))}^{(2)}h^{[2]}, w\rangle_H$$

and

$$\det\left\{\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\langle A_{\underline{\theta}}^{(2)}h, w\rangle_H\right)_{i,j=1,2}\right\} = \langle A_{\underline{\theta}}^{(2)}h^{[1]}, w\rangle_H \cdot \langle A_{\underline{\theta}}^{(2)}h^{[2]}, w\rangle_H.$$

Thus, denoting $d\underline{\theta} = d\theta_1 d\theta_2$,

$$J_2^{(2)} = \mathbf{E}[\delta_{\mathbf{x}}(\mathbf{X}_1^\varepsilon)\Phi(\varepsilon w)]$$

$$= \sum_{i=1}^{n} \mathbf{E}[\delta_{\mathbf{x}}(\mathbf{X}_1^\varepsilon)\Phi_i(\varepsilon w)]$$

$$= \sum_{i=1}^{n} \int_{|\underline{\theta}-\tilde{\underline{\theta}}_i|<\eta} \mathbf{E}[\delta_0(\varepsilon w_1)\delta_U(\varepsilon^2 S(1, w))$$
$$\times\ \delta_0(\langle A^{(2)}_{(\theta_1+(\pi/2),\theta_2)}h^{[1]}, \varepsilon w\rangle_H)\cdot\langle A^{(2)}_{\underline{\theta}}h^{[1]}, \varepsilon w\rangle_H$$
$$\times\ \delta_0(\langle A^{(2)}_{(\theta_1,\theta_2+(\pi/2))}h^{[2]}, \varepsilon w\rangle_H)\cdot\langle A^{(2)}_{\underline{\theta}}h^{[2]}, \varepsilon w\rangle_H\cdot\Phi_i(\varepsilon w)]\,d\underline{\theta}.$$

Note that $\langle A^{(2)}_{\underline{\theta}}h^{[1]}, \varepsilon w\rangle_H$ is a function of θ_1 and (w^1, w^2), and $\langle A^{(2)}_{\theta}h^{[2]}, \varepsilon w\rangle_H$ that of θ_2 and (w^3, w^4).

By the C-M transformation $w \to w + (A^{(2)}_{\underline{\theta}}h/\varepsilon)$,

$$J_2^{(2)} = \sum_{i=1}^{n} \int_{|\underline{\theta}-\tilde{\underline{\theta}}_i|<\eta} \exp(-\|A^{(2)}_{\underline{\theta}}h\|_H^2/2\varepsilon^2)\ \mathbf{E}[\exp(-\langle A^{(2)}_{\underline{\theta}}h, w\rangle_H/\varepsilon)$$
$$\times\ \delta_0(\varepsilon w_1)\delta_O(\varepsilon(A^{(2)}_{\underline{\theta}}h\otimes\dot{w} - \dot{w}\otimes A^{(2)}_{\underline{\theta}}h) + \frac{\varepsilon^2}{2}(w\otimes\dot{w} - \dot{w}\otimes w))$$
$$\times\ \delta_0(\langle A^{(2)}_{(\theta_1+(\pi/2),\theta_2)}h^{[1]}, A^{(2)}_{\underline{\theta}}h + \varepsilon w\rangle_H)$$
$$\times\ \delta_0(\langle A^{(2)}_{(\theta_1,\theta_2+(\pi/2))}h^{[2]}, A^{(2)}_{\underline{\theta}}h + \varepsilon w\rangle_H)$$
$$\times\ \langle A^{(2)}_{\underline{\theta}}h^{[1]}, A^{(2)}_{\underline{\theta}}h + \varepsilon w\rangle_H\cdot\langle A^{(2)}_{\underline{\theta}}h^{[2]}, A^{(2)}_{\underline{\theta}}h + \varepsilon w\rangle_H$$
$$\times\ \Phi_i(A^{(2)}_{\underline{\theta}}h + \varepsilon w)]\,d\underline{\theta}$$

and noting that $A^{(2)}_{\underline{\theta}} \in O(4)$ and Remark 5.2, this is equal to

$$\exp(-\|h\|_H^2/2\varepsilon^2)\sum_{i=1}^{n}\int_{|\underline{\theta}-\tilde{\underline{\theta}}_i|<\eta} \mathbf{E}[\exp(-\langle h, {}^tA^{(2)}_{\underline{\theta}}w\rangle_H/\varepsilon)$$
$$\times\ \delta_0(\varepsilon A^{(2)}_{\underline{\theta}}\cdot{}^tA^{(2)}_{\underline{\theta}}w_1)$$
$$\times\ \delta_O(A^{(2)}_{\underline{\theta}}\{\varepsilon(h\otimes{}^tA^{(2)}_{\underline{\theta}}w - {}^tA^{(2)}_{\underline{\theta}}w\otimes h)$$
$$+ \frac{\varepsilon^2}{2}({}^tA^{(2)}_{\underline{\theta}}w\otimes{}^tA^{(2)}_{\underline{\theta}}\dot{w} - {}^tA^{(2)}_{\underline{\theta}}\dot{w}\otimes{}^tA^{(2)}_{\underline{\theta}}w)\}{}^tA^{(2)}_{\underline{\theta}})$$
$$\times\ \delta_0(\langle{}^tA^{(2)}_{\underline{\theta}}\cdot A^{(2)}_{(\theta_1+(\pi/2),\theta_2)}h^{[1]}, h + \varepsilon{}^tA^{(2)}_{\underline{\theta}}w\rangle_H)$$
$$\times\ \delta_0(\langle{}^tA^{(2)}_{\underline{\theta}}\cdot A^{(2)}_{(\theta_1,\theta_2+(\pi/2))}h^{[2]}, h + \varepsilon{}^tA^{(2)}_{\underline{\theta}}w\rangle_H)$$
$$\times\ \langle h^{[1]}, h + \varepsilon{}^tA^{(2)}_{\underline{\theta}}w\rangle_H\cdot\langle h^{[2]}, h + \varepsilon{}^tA^{(2)}_{\underline{\theta}}w\rangle_H$$
$$\times\ \Phi_i(A^{(2)}_{\underline{\theta}}(h + \varepsilon{}^tA^{(2)}_{\underline{\theta}}w))]\,d\underline{\theta}.$$

By the invariance of Wiener measure under an orthogonal transformation, we see, noting also that $\langle A^{(2)}_{(\pi/2,0)}h^{[1]}, h\rangle_H = \langle A^{(2)}_{(0,\pi/2)}h^{[2]}, h\rangle_H = 0$,

$$J_2^{(2)} = \exp(-\|h\|_H^2/2\varepsilon^2)\sum_{i=1}^{n}\int_{|\underline{\theta}-\tilde{\underline{\theta}}_i|<\eta} \mathbf{E}[\exp(-\langle h, w\rangle_H/\varepsilon)$$
$$\times\ \delta_{[0,O]}\left(T(A^{(2)}_{\underline{\theta}})\left[\varepsilon w_1,\ \varepsilon(h\otimes\dot{w} - \dot{w}\otimes h) + \frac{\varepsilon^2}{2}(w\otimes\dot{w} - \dot{w}\otimes w)\right]\right)$$

$$\times\ \delta_0(\varepsilon\langle A^{(2)}_{(\pi/2,0)}h^{[1]}, w\rangle_H)\delta_0(\varepsilon\langle A^{(2)}_{(0,\pi/2)}h^{[2]}, w\rangle_H)$$
$$\times\ \langle h^{[1]}, h+\varepsilon w\rangle_H\cdot\langle h^{[2]}, h+\varepsilon w\rangle_H\cdot\Phi_i(A^{(2)}_{\underline{\theta}}(h+\varepsilon w))]\,d\underline{\theta}.$$

Since $T(A^{(2)}_{\underline{\theta}})\in O(10)$ and $-\langle h, w\rangle_H/\varepsilon = 2\pi\cdot S^{12}(1, w) + 4\pi\cdot S^{34}(1, w)$ under the condition that $(h\otimes\dot{w} - \dot{w}\otimes h) + \frac{\varepsilon}{2}(w\otimes\dot{w} - \dot{w}\otimes w) = 0$ and that $A^{(2)}_{\underline{\theta}}w_1 = 0$, we have finally,

$$J^{(2)}_2 = \exp(-\|h\|^2_H/2\varepsilon^2)\sum_{i=1}^{n}\int_{|\underline{\theta}-\underline{\tilde{\theta}}_i|<\eta}$$
$$\mathbf{E}[\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\}\delta_0(\varepsilon w_1)$$
$$\times\ \delta_0(\varepsilon(h\otimes\dot{w} - \dot{w}\otimes h) + \frac{\varepsilon^2}{2}(w\otimes\dot{w} - \dot{w}\otimes w))$$
$$\times\ \delta_0(\varepsilon\langle A^{(2)}_{(\pi/2,0)}h^{[1]}, w\rangle_H)\delta_0(\varepsilon\langle A^{(2)}_{(0,\pi/2)}h^{[2]}, w\rangle_H)$$
$$\times\ \langle h^{[1]}, h+\varepsilon w\rangle_H\cdot\langle h^{[2]}, h+\varepsilon w\rangle_H\cdot\Phi_i(A^{(2)}_{\underline{\theta}}(h+\varepsilon w))]\,d\underline{\theta}.$$

Therefore, setting $\mathbf{R}^{12}$-valued Wiener functional $g^{(2)}_0(w)$ by

$$g^{(2)}_0(w) = (w_1, (h\otimes\dot{w} - \dot{w}\otimes h)_{ij,1\le i<j\le 4}, \langle A^{(2)}_{(\pi/2,0)}h^{[1]}, w\rangle_H, \langle A^{(2)}_{(0,\pi/2)}h^{[2]}, w\rangle_H), \tag{5.6}$$

we have by Lemma 5.4 and Lemma 5.5 below,

$$J^{(2)}_2 \sim \exp(-\|h\|^2_H/2\varepsilon^2)\cdot\varepsilon^{-12}\cdot\|h^{[1]}\|^2_H\cdot\|h^{[2]}\|^2_H \tag{5.7}$$
$$\times\sum_{i=1}^{n}\int_{|\underline{\theta}-\underline{\tilde{\theta}}_i|<\eta}\Phi_i(A^{(2)}_{\underline{\theta}}h)\,d\underline{\theta}$$
$$\times\ \mathbf{E}[\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\}\delta_0(g^{(2)}_0(w))]\quad\text{as }\varepsilon\downarrow 0.$$

Lemma 5.3.B. $\mathbf{E}[\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\}\delta_0(g^{(2)}_0(w))]$

$$= \frac{3}{64\pi^4 u_1 u_2(u_1^2 - u_2^2)}.$$

Proof. Let $p_2(x)$ be the density of the law of $g^{(2)}_0(w)$. Then

$$\mathbf{E}[\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\}\delta_0(g^{(2)}_0(w))]$$
$$= \mathbf{E}[\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\}\,|\,g^{(2)}_0(w) = 0]\cdot p_2(0),$$

and it is easy to see that $p_2(0) = \dfrac{1}{16\pi^4 u_1 u_2(2u_1 + u_2)^2}$.

