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Spectral theory of magnetic Schrödinger operators with exploding potentials

Dedicated to Professor Teruo Ikebe on his sixtieth birthday

By

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1. Introduction

In the present paper we shall consider the Schrödinger operator

$$L=-\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}+ib_j(x)\right)^2+V(x),\ x\in \boldsymbol{R}^n \tag{1.1}$$

in the Hilbert space $L^2(\boldsymbol{R}^n)$, where $i=\sqrt{-1}$ and $V(x)$, $b_j(x)\,(1\leqq j\leqq n)$ are real-valued functions. The scalar potential $V(x)$ is assumed to satisfy

$$V(x)\longrightarrow -\infty \quad \text{as} \quad |x|\longrightarrow \infty \tag{1.2}$$

and

$$V(x)\geqq -C|x|^{\alpha} \qquad (|x|\geqq R_0) \tag{1.3}$$

for some positive constants C, R_0 and $\alpha<2$. The spectral theory of L satisfying (1.2) has been extensively investigated by many authors. For a class of $b_j(x)$ and $V(x)$ satisfying (1.1) and (1.2) with $\alpha=2$ one can see the following assertions;

(a) the symmetric operator L defined on $C_0^\infty(\boldsymbol{R}^n)$ is essentially self-adjoint, i.e., L has the unique self-adjoint extension H (cf. Ikebe-Kato [11]).

(b) the spectrum of H consisits of all real numbers, i.e., $\sigma(H)=\boldsymbol{R}$ (cf. Eastham [6], Eastham-Kalf [7]).

(c) H has no eigenvalues (cf. Eastham-Kalf [7], Uchiyama [24], Uchiyama-Yamada [25]).

In this paper we shall give a sufficient condition to assure that H is absolutely continuous, and to derive a spectral representation of H. To this end we study the limiting absorption method (principle) for L.

The limiting absorption method is, roughly speaking, to investigate the limit of the resolvent $R(z)=(H-z)^{-1}$ as the non-real z approaches the real axis. More precisely, it is to choose appropriate weghted L^2 spaces A, B such that $A\subset L^2(\boldsymbol{R}^n)\subset B$ and the strong limit

$$R(\lambda\pm i0)f=s-\lim_{\varepsilon\downarrow 0}R(\lambda\pm i\varepsilon)f \qquad \text{in} \quad B$$

exists for any $f\in A$. In order to study the limiting absorption method the *radiation*

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condition is often used as an important tool, by which one can find $u=R(\lambda\pm i0)f$ among many solutions of $Lu-\lambda u=f\in A$.

Eidus [8] shows the limiting absorption method for Schrödinger operators in exterior domains with short-range potentials, by making use of the radiation condition

$$\int_{|x|=R}\left|\frac{\partial u}{\partial r}\mp i\sqrt{\lambda}\, u\right|^2 dS \longrightarrow 0$$

as $R\to\infty$, where $\lambda>0$ and $r=|x|$. Ikebe-Saitō [12] develops the limiting absorption method for Schrödinger operators with long-range potentials, by putting

$$A\equiv L_s^2=\{f\in L_{\mathrm{loc}}^2\,;\ \|f\|_s=\|(1+|\cdot|)f\|_{L^2}<\infty\}$$

and $B=L_{-s}^2\,(s>1/2)$ and considering the radiation condition

$$\text{(1.4)}\qquad \|\mathcal{D}_j^{\mp}u\|_{s-1}\equiv\left\|\left(\frac{\partial}{\partial x_j}+ib_j(x)+\frac{x_j}{|x|}k^{\pm}(x,\lambda)\right)u\right\|_{s-1}<\infty\quad (j=1,2,\cdots n),$$

where

$$k^{\pm}(x,\lambda)=\mp i\sqrt{\lambda}+\frac{n-1}{2r}\qquad(\lambda>0).$$

Ikebe [9] defines the radiation condition by putting in (1.4)

$$k^{\pm}(x,\lambda)=\mp i\sqrt{\lambda-V(x)}+\frac{n-1}{2r}$$

in order to show the spectral representation of Schrödinger operators. Mochizuki-Uchiyama [20] treats oscillating long-range potentials and proposes that $k^{\pm}(x,\lambda)$ can be selected as an approximate solution of the following Riccati-type equation

$$\text{(1.5)}\qquad V(x)-\lambda+\frac{\partial}{\partial r}k(x,\lambda)+\frac{n-1}{r}k(x,\lambda)-k(x,\lambda)^2=0$$

at infinity. Saitō [21] solves the eikonal equation

$$|\nabla R|^2=1-\frac{V(x)}{\lambda}\qquad(\lambda>0)$$

and adopts

$$\mp i\sqrt{\lambda}\,\frac{\partial}{\partial x_j}R(x,\lambda)$$

instead of $(x_j/|x|)k^{\pm}(x,\lambda)$ in (1.4).

