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Complements on the Hilbert transform and the fractional derivative of Brownian local times

By

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I. Introduction

Hilbert transform and fractional derivative of order $\alpha\in(0; 1/2)$ of Brownian local times have been already studied by various authors (Yamada [14], [15], Yor [16], Biane et Yor [2]). It is easy to show that these processes have a bounded quadratic variation with regard to the dyadic subdivisions, but they have not a bounded variation. So, we cannot directly use the theory of stochastic integration for semimartingales to study them.

We first prove that they have almost surely a finite p-variation, for some suitable p; and the integration defined in [1] for such processes allows us to avoid the former difficulty. In particular, we obtain an Itô's formula and prove that they admit occupation densities (which are represented by a Tanaka's formula). The main results of this paper are two Ray-Knight's type theorems which describe these occupation densities taken at some first hitting times. Eventually, we translate these results to Bessel processes of dimension $d<1$, and give in Appendix some general consequences of this study.

In the present paper, we consider $(\Omega, \mathcal{F}, P)$, a complete probability space, endowed with a right continuous increasing family $(\mathcal{F}_t: t\geqq 0)$ of σ-field. Let $B=(B_t: t\geqq 0)$ be a real $(\mathcal{F}_t)$-Brownian motion, and $\{L_t^a: a\in\boldsymbol{R}, t\geqq 0\}$ a jointly continuous version of its local times. Using the Hölder property of $a\mapsto L_t^a$, we can define, for any $\alpha\in(0, 1/2)$

$$C(t)=\text{v.p.}\int_0^t \frac{ds}{B_s}=\lim_{\varepsilon\uparrow 0}\int_0^t \mathbf{1}_{|B_s|>\varepsilon}\frac{ds}{B_s}=\int_{\boldsymbol{R}} L_t^a a^{-1} da$$

and

$$H(-1-\alpha, t)=\text{p.f.}\int_0^t B_s^{-1-\alpha}\mathbf{1}_{B_s>0}ds=\lim_{\varepsilon\downarrow 0}\int_0^t B_s^{-1-\alpha}\mathbf{1}_{B_s>\varepsilon}ds$$

$$=\int_{\boldsymbol{R}^+}(L_t^a-L_t^0)a^{-1-\alpha}da$$

$C(t)$ and $H(-1-\alpha, t)$ are represented by the following generalizations of Itô's formula (Yamada [14], [15], Yor [16])

$$\text{(I.1)}: \qquad C(t)=2\Big(B_t\log|B_t|-B_t-\int_0^t \log|B_s|\,dB_s\Big)$$

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and

(I.2): $$H(-1-\alpha,\ t)=-\frac{2}{\alpha(1-\alpha)}(B_t^+)^{1-\alpha}+2\alpha^{-1}\int_0^t B_s^{-\alpha}\mathbf{1}_{B_s>0}dB_s\,.$$

$C(t)$ and $H(-1-\alpha,\ t)$ are additive functionals locally of zero energy in Fukushima's sense (see Fukushima [6], Yamada [15]), and therefore have almost surely a bounded quadratic variation with respect to the dyadic subdivisions of any compact interval of $\boldsymbol{R}_+$; but are not of bounded variation (see Wang [13]).

Eventually, let us notice the following scaling invariance properties for $C(\cdot)$ and $H(-1-\alpha,\ \cdot)$:

(I.3): For any $\lambda>0$, $(C(\lambda t):\ t\geqq 0)\overset{d}{=}(\lambda^{1/2}C(t):\ t\geqq 0)$

and $(H(-1-\alpha,\ \lambda t):\ t\geqq 0)\overset{d}{=}(\lambda^{(1-\alpha)/2}H(-1-\alpha,\ t):\ t\geqq 0)$

which are easy consequences of the scaling invariance property for B and of the definition of $C(\cdot)$ and $H(-1-\alpha,\ \cdot)$.

II. Processes locally of bounded p-variation

Let p be a positive real number, $f:\ \boldsymbol{R}_+\to\boldsymbol{R}$ a càdlàg function, and $\sigma=(s_0=0<s_1<\cdots<s_n=t)$ a finite subdivision of $[0;t]$. Then we denote by $V_\sigma^p(f)=|f(0)|^p+\sum_{i=0}^{n-1}|f(s_{i+1})-f(s_i)|^p$ the p-variation of f with regard to σ; and we say that f has a bounded p-variation on $[0;t]$ if $\{V_\sigma^p(f):\ \sigma$ finite subdivision of $[0;t]\}$ is bounded, and that f has an infinite p-variation on $[0;t]$ otherwise. Note that f has a bounded p'-variation on $[0;t']$ whenever f has a bounded p-variation on $[0;t]$ with $p\leqq p'$, $t'\leqq t$. We have

Theorem II.1. *$\boldsymbol{P}$ a.s., for every positive t,*

i) *$H(-1-\alpha,\ \cdot)$ has a bounded p-variation on $[0,\ t]$ if and only if $p>(1-\alpha)^{-1}$.*

ii) *C has a bounded p-variation on $[0;\ t]$ if and only if $p>1$.*

Proof. Let us denote by τ the right-continuous inverse of L^0 (i.e. $\tau_t=\inf\{s:\ L_s^0>t\}$). Then $H(-1-\alpha,\ \tau.)$ is a stable process of index $(1-\alpha)^{-1}$ (indeed, the strong Markov property implies that $H(-1-\alpha,\ \tau.)$ has stationary independent increments, and it remains to apply the scaling invariance property (I.3), see Biane and Yor [2]). According to Bretagnolle's Theorem III.b in [3], $\boldsymbol{P}$ a.s., for all positive t and all $p>(1-\alpha)^{-1}$, $H(-1-\alpha,\ \tau.)$ has bounded p-variation on $[0;t]$. Let us denote by $\mathcal{Z}=\{t:\ B_t=0\}$ and by $\mathcal{Z}_r$ the right-ends of the excursion intervales of B. Then, $\boldsymbol{P}$ a.s., for all positive t and $p>(1-\alpha)^{-1}$, $\{V_\nu^p(H(-1-\alpha,\ \cdot)):\ \nu$ finite subdivision of $[0;\tau_t]$ and $\nu\subset\mathcal{Z}_r\}$ is bounded; and since $H(-1-\alpha,\ \cdot)$ is continuous and $\mathcal{Z}$ is the closure of $\mathcal{Z}_r$, we have

(II.1): $$\mathrm{Sup}\{V_\nu^p(H(-1-\alpha,\ \cdot)):\ \nu\text{ finite subdivision of }[0;\tau_t]\text{ and }\nu\cup\mathcal{Z}\}<\infty$$

Let us prove by induction that for every non negative integer n,

$(\mathrm{I})_n$: *For* $\boldsymbol{P}$ *a.e.* $\omega\in\Omega$, *if* σ *is a finite subdivision of* $[0;\tau_t]$ *with* $\mathrm{Card}\{s\in\sigma: s\notin\mathcal{Z}\}\leqq n$, *then there exists* ν, *a finite subdivision of* $[0;\tau_t]$ *included in* $\mathcal{Z}$ *such that, for all* $p\geqq 1$,

$$V_\sigma^p(H(-1-\alpha,\ \cdot)(\omega))\leqq V_\nu^p(H(-1-\alpha,\ \cdot)(\omega)).$$

Indeed, $(\mathrm{I})_0$ is obvious, so we suppose that $(\mathrm{I})_n$ holds. Consider $\sigma=(0=s_0<\cdots<s_n=\tau_t)$ a finite subdivision of $[0;\tau_t]$ with $\mathrm{Card}(\sigma\setminus\mathcal{Z})=n+1$. Let us denote by $i=\inf\{k: s_k\notin\mathcal{Z}\}$. Then $i\geqq 1$, and

a) If $\inf\{H(-1-\alpha,\ s): s=s_{i\pm1}\}\leqq H(-1-\alpha,\ s_i)\leqq \mathrm{Sup}\{H(-1-\alpha,\ s): s=s_{i\pm1}\}$, then take $\sigma'=\sigma\setminus\{s_i\}$. For $p\geqq 1$, $V_\sigma^p(H(-1-\alpha,\ \cdot)(\omega))\leqq V_{\sigma'}^p(H(-1-\alpha,\ \cdot)(\omega))$, and it remains to apply $(\mathrm{I})_n$.

b) If $H(-1-\alpha,\ s_i)<\inf\{H(-1-\alpha,\ s): s=s_{i\pm1}\}$, then denote by $g(s_i)=\inf\{s<s_i: B_s=0\}$. Since according to its definition, $H(-1-\alpha,\ \cdot)$ increases on every interval on which B is never 0, $g(s_i)>s_{i-1}$ and $H(-1-\alpha,\ g(s_i))<H(-1-\alpha,\ s_i)$. So take $\sigma'=(s_0,\cdots,s_{i-1},g(s_i),s_{i+1},\cdots,s_m)$ and apply $(\mathrm{I})_n$.

c) If $H(-1-\alpha,\ s_i)>\mathrm{Sup}\{H(-1-\alpha,\ s): s=s_{i\pm1}\}$, then denote by $d(s_i)=\mathrm{Sup}\{s>s_i: B_s=0\}$. The same arguments as above imply that $d(s_i)<s_{i+1}$ and $H(-1-\alpha,\ d(s_i))>H(-1-\alpha,\ s_i)$. So take $\sigma'=(s_0,\cdots,s_{i+1},d(s_i),s_{i+1},\cdots,s_m)$ and apply $(\mathrm{I})_n$.