Let $\Xi^{(2)}_{ij}$, $1\le i<j\le 4$, be the (i, j)-component of $(h\otimes\dot{w} - \dot{w}\otimes h)$. Then

$$\Xi^{(2)}_{12} = \sqrt{u_1/2\pi}\,\xi^{(2)}_1 + \sqrt{u_1/2\pi}(\eta^{(1)}_1 - \sqrt{2}\cdot\eta^{(1)}_0),$$
$$\Xi^{(2)}_{13} = \sqrt{u_1/2\pi}\,\xi^{(3)}_1 - \sqrt{u_2/4\pi}\,\xi^{(1)}_2,$$

$$\Xi_{14}^{(2)} = \sqrt{u_1/2\pi}\,\xi_1^{(4)} + \sqrt{u_2/4\pi}\,(\eta_2^{(1)} - \sqrt{2}\cdot\eta_0^{(1)}),$$
$$\Xi_{23}^{(2)} = -\sqrt{u_1/2\pi}(\eta_1^{(3)} - \sqrt{2}\cdot\eta_0^{(3)}) - \sqrt{u_2/4\pi}\,\xi_2^{(2)},$$
$$\Xi_{24}^{(2)} = -\sqrt{u_1/2\pi}\,(\eta_1^{(4)} - \sqrt{2}\cdot\eta_0^{(4)}) + \sqrt{u_2/4\pi}\,(\eta_2^{(2)} - \sqrt{2}\cdot\eta_0^{(2)})$$

and

$$\Xi_{34}^{(2)} = \sqrt{u_2/4\pi}\,\xi_2^{(4)} + \sqrt{u_2/4\pi}\,(\eta_2^{(3)} - \sqrt{2}\cdot\eta_0^{(3)}).$$

Here $\xi_j^{(i)}$, $\eta_l^{(k)}$ are as in (5.5). Set

$$\Xi_1^{(2)} := \langle A_{(\pi/2,0)}^{(2)} h^{[1]}, w\rangle_H = -\sqrt{2\pi u_1}\,(\xi_1^{(1)} - \eta_1^{(2)})$$

and

$$\Xi_2^{(2)} := \langle A_{(0,\pi/2)}^{(2)} h^{[2]}, w\rangle_H = -\sqrt{4\pi u_2}\,(\xi_2^{(3)} - \eta_2^{(4)}).$$

Then

$$\begin{aligned}
&\mathbf{E}[\exp\{2\pi S^{12}(1,w) + 4\pi S^{34}(1,w)\}\,|\,g_0^{(2)}(w) = 0] \\
&= \mathbf{E}\Bigg[\exp\Bigg(\sum_{k=1}^{2}\sum_{m=1}^{\infty}\frac{k}{m}\{\xi_m^{(2k)}(\eta_m^{(2k-1)} - \sqrt{2}\cdot\eta_0^{(2k-1)}) \\
&\qquad\qquad - \xi_m^{(2k-1)}(\eta_m^{(2k)} - \sqrt{2}\cdot\eta_0^{(2k)})\}\Bigg)\,\Big|\,g_0^{(2)}(w) = 0\Bigg] \\
&= \prod_{k=1}^{2}\mathbf{E}[\exp(\xi_k^{(2k)}\eta_k^{(2k-1)} - \xi_k^{(2k-1)}\eta_k^{(2k)})\,|\,\tilde{\Xi}_{2k-1,2k}^{(2)} = 0,\ \Xi_k^{(2)} = 0] \\
&\quad\times\mathbf{E}\Bigg[\exp\left\{\frac{1}{2}(\xi_2^{(2)}\eta_2^{(1)} - \xi_2^{(1)}\eta_2^{(2)}) + 2(\xi_1^{(4)}\eta_1^{(3)} - \xi_1^{(3)}\eta_1^{(4)})\right\} \\
&\qquad\qquad |\,\tilde{\Xi}_{13}^{(2)} = \tilde{\Xi}_{14}^{(2)} = \tilde{\Xi}_{23}^{(2)} = \tilde{\Xi}_{24}^{(2)} = 0\Bigg] \\
&\quad\times\prod_{k=1}^{2}\prod_{m=3}^{\infty}\mathbf{E}\left[\exp\left\{\frac{k}{m}(\xi_m^{(2k)}\eta_m^{(2k-1)} - \xi_m^{(2k-1)}\eta_m^{(2k)})\right\}\right] \\
&= I_1 \times I_2 \times I_3.
\end{aligned}$$

Here $\tilde{\Xi}_{ij}^{(2)}$, $1 \le i < j \le 4$, denote random variables constructed by excluding the terms $\eta_0^{(k)}$ from $\Xi_{ij}^{(2)}$.

We see easily that $I_1 = \dfrac{1}{4}$ and that $I_3 = \prod_{k=1}^{2}\prod_{m=3}^{\infty}\left(1 - \dfrac{k^2}{m^2}\right)^{-1} = 9$. So all we must do is to compute I_2.

Define $X_i^{(2)}$, $i = 1,\cdots,4$, by

$$X_1^{(2)} = -\sqrt{u_2/2}\cdot\eta_1^{(3)} + \sqrt{u_1}\cdot\xi_2^{(2)},$$
$$X_2^{(2)} = \sqrt{u_2/2}\cdot\xi_1^{(4)} - \sqrt{u_1}\cdot\eta_2^{(1)},$$
$$X_3^{(2)} = \sqrt{u_2/2}\cdot\xi_1^{(3)} + \sqrt{u_1}\cdot\xi_2^{(1)}$$

and

$$X_4^{(2)} = -\sqrt{u_2/2}\cdot\eta_1^{(4)} - \sqrt{u_1}\cdot\eta_2^{(2)}.$$

Then

$$\exp\left\{\frac{1}{2}(\xi_2^{(2)}\eta_2^{(1)} - \xi_2^{(1)}\eta_2^{(2)}) + 2(\xi_1^{(4)}\eta_1^{(3)} - \xi_1^{(3)}\eta_1^{(4)})\right\}$$
$$= \exp\{(2(u_1 + 2u_2)/(2u_1 + u_2)^2)(-X_1^{(2)}X_2^{(2)} + X_3^{(2)}X_4^{(2)}) + P_2(\Xi)\}$$

where $P_2(\Xi)$ is a polynomial of degree 2 in 4 variables $\Xi = (\tilde{\Xi}_{13}^{(2)}, \tilde{\Xi}_{14}^{(2)}, \tilde{\Xi}_{23}^{(2)}, \tilde{\Xi}_{24}^{(2)})$ whose constant term is 0. This equality is obtained by the orthogonal decomposition in $L^2(P)$ of $\xi_j^{(i)}$ and $\eta_l^{(k)}$ with respect to $\ell.\mathcal{s}.\{\tilde{\Xi}_{13}^{(2)}, \tilde{\Xi}_{14}^{(2)}, \tilde{\Xi}_{23}^{(2)}, \tilde{\Xi}_{24}^{(2)}\}$, for example, $\xi_2^{(2)}$ is decomposed by

$$\xi_2^{(2)} = \frac{2}{(2u_1 + u_2)}(\sqrt{u_1}\cdot X_1^{(2)} - \sqrt{\pi u_2}\cdot\tilde{\Xi}_{23}^{(2)}).$$

Noting that $X_i^{(2)} \sim N(0, u_1 + (u_2/2))$, $i = 1,\cdots,4$,

$$I_2 = \mathbf{E}[\exp\{(2(u_1 + 2u_2)/(2u_1 + u_2)^2)(-X_1^{(2)}X_2^{(2)} + X_3^{(2)}X_4^{(2)})\}]$$
$$= (2u_1 + u_2)^2/3(u_1^2 - u_2^2).$$

Combined I_1, I_2 and I_3 with $p_2(0)$, we conclude this lemma.

It is easy to compute that $\|h\|_H^2 = 4\pi u_1 + 8\pi u_2$, $\|h^{[1]}\|_H^2 = 4\pi u_1$ and $\|h^{[2]}\|_H^2 = 8\pi u_2$, and we can show that

$$\sum_{i=1}^{n}\int_{|\underline{\theta}-\tilde{\underline{\theta}}_i|<\eta}\Phi_i(A_{\underline{\theta}}^{(2)}h)\,d\underline{\theta} = 4\pi^2$$

in the same way as in Case A. Therefore we have

$$J_2^{(2)} \sim \exp(-2\pi(u_1 + 2u_2)/\varepsilon^2)\varepsilon^{-12}\frac{6}{u_1^2 - u_2^2} \qquad \text{as } \varepsilon\downarrow 0.$$

In conclusion, we have

Theorem 5.1.B. *In* Case B, *i.e.*, $\mathbf{x} = [0, U]$, $U \sim u_1(\delta_{12} - \delta_{21}) + u_2(\delta_{34} - \delta_{43})$, $u_1 > u_2 > 0$,

$$p(\varepsilon^2, \mathbf{0}, \mathbf{x}) \sim \exp(-2\pi(u_1 + 2u_2)/\varepsilon^2)\varepsilon^{-12}\frac{6}{u_1^2 - u_2^2} \qquad \textit{as } \varepsilon\downarrow 0.$$

(Case C) $U = u(\delta_{12} - \delta_{21} + \delta_{34} - \delta_{43})$, $u > 0$.

In this case every element $h^{\underline{\theta}}$ of $K_{\min}^{\mathbf{0},\mathbf{x}}$ is obtained as in (4.14):

$$h^{\underline{\theta}} = A_{\underline{\theta}}^{(4)}h, \ \underline{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)\in[0, \pi/2]\times[0, 2\pi)^3,$$

where $A_{\underline{\theta}}^{(4)}$ and h are as in (4.7) and (4.15), respectively.

Now we need a lemma corresponding to Lemma 5.1.A or 5.1.B, but we must take care that the Malliavin covariance Σ of $\left(\frac{\partial}{\partial\theta_i}\langle A_{\underline{\theta}}^{(4)}h, w\rangle_H\right)_{i=1,\cdots,4}$ is degenerate at $\theta_1=0$ or $\theta_1=\pi/2$ since $\det\Sigma = 3^2\cdot 2^9\cdot\pi^4\cdot u^4\cdot\cos^2\theta_1\cdot\sin^2\theta_1$. Thus the corresponding lemma is as follows.