Suggested by Mochizuki-Uchiyama [20] we shall define $\mathcal{D}_j^{\mp}$ in (1.4) by setting

$$k^{\pm}(x,\lambda)=\mp i\sqrt{\lambda-(Vx)}+\frac{n-1}{2r}-\frac{\partial V/\partial r}{4(\lambda-V)}.$$

Then we see that the left-hand side of (1.5) is equal to

$$\text{(1.6)}\qquad \frac{(n-1)(n-3)}{4r^2}-\frac{V_{rr}}{4(\lambda-V)}-\frac{5}{16}\left(\frac{V_r}{\lambda-V}\right)^2,$$

which will be a short-range function by assuming an appropriate condition on $V(x)$.

There are also many papers concerning with the spectral representation of

Schrödinger operators (e.g., Agmon [1], Ikebe [9], [10], Isozaki [13], Iwatsuka [14], Jäger [15], Mochizuki-Uchiyama [20]). Our purpose in this paper is to construct a generalized Fourier transform $\mathcal{F}$ satisfying the following properties;

(a) $\mathcal{F}$ is a unitary operator on $L^2(\boldsymbol{R}^n)$ onto $L^2(\boldsymbol{R};\boldsymbol{h})$ ($\boldsymbol{h}=L^2(S^{n-1})$)

(b) $\mathcal{F}$ diagonalizes H in the sense that

$$(\mathcal{F}Hf)(\lambda)=\lambda(\mathcal{F}f)(\lambda) \qquad \text{for any} \quad f\in D(H),$$

where $L^2(\boldsymbol{R};\boldsymbol{h})$ is the Hilbert space consisting of all the $\boldsymbol{h}$-valued square integrable functions $\hat{f}(\lambda)$ with the norm

$$\|\hat{f}\|_{L^2(\boldsymbol{R};\boldsymbol{h})}=\left(\int_{\boldsymbol{R}}\|\hat{f}(\lambda)\|_{\boldsymbol{h}}^2 d\lambda\right)^{1/2}.$$

Recently, several authors studied the limiting absorption method and the spectral representation for Schrödinger operators with exploding potentials. The terminology of "exploding potentials", which seems to be originated by Ben-Artzi [3] or Jäger-Rejto [16], is used in the sense that the potential $V(x)$ is unbounded below at infinity. We do not know whether such a potential is an important object in Quantum Mechanics, althogh it seems to be interesting in Mathematics. For the works of second-order ordinary differential operators with exploding potentials one can be referred to Chapter XIII and References in Dunford-Schwartz [5].

Jäger-Rejto [16] gives a sufficient condition for H to be absolutely continuous under the condition (1.3) with $\alpha=1$. The proof is given along the line of Jäger [15], which shows resolvent estimates of Schrödinger operator $-\Delta+V(x)$ by studying second-order ordinary differential equations in the Hilbert space $L^2(S^{n-1})$. Ben-Artzi [3] and Schwartzman [22] obtain the limiting absorption method for a class of spherically symmetric potentials. Ben-Artzi [4] extends his result [3] to the case with short-range perturbations with respect to $V(r)$ and shows that the spectrum of H is absolutely continuous in any open intervals containing no eigenvalues. Schwartzman [23] gives the spectral representation of H with spherically symmetric exploding potentials $V(r)$ and short-range perturbations with respect to V. The existence or the non-existence of eigenvalues of H is not discussed in [4] and [23].

Our method is based on Ikebe [9], Ikebe-Saitō [12] and Mochizuki-Uchiyama [20], which enable us to adopt magnetic potentials $b_j(x)$. Our proof of the limiting absorption method is developed along the line of Ikebe-Saitō [12], by using $\mathcal{D}_j^{\pm}$ suggested by Mochizuki-Uchiyama [20].