Hence $(\mathrm{I})_n$ is proven by induction, and according to (II.1), the sufficient condition of i) is established.

Since Theorem III.b of Bretagnolle [3] implies that $\boldsymbol{P}$ a.s., for all positive t and $p\leqq(1+\alpha)^{-1}$, $H(-1-\alpha,\ \tau.)$ has an infinite p-variation on $[0;t]$, the same property obviously holds for $H(-1-\alpha,\ \cdot)$; so i) is proved.

ii) is easily obtained by the same arguments as for i).

The definifion of p-variation for processes is slightly different: a *continuous* adapted process Z has a bounded p-variation if

$$\mathrm{Sup}\{\boldsymbol{E}(V_\sigma^p(Z)): \sigma\in\mathbf{S}\}<\infty,$$

where $\mathbf{S}$ denotes the set of random subdivisions $\sigma=(0\equiv S_0\leqq S_1\leqq\cdots\leqq S_n)$ of *optional times* (such subdivisions are called "optional subdivisions"). We say that Z has a locally bounded p-variation if there exists an increasing sequence of optional times $(T_n: n\in\mathbf{N})$, $T_n\uparrow+\infty$ P a.s., such that, for all n, Z^{T_n} has a bounded p-variation. It is easy to prove that if Y is an adapted continuous process, such that for $\boldsymbol{P}$ a.s. $\omega\in\Omega$ and for all positive t, the function $Y(\omega)$ has a bounded p-variation on $[0;t]$, then the process Y has a locally bounded p-variation; but the converse is not true (since a Brownian motion has a bounded 2-variation in our sense, but its paths have a.s. an infinite 2-variation on $[0;1]$).

Let us now recall some results established in [1]. When X (respectively Y) is a continuous adapted process locally of bounded p-variation (respectively q-variation), where $p<q$ verify $1/p+1/q>1$, then we can define $Z.=\int_0^\cdot Y_s dX_s$ which generalizes the usual integral when X or Y are semimartingales. Z has a locally bounded p-variation, and we have

Theorem II.2. (**associativity**). *Let Y' be an adapted continuous process which has a locally bounded q-variation. Then $Y\cdot Y'$ has a locally bounded q-variation too, and* $\int_0^{\cdot} Y'_s\cdot Y_s dX_s=\int_0^{\cdot} Y'_s dZ_s$.

Suppose now that $p<2$, and that $Y^i=M^i+X^i$ ($i\in\{1;\cdots;d\}$), where M^i is a continuous local martingale and X^i has a locally bounded p-variation. Then we may take $q=2$ and we have (see also Föllmer [5])

Theorem II.3. (**Itô's formula**). *Let $f: \boldsymbol{R}^d\to\boldsymbol{R}$ be a function of class C^2. Then $\frac{\partial f}{\partial x_i}(Y.)$ has a locally bounded 2-variation, and*

$$f(Y_t)-f(Y_0)=\sum_{i=1}^{d}\left(\int_0^t \frac{\partial f}{\partial x_i}(Y_s)dM_s^i+\int_0^t \frac{\partial f}{\partial x_i}(Y_s)\,dX_s^i\right)$$
$$+\frac{1}{2}\sum_{i,j}\int_0^t \frac{\partial^2 f}{\partial x_i\partial x_j}(Y_s)d\langle M^i, M^j\rangle_s .$$

III. Stochastic calculus and occupation measures

Using the integral defined in [1] for processes of locally bounded p-variation, we are now able to develop a stochastic calculus for C and $H(-1-\alpha, \cdot)$.

III.1. Study for C. The main tool for the stochastic calculus for C is the following

Proposition III.1. *If f is a function with continuous derivative, then we have the following Itô's formula*:

$$B_t\cdot f(C(t))=\int_0^t f(C(s))dB_s+\int_0^t f'(C(s))ds .$$

Proof. Formula (I.1) implies

$$B_t\cdot C(t)=2\left(B_t^2\log|B_t|-B_t^2-B_t\int_0^t\log|B_s|\,dB_s\right),$$

and from the usual Itô's formula, we deduce

$$B_t\cdot C(t)=\int_0^t C(s)dB_s+t .$$

Since C is a continuous process locally of bounded p-variation for all $p>1$, the comparison with Theorem II.3 gives

$$\int_0^t B_s dC(s)=t .$$

Then, Theorem II.2 implies that for any Z locally of bounded β-variation ($\beta>1$), we have

$$\int_0^t Z(s)B_s dC(s)=\int_0^t Z(s)ds\,.$$

Applying again Theorem II.3, we deduce that for any f of class C^2,

$$B_t\cdot f(C(t))=\int_0^t f(C(s))dB_s+\int_0^t f'(C(s))ds\,.$$

and by density, this formula holds for f of class C^1.

Let us now define the family of processes $\{\lambda_t^a : a\in \boldsymbol{R},\ t\geqq 0\}$ by the following Tanaka's formula

$$B_t\cdot \mathbf{1}_{(C(t)>a)}=\int_0^t \mathbf{1}_{(C(s)>a)}dB_s+\lambda_t^a\,. \tag{III.1}$$

Then we have

Theorem III.2. *$\boldsymbol{P}$ a.s., for any bounded Borelian function f,*

$$\int_0^t f(C(s))ds=\int_{\boldsymbol{R}} f(a)\lambda_t^a da\,.$$

Proof. Let $\varphi\colon \boldsymbol{R}\to\boldsymbol{R}$ be a function of class C^1 with compact support. Definition (III.1) implies

$$\int_{\boldsymbol{R}} B_t \mathbf{1}_{(C(t)>a)}\varphi(a)da=\int_{\boldsymbol{R}} da\varphi(a)\int_0^t \mathbf{1}_{(C(s)>a)}dB_s+\int_{\boldsymbol{R}}\varphi(a)\lambda_t^a da\,.$$

If we denote by Φ the primitive of φ with $\lim_{-\infty}\Phi=0$, then we have

$$\int_{\boldsymbol{R}} B_t \mathbf{1}_{(C(t)>a)}\varphi(a)da=B_t\Phi(C(t))\,,$$

and the stochastic Fubini's theorem (see Lemma 4.1, p. 116 in Ikeda and Watanabe [8]) implies

$$\int_{\boldsymbol{R}} da\varphi(a)\int_0^t \mathbf{1}_{(C(s)>a)}dB_s=\int_0^t dB_s\int_{\boldsymbol{R}} da\varphi(a)\mathbf{1}_{(C(s)>a)}=\int_0^t \Phi(C(s))dB_s\,.$$

Hence we have

$$B_t.\Phi(C(t))=\int_0^t \Phi(C(s))dB_s+\int_{\boldsymbol{R}}\varphi(a)\lambda_t^a da\,.$$

The comparison with Proposition III.1 finishes the proof.

Remark. λ_t^a can also be described by the following formulas

$$B_t \mathbf{1}_{(C(t)<a)}=\int_0^t \mathbf{1}_{(C(s)<a)}dB_s-\lambda_t^a\,,\qquad \text{and}$$

$$B_t\,\mathrm{sgn}(C(t)-a)=\int_0^t \mathrm{sgn}(C(s)-a)dB_s+2\lambda_t^a\,,$$

where $\mathrm{sgn}=\mathbf{1}_{\boldsymbol{R}_+}-\mathbf{1}_{\boldsymbol{R}_-}$.