Lemma 5.1.C. *For every $\underline{\theta}_0\in(0,\pi/2)\times[0,2\pi)^3$, there exists $\eta_0>0$ such that for each $\eta\in(0,\eta_0)$, there exists $\gamma=\gamma(\eta)>0$ satisfying*

$$\int_{|\underline{\theta}-\underline{\theta}_0|<\eta}\delta_0\left(\left(\frac{\partial}{\partial\theta_i}\langle A_{\underline{\theta}}^{(4)}h, w\rangle_H\right)_{i=1,\cdots,4}\right)\det\left\{\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\langle A_{\underline{\theta}}^{(4)}h, w\rangle_H\right)_{i,j=1,\cdots,4}\right\}d\underline{\theta}=1$$
$$\text{on } \{w;\ \|w-A_{\underline{\theta}_0}^{(4)}h\|_2<\gamma\},$$

and

$$\{\underline{\theta};\ \|A_{\underline{\theta}}^{(4)}h-A_{\underline{\theta}_0}^{(4)}h\|_2<\gamma\}\subset\{\underline{\theta};\ |\underline{\theta}-\underline{\theta}_0|<\eta\}$$

where $d\underline{\theta}=d\theta_1d\theta_2d\theta_3d\theta_4$.

Remark 5.4. Now we fix $\underline{\theta}_0\in(0,\pi/2)\times[0,2\pi)^3$, then for every 4×4 matrix A we have

$$\int_{|\underline{\theta}-\underline{\theta}_0|<\eta}\delta_0\left(\left(\frac{\partial}{\partial\theta_i}\langle A_{\underline{\theta}}^{(4)}h, Aw\rangle_H\right)_{i=1,\cdots,4}\right)\det\left\{\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\langle A_{\underline{\theta}}^{(4)}h, Aw\rangle_H\right)_{i,j=1,\cdots,4}\right\}d\underline{\theta}=1$$
$$\text{on } \{w;\ \|Aw-A_{\underline{\theta}_0}^{(4)}h\|_2<\gamma\}.$$

Moreover if $A\in O(4)$,

$$\int_{|\underline{\theta}-\underline{\theta}_0|<\eta}\delta_0\left(\left(\frac{\partial}{\partial\theta_i}\langle {}^tA\cdot A_{\underline{\theta}}^{(4)}h, w\rangle_H\right)_{i=1,\cdots,4}\right)\det\left\{\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\langle {}^tA\cdot A_{\underline{\theta}}^{(4)}h, w\rangle_H\right)_{i,j=1,\cdots,4}\right\}d\underline{\theta}=1 \tag{5.8}$$
$$\text{on } \{w;\ \|w-{}^tA\cdot A_{\underline{\theta}_0}^{(4)}h\|_2\langle\gamma\}.$$

Especially let A be $A_{\underline{\theta}_0}^{(4)}\cdot A_{\underline{\theta}'}^{(4)}$, $\underline{\theta}'\in[0,\pi/2]\times[0,2\pi)^3$. Then ${}^tA\cdot A_{\underline{\theta}_0}^{(4)}h = A_{\underline{\theta}'}^{(4)}h$. Therefore Lemma 5.1.C is extended in the form (5.8) for all elements of $K_{\min}^{\mathbf{0},\mathbf{x}}$.

Now we define Φ, Φ_i, etc. in the same way as in Case A or in Case B, and it is enough to treat

$$J_2^{(3)} := \mathbf{E}[\delta_{\mathbf{x}}(\mathbf{X}_1^\varepsilon)\Phi(\varepsilon w)].$$

By (5.8), the definition of $\Phi_i(w)$ and the transformation $w \to A^{(4)}_{\underline{\theta}_i} \cdot {}^tA^{(4)}_{\underline{\theta}_0} w$, we have

$$\int_{|\underline{\theta}-\underline{\theta}_0|<\eta} \delta_0\left(\left(\frac{\partial}{\partial\theta_i}\langle A^{(4)}_{\underline{\theta}}h, w\rangle_H\right)_{i=1,\cdots,4}\right)$$
$$\det\left\{\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\langle A^{(4)}_{\underline{\theta}}h, w\rangle_H\right)_{i,j=1,\cdots,4}\right\} d\underline{\theta}\cdot \Phi_i(A^{(4)}_{\underline{\theta}_i}\cdot {}^tA^{(4)}_{\underline{\theta}_0} w)$$
$$= \Phi_i(A^{(4)}_{\underline{\theta}_i}\cdot {}^tA^{(4)}_{\underline{\theta}_0} w).$$

So

$$J_2^{(3)} = \sum_{i=1}^n \mathbf{E}[\delta_{\mathbf{x}}(\mathbf{X}_1^\varepsilon)\Phi_i(\varepsilon w)]$$
$$= \sum_{i=1}^n \int_{|\underline{\theta}-\underline{\theta}_0|<\eta} \mathbf{E}[\delta_0(\varepsilon w_1)\delta_U(\varepsilon^2 S(1, w))$$
$$\times \delta_0\left(\left(\frac{\partial}{\partial\theta_i}\langle A^{(4)}_{\underline{\theta}}h, \varepsilon w\rangle_H\right)_{i=1,\cdots,4}\right)$$
$$\times \det\left\{\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\langle A^{(4)}_{\underline{\theta}}h, \varepsilon w\rangle_H\right)_{i,j=1,\cdots,4}\right\}$$
$$\times \Phi_i(A^{(4)}_{\underline{\theta}_i}\cdot {}^tA^{(4)}_{\underline{\theta}_0} w)]\, d\underline{\theta}$$
$$= \sum_{i=1}^n \int_{|\underline{\theta}-\underline{\theta}_0|<\eta} \exp(-\|A^{(4)}_{\underline{\theta}}h\|_H^2/2\varepsilon^2)\,\mathbf{E}[\exp(-\langle A^{(4)}_{\underline{\theta}}h, w\rangle_H/\varepsilon)$$
$$\times \delta_0(\varepsilon w_1)\,\delta_O(\varepsilon(A^{(4)}_{\underline{\theta}}h \otimes \dot{w} - \dot{w}\otimes A^{(4)}_{\underline{\theta}}h) + \frac{\varepsilon^2}{2}(w\otimes\dot{w} - \dot{w}\otimes w))$$
$$\times \delta_0\left(\left(\frac{\partial}{\partial\theta_i}\langle A^{(4)}_{\underline{\theta}}h, h'\rangle_H\Big|_{h' = A^{(4)}_{\underline{\theta}}h + \varepsilon w}\right)_{i=1,\cdots,4}\right)$$
$$\times \det\left\{\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\langle A^{(4)}_{\underline{\theta}}h, h'\rangle_H\Big|_{h' = A^{(4)}_{\underline{\theta}}h + \varepsilon w}\right)_{i,j=1,\cdots,4}\right\}$$
$$\times \Phi_i(A^{(4)}_{\underline{\theta}_i}\cdot {}^tA^{(4)}_{\underline{\theta}_0}\cdot A^{(4)}_{\underline{\theta}}h + \varepsilon A^{(4)}_{\underline{\theta}_i}\cdot {}^tA^{(4)}_{\underline{\theta}_0} w)]\, d\underline{\theta}.$$

where the last equality is obtained by a C-M transformation $w \to w + A^{(4)}_{\underline{\theta}}h/\varepsilon$. Noting that $A^{(4)}_{\underline{\theta}} \in O(4)$ and Remark 5.2, this is equal to

$$\sum_{i=1}^n \int_{|\underline{\theta}-\underline{\theta}_0|<\eta} \exp(-\|h\|_H^2/2\varepsilon^2)\,\mathbf{E}[\exp(-\langle h, {}^tA^{(4)}_{\underline{\theta}}w\rangle_H/\varepsilon)$$
$$\times \delta_0(\varepsilon A^{(4)}_{\underline{\theta}}\cdot {}^tA^{(4)}_{\underline{\theta}} w_1)$$
$$\times \delta_O(A^{(4)}_{\underline{\theta}}\{\varepsilon(h\otimes {}^tA^{(4)}_{\underline{\theta}}\dot{w} - {}^tA^{(4)}_{\underline{\theta}}\dot{w}\otimes h)$$
$$+ \frac{\varepsilon^2}{2}({}^tA^{(4)}_{\underline{\theta}}w \otimes {}^tA^{(4)}_{\underline{\theta}}\dot{w} - {}^tA^{(4)}_{\underline{\theta}}\dot{w}\otimes {}^tA^{(4)}_{\underline{\theta}}w)\}\,{}^tA^{(4)}_{\underline{\theta}})$$

$$\times\ \delta_0\left(\left(\frac{\partial}{\partial\theta_i}\langle A_{\underline{\theta}}^{(4)}h,\ h'\rangle_H\Big|_{h'=A_{\underline{\theta}}^{(4)}h+\varepsilon A_{\underline{\theta}}^{(4)}\cdot{}^tA_{\underline{\theta}}^{(4)}w}\right)_{i=1,\cdots,4}\right)$$

$$\times\ \det\left\{\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\langle A_{\underline{\theta}}^{(4)}h,\ h'\rangle_H\Big|_{h'=A_{\underline{\theta}}^{(4)}h+\varepsilon A_{\underline{\theta}}^{(4)}\cdot{}^tA_{\underline{\theta}}^{(4)}w}\right)_{i,j=1,\cdots,4}\right\}$$

$$\times\ \Phi_i(A_{\tilde{\underline{\theta}}_i}^{(4)}\cdot{}^tA_{\underline{\theta}_0}^{(4)}\cdot A_{\underline{\theta}}^{(4)}h+\varepsilon A_{\tilde{\underline{\theta}}_i}^{(4)}\cdot{}^tA_{\underline{\theta}_0}^{(4)}\cdot A_{\underline{\theta}}^{(4)}\cdot{}^tA_{\underline{\theta}}^{(4)}w)]\,d\underline{\theta}.$$