The contents hereafter are as follows;

In §2 our assumptions and some notations are explained. In §3 we give the result of the limiting absorption method. Our resolvent estimate is similar to the one obtained in Schwartzman [23]. In § 4 preliminary proposition are prepared to show the proof of Theorems in § 3. The proof of Theorems in § 3 are seen in § 5. In § 6 we construct a generalized Fourier transform $\mathcal{F}$ diagonalizing H. Finally, we prove the unitarity of $\mathcal{F}$ in § 7.

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§ 2. The main assumption

Throughout this paper we shall assume the following condition (A);

(A.1) $V(x)$ can be decomposed as $V(x)=V_0(x)+V_1(x)$ such that $V_0(x)$ is a real-valued C^2 function satisfying

$$V_0(x) \longrightarrow -\infty \quad \text{as} \quad |x| \longrightarrow \infty$$

and $V_1(x)$ is a real-valued measurable Q_β function ($\beta>0$), i.e.,

$$\int_{|x-y|\leq 1} \frac{|V_1(y)|^2}{|x-y|^{n-4+\beta}} dy$$

is a bounded function of $x \in \boldsymbol{R}^n$.

(A.2) There exist positive constants C and $\alpha<2$ such that

$$-C(1+|x|)^\alpha \leqq V_0(x) \leqq -1,$$

(A.3) $\dfrac{\partial V_0}{\partial r} \leqq 0$.

(A.4) Each $b_j(x)$ is a real-valued C^1 function.

(A.5) There exists a positive constant δ such that

(A.5.1) $V_1(x)/(-V_0(x))^{1/4}=O(|x|^{-1-\delta})$,

(A.5.2) $\left(\dfrac{\partial b_k}{\partial x_j}-\dfrac{\partial b_j}{\partial x_k}\right)/(-V_0(x))^{1/4}=O(|x|^{-1-\delta})$,

(A.5.3) $\left(r\dfrac{\partial V_0}{\partial x_j}-x_j\dfrac{\partial V_0}{\partial r}\right)/(-V_0(x))^{3/4}=O(|x|^{-\delta})$

and

(A.5.4) $\dfrac{(\partial/\partial x)^\alpha V_0(x)}{V_0(x)}=O(r^{-(|\alpha|/2)-\delta})$

as $r\to\infty$ for $1\leqq j,\ k\leqq n$ and $1\leqq|\alpha|\leqq 2$, where

$$(\partial/\partial x)^\alpha:=(\partial/\partial x_1)^{\alpha_1}(\partial/\partial x_2)^{\alpha_2}\cdots(\partial/\partial x_n)^{\alpha_n}$$

for a multi-index $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)$, and

$$|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n.$$

(A.6) The unique continuation property holds for L.

The potential $V_0(x)$ satisfying (A.3) is called to be repulsive. One may be referred to Arai [2] and Lavine [18] for the work treating repulsive potentials bounded at infinity.

Under the condition (A) we investigate the limiting absorption method for L. Some additional conditions, roughly speaking, $\delta>\frac{1}{2}$ in (A.5) will be imposed on the condition (A) in order to study the spectral representation of H.

We list some notations used in this paper;

$$\partial_j=\frac{\partial}{\partial x_j},\ \partial_r=\frac{\partial}{\partial r},\ D_j=\frac{\partial}{\partial x_j}+ib_j(x),$$

$$\hat{x}_j=x_j/|x|,\ D_r=\sum_{j=1}^{n}\hat{x}_jD_j,\ L_j=r\partial_j-x_j\partial_r$$

$$Du=(D_1u,\ D_2u,\ \cdots,\ D_nu),\ |Du|=\Big(\sum_{j=1}^{n}|D_ju|^2\Big)^{1/2}$$

$$B(R)=\{x\in\boldsymbol{R}^n;\ |x|<R\},$$

$$S(R)=\{x\in\boldsymbol{R}^n;\ |x|=R\},$$

$$E(R)=\{x\in\boldsymbol{R}^n;\ |x|>R\},$$

$$B(R,\ \rho)=\{x\in\boldsymbol{R}^n;\ R\leqq|x|\leqq\rho\},$$

$$K^{\pm}(a,\ b)=\{z\in\boldsymbol{C}^n;\ a\leqq\mathrm{Re}\,z\leqq b,\ 0<\pm\mathrm{Im}\,z\leqq1\},$$