Let us now study the regularity of $(a, t)\mapsto\lambda_t^a$:

Proposition III.3. *There is a version of* $(a, t)\mapsto\lambda_t^a$ *such that,* $\boldsymbol{P}$ *a.s.,*

i) *for all* $a\in\boldsymbol{R}$, $t\mapsto\lambda_t^a$ *admits left and right limits at all* t.

ii) *for all* $t\geqq 0$, $a\mapsto\lambda_t^a$ *admits left and right limits at all* a.

iii) *for all* $a\in\boldsymbol{R}$ *and* $t\geqq 0$,

$$\lambda_{t+}^a+\lambda_{t-}^a=2\lambda_t^a\,,\qquad \lambda_{t+}^a-\lambda_{t-}^a=|B_t|\mathbf{1}_{(C(t)=a)}\,,$$

$$\lambda_t^{a+}+\lambda_t^{a-}=2\lambda_t^a\,,\qquad \lambda_t^{a+}-\lambda_t^{a-}=B_t\mathbf{1}_{(C(t)=a)}\,.$$

Proof. From the Burkholder, Davis, Gundy inequalities and the Kolmogorov criterium, we easily obtain that there is a continuous version of

$$(a, t)\longmapsto\int_0^t\mathbf{1}_{(C(s)>a)}dB_s\,.$$

Therefore $(a, t)\mapsto\lambda_t^a$ is continuous at every point (a, t) verifying $B_t=0$ or $C(t)\neq a$. If $B_t\neq 0$ and $C(t)=a$, then, Tanaka's formula (III.1) implies

$$\lambda_t^{a+}+\lambda_t^{a-}=2\lambda_t^a \qquad\text{and}\qquad \lambda_t^{a+}-\lambda_t^{a-}=B_t\,.$$

If we suppose furthermore that $B_t>0$, it is clear that $\lim\limits_{s\uparrow t}C(s)=a^-$ and $\lim\limits_{s\downarrow t}C(s)=a^+$; hence we get

$$\lambda_{t+}^a-\lambda_{t-}^a=B_t \qquad\text{and}\qquad \lambda_{t+}^a+\lambda_{t-}^a=2\lambda_t^a\,.$$

The proof is the same when $B_t<0$.

We will now always consider such a version of $(a, t)\mapsto\lambda_t^a$; and we are able to give an other description of λ_t^0:

Proposition III.4. $\boldsymbol{P}$ *a.s., for every positive* t,

$$\lambda_t^0=\sum_{s<t}|B_s|\mathbf{1}_{(C(s)=0)}+\frac{1}{2}|B_t|\mathbf{1}_{(C(t)=0)}\,.$$

Proof. Let us first establish the following

Lemma III.5. $\boldsymbol{P}$ *a.s.,* $\{s\in\boldsymbol{R}_+: C(s)=0\}$ *is a closed countable set, and all its points, except* 0, *are isolated.*

Proof of the lemma III.5. On the one hand, if $\tau_u=\inf\{v\geqq 0: L_v^0>u\}$, then $\{s: B_s=C(s)=0\}=\{s=\tau_u: C(\tau_u)=0\}\cup\{s=\tau_{u-}: C(\tau_{u-})=0\}$, and since $C(\tau.)$ is a Cauchy process (see Biane and Yor [2]), $\boldsymbol{P}$ a.s., $\{u\geqq 0: C(\tau_u)=0$ or $C(\tau_{u-})=0\}=\{0\}$, and hence $\boldsymbol{P}$ a.s., $\{s\geqq 0: B_s=C(s)=0\}=\{0\}$.

On the other hand, if we pick $s\in\boldsymbol{R}_+$ such that $B_s\neq 0$ and $C(s)=0$, then there exists a positive ε such that for every $u\in(s-\varepsilon, s+\varepsilon)$, $|B_u|>\varepsilon$; and since $C(u)=\int_s^u dv/B_v$, s is an isolated zero of C. In particular, $\{s: C(s)=0\}$ is finite or countable. Usual scaling arguments and the $0-1$ law imply that $\{s: C(s)=0\}$ is countable.

Proof of proposition III.4. (*end*) Theorem III.2 and Proposition III.3 imply that $\boldsymbol{P}$ a. s., on any interval $[s, t]$ on which C is never zero,

$$\lambda_s^0=\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\int_0^s \mathbf{1}_{[0,\varepsilon)}(C(v))dv=\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\int_0^t \mathbf{1}_{[0,\varepsilon)}(C(v))dv=\lambda_t^0 .$$

Since $t\mapsto\lambda_t^0$ increase only on $\{s: C(s)=0\}$ which is a closed set whose the only accumulation point is 0, Proposition III.4 is a direct consequence of Proposition III.3.

Another easy consequence of Proposition III.3 is

Proposition III.6. *If f is a continuous function with compact support, then the sequence of processes* $\left\{n^{-1/2}\int_0^{nt} f(C(s))ds : t\geqq 0\right\}$ *converges in the sense of the finite dimensional distributions as* $n\uparrow+\infty$ *to*

$$\left\{\left(\int f(a)\,da\right)\lambda_t^0 : t\geqq 0\right\}.$$

Proof.

$$n^{-1/2}\int_0^{nt} f(C(s))ds=n^{1/2}\int_0^t f(C(ns))ds\overset{(d)}{=}n^{1/2}\int_0^t f(n^{1/2}C(s))ds .$$

Since

$$n^{1/2}\int_0^t f(n^{1/2}C(s))ds=n^{1/2}\int_{\boldsymbol{R}} f(n^{1/2}a)\lambda_t^a da=\int_{\boldsymbol{R}} f(a)\lambda_t^{n^{-1/2}a}da$$

and since, according to Proposition III.3, $\lim_{n\uparrow\infty}\lambda_t^{n^{-1/2}a}=\lambda_t^0$ uniformly on any compact neighbourhood of 0, except when $C(t)=0$, we obtain

$$\lim_{n\uparrow\infty}\int_{\boldsymbol{R}} f(a)\lambda_t^{n^{-1/2}a}da=\left(\int_{\boldsymbol{R}} f(a)da\right)\lambda_t^0 .$$

III.2. Study for $H(-1-\alpha, \cdot)$. Henceforth, let us make the *convention* $0^\beta=0$ *for all real number* β (*even the negative ones*). Let us introduce

Notation. $S_t=\mathrm{Sup}\{H(-1-\alpha, s): s\leqq t\}$, $I_t=\mathrm{Inf}\{H(-1-\alpha, s): s\leqq t\}$

$$T(x)=\inf\{t\geqq 0: H(-1-\alpha, t)=x\}\ (x\in\boldsymbol{R}),\quad\text{and}\quad A_t=\int_0^t (B_s^+)^{-2\alpha}ds .$$

An important and direct consequence of the definition of $H(-1-\alpha, t)$ is the following

Lemma III.7. $\boldsymbol{P}$ *a.s., for every* $x\in\boldsymbol{R}$, $T(x)$ *is finite; and if* $x<0$, *then* $B_{T(x)}^+=0$.

Proof. With help from the scaling invariance property for $H(-1-\alpha, \cdot)$, we just have to show that $T(-1)$ and $T(1)$ are finite $\boldsymbol{P}$ a. s..

If $\boldsymbol{P}(\{T(-1)=+\infty\})=\delta>0$, then, applying (I.3), for any $\varepsilon>0$,

$$P(\{\forall t<\varepsilon,\ H(-1-\alpha,\ t)\geqq 0\})=\delta\,.$$

So

$$P(\bigcap_{\varepsilon>0}\{\forall t<\varepsilon,\ H(-1-\alpha,\ t)\geqq 0\})=\delta\,,$$

and Blumenthal's 0-1 law implies that $\delta=1$. Therefore $H(-1-\alpha,\ \cdot)$ would be a positive additive functional of B; and we know it is not true. The proof is the same for $T(1)$.

Eventually, if $B^{+}_{T(x)}$ were positive for a negative x then $H(-1-\alpha,\ \cdot)$ would increase on a neighbourhood of $T(x)$; so $T(x)$ would not be the first hitting time of x by $H(-1-\alpha,\ \cdot)$.