By the invariance of Wiener measure under an orthogonal transformation, $\frac{\partial}{\partial\theta_i}\langle A_{\underline{\theta}}^{(4)}h,\ h'\rangle_H\Big|_{h'=A_{\underline{\theta}}^{(4)}h}=0$ and $T(A_{\underline{\theta}}^{(4)})\in O(10)$, we finally see, noting that $-\langle h, w\rangle_H/\varepsilon = 2\pi S^{12}(1, w)+4\pi S^{34}(1, w)$ under the condition $(h\otimes\dot{w}-\dot{w}\otimes h)+\frac{\varepsilon}{2}(w\otimes\dot{w}-\dot{w}\otimes w)=0$ and $w_1=0$,

$$J_2^{(3)}=\exp(-\|h\|_H^2/2\varepsilon^2)\sum_{i=1}^{n}\int_{|\underline{\theta}-\underline{\theta}_0|<\eta}\mathbf{E}[\exp\{2\pi S^{12}(1, w)+4\pi S^{34}(1, w)\}$$

$$\times\ \delta_0(\varepsilon w_1)\delta_O\left(\varepsilon(h\otimes\dot{w}-\dot{w}\otimes h)+\frac{\varepsilon^2}{2}(w\otimes\dot{w}-\dot{w}\otimes w)\right)$$

$$\times\ \delta_0\left(\left(\varepsilon\cdot\frac{\partial}{\partial\theta_i}\langle A_{\underline{\theta}}^{(4)}h,\ h'\rangle_H\Big|_{h'=A_{\underline{\theta}}^{(4)}w}\right)_{i=1,\cdots,4}\right)$$

$$\times\ \det\left\{\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\langle A_{\underline{\theta}}^{(4)}h,\ h'\rangle_H\Big|_{h'=A_{\underline{\theta}}^{(4)}h+\varepsilon A_{\underline{\theta}}^{(4)}w}\right)_{i,j=1,\cdots,4}\right\}$$

$$\times\ \Phi_i(A_{\tilde{\underline{\theta}}_i}^{(4)}\cdot{}^tA_{\underline{\theta}_0}^{(4)}\cdot A_{\underline{\theta}}^{(4)}h+\varepsilon A_{\tilde{\underline{\theta}}_i}^{(4)}\cdot{}^tA_{\underline{\theta}_0}^{(4)}\cdot A_{\underline{\theta}}^{(4)}w)]\,d\underline{\theta}.$$

Define $\mathbf{R}^{14}$-valued Wiener functional $g_{0,\underline{\theta}}^{(3)}(w)$ by

$$g_{0,\underline{\theta}}^{(3)}(w)=\left(w_1,\ (h\otimes\dot{w}-\dot{w}\otimes h)_{ij,1\le i<j\le 4},\ \left(\frac{\partial}{\partial\theta_i}\langle A_{\underline{\theta}}^{(4)}h,\ h'\rangle_H\Big|_{h'=A_{\underline{\theta}}^{(4)}w}\right)_{i=1,\cdots,4}\right). \tag{5.9}$$

Then, by Lemma 5.4 and Lemma 5.5, given below, we have

$$J_2^{(3)}\sim\exp(-\|h\|_H^2/2\varepsilon^2)\varepsilon^{-14}\sum_{i=1}^{n}\int_{|\underline{\theta}-\underline{\theta}_0|<\eta}\mathbf{E}[\exp\{2\pi S^{12}(1, w)+4\pi S^{34}(1, w)\}\,\delta_0(g_{0,\underline{\theta}}^{(3)}(w))]$$
$$\times\ \det\left\{\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\langle A_{\underline{\theta}}^{(4)}h,\ h'\rangle_H\Big|_{h'=A_{\underline{\theta}}^{(4)}h}\right)_{i,j=1,\cdots,4}\right\}$$
$$\times\ \Phi_i(A_{\tilde{\underline{\theta}}_i}^{(4)}\cdot{}^tA_{\underline{\theta}_0}^{(4)}\cdot A_{\underline{\theta}}^{(4)}h)\,d\underline{\theta}. \tag{5.10}$$

Lemma 5.3.C. $\mathbf{E}[\exp\{2\pi S^{12}(1, w)+4\pi S^{34}(1, w)\}\,\delta_0(g_{0,\underline{\theta}}^{(3)}(w))]$

$$=\frac{1}{2^{10}\cdot 3\cdot u^5\cdot\pi^6\cdot\sin\theta_1\cdot\cos\theta_1}.$$

Proof. Let $p_3^{\underline{\theta}}(x)$ be the density of the law of $g_{0,\underline{\theta}}^{(3)}(w)$, then

$$\begin{aligned}
&\mathbf{E}[\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\}\,\delta_0(g_{0,\underline{\theta}}^{(3)}(w))] \\
&\quad = \mathbf{E}[\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\}\,|\,g_{0,\underline{\theta}}^{(3)}(w) = 0]\cdot p_3^{\underline{\theta}}(0),
\end{aligned}$$

and it is easy to see that $p_3^{\underline{\theta}}(0) = \dfrac{1}{2^7\cdot 3^3\cdot u^5\cdot \pi^6\cdot \sin\theta_1\cdot\cos\theta_1}$. Let $\Xi_{ij}^{(3)} = (h\otimes\dot{w} - \dot{w}\otimes h)_{ij}$, $1\le i<j\le 4$, and $\Xi_k^{(3)} = \dfrac{\partial}{\partial\theta_k}\langle A_{\underline{\theta}}^{(4)}h, h'\rangle_H\Big|_{h'=A_{\underline{\theta}}^{(4)}w}$, $k = 1,\cdots,4$. Then using $\xi_j^{(i)}$, $\eta_l^{(k)}$ in (5.5), we have

$$\begin{aligned}
\Xi_{12}^{(3)} &= \sqrt{u/2\pi}\,\xi_1^{(2)} + \sqrt{u/2\pi}\,(\eta_1^{(1)} - \sqrt{2}\cdot\eta_0^{(1)}),\\
\Xi_{13}^{(3)} &= \sqrt{u/2\pi}\,\xi_1^{(3)} - \sqrt{u/4\pi}\,\xi_2^{(1)},\\
\Xi_{14}^{(3)} &= \sqrt{u/2\pi}\,\xi_1^{(4)} + \sqrt{u/4\pi}\,(\eta_2^{(1)} - \sqrt{2}\cdot\eta_0^{(1)}),\\
\Xi_{23}^{(3)} &= -\sqrt{u/2\pi}\,(\eta_1^{(3)} - \sqrt{2}\cdot\eta_0^{(3)}) - \sqrt{u/4\pi}\,\xi_2^{(2)},\\
\Xi_{24}^{(3)} &= -\sqrt{u/2\pi}\,(\eta_1^{(4)} - \sqrt{2}\cdot\eta_0^{(4)}) + \sqrt{u/4\pi}\,(\eta_2^{(2)} - \sqrt{2}\cdot\eta_0^{(2)}),\\
\Xi_{34}^{(3)} &= \sqrt{u/4\pi}\,\xi_2^{(4)} + \sqrt{u/4\pi}\,(\eta_2^{(3)} - \sqrt{2}\cdot\eta_0^{(3)}),\\
\Xi_1^{(3)} &= -\sqrt{4\pi u}\cos(\theta_2-\theta_3)(\eta_2^{(1)} + \xi_2^{(2)})\\
&\quad - \sqrt{4\pi u}\sin(\theta_2-\theta_3)(\xi_2^{(1)} - \eta_2^{(2)})\\
&\quad + \sqrt{2\pi u}\cos(\theta_2-\theta_3)(\eta_1^{(3)} + \xi_1^{(4)})\\
&\quad - \sqrt{2\pi u}\sin(\theta_2-\theta_3)(\xi_1^{(3)} - \eta_1^{(4)}),\\
\Xi_2^{(3)} &= -\sqrt{2\pi u}\cos^2\theta_1\,(\xi_1^{(1)} - \eta_1^{(2)})\\
&\quad - \sqrt{4\pi u}\sin\theta_1\cos\theta_1\sin(\theta_2-\theta_3)(\eta_2^{(1)} + \xi_2^{(2)})\\
&\quad + \sqrt{4\pi u}\sin\theta_1\cos\theta_1\cos(\theta_2-\theta_3)(\xi_2^{(1)} - \eta_2^{(2)})\\
&\quad + \sqrt{2\pi u}\sin\theta_1\cos\theta_1\sin(\theta_2-\theta_3)(\eta_1^{(3)} + \xi_1^{(4)})\\
&\quad + \sqrt{2\pi u}\sin\theta_1\cos\theta_1\cos(\theta_2-\theta_3)(\xi_1^{(3)} - \eta_1^{(4)})\\
&\quad + \sqrt{4\pi u}\cos^2\theta_1\,(\xi_2^{(3)} - \eta_2^{(4)}),\\
\Xi_3^{(3)} &= -\sqrt{4\pi u}\,(\xi_2^{(3)} - \eta_2^{(4)})
\end{aligned}$$

and

$$\begin{aligned}
\Xi_4^{(3)} &= -\sqrt{2\pi u}\sin^2\theta_1\,(\xi_1^{(1)} - \eta_1^{(2)})\\
&\quad + \sqrt{4\pi u}\sin\theta_1\cos\theta_1\sin(\theta_2-\theta_3)(\eta_2^{(1)} + \xi_2^{(2)})\\
&\quad - \sqrt{4\pi u}\sin\theta_1\cos\theta_1\cos(\theta_2-\theta_3)(\xi_2^{(1)} - \eta_2^{(2)})\\
&\quad - \sqrt{2\pi u}\sin\theta_1\cos\theta_1\sin(\theta_2-\theta_3)(\eta_1^{(3)} + \xi_1^{(4)})
\end{aligned}$$

$$- \sqrt{2\pi u}\sin\theta_1\cos\theta_1\cos(\theta_2-\theta_3)(\xi_1^{(3)}-\eta_1^{(4)})$$
$$- \sqrt{4\pi u}\cos^2\theta_1(\xi_2^{(3)}-\eta_2^{(4)}).$$

Now set $\tilde{\Xi}_4^{(3)} = \Xi_2^{(3)} + \Xi_4^{(3)}$ and $\tilde{\tilde{\Xi}}_2^{(3)} = \Xi_2^{(3)} + \cos^2\theta_1(\Xi_3^{(3)} - \tilde{\Xi}_4^{(3)})$, i.e.