$L^2_{s,\Omega}$ is the weighted L^2 space with the norm

$$\|f\|_{s,\Omega}=\Big(\int_{\Omega}(1+|x|)^{2s}|f(x)|^2dx\Big)^{1/2},$$

$$L^2_s=L^2_{s,\boldsymbol{R}^n},\ \|f\|_s=\|f\|_{s,\boldsymbol{R}^n},$$

$L^2_{V_0:s,\Omega}$ denotes the weighted L^2 space of all functions $f(x)$ such that $\sqrt[4]{-V_0}\,f\in L^2_{s,\Omega}$ with the norm

$$\|f\|_{V_0:s,\Omega}=\|\sqrt[4]{-V_0}\,f\|_{s,\Omega},$$

$$L^2_{V_0;s}=L^2_{V_0:s,\boldsymbol{R}^n},\ \|f\|_{V_0:s}=\|f\|_{V_0:s,\boldsymbol{R}^n},$$

$H_m(\Omega)$ is the Sobolev space of all $L^2(\Omega)$ functions with $L^2(\Omega)$ distribution derivatives up to the m-th order, inclusive,

$$H_m=H_m(\boldsymbol{R}^n).$$

$H_{m,\mathrm{loc}}$ is the class of all locally H_m functions, $B_{jk}=\frac{\partial}{\partial x_j}b_k-\frac{\partial}{\partial x_k}b_j$ for a vector potential $b(x)=(b_1(x),\ b_2(x),\ \cdots,\ b_n(x))$.

Under the condition (A) the Schrödinger operator L has the unique self-adjoint operator H with the domain

$$D(H)=\{u\in H_{2,\mathrm{loc}};\ Lu\in L^2(\boldsymbol{R}^n)\} \tag{2.1}$$

(cf. Ikebe-Kato [11]), and H has no eigenvalues (cf. Uchiyama [24], Uchiyama-Yamada [25]).

§ 3. The limiting absorption method

Let us take any $K^{\pm}(a, b)$

$$K^{\pm}(a, b)=\{z\in C;\ a\leqq \mathrm{Re}\, z\leqq b,\ 0<\pm\mathrm{Im}\, z\leqq 1\}$$

and choose a positive number R sufficiently large so that

$$a-V_0(x)\geqq 1, \qquad x\in E(R) \tag{3.1}$$

in view of (A.1). Then put

$$k=k(x, z)=-i\sqrt{z-V_0(x)}+\frac{n-1}{2r}-\frac{\partial_r V_0}{4(z-V_0)} \tag{3.2}$$

for $z\in K^{\pm}(a, b)$ and $x\in E(R)$, where we take the square root z as $\mathrm{Im}\sqrt{z}>0$ for a non-real z. Then we have

$$\pm\mathrm{Re}\sqrt{z-V_0(x)}>0 \qquad \text{for} \quad z\in K^{\pm}(a, b) \tag{3.3}$$

and $x\in E(R)$. For a real number λ define

$$\begin{aligned} k^{\pm}(x, \lambda)&=\lim_{\varepsilon\downarrow 0} k(x, \lambda\pm i\varepsilon)\\ &=\mp i\sqrt{\lambda-V_0(x)}+\frac{n-1}{2r}-\frac{\partial_r V_0}{4(\lambda-V_0)}. \end{aligned}$$

Now we introduce

$$\begin{aligned} \mathcal{D}_j&=\mathcal{D}_j(z)=D_j+\hat{x}_j k(x, z),\\ \mathcal{D}_r&=\mathcal{D}_r(z)=\sum_{j=1}^{n}\hat{x}_j\mathcal{D}_j \end{aligned} \tag{3.4}$$

for a non-real z and

$$\begin{aligned} \mathcal{D}_j^{\pm}&=\mathcal{D}_j^{\pm}(\lambda)=D_j+\hat{x}_j k^{\pm}(x, \lambda),\\ \mathcal{D}_r^{\pm}&=\mathcal{D}_r^{\pm}(\lambda)=\sum_{j=1}^{n}\hat{x}_j\mathcal{D}_j^{\pm} \end{aligned} \tag{3.5}$$

for a real λ.