As in § III.1, let us prove

Proposition III.8. *If f is a function with continuous derivative, and if F is a primitive of f, then we have the following Itô's formula:*

$$(B_t^+)^{1-\alpha}f(H(-1-\alpha,\ t))=(1-\alpha)\int_0^t f(H(-1-\alpha,\ s))(B_s^+)^{-\alpha}dB_s$$
$$+\int_0^t f'(H(-1-\alpha,\ s))(B_s^+)^{-2\alpha}ds-\frac{\alpha(1-\alpha)}{2}[F(H(-1-\alpha,\ t))-F(0)]\,.$$

Proof. With no loss of generality, we may assume that f is C^2 with compact support. We first notice that

(III.2):
$$(B_t^+)^{1-\alpha}H(-1-\alpha,\ t)=(1-\alpha)\int_0^t H(-1-\alpha,\ s)(B_s^+)^{-\alpha}dB_s$$
$$+\int_0^t (B_s^+)^{-2\alpha}ds-\frac{\alpha(1-\alpha)}{4}H^2(-1-\alpha,\ t)\,.$$

Indeed, Formula (I.2) implies

$$\left[H(-1-\alpha,\ t)+\frac{2}{\alpha(1-\alpha)}(B_t^+)^{1-\alpha}\right]^2$$
$$=4\alpha^{-2}\int_0^t(B_s^+)^{-2\alpha}ds+4\alpha^{-1}\int_0^t\left[H(-1-\alpha,\ s)+\frac{2}{\alpha(1-\alpha)}(B_s^+)^{1-\alpha}\right](B_s^+)^{-\alpha}dB_s\,,$$

therefore

$$\frac{4}{\alpha(1-\alpha)}(B_t^+)^{1-\alpha}H(-1-\alpha,\ t)=4\alpha^{-1}\int_0^t\left[H(-1-\alpha,\ s)+\frac{2}{\alpha(1-\alpha)}(B_s^+)^{1-\alpha}\right](B_s^+)^{-\alpha}dB_s$$
$$+4\alpha^{-2}\int_0^t(B_s^+)^{-2\alpha}ds-4(\alpha(1-\alpha))^{-2}(B_t^+)^{2-2\alpha}-H^2(-1-\alpha,\ t)\,.$$

On the other hand,

$$(B_t^+)^{2-2\alpha}=(2-2\alpha)\int_0^t(B_s^+)^{1-2\alpha}dB_s+(1-\alpha)(1-2\alpha)\int_0^t(B_s^+)^{-2\alpha}ds\,,$$

and hence

$$4[\alpha(1-\alpha)]^{-1}(B_t^+)^{1-\alpha}H(-1-\alpha,\,t)=4\alpha^{-1}\int_0^t H(-1-\alpha,\,s)(B_s^+)^{-\alpha}dB_s$$
$$-4[\alpha(1-\alpha)]^{-1}\int_0^t (B_s^+)^{-2\alpha}ds-H^2(-1-\alpha,\,t)\,,$$

and (III.2) is proved.

According to Theorem II.1, there exists $p<2$ such that $H(-1-\alpha,\,\cdot)$ is a process locally of bounded p-variation. The comparison of (III.2) with Theorem II.3 for $(B_t^+)^{1-\alpha}H(-1-\alpha,\,t)$ gives

$$\int_0^t (B_s^+)^{1-\alpha}dH(-1-\alpha,\,s)=\int_0^t (B_s^+)^{-2\alpha}ds\,.$$

The application of Theorem II.2 and II.3 to $(B_t^+)^{1-\alpha}f(H(-1-\alpha,\,t))$ imply the proposition.

Using the same arguments as for C, we obtain

Theorem III.9. *For every $a\in\boldsymbol{R}$ and $t\in\boldsymbol{R}_+$, let $\lambda_t^a(\alpha)$ be the process defined by Tanaka's formula:*

$$(B_t^+)^{1-\alpha}\mathbf{1}_{\{H(-1-\alpha,\,t)>a\}}=(1-\alpha)\int_0^t \mathbf{1}_{\{H(-1-\alpha,\,s)>a\}}(B_s^+)^{-\alpha}dB_s$$
$$-\frac{\alpha(1-\alpha)}{2}[(H(-1-\alpha,\,t)-a)^+-a^+]+\lambda_t^a(\alpha)\,.$$

Then, $\boldsymbol{P}$ a.s., for every bounded Borelian function φ and $t>0$, we have

$$\int_0^t \varphi(H(-1-\alpha,\,s))(B_s^+)^{-2\alpha}ds=\int_{\boldsymbol{R}}\varphi(a)\lambda_t^a(\alpha)da\,.$$

Remark. At first sight, the choice of the definition of the occupation measure, and consequently of $\lambda_t^a(\alpha)$, may look strange, and we could think. it would be more natural to consider $\bar{\lambda}_t^a(\alpha)$ defined by

$$(B_t^+)^{1+\alpha}\mathbf{1}_{\{H(-1-\alpha,\,t)>a\}}=\int_0^t \mathbf{1}_{\{H(-1-\alpha,\,t)>a\}}d(B_s^+)^{1+\alpha}+\bar{\lambda}_t^a(\alpha)\,,$$

and to prove with the same arguments as the former ones that

$$\int_0^t \varphi(H(-1-\alpha,\,s))ds=\int_{\boldsymbol{R}}\varphi(a)\bar{\lambda}_t^a(\alpha)da\,.$$

The justification of our choice is that the extension of Itô's formula that we gave in Proposition III.8 is better adapted to the stochastic calculus than the one we would have obtained if we had considered $(B_t^+)^{1+\alpha}$ instead of $(B_t^+)^{1-\alpha}$: indeed, the bracket of the martingale part of $(B_t^+)^{1-\alpha}f(H(-1-\alpha,\,t))$, $(1-\alpha)^2\int_0^t f^2(H(-1-\alpha,\,s))(B_s^+)^{-2\alpha}ds$ is of the same type as its part of bounded variation, $\int_0^t f'(H(-1-\alpha,\,s))(B_s^+)^{-2\alpha}ds$. § IV is a direct consequence of this fact. Furthermore, this choice will allow us to translate our results to Bessel processes of dimension $0<d<1$ (see § V).

As in § III.1, we have

Corollary III.10. i) *There exists a jointly continuons version of*

$$(a, t) \longmapsto \lambda_t^a(\alpha)-(B_t^+)^{1-\alpha}\mathbf{1}_{\{H(-1-\alpha,t)>a\}}\,.$$

Henceforth, $\{\lambda_t^a : a\in\boldsymbol{R},\ t\geqq 0\}$ *will always denote such a version.*

ii) $\boldsymbol{P}$ *a.s., for every* $t\in\boldsymbol{R}_+$,

$$\lambda_t^0(\alpha)=\sum_{s<t}(B_s^+)^{1-\alpha}\mathbf{1}_{\{H(-1-\alpha,t)>0\}}+1/2(B_t^+)^{1-\alpha}\mathbf{1}_{\{H(-1-\alpha,t)>0\}}\,.$$

Proof. i) can be proven by the same methods as Proposition III.3.

ii) Let us set

$$\nu_t^a(\alpha)=\sum_{s<t}(B_s^+)\mathbf{1}_{\{H(-1-\alpha,t)>a\}}=\int_0^{t-}\mathbf{1}_{\{B_s>0\}}d\lambda_s^a.$$

It is easy to see that $\boldsymbol{P}$ a.s., for every bounded Borelian function φ and every $t\geqq 0$,

$$\int_{\boldsymbol{R}}\varphi(a)\lambda_t^a da=\int_0^t\varphi(H(-1-\alpha,s))(B_s^+)^{-2\alpha}ds=\int_{\boldsymbol{R}}\varphi(a)\nu_t^a da\,.$$

and hence we have

(III.7): $\boldsymbol{P}$ a.s., for every $t\geqq 0$, da a.s., $\lambda_t^a=\nu_t^a$;

and we want to prove that this equality holds for $a=0$. So, let us suppose that there exists $\varepsilon>0$ so that $\boldsymbol{P}(\lambda_{T(-1)}^0-\nu_{T(-1)}^0>\varepsilon)>\varepsilon$. For every $a\in(-1;0)$, we set $\tilde{H}(-1-\alpha,t)=-a+H(-1-\alpha,t+T(-2-a))$, $\tilde{B}_t=B_{t+T(-2-a)}$; and $\tilde{\lambda}_t^0$ and $\tilde{\nu}_t^0$ the corresponding versions of the occupation densities for $\tilde{H}(-1-\alpha,\cdot)$. We have

$$\lambda_{T(-2)}^a=\tilde{\lambda}_{T(-2-a)}^0\,,\qquad \nu_{T(-2)}^a=\tilde{\nu}_{T(-2-a)}^0\,,$$

and, according to Lemma III.7,

$$\begin{aligned}\boldsymbol{P}(\lambda_{T(-2)}^a-\nu_{T(-2)}^a>\varepsilon)&=\boldsymbol{P}(\lambda_{T(-2-a)}^0-\nu_{T(-2-a)}^0>\varepsilon)\\&\geqq\boldsymbol{P}(\lambda_{T(-1)}^0-\nu_{T(-1)}^0>\varepsilon)>\varepsilon\,.\end{aligned}$$

Hence $\int_{-1}^0 da\boldsymbol{P}(\lambda_{T(-2)}^a-\nu_{T(-2)}^a>\varepsilon)\geqq\varepsilon$, which leads to a contradiction with (III.7). Hence $\lambda_{T(-1)}^0=\nu_{T(-1)}^0$ $\boldsymbol{P}$ a.s., and by scaling invariance, $\lambda_{T(-n)}^0=\nu_{T(-n)}^0$ for all $n\in\boldsymbol{N}^*$. To finish the proof, we just have to notice that $\boldsymbol{P}$ a.s., $\forall T\in\boldsymbol{R}_+$, if $\lambda_T^0=\nu_T^0$ and $H(-1-\alpha,T)\neq 0$, then $\forall t\leqq T$, $\lambda_t^0=\nu_t^0$, provided that $H(-1-\alpha,t)$ is not 0; and to apply i).