$$\tilde{\Xi}_4^{(3)} = -\sqrt{2\pi u}(\xi_1^{(1)} - \eta_1^{(2)})$$

and

$$\begin{aligned}\tilde{\tilde{\Xi}}_2^{(3)} = & -\sqrt{4\pi u}\sin\theta_1\cos\theta_1\sin(\theta_2-\theta_3)(\eta_2^{(1)}+\xi_2^{(2)})\\ & +\sqrt{4\pi u}\sin\theta_1\cos\theta_1\cos(\theta_2-\theta_3)(\xi_2^{(1)}-\eta_2^{(2)})\\ & +\sqrt{2\pi u}\sin\theta_1\cos\theta_1\sin(\theta_2-\theta_3)(\eta_1^{(3)}+\xi_1^{(4)})\\ & +\sqrt{2\pi u}\sin\theta_1\cos\theta_1\cos(\theta_2-\theta_3)(\xi_1^{(3)}-\eta_1^{(4)}),\end{aligned}$$

and let $\tilde{\Xi}_{ij}^{(3)}$, $1 \le i < j \le 4$, be random variables obtained by excluding the terms $\eta_0^{(k)}$, $k = 1,\cdots,4$, from $\Xi_{ij}^{(3)}$. Then

$$\begin{aligned}
&\mathbf{E}[\exp\{2\pi S^{12}(1,w) + 4\pi S^{34}(1,w)\}\,|\,g_{0,\underline{\theta}}^{(3)}(w) = 0]\\
&= \mathbf{E}\Big[\exp\Big(\sum_{k=1}^{2}\sum_{m=1}^{\infty}\frac{k}{m}\{\xi_m^{(2k)}(\eta_m^{(2k-1)} - \sqrt{2}\cdot\eta_0^{(2k-1)})\\
&\qquad - \xi_m^{(2k-1)}(\eta_m^{(2k)} - \sqrt{2}\cdot\eta_0^{(2k)})\}\Big)\,|\,g_{0,\underline{\theta}}^{(3)}(w) = 0\Big]\\
&= \mathbf{E}\Big[\exp\Big(\sum_{k=1}^{2}\sum_{m=1}^{\infty}\frac{k}{m}(-\xi_m^{(2k-1)}\eta_m^{(2k)} + \xi_m^{(2k)}\eta_m^{(2k-1)})\Big)\,|\\
&\qquad \tilde{\Xi}_{ij}^{(3)} = 0,\ 1 \le i < j \le 4,\ \Xi_1^{(3)} = \tilde{\tilde{\Xi}}_2^{(3)} = \Xi_3^{(3)} = \tilde{\Xi}_4^{(3)} = 0\Big]\\
&= \mathbf{E}[\exp\{(-\xi_1^{(1)}\eta_1^{(2)} + \xi_1^{(2)}\eta_1^{(1)}) + (-\xi_2^{(3)}\eta_2^{(4)} + \xi_2^{(4)}\eta_2^{(3)})\}\,|\\
&\qquad \tilde{\Xi}_4^{(3)} = \tilde{\Xi}_{12}^{(3)} = \tilde{\Xi}_{34}^{(3)} = \Xi_3^{(3)} = 0]\\
&\quad\times \mathbf{E}\Big[\exp\Big\{\frac{1}{2}(\xi_2^{(2)}\eta_2^{(1)} - \xi_2^{(1)}\eta_2^{(2)}) + 2(\xi_1^{(4)}\eta_1^{(3)} - \xi_1^{(3)}\eta_1^{(4)})\Big\}\\
&\qquad |\,\tilde{\Xi}_{13}^{(3)} = \tilde{\Xi}_{14}^{(3)} = \tilde{\Xi}_{23}^{(3)} = \tilde{\Xi}_{24}^{(3)} = \Xi_1^{(3)} = \tilde{\tilde{\Xi}}_2^{(3)} = 0\Big]\\
&\quad\times \prod_{k=1}^{2}\prod_{m=3}^{\infty}\mathbf{E}\Big[\exp\Big\{\frac{k}{m}(\xi_m^{(2k)}\eta_m^{(2k-1)} - \xi_m^{(2k-1)}\eta_m^{(2k)})\Big\}\Big]\\
&= I_1 \times I_2 \times I_3.
\end{aligned}$$

Here the second equality is obtained by that $\eta_0^{(i)} = 0$, $i = 1,\cdots,4$, and that $\Xi_2^{(3)} = \Xi_3^{(3)} = \Xi_4^{(3)} = 0$ if and only if $\tilde{\tilde{\Xi}}_2^{(3)} = \Xi_3^{(3)} = \tilde{\Xi}_4^{(3)} = 0$, and it is easy to see that $I_3 = \prod_{k=1}^{2}\prod_{m=3}^{\infty}\left(1 - \frac{k^2}{m^2}\right)^{-1} = 9$ and that

$$I_1 = \mathbf{E}[\exp(-\xi_1^{(1)}\eta_1^{(2)})|\tilde{\Xi}_4^{(3)}=0] \times \mathbf{E}[\exp(\xi_1^{(2)}\eta_1^{(1)})|\tilde{\Xi}_{12}^{(3)}=0]$$
$$\times \mathbf{E}[\exp(-\xi_2^{(3)}\eta_2^{(4)})|\Xi_3^{(3)}=0] \times \mathbf{E}[\exp(\xi_2^{(4)}\eta_2^{(3)})|\tilde{\Xi}_{34}^{(3)}=0]$$
$$= \left((1/\sqrt{2\pi})\int_{-\infty}^{\infty} e^{-x^2}dx\right)^4 = \frac{1}{4}.$$

Define $X_i^{(3)}$, $i=1,\cdots,4$, by

$$X_1^{(3)} = -\sqrt{2}\cdot\eta_1^{(3)} + 2\cdot\xi_2^{(2)},$$
$$X_2^{(3)} = \sqrt{2}\cdot\xi_1^{(4)} - 2\cdot\eta_2^{(1)},$$
$$X_3^{(3)} = \sqrt{2}\cdot\xi_1^{(3)} + 2\cdot\xi_2^{(1)}$$

and

$$X_4^{(3)} = -\sqrt{2}\cdot\eta_1^{(4)} - 2\cdot\eta_2^{(2)}.$$

Then

$$\exp\left\{\frac{1}{2}(\xi_2^{(2)}\eta_2^{(1)} - \xi_2^{(1)}\eta_2^{(2)}) + 2(\xi_1^{(4)}\eta_1^{(3)} - \xi_1^{(3)}\eta_1^{(4)})\right\} \tag{5.11}$$
$$= \exp\left\{-\frac{1}{6}(X_1^{(3)}X_2^{(3)} - X_3^{(3)}X_4^{(3)}) + P_3(\Xi)\right\}$$

where $P_3(\Xi)$ is a polynomial of degree 2 in 4 variables $\Xi = (\tilde{\Xi}_{13}^{(3)}, \tilde{\Xi}_{14}^{(3)}, \tilde{\Xi}_{23}^{(3)}, \tilde{\Xi}_{24}^{(3)})$ whose constant term is 0. This equality is obtained in the same way as in Case B. Noting that

$$\Xi_1^{(3)} = \sqrt{\pi u}\,\{(X_2^{(3)} - X_1^{(3)})\cos(\theta_2-\theta_3) - (X_3^{(3)} + X_4^{(3)})\sin(\theta_2-\theta_3)\}$$

and that

$$\tilde{\Xi}_2^{(3)} = \sqrt{\pi u}\sin\theta_1\cos\theta_1$$
$$\times \{(X_2^{(3)} - X_1^{(3)})\sin(\theta_2-\theta_3) + (X_3^{(3)} + X_4^{(3)})\cos(\theta_2-\theta_3)\},$$

we have $\Xi_1^{(3)} = \tilde{\Xi}_2^{(3)} = 0$ if and only if $X_1^{(3)} - X_2^{(3)} = 0$ and $X_3^{(3)} + X_4^{(3)} = 0$. Thus

$$I_2 = \mathbf{E}\left[\exp\left\{-\frac{1}{6}(X_1^{(3)}X_2^{(3)} - X_3^{(3)}X_4^{(3)})\right\}\right|$$
$$\left. X_1^{(3)} - X_2^{(3)} = 0,\ X_3^{(3)} + X_4^{(3)} = 0\right]$$
$$= \mathbf{E}\left[\exp\left(-\frac{1}{6}X_1^{(3)}X_2^{(3)}\right)\middle| X_1^{(3)} - X_2^{(3)} = 0\right]$$
$$\times \mathbf{E}\left[\exp\left(\frac{1}{6}X_3^{(3)}X_4^{(3)}\right)\middle| X_3^{(3)} + X_4^{(3)} = 0\right]$$
$$= \frac{1}{2},$$

for $X_1^{(3)} + X_2^{(3)}, X_3^{(3)} - X_4^{(3)} \sim N(0, 12)$.

Combined I_1, I_2 and I_3 with $p_3^{\underline{\theta}}(0)$, the proof is completed.

It is easy to compute that

$$\det\left\{\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\langle A_{\underline{\theta}}^{(4)}h, h'\rangle_H\bigg|_{h'=A_{\underline{\theta}}^{(4)}h}\right)_{i,j=1,\cdots,4}\right\}$$
$$= 3^2\cdot 2^9\cdot\pi^4\cdot u^4\cdot\cos^2\theta_1\sin^2\theta_1$$

and that $\|h\|_H^2 = 12\pi u$, so we have

$$J_2^{(3)} \sim \exp(-6\pi u/\varepsilon^2)\varepsilon^{-14}\frac{3}{2u\pi^2}\sum_{i=1}^{n}\int_{|\underline{\theta}-\underline{\theta}_0|<\eta}$$
$$\Phi_i(A_{\tilde{\underline{\theta}}_i}^{(4)}\cdot {}^tA_{\underline{\theta}_0}^{(4)}\cdot A_{\underline{\theta}}^{(4)}h)\cdot\sin\theta_1\cos\theta_1\,d\underline{\theta} \qquad \text{as } \varepsilon\downarrow 0.$$

Proposition 5.2. *Define a metric g on $K_{\min}^{\mathbf{0},\mathbf{x}}$ by*

$$g = \sum g_{ij}\,d\theta^i\,d\theta^j$$

where

$$g_{ij} = \left\langle\frac{\partial}{\partial\theta_i}A_{\underline{\theta}}^{(4)}h, \frac{\partial}{\partial\theta_j}A_{\underline{\theta}}^{(4)}h\right\rangle_H.$$

If we introduce another metric g' on $K_{\min}^{\mathbf{0},\mathbf{x}}$ by

$$g' = \sum g'_{ij}\,d\theta'^i d\theta'^j$$

where

$$g'_{ij} = \left\langle\frac{\partial}{\partial\theta'_i}A_{\underline{\theta}'}^{(4)}h, \frac{\partial}{\partial\theta'_j}A_{\underline{\theta}'}^{(4)}h\right\rangle_H$$

and, for some $\underline{\alpha}\in[0,\pi/2]\times[0,2\pi)^3$, $A_{\underline{\theta}'}^{(4)} = A_{\underline{\alpha}}^{(4)}\cdot A_{\underline{\theta}}^{(4)}$, then $g = g'$.