Let $z_0\in\overline{K^{+}(a, b)}$ $(z_0\in\overline{K^{-}(a, b)})$ and let $u(x)$ be an $H_{2,\mathrm{loc}}$ solution of

$$Lu-z_0u=f\in L_s^2 \quad \left(s>\frac{1}{2}\right).$$

Then the solution $u(x)$ is said, following Ikebe-Saitō [12] and Mochizuki-Uchiyama [20], to satisfy the *outgoing* (*incoming*) *radiation condition*, if $z_0\in K^{+}(a, b)$ $(z_0\in K^{-}(a, b))$ and

$$\|\mathcal{D}_j(z_0)u\|_{s-1, E(R)}<\infty \qquad (1\leqq j\leqq n) \tag{3.6}$$

or if z_0 is real and

$$\begin{aligned} &\|\mathcal{D}_j^{+}(z_0)u\|_{s-1, E(R)}<\infty\\ &(\|\mathcal{D}_j^{-}(z_0)u\|_{s-1, E(R)}<\infty) \qquad (1\leqq j\leqq n) \end{aligned} \tag{3.7}$$

Under the condition (A) we have the following theorems, which will be proved in § 5.

Theorem 3.1. *For any* $K^{\pm}(a, b)$ *and any* s *such that* $\frac{1}{2}<s\leqq\min\left(1, \frac{1+\delta}{2}\right)$ *there exists a positive constant* C *such that*

$$\|R(z)f\|_{V_0;-s}\leqq C\|f\|_s,$$

and

$$\|\mathcal{D}_j R(z)f\|_{s-1,E(R)}\leqq C\|f\|_s \qquad (1\leqq j\leqq n)$$

for any $f\in L_s^2$ *and* $z\in K^{\pm}(a, b)$, *where* $R(z)=(H-z)^{-1}$.

Theorem 3.2. *Let* s *be as in Theorem* 3.1 *and* $f\in L_s^2$. *Then for any real number* λ *there exists a unique solution* $u^+=u^+(\lambda, f)(u^-=u^-(\lambda, f))\in H_{2,\mathrm{loc}}\cap L^2_{V_0;-s}$ *of*

$$Lu-\lambda u=f$$

such that $u^+(u^-)$ *satisfies the outgoing (incoming) radiation condition. For any sequence* $\{z_m\}$ *in* $\boldsymbol{C}$ *satisfying*

$$\lim_{m\to\infty} z_m=\lambda \quad \textit{and} \quad \mathrm{Im}\, z_m>0, \quad m=1, 2, \cdots$$

$$(\mathrm{Im}\, z_m<0, \quad m=1, 2, \cdots)$$

we have

$$R(z_m)f \longrightarrow u^+(u^-) \qquad \textit{strongly in} \quad L^2_{V_0;-s}.$$

The solution $u^+(u^-)$ *as above is denoted by* $R(\lambda+i0)f\,(R(\lambda-i0)f)$.

Theorem 3.3. *Let* s *be as in Theorem* 3.1 *and* $f\in L_s^2$. *Then* $R(\lambda+i0)f$ *and* $R(\lambda-i0)f$ *are* $L^2_{V_0;-s}$ *valued strongly continuous functions with respect to* $\lambda\in\boldsymbol{R}$.

Since (A.2) gives

$$\|u\|_{-s}\leqq\|u\|_{V_0;-s} \tag{3.8}$$

for any real s, the continuity of $R(\lambda+i0)\, f$ in $L^2_{V_0;-s}$ with respect to λ in Theorem 3.3 implies the continuity in L^2_{-s}. Therefore the following assertion can be proved by the same argument as in § 3 of Ikebe-Saitō [12].

Corollary 3.4. *Let* E *be the spectral measure of the self-adjoint operator* H, *i.e.*, $H=\int_{-\infty}^{+\infty}\lambda\, dE(\lambda)$. *Let* s *be as in Theorem* 3.1. *Then for any real numbers* a, b *and* $f\in L_s^2$ *we have*

$$\begin{aligned}&(E((a, b))f, f)_{L^2}\\ &\quad =\frac{1}{2\pi i}\int_a^b\langle R(\lambda+i0)f-R(\lambda-i0)f, f\rangle d\lambda,\end{aligned}$$

where $\langle u, v\rangle=\int_{R^n}u(x)\overline{v(x)}dx$ *for* $u\in L^2_{-s}$ *and* $v\in L_s^2$.

Therefore, H *is an absolutely continuous operator.*

§ 4. Preliminary propositions

Throughout this section we assume the condition (A), while some proposition may hold under a weaker condition.