Eventually, our main tool for § IV will be the following

Proposition III.11. *If* f *is a* C^1 *function, and* F *a primitive of* f, *then*

$$\begin{aligned}\mathcal{E}_t^f=\exp\Big\{&2(1-\alpha)^2(B_s^+)^{1-\alpha}f(H(-1-\alpha,t))+\alpha(1-\alpha)[F(H(-1-\alpha,t))-F(0)]\\&-2(1-\alpha)^{-2}\int_0^t(f'+f^2)(H(-1-\alpha,s))(B_s^+)^{-2\alpha}ds\end{aligned}$$

is a continuous local martingale.

Proof. We deduce from Proposition III.8 that

$$\mathcal{E}_t^f=\exp\left\{2(1-\alpha)^{-1}\int_0^t f(H(1-\alpha,\ s))(B_s^+)^{-\alpha}dB_s-2(1-\alpha)^{-2}\int_0^t f'(H(-1-\alpha,\ s))(B_s^+)^{-2\alpha}ds\right\},$$

so $\mathcal{E}_t^f$ is a continuous local martingale.

IV. Ray-Knight's type results

Let us fix $\alpha\in(0,\ 1/2)$ and henceforth, we will omit α in $\lambda_t^a(\alpha)$ and denote it by λ_t^a.

IV.1. A Ray-Knight's theorem for $\lambda_{T(-1)}^a$. The main result of this paragraph is the following

Theorem IV.1. i) $\{4(1-\alpha)^{-2}\lambda_{T(-1)}^{a-1} : 0\leqq a\leqq 1\}$ *is the square of a Bessel process starting from* 0 *and of dimension* $2\alpha/(1-\alpha)$.

ii) $\{4(1-\alpha)^{-2}\lambda_{T(-1)}^a : 0\leqq a\}$ *is the square of a Bessel process of dimension* 0.

Proof. Let $g:[-1,\ +\infty)\to\boldsymbol{R}_+$, be a continuous function with compact support, and $\Phi_g:[-1,\ +\infty)\to\boldsymbol{R}_+$, the non-increasing solution of the Sturm-Liouville equation $\Phi''=g\Phi$, with $\Phi(-1)=1$. Then we have

$$\text{(IV.1)}:\qquad \boldsymbol{E}\left[\exp\left\{-2(1-\alpha)^{-2}\int_0^{T(-1)}g(H(-1-\alpha,\ s))(B^+)^{-2\alpha}ds\right\}\right]=[\Phi_g(0)]^{\alpha/(1-\alpha)}.$$

Indeed, if we take $f=\Phi_g'/\Phi_g$, $F=\log\Phi_g$ and $f'+f^2=g$ in Proposition III.11, then $\mathcal{E}^f$ is a continuous bounded martingale on $\{t\leqq T(-1)\}$, and the optional sampling theorem implies that

$$\boldsymbol{E}\left[\exp\left\{2(1-\alpha)^{-2}(B_{T(-1)}^+)^{1-\alpha}f(-1)+\alpha(1-\alpha)[(F(-1)-F(0)]\right.\right.$$
$$\left.\left.-2(1-\alpha)^{-2}\int_0^{T(-1)}g(H(-1-\alpha,\ s))(B_s^+)^{-2\alpha}ds\right\}\right]=1.$$

Since $B_{T(-1)}^+=0$ and $F(-1)=0$, we have

$$\boldsymbol{E}\left[\exp\left\{-2(1-\alpha)^{-2}\int_0^{T(-1)}g(H(-1-\alpha,\ s))(B_s^+)^{-2\alpha}ds\right\}\right]=\exp(\alpha F(0)/(1-\alpha))$$
$$=[\Phi_g(0)]^{\alpha/(1-\alpha)}.$$

On the other hand, it is well known (see Durrett [4], §8.7 for example) that

$$\boldsymbol{E}\left[\exp\left\{-\frac{1}{2}\int_0^{\sigma(-1)}g(B_s)ds\right\}\right]=\Phi_g(0)\qquad\text{where } \sigma(-1)=\inf\{t:\ B_t=-1\}.$$

Using (IV.1), we obtain

$$\boldsymbol{E}\left[\exp\left\{-\frac{1}{2}\int_{-1}^{+\infty}g(a)4(1-\alpha)^{-2}\lambda_{T(-1)}^a da\right\}\right]=\boldsymbol{E}\left[\exp\left\{-\frac{1}{2}\int_{-1}^{+\infty}g(a)L_{\sigma(-1)}^a da\right\}\right]^{\alpha/(1-\alpha)}$$

A famous due to Ray [11] and Knight [8] claims that $\{L^{a-1}_{\sigma(-1)}: 0\leqq a\leqq 1\}$ is the square of a Bessel process of dimension 2 and starting from 0. The additive property of the squares of Bessel processes (Shiga and Watanabe [12]) implies that $\{4(1-\alpha)^{-2}\lambda^{a-1}_{T(-1)}: 0\leqq a\leqq 1\}$ is the square of a Bessel process of dimension $2\alpha/(1-\alpha)$ and starting from 0.

On the other hand, $\{L^{a}_{\sigma(-1)}: 0\leqq a\}$ is the square of a Bessel process of zero dimension, and the same arguments imply that $\{4(1-\alpha)^{-2}\lambda^{a}_{T(-1)}: 0\leqq a\}$ is the square of a Bessel process of dimension 0.

Proposition III.11 also allows us to compute the law of $(S_{T(-a)}, A_{T(-a)})$ for every positive a:

Proposition IV.2. *For every $\theta, b>0$,*

$$E\left[\exp\left\{-2\theta^2(1-\alpha)^{-2}A_{T(-a)}\right\}\mathbf{1}_{(S_{T(-a)}<b)}\right]=\left(\frac{\operatorname{sh}\theta b}{\operatorname{sh}\theta(a+b)}\right)^{\alpha/(1-\alpha)}$$

In particular, $P(S_{T(-a)}<b)=(b/(a+b))^{\alpha/(1-\alpha)}$.

Proof. Let us take $f(x)=\theta\coth\theta(x-b)$ and $F(x)=\log|\operatorname{sh}\theta(x-b)|$. Then $\mathcal{E}^f$ is a bounded martingale on $\{t<T(-a)\wedge T(b)\}$; and since

$$\lim_{t\uparrow T(b)} f(H(-1-\alpha-t))=\lim_{t\uparrow T(b)} F(H(-1-\alpha, t))=-\infty,$$

the optional sampling theorem gives

$$E\left[\exp\left\{\frac{\alpha}{1-\alpha}[F(-a)-F(0)]-2\theta^2(1-\alpha)^{-2}A_{T(-a)}\right\}\mathbf{1}_{T(-a)<T(b)}\right]=1;$$

this proves the first part of the proposition. The second follows by taking the limit as $\theta\downarrow 0$.

Remark. In particular $A_{T(-a)}$ has the same distribution as $\sigma(a/2)=\inf\{t: B_t=a/2\}$. This can also be seen directly noticing that, according to Lemma III.7 and Formula (I.2), $I_t=\inf\left\{2\alpha^{-1}\int_0^t(B_s^+)^{-\alpha}dB_s: s\leqq t\right\}$.

We are now going to study by the same methods $\lambda^{a}_{T(1)}$, but the results shall be quite different, because this time, $B^+_{T(1)}\neq 0$.