Proof.

$$g'_{ij} = \left\langle\sum_k\frac{\partial}{\partial\theta_k}\frac{\partial\theta_k}{\partial\theta'_i}A_{\underline{\alpha}}^{(4)}\cdot A_{\underline{\theta}}^{(4)}h, \sum_l\frac{\partial}{\partial\theta_l}\frac{\partial\theta_l}{\partial\theta'_j}A_{\underline{\alpha}}^{(4)}\cdot A_{\underline{\theta}}^{(4)}h\right\rangle_H$$
$$= \sum_{k,l}\frac{\partial\theta_k}{\partial\theta'_i}\frac{\partial\theta_l}{\partial\theta'_j}\left\langle\frac{\partial}{\partial\theta_k}A_{\underline{\alpha}}^{(4)}\cdot A_{\underline{\theta}}^{(4)}h, \frac{\partial}{\partial\theta_l}A_{\underline{\alpha}}^{(4)}\cdot A_{\underline{\theta}}^{(4)}h\right\rangle_H$$
$$= \sum_{k,l}\frac{\partial\theta_k}{\partial\theta'_i}\frac{\partial\theta_l}{\partial\theta'_j}\left\langle\frac{\partial}{\partial\theta_k}A_{\underline{\theta}}^{(4)}h, \frac{\partial}{\partial\theta_l}A_{\underline{\theta}}^{(4)}h\right\rangle_H$$
$$= \sum_{k,l}\frac{\partial\theta_k}{\partial\theta'_i}\frac{\partial\theta_l}{\partial\theta'_j}g_{kl}.$$

So it is easy to see that $g = g'$.

Since $\|A_{\underline{\theta}}^{(4)}h\|_H^2$ is independent of $\underline{\theta}$, it is clear that

$$\left\langle \frac{\partial}{\partial\theta_i} A_{\underline{\theta}}^{(4)} h, \frac{\partial}{\partial\theta_j} A_{\underline{\theta}}^{(4)} h \right\rangle_H = - \frac{\partial^2}{\partial\theta_i \partial\theta_j} \langle A_{\underline{\theta}}^{(4)} h, h' \rangle_H \Big|_{h' = A_{\underline{\theta}}^{(4)} h} .$$

Thus

$$\begin{aligned}
&\sum_{i=1}^{n} \int_{|\underline{\theta} - \underline{\theta}_0| < \eta} \Phi_i(A_{\tilde{\underline{\theta}}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)} h) \sin\theta_1 \cos\theta_1 \, d\underline{\theta} \\
&= \sum_{i=1}^{n} \int_{|\underline{\theta} - \underline{\theta}_0| < \eta} (I_{U_i} \cdot \Phi_i)(A_{\tilde{\underline{\theta}}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)} h) \sin\theta_1 \cos\theta_1 \, d\underline{\theta} \\
&= \sum_{i=1}^{n} \int_{\underline{\theta} \in [0,\pi/2] \times [0,2\pi)^3} (I_{U_i} \cdot \Phi_i)(A_{\tilde{\underline{\theta}}_i}^{(4)} \cdot {}^t A_{\underline{\theta}_0}^{(4)} \cdot A_{\underline{\theta}}^{(4)} h) \sin\theta_1 \cos\theta_1 \, d\underline{\theta} \\
&= \sum_{i=1}^{n} \int_{\underline{\theta} \in [0,\pi/2] \times [0,2\pi)^3} (I_{U_i} \cdot \Phi_i)(A_{\underline{\theta}}^{(4)} h) \sin\theta_1 \cos\theta_1 \, d\underline{\theta} \\
&= \int_{\underline{\theta} \in [0,\pi/2] \times [0,2\pi)^3} \sum_{i=1}^{n} \Phi_i(A_{\underline{\theta}}^{(4)} h) \sin\theta_1 \cos\theta_1 \, d\underline{\theta} \\
&= 4\pi^3 .
\end{aligned}$$

Here the third equality is due to Proposition 5.2. Therefore

$$J_2^{(3)} \sim \exp(-6\pi u/\varepsilon^2) \varepsilon^{-14} \frac{6\pi}{u} \qquad \text{as } \varepsilon \downarrow 0.$$

In conclusion, we have

Theorem 5.1.C. *In* Case C, *i.e.*, $\mathbf{x} = [0, U]$, $U \sim u(\delta_{12} - \delta_{21} + \delta_{34} - \delta_{43})$, $u > 0$,

$$p(\varepsilon^2, \mathbf{0}, \mathbf{x}) \sim \exp(-6\pi u/\varepsilon^2) \varepsilon^{-14} \frac{6\pi}{u} \qquad \textit{as } \varepsilon \downarrow 0.$$

We finish this section by proving two lemmas quoted above which assured the asymptotics (5.4), (5.7) and (5.10).

Let $\chi_i \colon \mathbf{R}^{n(i)} \to \mathbf{R}$, $i = 1, 2, 3$, be C^∞-functions satisfying $\operatorname{Supp} \chi_i \subset \{|x| \leq 1\}$ where $n(1) = 11$, $n(2) = 12$ and $n(3) = 14$.

Lemma 5.4. A) *We can choose* $\eta > 0$ *such that for all* $i = 1, \cdots, n$,

$$\begin{aligned}
&\exp\{2\pi S^{34}(1, w)\} \chi_1(g_\varepsilon^{(1)}(w)) \Phi_i(A_\theta^{(1)}(\tilde{h} + \varepsilon w)) \\
&= \exp\{2\pi S^{34}(1, w)\} \chi_1(g_0^{(1)}(w)) \Phi_i(A_\theta^{(1)} \tilde{h}) + O(\varepsilon) \\
&\qquad \textit{as } \varepsilon \downarrow 0 \textit{ in } \tilde{\mathbf{D}}^\infty \textit{ if } |\theta - \tilde{\theta}_i| < \eta.
\end{aligned}$$

Furthermore $O(\varepsilon)$ *is uniform on* $\{\theta; |\theta - \tilde{\theta}_i| < \eta\}$. *Here* Φ_i, $\tilde{\theta}_i$, $i = 1, \cdots, n$, *are as in the statement after* Lemma 5.1.A.

B) *We can choose* $\eta > 0$ *such that for all* $i = 1, \cdots, n$,

$$\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\}\chi_2(g_\varepsilon^{(2)}(w))\,\Phi_i(A_{\underline{\theta}}^{(4)}(h+\varepsilon w))$$
$$= \exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\}\chi_2(g_0^{(2)}(w))\,\Phi_i(A_{\underline{\theta}}^{(4)}h) + O(\varepsilon)$$
$$\text{as } \varepsilon\downarrow 0 \text{ in } \tilde{\mathbf{D}}^\infty \text{ if } |\underline{\theta}-\tilde{\underline{\theta}}_i| < \eta.$$

Furthermore $O(\varepsilon)$ *is uniform on* $\{\underline{\theta};\, |\underline{\theta}-\tilde{\underline{\theta}}_i| < \eta\}$.

C) *For all* $\underline{\theta}_0 \in (0, \pi/2)\times[0, 2\pi)^3$ *we can choose* $\eta > 0$ *such that for all* $i = 1, \cdots, n$,

$$\begin{aligned}(5.12)\qquad &\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\}\chi_3(g_{\varepsilon,\underline{\theta}}^{(3)}(w))\\ &\quad\cdot\Phi_i(A_{\tilde{\underline{\theta}}_i}^{(4)}\cdot {}^tA_{\underline{\theta}_0}^{(4)}\cdot A_{\underline{\theta}}^{(4)}(h+\varepsilon w))\\ &= \exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\}\chi_3(g_{0,\underline{\theta}}^{(3)}(w))\\ &\quad\cdot\Phi_i(A_{\tilde{\underline{\theta}}_i}^{(4)}\cdot {}^tA_{\underline{\theta}_0}^{(4)}\cdot A_{\underline{\theta}}^{(4)}h) + O(\varepsilon)\\ &\qquad\text{as } \varepsilon\downarrow 0 \text{ in } \tilde{\mathbf{D}}^\infty \text{ if } |\underline{\theta}-\tilde{\underline{\theta}}_i| < \eta.\end{aligned}$$

Furthermore $O(\varepsilon)$ *is uniform on* $\{\underline{\theta};\, |\underline{\theta}-\tilde{\underline{\theta}}_i| < \eta\}$.

Here $g_0^{(1)}(w)$, $g_0^{(2)}(w)$ *and* $g_{0,\underline{\theta}}^{(3)}(w)$ *are as in* (5.3), (5.6) *and* (5.9), *respectively, and we define* $g_\varepsilon^{(1)}(w)$, $g_\varepsilon^{(2)}(w)$ *and* $g_{\varepsilon,\underline{\theta}}^{(3)}(w)$ *by*

$$\begin{aligned}g_\varepsilon^{(1)}(w) = (&w_1,\, S^{12}(1, w),\\ &((\tilde{h}\otimes\dot{w} - \dot{w}\otimes\tilde{h}) + \varepsilon S(1, w))_{ij,\, 1\le i<j\le 4,\, (i,j)\ne(1,2)},\ \langle A_{\pi/2}^{(1)}\tilde{h}, w\rangle_H),\end{aligned}$$

$$\begin{aligned}g_\varepsilon^{(2)}(w) = (&w_1,\, ((h\otimes\dot{w} - \dot{w}\otimes h) + \varepsilon S(1, w))_{ij,\,1\le i<j\le 4},\\ &\langle A_{(\pi/2,0)}^{(2)}h, w\rangle_H,\ \langle A_{(0,\pi/2)}^{(2)}h, w\rangle_H)\end{aligned}$$

and

$$\begin{aligned}g_{\varepsilon,\underline{\theta}}^{(3)}(w) = \Bigg(&w_1,\, ((h\otimes\dot{w} - \dot{w}\otimes h) + \varepsilon S(1, w))_{ij,\,1\le i<j\le 4},\\ &\left(\frac{\partial}{\partial\theta_i}\langle A_{\underline{\theta}}^{(4)}h, h'\rangle_H\Big|_{h' = A_{\underline{\theta}}^{(4)}w}\right)_{i=1,\cdots,4}\Bigg).\end{aligned}$$

Proof. We prove only C), the others being similarly proved. We use the same notations as in Lemma 5.3.C.