Lemma 4.1. *For any compact set K in C and any $R>0$ there exists a positive constant C such that*

$$\sum_{j=1}^{n}\int_{B(R)}\left|\frac{\partial u}{\partial x_j}\right|^2 dx \leq C\int_{B(R+1)}\{|u(x)|^2+|(L-z)u|^2\}dx$$

for any $u\in C_0^\infty(\boldsymbol{R}^n)$ and $z\in K$(for the proof see, e.g., Ikebe-Saitō [12], Lemma 2.1 or Weidmann [26], Auxiliary theorem 10.26).

Lemma 4.2. *Let z be a non-real number and R be sufficiently large so that* (3.1) *holds. Assume that $\varphi=\varphi(r)$ is a real-valued C^1 function such that $\varphi(R)=0$. Then we have*

$$\begin{aligned}(4.1)\qquad &\mathrm{Re}\int_{E(R)}\varphi(r)(L-z)u\overline{\mathcal{D}_r u}\,dx\\ &=\int_{E(R)}\Big\{\Big[\frac{1}{2}\varphi'(r)+\varphi(\mathrm{Im}\sqrt{z-V_0}+\mathrm{Re}\,h)|\mathcal{D}u|^2\Big]+\Big(\frac{\varphi}{r}-\varphi'\Big)(|\mathcal{D}u|^2-|\mathcal{D}_r u|^2)\\ &+\varphi\mathrm{Re}\Big[\overline{\mathcal{D}_r u}\Big(V_0-z+\frac{\partial k}{\partial r}+\frac{n-1}{r}k-k^2+V_1\Big)u\Big]\\ &+\mathrm{Re}\Big[(\varphi u)\sum_{j=1}^{n}\overline{(\mathcal{D}_j u)}\Big(\frac{i}{2\sqrt{z-V_0}}\frac{L_jV_0}{r}+\frac{L_jh}{r}\Big)\Big]\\ &+\mathrm{Im}\Big[(\varphi u)\sum_{j,l=1}^{n}\overline{(\mathcal{D}_j u)}\hat{x}_l B_{jl}(x)\Big]\Big\}dx\end{aligned}$$

for every $u\in C_0^\infty(\boldsymbol{R}^n)$, where

$$h=-\frac{\partial_r V_0}{4(z-V_0)},$$

$$L_j=r\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial r},\quad B_{jl}=\frac{\partial b_l}{\partial x_j}-\frac{\partial b_j}{\partial x_l},$$

$$|\mathcal{D}u|^2=\sum_{j=1}^{n}|\mathcal{D}_j u|^2.$$

Proof. It follows from (1.1), (3.4) and (A.1) that

$$\begin{aligned}(4.2)\qquad (L-z)u&=-\sum_{j=1}^{n}D_j^2u+Vu-zu\\ &=-\sum_{j=1}^{n}D_j(\mathcal{D}_j-\hat{x}_jk)u+V_0u+V_1u-zu\\ &=-\sum_{j=1}^{n}D_j\mathcal{D}_ju+kD_ru+\Big(\frac{\partial k}{\partial r}+\frac{n-1}{r}k+V_0+V_1-z\Big)u\\ &=-\sum_{j=1}^{n}D_j\mathcal{D}_ju+k\mathcal{D}_ru+\Big(\frac{\partial k}{\partial r}+\frac{n-1}{r}k-k^2+V_0+V_1-z\Big)u.\end{aligned}$$

Integration by parts us to obtain an identity

$$(4.3)\qquad -\mathrm{Re}\int_{E(R)}\varphi\sum_{j=1}^{n}(D_j\mathcal{D}_ju)\overline{\mathcal{D}_ru}\,dx$$

$$=-\mathrm{Re}\int_{E(R)}\varphi\sum_{j,l=1}^{n}(D_j\mathcal{D}_ju)\hat{x}_l\overline{\mathcal{D}_ru}\,dx$$

$$=\int_{E(R)}\Big\{\varphi'|\mathcal{D}_ru|^2+\frac{\varphi}{r}(|\mathcal{D}u|^2-|\mathcal{D}_ru|^2)+\varphi\,\mathrm{Re}\sum_{j,l=1}^{n}(\mathcal{D}_ju)\hat{x}_l\overline{D_j\mathcal{D}_lu}\Big\}dx$$

$$=\int_{E(R)}\Big\{\Big(\frac{\varphi}{r}-\frac{\varphi'}{2}-\frac{n-1}{2r}\varphi\Big)|\mathcal{D}u|^2+\Big(\varphi'-\frac{\varphi}{r}\Big)|\mathcal{D}_ru|^2\Big\}dx$$

$$+\mathrm{Re}\int_{E(R)}\varphi\sum_{j,l=1}^{n}(\mathcal{D}_ju)\hat{x}_l\overline{(D_j\mathcal{D}_lu-D_l\mathcal{D}_ju)}dx\,.$$