IV.2. Results for $T(1)$. Let $g:[-1,+\infty)\to\boldsymbol{R}_+$ be a continuous function with compact support, and φ_g and ψ_g the system of fundamental solutions of the Sturm-Liouville equation $\Phi''=g\Phi$, with $\varphi_g(-1)=1$, $\varphi'_g(-1)=0$, $\psi_g(-1)=0$ and $\psi'_g(-1)=1$. φ_g and ψ_g are both non-decreasing functions, and positive on $(-1,+\infty)$. If Φ_g is the unique non-negative, non-increasing solution with $\Phi_g(-1)=1$, then there exists $r(g)$ so $\Phi_g=\varphi_g-r(g)\psi_g$. Since $\lim_{+\infty}\psi_g=+\infty$, we have $\lim_{+\infty}\Phi_g/\psi_g=0$; and $\lim_{+\infty}\varphi_g/\psi_g=r(g)$.

Theorem IV.3. *With the former notations, for every $\theta\geqq 0$, we have*

$$E\left[\exp\left\{-2\theta(1-\alpha)^{-2}(B^+_{T(1)})^{1-\alpha}-2(1-\alpha)^{-2}\int_R g(-a)\lambda^a_{T(1)}ds\right\}\right]$$
$$=[\varphi_g(0)+\theta\psi_g(0)]^{\alpha/(1-\alpha)}-[(r(g)+\theta)\psi_g(0)]^{\alpha/(1-\alpha)}.$$

Proof. According to Proposition III.11, if f is C^1 and F is a primitive of f, then

$$\mathcal{E}^f_t=\exp\left\{-2(1-\alpha)^{-2}(B^+_t)^{1-\alpha}f(-H(-1-\alpha,t))+\alpha(1-\alpha)[F(-H(-1-\alpha,t))-F(0)]\right.$$
$$\left.-2(1-\alpha)^{-2}\int_0^t(f'+f^2)(-H(-1-\alpha,s))(B^+_s)^{-2\alpha}ds\right\}$$

is a local martingale on $\{t<T(1)\}$. On the one hand, if we take $f=\psi'_g/\psi_g$ and $F=\log\psi_g$, then $\lim_{t\uparrow T(1)}\mathcal{E}^f_t=0$ (because $\lim_{-1+}\psi_g=0$), and hence, for every $b>0$, $\mathcal{E}^f$ is a bounded martingale on $\{t<T(1)\wedge T(-b)\}$. The optional sampling theorem applied to $T(1)\wedge T(-b)$ gives

(IV.2): $$E\left[\exp\left\{-2(1-\alpha)^{-2}\int_0^{T(-b)}g(-H(-1-\alpha,s))(B^+_s)^{-2\alpha}ds\right\}\mathbf{1}_{(T(-b)<T(1))}\right]$$
$$=\left(\frac{\psi_g(0)}{\psi_g(b)}\right)^{\alpha/(1-\alpha)}.$$

On the other hand, if we set $\phi=\varphi_g+\theta\psi_g$, $f=\phi'/\phi$ and $F=\log\phi$ and if we apply the optional sampling theorem to $T(1)\wedge T(-b)$, then

$$E\left[\exp\left\{-2\theta(1-\alpha)^{-2}(B^+_{T(1)})^{1-\alpha}-2(1-\alpha)^{-2}\int_0^{T(1)}g(-H(-1-\alpha,s))(B^+_s)^{-2\alpha}ds\right\}\right.$$
$$\left.\times\left(\frac{\phi(-1)}{\phi(0)}\right)^{\alpha/(1-\alpha)}\mathbf{1}_{(T(1)<T(-b))}\right]$$
$$+E\left[\exp\left\{-2(1-\alpha)^{-2}\int_0^{T(-b)}g(-H(-1-\alpha,s))(B^+_s)^{-2\alpha}ds\right\}\left(\frac{\phi(b)}{\phi(0)}\right)^{\alpha/(1-\alpha)}\mathbf{1}_{(T(1)>T(-b))}\right]=1.$$

We deduce from (IV.2) that

$$E\left[\exp\left\{-2\theta(1-\alpha)^{-2}(B^+_{T(1)})^{1-\alpha}-2(1-\alpha)^{-2}\int_0^{T(1)}g(-H(-1-\alpha,s))(B^+_s)^{-2\alpha}ds\right\}\mathbf{1}_{(T(1)<T(-b))}\right]$$
$$=(1-[\phi(b)\psi_g(0)(\phi(0)\psi_g(b))^{-1}]^{\alpha/(1-\alpha)})[\phi(0)]^{\alpha/(1-\alpha)},$$

and since $\phi(-1)=1$, we finally obtain

$$E\left[\exp\left\{-2\theta(1-\alpha)^{-2}(B^+_{T(1)})^{1-\alpha}-2(1-\alpha)^{-2}\int_0^{T(1)}g(-H(-1-\alpha,s))(B^+_s)^{-2\alpha}ds\right]\mathbf{1}_{(T(1)<T(-b))}\right]$$
$$=[\varphi_g(0)+\theta\psi_g(0)]^{\alpha/(1-\alpha)}-\left[\frac{\varphi_g(b)+\theta\psi_g(b)}{\psi_g(b)}\psi_g(0)\right]^{\alpha/(1-\alpha)}.$$

and the theorem is proved by taking the limit as $b\uparrow+\infty$.

Remark. According to Tanaka's formula $\lambda^1_{T(1+)}=(B^+_{T(1)})^{1-\alpha}$, and we could think that, as in §IV.1., conditionally on $(B^+_{T(1)})^{1-\alpha}=x$, the process $\{4(1-\alpha)^{-2}\lambda^{1-a}_{T(1+)}:0\leqq a\leqq 1\}$ is the square of a Bessel process starting from x and of dimension $2\alpha/(1-\alpha)$. This is not true, because if it were, we would have

$$E\left[\exp\left\{-2(1-\alpha)^{-2}\int_0^{T(1)} g(-H(-1-\alpha,\ s))(B_s^+)^{-2\alpha}ds\right\}\right.$$

$$\left.=E\left[\exp\left\{-(B_{T(1)}^+)^{1-\alpha}r(g)/2\right\}\right](\Phi_g(0))^{\alpha/(1-\alpha)}.\right.$$

and the right side of this equality, which can be computed using Proposition IV.5, is not the same as the one we have found in Theorem IV.3. However, we have the following

Theorem IV.4. i) *The law of* $\lambda_{T(1)}^0$ *is given by: for every positive* k,

$$E\left[\exp\left\{-\frac{1}{2}4(1-\alpha)^{-2}k\lambda_{T(1)}^0\right\}\right]=1-(k/(k+1))^{\alpha/(1-\alpha)}.$$

ii) *Conditionally on* $4(1-\alpha)^{-2}\lambda_{T(1)}^0=x$ $(x>0)$, $\{4(1-\alpha)^{-2}\lambda_{T(1)}^{-a}:0\leqq a\}$ *is the square of a Bessel process of dimension* 0 *and starting from* x.

Proof. i) is an easy consequence of Theorem IV.3.

ii) If $g\equiv 0$ on $[-1;0]$, then $\varphi_g(0)=\psi_g(0)=1$, so

$$E\left[\exp\left\{-\frac{1}{2}\int_0^{+\infty} g(a)4(1-\alpha)^{-2}\lambda_{T(1)}^{-a}da\right\}\right]=1-[r(g)]^{\alpha/(1-\alpha)}.$$

Notice that if $\Phi_g=\varphi_g-r(g)\psi_g$ is the non-increasing, nongative solution of $\Phi''=g\Phi$ with $\Phi(-1)=1$, then $\Phi_g'(0)/\Phi_g(0)=-r(g)/(1-r(g))$, and hence

$$E\left[\exp\left\{-\frac{1}{2}\int_0^{+\infty} g(a)4(1-\alpha)^{-2}\lambda_{T(1)}^{-a}da\right\}\right]=E\left[\exp\left(-\frac{1}{2}4(1-\alpha)^{-2}\Phi_g'(0)\Phi_g^{-1}(0)\lambda_{T(1)}^0\right\}\right],$$

According to Pitman and Yor [10], this proves our assertion.