It is enough to prove that we can choose $\eta > 0$ such that

$$\begin{aligned}(5.13)\qquad &\sup_{0<\varepsilon\le 1,\, |\underline{\theta}-\tilde{\underline{\theta}}_i|<\eta} \|\exp\{2\pi S^{12}(1, w) + 4\pi S^{34}(1, w)\}\ \chi_3(g_{\varepsilon,\underline{\theta}}^{(3)}(w))\\ &\quad\cdot\Phi_i(A_{\tilde{\underline{\theta}}_i}^{(4)}\cdot {}^tA_{\underline{\theta}_0}^{(4)}\cdot A_{\underline{\theta}}^{(4)}(h+\varepsilon w))\|_{L^p(P)} < \infty\end{aligned}$$

for some $p > 1$. This is because the estimate (5.12) is true for almost all w and (5.13) guarantees the uniformly integrability, thus (5.12) is valid in the sense of L^p for some $p > 1$: The L^p-estimate of its higher order H-derivatives can be obtained in the same way.

Using $\xi_j^{(i)}$, $\eta_j^{(i)}$ in (5.5) the integrand of (5.12) is expressed by

$$\exp\{-\xi_1^{(1)}(\eta_1^{(2)}-\sqrt{2}\cdot\eta_0^{(2)})+\xi_1^{(2)}(\eta_1^{(1)}-\sqrt{2}\cdot\eta_0^{(1)})$$
$$-\xi_2^{(3)}(\eta_2^{(4)}-\sqrt{2}\cdot\eta_0^{(4)})+\xi_2^{(4)}(\eta_2^{(3)}-\sqrt{2}\cdot\eta_0^{(3)})\}$$
$$\cdot\chi_3(g_{\varepsilon,\underline{\theta}}^{(3)}(w))\cdot\Phi_i(A_{\tilde{\underline{\theta}}_i}^{(4)}\cdot{}^tA_{\underline{\theta}_0}^{(4)}\cdot A_{\underline{\theta}}^{(4)}(h+\varepsilon w))$$
$$\times\exp\left(\frac{1}{2}\{-\xi_2^{(1)}(\eta_2^{(2)}-\sqrt{2}\cdot\eta_0^{(2)})+\xi_2^{(2)}(\eta_2^{(1)}-\sqrt{2}\cdot\eta_0^{(1)})\right)\}$$
$$+2\{-\xi_1^{(3)}(\eta_1^{(4)}-\sqrt{2}\cdot\eta_0^{(4)})+\xi_1^{(4)}(\eta_1^{(3)}-\sqrt{2}\cdot\eta_0^{(3)})\})$$
$$\cdot\chi_3(g_{\varepsilon,\underline{\theta}}^{(3)}(w))\cdot\Phi_i(A_{\tilde{\underline{\theta}}_i}^{(4)}\cdot{}^tA_{\underline{\theta}_0}^{(4)}\cdot A_{\underline{\theta}}^{(4)}(h+\varepsilon w))$$
$$\times\exp\left(\sum_{k=1}^{2}\sum_{m=3}^{\infty}\frac{k}{m}\{\xi_m^{(2k)}(\eta_m^{(2k-1)}-\sqrt{2}\cdot\eta_0^{(2k-1)})\right.$$
$$\left.-\xi_m^{(2k-1)}(\eta_m^{(2k)}-\sqrt{2}\cdot\eta_0^{(2k)})\}\right)$$
$$\cdot\chi_3(g_{\varepsilon,\underline{\theta}}^{(3)}(w))$$
$$=I_1\times I_2\times I_3.$$

It is easy to see that $\sup_{\varepsilon,\underline{\theta}}\mathbf{E}[I_3^p]<\infty$, $1<p<3/2$, thus all we must do is to verify

(5.14) $$\sup_{\varepsilon,\underline{\theta}}\mathbf{E}[(I_1\times I_2)^q]<\infty,\text{ for some } q>3.$$

If $\Phi_i(A_{\tilde{\underline{\theta}}_i}^{(4)}\cdot{}^tA_{\underline{\theta}_0}^{(4)}\cdot A_{\underline{\theta}}^{(4)}(h+\varepsilon w))>0$, then we have

$$\| A_{\tilde{\underline{\theta}}_i}^{(4)}\cdot{}^tA_{\underline{\theta}_0}^{(4)}\cdot A_{\underline{\theta}}^{(4)}(h+\varepsilon w)-A_{\underline{\theta}_i}^{(4)}h\|_2<\gamma$$

where $\gamma=\gamma(\eta)$ is as in Lemma 5.1.C. Hence

$$\varepsilon^2\int_0^1|w_t|^2\,dt-2\int_0^1|A_{\tilde{\underline{\theta}}_i}^{(4)}{}^tA_{\underline{\theta}_0}^{(4)}(A_{\underline{\theta}}^{(4)}-A_{\underline{\theta}_0}^{(4)})h_t|^2\,dt<2\gamma^2,$$

i.e.

$$\varepsilon^2\int_0^1|w_t|^2\,dt-2\int_0^1|(A_{\underline{\theta}}^{(4)}-A_{\underline{\theta}_0}^{(4)})h_t|^2\,dt<2\gamma^2.$$

For all $\eta>0$, there exist $\gamma'=\gamma'(\eta)$ such that $\|A_{\underline{\theta}}^{(4)}-A_{\underline{\theta}_0}^{(4)}\|_{\mathrm{op}}<\gamma'$ if $|\underline{\theta}-\underline{\theta}_0|<\eta$ and $\gamma'\downarrow 0$ as $\eta\downarrow 0$. So there exists a constant $K>0$ satisfying

(5.15) $$\varepsilon^2\int_0^1|w_t|^2\,dt<2\gamma^2+2K\gamma'^2$$

for all $\varepsilon\in(0,1]$. On the other hand, $\chi_3(g_{\varepsilon,\underline{\theta}}^{(3)}(w))>0$ implies that

$$|w_1|<\delta,$$
$$|\Xi_{ij}^{(3)}+\varepsilon S^{ij}(1,w)|<\delta$$

and

$$\left| \frac{\partial}{\partial \theta_i} \langle A^{(4)}_{\underline{\theta}} h, h' \rangle_H \bigg|_{h' = A^{(4)}_{\underline{\theta}} w} \right| < \delta$$

for some $\delta > 0$. Clearly, for any constant $c_1 \in \mathbf{R}$,

$$\begin{aligned} &\exp(c_1 \{\xi^{(i)}_m (\eta^{(j)}_m - \sqrt{2} \cdot \eta^{(j)}_0)\}) \chi_3(g^{(3)}_{\varepsilon,\underline{\theta}}(w)) \\ &\quad = \exp(c_1 \xi^{(i)}_m \eta^{(j)}_m) \exp(-\sqrt{2}\, c_1 \xi^{(i)}_m \eta^{(j)}_0) \chi_3(g^{(3)}_{\varepsilon,\underline{\theta}}(w)) \\ &\quad \le \exp(c_1 \xi^{(i)}_m \eta^{(j)}_m) \exp(|c_1| \cdot \sqrt{2} \cdot \delta \cdot \xi^{(i)}_m) \end{aligned}$$

and $\exp(|c_1| \cdot \sqrt{2} \cdot \delta \cdot \xi^{(j)}_m) \in L^q$ for all $q > 0$. Therefore we can assume $\eta^{(i)}_0 = 0$ in (5.13).

First we treat with the term I_1. Clearly

$$\begin{aligned} &\exp(-\xi^{(1)}_1 \eta^{(2)}_1) \chi_3(g^{(3)}_{\varepsilon,\underline{\theta}}(w)) \\ &= \exp\left(\frac{1}{2} \{((\xi^{(1)}_1 - \eta^{(2)}_1)/\sqrt{2})^2 - ((\xi^{(1)}_1 + \eta^{(2)}_1)/\sqrt{2})^2\} \right) \chi_3(g^{(3)}_{\varepsilon,\underline{\theta}}(w)) \\ &= \exp\left\{ -\frac{1}{2} ((\xi^{(1)}_1 + \eta^{(2)}_1)/\sqrt{2})^2 \right\} \exp\{(\Xi^{(3)}_4)^2 / 8\pi u\} \chi_3(g^{(3)}_{\varepsilon,\underline{\theta}}(w)) \\ &\le \exp(\delta^2 / 2\pi u) \exp\left\{ -\frac{1}{2} ((\xi^{(1)}_1 + \eta^{(2)}_1)/\sqrt{2})^2 \right\} \in L^q \text{ for all } q > 0. \end{aligned}$$

Similarly we can prove $\exp(-\xi^{(3)}_2 \eta^{(4)}_2) \chi_3(g^{(3)}_{\varepsilon,\underline{\theta}}) \in L^q$ for all $q > 0$. Next

$$\exp(\xi^{(2)}_1 \eta^{(1)}_1) = \exp\left\{ -\frac{1}{2} ((\xi^{(2)}_1 - \eta^{(1)}_1)/\sqrt{2})^2 \right\} \exp\left\{ \frac{\pi}{u} (\tilde{\Xi}^{(3)}_{12})^2 \right\}$$

and

$$\begin{aligned} &\exp\left\{ \frac{\pi}{u} (\tilde{\Xi}^{(3)}_{12})^2 \right\} \Phi_i(A^{(4)}_{\tilde{\underline{\theta}}_i} \cdot {}^t A^{(4)}_{\underline{\theta}_0} \cdot A^{(4)}_{\underline{\theta}}(h + \varepsilon w)) \\ &\quad \le \exp\left(\frac{\pi}{u} \{(1 + \sqrt{u/\pi})\delta + |\varepsilon S^{12}(1, w)|^2\} \right). \end{aligned}$$

It is easy to show that there exists a Brownian motion $B(t)$ on W^1_0 such that

$$\varepsilon S^{12}(1, w) = B\left(\varepsilon^2 \int_0^1 \{(w^1_t)^2 + (w^2_t)^2\}\, dt \right).$$

Appealing to (5.15), for each $q > 1$ we can choose η such that

$$\sup_{\varepsilon,\underline{\theta}} \mathbf{E}[\exp\{q \xi^{(2)}_1 \eta^{(1)}_1\} \Phi_i(A^{(4)}_{\tilde{\underline{\theta}}_i} \cdot {}^t A^{(4)}_{\underline{\theta}_0} \cdot A^{(4)}_{\underline{\theta}}(h + \varepsilon w))] < \infty.$$

Similarly, for each $q > 1$ we can choose η such that

$$\sup_{\varepsilon,\underline{\theta}} \mathbf{E}[\exp\{q\xi_2^{(4)}\eta_2^{(3)}\}\, \Phi_i(A_{\underline{\tilde{\theta}}_i}^{(4)}\cdot {}^tA_{\underline{\theta}_0}^{(4)}\cdot A_{\underline{\theta}}^{(4)}(h+\varepsilon w))] < \infty.$$