Noting that

$$D_j\mathcal{D}_l-D_l\mathcal{D}_j=k(\hat{x}_lD_j-\hat{x}_jD_l)+\Big(\hat{x}_l\frac{\partial k}{\partial x_j}-\hat{x}_j\frac{\partial k}{\partial x_l}\Big)+i\Big(\frac{\partial}{\partial x_j}b_l-\frac{\partial}{\partial x_l}b_j\Big)$$

$$=k(\hat{x}_l\mathcal{D}_j-\hat{x}_j\mathcal{D}_l)+\Big(\hat{x}_l\frac{\partial k}{\partial x_j}-\hat{x}_j\frac{\partial k}{\partial x_l}\Big)+iB_{jl}$$

and

$$(4.4)\qquad L_j=r\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial r}=r\sum_{l=1}^{n}\hat{x}_l\Big(\hat{x}_l\frac{\partial}{\partial x_j}-\hat{x}_j\frac{\partial}{\partial x_l}\Big)$$

we have from (4.3) that

$$(4.5)\qquad -\mathrm{Re}\int_{E(R)}\varphi\sum_{j=1}^{n}(D_j\mathcal{D}_ju)\overline{\mathcal{D}_ru}\,dx$$

$$=\int_{E(R)}\Big\{\Big(\frac{\varphi}{r}-\frac{\varphi'}{2}-\frac{n-1}{2r}\varphi+[\mathrm{Re}\,k]\varphi\Big)|\mathcal{D}u|^2+\Big(\varphi'-\frac{\varphi}{r}-[\mathrm{Re}\,k]\varphi\Big)|\mathcal{D}_ru|^2$$

$$+\varphi\,\mathrm{Re}\sum_{j=1}^{n}\frac{L_jk}{r}\overline{(\mathcal{D}_ju)}u+\varphi\,\mathrm{Im}\sum_{j,l=1}^{n}\hat{x}_lB_{jl}\overline{(\mathcal{D}_ju)}u\Big\}dx\,.$$

The definition (3.2) of $k(x,z)$ gives

$$(4.6)\qquad L_jk=\frac{iL_jV_0}{2\sqrt{z-V_0}}+L_jh\,.$$

Therefore (4.1) follows from (4.2), (4.3), (4.5) and (4.6). (cf. Lemma 2.2 in Ikebe-Saitō [12], which shows a similar identity.) Q.E.D.

Lemma 4.3. *Let* $\frac{1}{2}<s\leqq 1$. *For each* $K^{\pm}(a,b)$ *there exist positive constants* C *and* R *such that*

$$(4.7)\qquad \|\mathcal{D}u\|^2_{s-1,E(R)}\equiv\sum_{j=1}^{n}\|\mathcal{D}_ju\|^2_{s-1,E(R)}$$

$$\leqq C\{\|u\|^2_{V_0;\,s-1-\delta}+\|(L-z)u\|^2_s\}$$

for any $u\in C_0^\infty(\boldsymbol{R}^n)$ *and* $z\in K^{\pm}(a,b)$.

Proof. Choose R so large by means of (A.1), (A.2) and (A.5.1) that

(4.8) $$a-V_0(x)>1, \qquad x\in E(R)$$

and

(4.9) $$V_1(x) \text{ is bounded in } E(R).$$

In order to prove Lemma 4.3 we make use of Lemma 4.2 by putting

$$\varphi(r)=\zeta(r-R)(1+r)^{2s-1},$$

where ζ is a C^∞ function on $\boldsymbol{R}$ such that $\zeta'\geqq 0$

(4.10) $$\zeta(t)=\begin{cases} 1, & t\geqq 1 \\ 0, & t\leqq 0. \end{cases}$$

Then the left-hand side of (4.1) is estimated by Schwarz inequality as

(4.11) $$\begin{aligned} &\mathrm{Re}\int_{E(R)}\varphi(L-z)u\overline{\mathcal{D}_r u}\,dx \\ &\leqq \|(L-z)u\|_s\|\mathcal{D}_r u\|_{s-1,E(R)}. \end{aligned}$$