As in §IV.1, we are also able to obtain the distribution of $(B_{T(1)}, A_{T(1)}, I_{T(1)})$:

Proposition IV.5. *If* a, γ, δ *are three positive real numbers, and if we set* $\theta=\delta(1-\alpha)/2$, *and* $b=1+\delta(1-\alpha)^{-1}\log|(\gamma(1-\alpha)+\delta)(\alpha(1-\alpha)-\delta)^{-1}|$, *then*

$$E\left[\exp\left(-\gamma(B_{T(1)}^+)^{1-\alpha}-\frac{\delta^2}{2}A_{T(1)}\right\}\mathbf{1}_{(I_{T(1)}>-a)}\right]=$$

$$\begin{cases}\left(\dfrac{\operatorname{sh}\theta b}{\operatorname{sh}\theta(b-1)}\right)^{\alpha/(1-\alpha)}-\left(\dfrac{\operatorname{sh}\theta(a+b)\operatorname{sh}\theta}{\operatorname{sh}\theta(b-1)\operatorname{sh}\theta(a+1)}\right)^{\alpha/(1-\alpha)} & \textit{when } \gamma(1-\alpha)>\delta\\[2ex] \left(\dfrac{\operatorname{ch}\theta b}{\operatorname{ch}\theta(b-1)}\right)^{\alpha/(1-\alpha)}-\left(\dfrac{\operatorname{ch}\theta(a+b)\operatorname{sh}\theta}{\operatorname{ch}\theta(b-1)\operatorname{sh}\theta(a+1)}\right)^{\alpha/(1-\alpha)} & \textit{when } \gamma(1-\alpha)<\delta.\end{cases}$$

In particular, for every $b>1$,

$$E[\exp\{-2(1-\alpha)^{-2}(b-1)^{-1}(B_{T(1)}^+)^{1-\alpha}\}\mathbf{1}_{(I_{T(1)}>-a)}]$$

$$\left(\frac{b}{b-1}\right)^{\alpha/(1-\alpha)}-\left(\frac{a+b}{(a+1)(b-1)}\right)^{\alpha/(1-\alpha)}.$$

Proof. The arguments are the same as in the proof of Theorem IV.3, with firstly $f(x)=\theta \coth \theta(x-b)$ and $F(x)=\log|\operatorname{sh} \theta(x-b)|$ and secondly $f(x)=\theta \operatorname{th} \theta(x-b)$ and $F(x)=\log \operatorname{ch} \theta(x-b)$.

V. Applications to the Bessel process of dimension d $(0<d<1)$

According to a result of Biane and Yor (Lemma 3.1 in [2], with a slight modification), if we set $\sigma(t)=\inf\{s: A_s>t\}$, then $X_t=(1-\alpha)^{-1}(B^+_{\sigma(t)})^{1-\alpha}$ is a Bessel process of dimension $d=(1-2\alpha)/(1-\alpha)$, with an instantaneous reflecting barrier at 0. Hence, the canonic decomposition of X as a sum of a martingale and a zero quadratic variation process is

$$\text{(V.1)}: \qquad X_t=\beta_t-\frac{\alpha}{2}H(-1-\alpha, \sigma(t))=\beta_t+(d-1)H(t),$$

where β is a Brownian motion and $H(t)=\frac{1}{2}$ v.p. $\int_0^t \frac{ds}{X_s}$. The previous results for $H(-1-\alpha, \cdot)$ can be translated to H by easy change-of-time arguments:

If f is a function with continuous derivative, and if F is a primitive of f, then we have

$$\text{(V.2)}: \qquad X_t f(H(t))=\int_0^t f(H(s))d\beta_s+(d-1)[F(H(t))-F(0)]+\frac{1}{2}\int_0^t f'(H(s))ds.$$

In particular

$$\text{(V.3)}: \qquad \exp\left\{X_t f(H(t))+(1-d)[F(H(t))-F(0)]-\frac{1}{2}\int_0^t (f'+f^2)(H(s))ds\right\}$$

is a continuous local martingale.

The family of random variable $\{\lambda_t^a: a\in \boldsymbol{R},\ t\geqq 0\}$ defined by Tanaka's formula

$$\text{(V.4)}: \qquad X_t \mathbf{1}_{(H(t)>a)}=\int_0^t \mathbf{1}_{(H(s)>a)}d\beta_s+(d-1)[(H(t)-a)^+-a^+]+\frac{1}{2}\lambda_t^a$$

is a version of the occupation densities of H:

$$\text{for every Borelian bounded function } \varphi,\ \int_0^t \varphi(H(s))ds=\int_{\boldsymbol{R}} \varphi(a)\lambda_t^a da.$$

If we set $T(x)=\inf\{t: H(t)=x\}$, $I_t=\inf\{H(s): s\leqq t\}$ and $S_t=\operatorname{Sup}\{H(s): s\leqq t\}$, then the main results are

Theorem V.1. i) $\{\lambda_{T(-1)}^{a-1}: 0\leqq a\leqq 1\}$ *is the square of a Bessel process starting from* 0 *and of dimension* $2-2d$.

ii) $\{\lambda_{T(-1)}^a: 0\leqq a\}$ *is the square of a Bessel process of dimension* 0.

Theorem V.2. *For every* $\theta\geqq 0$,

$$\boldsymbol{E}\left[\exp\left\{-\theta X_{T(1)}-\frac{1}{2}\int g(-a)\lambda_{T(1)}^a da\right\}\right]=(\varphi_g(0)+\theta\psi_g(0))^{1-d}-((r(g)+\theta)\psi_g(0))^{1-d},$$

where g, φ_g, ψ_g, *and* $r(g)$ *are defined in* § IV.2.

Theorem V.3. i) *The law of* $\lambda^0_{T(1)}$ *is given by: for every positive* k,

$$E\left[\exp\left\{-\frac{1}{2}k\lambda^0_{T(1)}\right\}\right]=1-(k/(k+1))^{1-d}.$$

ii) *Conditionally on* $\lambda^0_{T(1)}=x$ $(x>0)$, $\{\lambda^{-a}_{T(1)}: 0\leqq a\}$ *is the square of a Bessel process of dimension* 0 *and starting from* x.

Proposition V.4. i) *For every positive* θ, a *and* b,

$$E\left[\exp\left(-\frac{\theta^2}{2}T(-a)\right)1_{(S_{T(-a)}<b)}\right]=\left(\frac{\operatorname{sh}\theta b}{\operatorname{sh}\theta(a+b)}\right)^{1-d}.$$

ii) *For every positive* a, γ *and* θ, *we have*

$$E\left[\exp\left\{-\gamma X_{T(1)}-\frac{\theta^2}{2}T(1)\right\}1_{(I_{T(1)}>-a)}\right]$$

$$=\begin{cases}\left(\dfrac{\operatorname{sh}\theta b}{\operatorname{sh}\theta(b-1)}\right)^{1-d}-\left(\dfrac{\operatorname{sh}\theta(a+b)\operatorname{sh}\theta}{\operatorname{sh}\theta(b-1)\operatorname{sh}\theta(a+1)}\right)^{1-d} & \text{when } \gamma>\theta\\ \left(\dfrac{\operatorname{ch}\theta b}{\operatorname{ch}\theta(b-1)}\right)^{1-d}-\left(\dfrac{\operatorname{ch}\theta(a+b)\operatorname{sh}\theta}{\operatorname{ch}\theta(b-1)\operatorname{sh}\theta(a+1)}\right)^{1-d} & \text{when } \gamma<\theta,\end{cases}$$

where b *is defined by* $\gamma=\theta\coth\theta(b-1)$ *if* $\gamma>\theta$, *and* $\gamma=\theta\operatorname{th}\theta(b-1)$ *if* $\gamma<\theta$.

Appendix

A.1. Applications to the measure B_1P. This part is inspirated by the Appendix A of Biane and Yor [2]. Our purpose is to propose simplified proofs and generalizations of these authors' results.

Let Z be an adapted continuous and locally of bounded β-variation process $(\beta>1)$. According to Theorem II.1 and to the results of [1], we are able to define $\int_0^\cdot Z_s dC(s)$ which is a continuous process, locally of bounded $(1+\eta)$-variation for every positive η. We have

Proposition A.1. *The process* $Y_\cdot=\int_0^\cdot Z_s dB_s-\int_0^\cdot Z_s dC(s)$ *is orthogonal to* B *, i.e.* $B.Y$ *is a local martingale.*

Proof. According to Theorems II.2 and II.3, we have

$$B_tY_t=\int_0^t Y_s dB_s+\int_0^t Z_s.B_s dB_s-\int_0^t Z_s.B_s dC(s)+\int_0^t Z_s ds.$$

On the other and, we saw in §III.1 that

$$\int_0^t Z_s.B_s dC(s)=\int_0^t Z_s ds,$$

and this prove the proposition.