Therefore it is easy to check that

$$\sup_{\varepsilon,\underline{\theta}} \mathbf{E}[I_1^q] < \infty,\ q > 1.$$

As for the term I_2 it is enough by (5.11) to treat with the terms $\exp\left\{-\frac{1}{6}(X_1X_2 - X_3X_4)\right\}$ and $\exp\{P_3(\Xi)\}$. Clearly

$$\begin{aligned}|P_3(\Xi)| &\le \sum c_2|\tilde{\Xi}_{ij}^{(3)}| + \sum c_3|\tilde{\Xi}_{ij}^{(3)}|\cdot|\tilde{\Xi}_{kl}^{(3)}| \\ &\le \sum c_2|\tilde{\Xi}_{ij}^{(3)}| + \sum c_3\frac{1}{2}(|\tilde{\Xi}_{ij}^{(3)}|^2 + |\tilde{\Xi}_{kl}^{(3)}|^2).\end{aligned}$$

Noting that $|\tilde{\Xi}_{ij}^{(3)}| \le \delta + |\varepsilon S^{ij}(1, w)|$ on $\chi_3 > 0$, we can control the term $\exp\{P_3(\Xi)\}$ in the same way as in I_1. Furthermore, noting that $|\Xi_1^{(3)}| < \delta$ and $|\tilde{\Xi}_2^{(3)}| < \delta$ if $\chi_3 > 0$, we can easily show that $|X_1 - X_2|^2 + |X_3 + X_4|^2 < 2\delta^2$, the term $\exp\left\{-\frac{1}{6}(X_1X_2 - X_3X_4)\right\}$ is also controled in the same way as in I_1. Therefore

$$\sup_{\varepsilon,\underline{\theta}} \mathbf{E}[I_2^q] < \infty \text{ for all } q > 1$$

and this completes the proof.

Lemma 5.5. *All of $g_\varepsilon^{(1)}$, $g_\varepsilon^{(2)}$ and $g_{\varepsilon,\underline{\theta}}^{(3)}$ are uniformly non-degenerate.*

Remark 5.5. The above lemma ensures the asymptotic expansions of $\delta_0(g_\varepsilon^{(1)})$, $\delta_0(g_\varepsilon^{(2)})$ and $\delta_0(g_{\varepsilon,\underline{\theta}}^{(3)})$, thus, combined with Lemma 5.4, we can justify the asymptotics (5.4), (5.7) and (5.10) and furthermore the asymptotic expansions of $J_2^{(1)}$, $J_2^{(2)}$ and $J_2^{(3)}$. Hence, we can conclude that $p(t, \mathbf{0}, \mathbf{x})$ has the expansion of the form (0.1), the main term of which is given by Theorem 5.1.A, B and C respectively.

Proof of Lemma 5.5. Here we treat only $g_\varepsilon^{(1)}$ since the others can be proved in a similar way.

Let $g_{\varepsilon,t}^{(1)}(w)$ be the $\mathbf{R}^{11}$-valued Wiener process given by

$$g_{\varepsilon,t}^{(1)}(w) = (w_t,\ S^{12}(t, w),$$

$$\left(\int_0^t (\tilde{h}_t^i dw_t^j - \tilde{h}_t^j dw_t^i) + \varepsilon S^{ij}(t, w)\right)_{\substack{1\le i<j\le 4\\(i,j)\ne(1,2)}},\ \sum_i \int_0^t (A_{\pi/2}^{(1)}\dot{\tilde{h}})_t^i dw_t^i).$$

Clearly $g_{\varepsilon,1}^{(1)}(w) = g_\varepsilon^{(1)}(w)$. Then $g_{\varepsilon,t}^{(1)}(w)$ satisfies the following S.D.E.:

$$dg_{\varepsilon,t}^{(1)}(w) = \sum_{\alpha=1}^{4} L_\alpha(\varepsilon, t, g_{\varepsilon,t}^{(1)}(w))\circ dw_t^\alpha$$

where $L_\alpha(\varepsilon, t, \zeta)$, $\alpha = 1,\cdots,4$, $\zeta = (\zeta_1,\cdots,\zeta_{11}) = (\mathbf{x}, x^1) \in \mathbf{R}^{11}$, are given by

$$L_1(\varepsilon, t, \zeta) = \frac{\partial}{\partial x_1} - \frac{1}{2}\left(x_2 \cdot \frac{\partial}{\partial x_{(12)}} + (\varepsilon x_3 + 2\tilde{h}_t^3) \cdot \frac{\partial}{\partial x_{(13)}} + (\varepsilon x_4 + 2\tilde{h}_t^4) \cdot \frac{\partial}{\partial x_{(14)}} \right),$$

$$L_2(\varepsilon, t, \zeta) = \frac{\partial}{\partial x_2} + \frac{1}{2}\left(x_1 \cdot \frac{\partial}{\partial x_{(12)}} - (\varepsilon x_3 + 2\tilde{h}_t^3) \cdot \frac{\partial}{\partial x_{(23)}} - (\varepsilon x_4 + 2\tilde{h}_t^4) \cdot \frac{\partial}{\partial x_{(24)}} \right),$$

$$L_3(\varepsilon, t, \zeta) = \frac{\partial}{\partial x_3} + \frac{1}{2}\left(\varepsilon x_1 \cdot \frac{\partial}{\partial x_{(13)}} + \varepsilon x_2 \cdot \frac{\partial}{\partial x_{(23)}} - (\varepsilon x_4 + 2\tilde{h}_t^4) \cdot \frac{\partial}{\partial x_{(34)}} \right) - \sqrt{4\pi u}\,\sin 2\pi t \frac{\partial}{\partial x^1}$$

and

$$L_4(\varepsilon, t, \zeta) = \frac{\partial}{\partial x_4} + \frac{1}{2}\left(\varepsilon x_1 \cdot \frac{\partial}{\partial x_{(14)}} + \varepsilon x_2 \cdot \frac{\partial}{\partial x_{(24)}} + (\varepsilon x_3 + 2\tilde{h}_t^3) \cdot \frac{\partial}{\partial x_{(34)}} \right) + \sqrt{4\pi u}\,\cos 2\pi t \frac{\partial}{\partial x^1}.$$

Let Y_t^ε be the 11×11 matrix given by

$$dY_t^\varepsilon = \partial L_\alpha(\varepsilon, t, g_{\varepsilon,t}^{(1)})\, Y_t^\varepsilon \circ dw_t^\alpha$$

where $\partial L_\alpha(\varepsilon, t, \zeta)$ is the 11×11 matrix given by $(\partial L_\alpha(\varepsilon, t, \zeta))_{ij} = \dfrac{\partial}{\partial \zeta_j} L_\alpha^i(\varepsilon, t, \zeta)$. Then we have

$$\langle Dg_\varepsilon^{(1)}(w), Dg_\varepsilon^{(1)}(w) \rangle_H = Y_1^\varepsilon \sum_{\alpha=1}^{4} \int_0^1 (Y_t^\varepsilon)^{-1} L_\alpha(\varepsilon, t, g_{\varepsilon,t}^{(1)}(w)) \otimes (Y_t^\varepsilon)^{-1} L_\alpha(\varepsilon, t, g_{\varepsilon,t}^{(1)}(w))\, dt\, {}^tY_1^\varepsilon.$$

By a slight computation, we know that $\det Y_1^\varepsilon = 1$. Therefore we will only evaluate the integral part which will be denoted by $\sigma(\varepsilon, w)$.

Let $l = {}^t(l_i, l_{jk}, l^1)_{\substack{i=1,\cdots,4 \\ 1 \le j < k \le 4}} \in \mathbf{R}^{11}$. Then we can easily compute that

$$
\begin{aligned}
&{}^t l\sigma(\varepsilon, w)l \\
&= \int_0^1 (\{l_1 - l_{12}\cdot w_t^2 - l_{13}(\varepsilon w_t^3 + \sqrt{u/\pi}\sin 2\pi t) \\
&\qquad - l_{14}(\varepsilon w_t^4 + \sqrt{u/\pi}(1-\cos 2\pi t))\}^2 \\
&\quad + \{l_2 + l_{12}\cdot w_t^1 - l_{23}(\varepsilon w_t^3 + \sqrt{u/\pi}\sin 2\pi t) \\
&\qquad - l_{24}(\varepsilon w_t^4 + \sqrt{u/\pi}(1-\cos 2\pi t))\}^2 \\
&\quad + \{(l_3 + l_{13}\cdot w_t^1 + l_{23}\cdot w_t^2 \\
&\qquad - l_{34}(\varepsilon w_t^4 + \sqrt{u/\pi}(1-\cos 2\pi t)) - l^1\cdot\sqrt{4\pi u}\sin 2\pi t\}^2 \\
&\quad + \{l_4 + l_{14}\cdot w_t^1 + l_{24}\cdot w_t^2 \\
&\qquad + l_{34}(\varepsilon w_t^3 + \sqrt{u/\pi}\sin 2\pi t)) + l^1\cdot\sqrt{4\pi u}\cos 2\pi t\}^2)\,dt.
\end{aligned}
$$

Now we will prove that for any T large enough,

$$
P\left(\inf_{|l|=1} {}^t l\sigma(\varepsilon, w)l < \frac{1}{T}\right) \le c_1 e^{-c_2 T^{c_3}} \tag{5.16}
$$

for some positive constants c_1, c_2 and c_3 all of which are independent of ε. We know easily that

$$
P\left(\sup_{|l|=1} {}^t l\sigma(\varepsilon, w)l \ge T\right) \le c_4 e^{-c_5 T}
$$

for some positive constants c_4 and c_5 which are independent both of ε and l. Thus it is enough to estimate

$$
P\left({}^t l\sigma(\varepsilon, w)l < \frac{1}{T}\right)
$$

uniformly in l (cf. S.Kusuoka-D.W.Stroock [12], Appendix). Appealing to J.Norris [18] or N.Ikeda-S.Watanabe [10], however, it is easy to check that

$$
P\left({}^t l\sigma(\varepsilon, w)l < \frac{1}{T}\right) \le c_6 e^{-c_7 T^{c_8}}
$$

where c_6, c_7 and c_8 are positive constants all of which are independent of l. Thus (5.16) is concluded.

Then it is easy to see that

$$
\varlimsup_{\varepsilon\downarrow 0} \mathbf{E}[|\det\langle g_\varepsilon^{(1)}(w), g_\varepsilon^{(1)}(w)\rangle_H|^{-p}] < \infty
$$

for all $p > 0$, and this completes the proof.

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