Let $\mathscr{P}$ be the signed measure $\mathscr{P}=B_1P$. We will say that a process Y is a $\mathscr{P}$-

martingale. If $B.Y$ is a $\boldsymbol{P}$-local martingale. A translation of Proposition A.1 is the following analogous of Girsanov's theorem:

Corollary A.2. *If M denote the $\boldsymbol{P}$-local martingale $M_t=\int_0^t Z_s dB_s$, then*

$$\hat{M}_t=M_t-\int_0^t Z_s dC(s) \text{ is a } \mathcal{P}\text{-martingale.}$$

If we set $\hat{B}=B-C$, then $\hat{B}$ is a $\mathcal{P}$-martingale, and more precisely we have

Corollary A.3. (Biane and Yor) *If $h: \boldsymbol{R}\times\boldsymbol{R}_+\to\boldsymbol{R}$ is solution of the heat equation: $(1/2)(\partial^2 h/\partial x^2)+(\partial h/\partial y)=0$, then $h(\hat{B}_t, t)$ is a $\mathcal{P}$-martingale.*

Proof. According to Theorem II.3,

$$h(\hat{B}_t, t)=h(0, 0)+\int_0^t \frac{\partial h}{\partial x}(B_s-C(s), s)dB_s-\int_0^t \frac{\partial h}{\partial x}(B_s-C(s), s)dC(s),$$

Corollary A.3 is then a consequence of Corollary A.2 for

$$M_t=\int_0^t \frac{\partial h}{\partial x}(\hat{B}_s, s)dB_s.$$

Eventually, we have the following generalization of the second part of Theorem A1 in [2]:

Corollary A.4. *The process $\mathcal{E}_t^Z=\exp\left\{i\int_0^t Z_s dB_s-i\int_0^t Z_s dC(s)-\frac{1}{2}\int_0^t Z_s^2 ds\right\}$ is a $\mathcal{P}$-martingale.*

Proof. Theorems II.2 and II.3 imply

$$\mathcal{E}_t^Z=1+i\int_0^t Z_s\mathcal{E}_s^Z dB_s-i\int_0^t Z_s\mathcal{E}_s^Z dC(s),$$

and Corollary A.4 is once again a consequence of Corollary A.2 for

$$M_t=i\int_0^t Z_s\mathcal{E}_s^Z dB_s.$$

A.2. Existence of decreasing times for $H(-1-\alpha, \cdot)$. Since $2\alpha/(1-\alpha)<2$, an interesting consequence of Theorem IV.1 is that, contrary to what happens for the Brownian motion, $\boldsymbol{P}$ a.s. there exist points $x\in(-1, 0)$ with $\lambda_{T(-1)}^x=0$. We will see that it implies the existence of decreasing times for $H(-1-\alpha, \cdot)$.

Definition. Let I be an open interval, $f: I\to\boldsymbol{R}$ a function, and t a point of I. We will say that t is a non-increasing time for f on I if

$$\forall(u, v)\in I\times I, \quad u\leqq t\leqq v \Longrightarrow f(v)\leqq f(t)\leqq f(u).$$

We will say that t is a decreasing time for f on I if

$$\forall(u, v)\in I\times I, \quad u<t<v \Longrightarrow f(v)<f(t)<f(u).$$

We have

Proposition A.5. *P a.s., there exist non-increasing times for $H(-1-\alpha, \cdot)$ on* $(0, T(-1))$.

Proof. Let us prove first that P a. s., for every rational number q in $(-1; 0)$, and every y in $(q; 0)$,

$$\text{(A.1):} \qquad \lambda^y_{T(1)}=0 \Longrightarrow \operatorname{Sup}\{H(-1-\alpha, t): T(q)\leqq t\leqq T(-1)\}\leqq y\,.$$

Indeed, if $\tilde{B}_t=B_{t+T(q)}$ and $\tilde{H}(-1-\alpha, t)=H(-1-\alpha, t+T(q))-q$, since $B^+_{T(q)}=0$, and since $H(-1-\alpha, \cdot)$ is an additive functional, we have

$$\tilde{H}(-1-\alpha, t)=\text{p. f.}\int_0^t(\tilde{B}^+_s)^{-1-\alpha}ds\,.$$

If $\tilde{T}(-1-q)=\inf\{t: \tilde{H}(-1-\alpha, t)=-1-q\}$ and if we denote by $\{\tilde{\lambda}^a_t: a\in \boldsymbol{R}, t\geqq 0\}$ the family of the local times of $\tilde{H}(-1-\alpha, \cdot)$, then, according to the scaling invariance property,

(A.2): $\{4(1-\alpha)^{-2}\tilde{\lambda}^a_{\tilde{T}(-1-q)} : 0\leqq a\}$ is the square of a Bessel process of dimension 0:

and we obviously have $\lambda^{y-q}_{T(-1)}=\lambda^{y-q}_{T(q)}+\tilde{\lambda}^y_{\tilde{T}(-1-q)}$. Particularly,

$$\text{(A.3):} \qquad \lambda^{y-q}_{T(-1)}=0 \Longleftrightarrow \lambda^{y-q}_{T(q)}=0 \quad\text{and}\quad \tilde{\lambda}^y_{\tilde{T}(-1-q)}=0\,.$$

On the other hand, for every $y\geqq 0$,

$$\text{(A.4):} \qquad \tilde{\lambda}^y_{\tilde{T}(-1-q)}=0 \Longleftrightarrow \forall t\leqq \tilde{T}(-1-q)\,, \qquad \tilde{H}(-1-\alpha, t)\leqq y\,.$$

Indeed, according to (A.2), if $\tilde{\lambda}^y_{\tilde{T}(-1-q)}=0$, then, for every $z>y$, $\tilde{\lambda}^z_{\tilde{T}(-1-q)}=0$, hence $\int_0^{\tilde{T}(-1-q)}\mathbf{1}_{(\tilde{H}(-1-\alpha, s)>y)}(B^+_s)^{-2\alpha}ds=0$, and (A.4) is proved. (A.1) is then a consequence of (A.3) and (A.4). Now take $x\in(-1; 0)$ so that $\lambda^x_{T(-1)}=0$; $T(x-)=\inf\{t: H(-1-\alpha, t)<x\}$, and $(x_n: n\in \boldsymbol{N})$ an increasing sequence of rational numbers converging to x. Then $T(x_n)\downarrow T(x-)$, $H(-1-\alpha, t)\geqq x_n$ if $0\leqq t\leqq T(x_n)$ and $H(-1-\alpha, t)<x$ if $T(x_n)\leqq t\leqq T(-1)$. Hence, taking the limit as $n\uparrow +\infty$, we obtain $H(-1-\alpha, t)\geqq x$ if $t\leqq T(x-)$ and $H(-1-\alpha, t)\leqq x$ if $T(x-)\leqq t\leqq T(-1)$; and Proposition A.5 is proved.

Remark. This result may look surprising, because $H(-1-\alpha, \cdot)$ increases on every positive excursion of B, and is constant on the negative ones.

The structure of the set of the zero of a Bessel process allows us to claim.

Theorem A.6. *P a.s., there exist points $x\in(-1; 0)$ that $H(-1-\alpha, \cdot)$ hits only once before $T(-1)$. Particularly, the hitting times of such x are decreasing times for $H(-1-\alpha, \cdot)$ on* $[0, T(-1)]$.

Proof. Let us denote by $Z_\omega=\{x\in[-1; 0]: \lambda^x_{T(-1)}=0\}$. According to Theorem IV.2, P a.s., Z_ω is a non-empty perfect set. Let x be a left-and-right accumulation point (i.e. $\forall\varepsilon>0$, there exist y and z in Z_ω so that $x-\varepsilon<y<x<z<x+\varepsilon$). According to (A.2), if x_1, $x_2\in Z_\omega$ and $-1<x_2<x<x_1<0$, then

$$x_1<H(-1-\alpha, t) \quad \text{when } 0\leqq t\leqq T(x_1),$$

$$x_1\leqq H(-1-\alpha, t)\leqq x_2 \quad \text{when } T(x_1)\leqq t\leqq T(x_2), \quad \text{and}$$

$$x_2\leqq H(-1-\alpha, t) \quad \text{when } T(x_2)\leqq t\leqq T(-1).$$

We deduce that $H(-1-\alpha, t)>x$ on $[0; T(x))$, $H(-1-\alpha, t)=x$ on $[T(x); T(x-)]$ and $H(-1-\alpha, t)<x$ on $(T(x-); T(-1)]$. Hence, for all x left-and-right accumulation point of Z_ω, $N(x)=\{t\leqq T(-1): H(-1-\alpha, t)=x\}$ is a closed interval. So, there exist at most countable many points in Z_ω for which $N(x)$ is not a single point. Since the set of the left-and-right accumulation points of Z_ω is not countable, there exist points in Z_ω that $H(-1-\alpha, \cdot)$ hits once and only once on $[0; T(-1)]$.

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