Let us consider the right-hand side of (4.1). Since

$$\begin{aligned} \varphi'&=(2s-1)(1+r)^{2s-2}\zeta(r-R)+(1+r)^{2s-1}\zeta'(r-R) \\ &\geqq(2s-1)(1+r)^{2s-2} \qquad \text{for} \quad r\geqq R+1, \end{aligned}$$

$$\mathrm{Im}\sqrt{z-V_0(x)}>0,$$

$$\mathrm{Re}\, h=\mathrm{Re}\left[-\frac{\partial_r V_0}{4(z-V_0)}\right]=-\frac{1}{4}\frac{(\partial_r V_0)(\mathrm{Re}\, z-V_0)}{(\mathrm{Re}\, z-V_0)^2+(\mathrm{Im}\, z)^2}\geqq 0$$

as a result of (A.3) and (4.8), we see

(4.12) $$\left[\frac{1}{2}\varphi'(r)+(\mathrm{Im}\sqrt{z-V_0}+\mathrm{Re}\, h)\varphi(r)\right]|\mathcal{D}u|^2\geqq\left(s-\frac{1}{2}\right)(1+r)^{2s-2}|\mathcal{D}u|^2 \quad (r\geqq R+1).$$

It follows from the condition $s\leqq 1$ that

$$\frac{\varphi}{r}-\varphi'=(1+r)^{2s-2}\left(2-2s+\frac{1}{r}\right)>0 \qquad (r\geqq R+1),$$

which and (3.4) yield

(4.13) $$\left(\frac{\varphi}{r}-\varphi'\right)(|\mathcal{D}u|^2-|\mathcal{D}_r u|^2)\geqq 0 \qquad (r\geqq R+1).$$

The conditions (A.1), (A.5.3), (A.5.4) and (3.2) imply

(4.14) $$\begin{aligned} &V_0-z+\frac{\partial k}{\partial r}+\frac{n-1}{r}k-k^2 \\ &=\frac{(n-1)(n-3)}{4r^2}-\frac{\partial_r^2 V_0}{4(z-V_0)}-\frac{5}{16}\left(\frac{\partial_r V_0}{z-V_0}\right)^2 \\ &=O(r^{-1-\delta}), \end{aligned}$$

$$\frac{L_j V_0}{r\sqrt[4]{-V_0}\sqrt{z-V_0}}=O(r^{-1-\delta})$$

and

$$\frac{L_j h}{r}=-\frac{1}{4r}L_j\Big(\frac{\partial_r V_0}{z-V_0}\Big)$$
$$=\frac{1}{4}\frac{(\partial_j V_0-\hat{x}_j\partial_r V_0)\partial_r V_0}{(z-V_0)^2}-\frac{1}{4}\frac{\partial_j\partial_r V_0-\hat{x}_j\partial_r^2 V_0}{z-V_0}=O(r^{-1-\delta})$$

as $|x|\to\infty$ uniformly in $z\in K^{\pm}(a, b)$. Thus we have from Lemma 4.2, (4.8), (4.9), (4.11), (4.12), (4.13), (A.2), (A.5.1), (A.5.2) that

$$(4.15)\qquad \Big(s-\frac{1}{2}\Big)\int_{E(R+1)}(1+r)^{2s-2}|\mathcal{D}u|^2dx$$
$$\leqq C_1\Big\{\int_{B(R,R+1)}|\mathcal{D}u|^2dx+\int_{E(R)}(1+r)^{2s-1}|\mathcal{D}u||u|(1+r)^{-1-\delta}\sqrt[4]{-V_0}\,dx\Big\}$$
$$\leqq C_1\int_{B(R,R+1)}|\mathcal{D}u|^2dx+C_1\|\mathcal{D}u\|_{s-1,E(R)}\|u\|_{V_0;\,s-1-\delta},$$

where C_1 is a positive constant independent of $u\in C_0^\infty(\boldsymbol{R}^n)$ and $z\in K^{\pm}(a, b)$. In view of Lemma 4.1 we obtain

$$(4.16)\qquad \int_{B(R,R+1)}|\mathcal{D}u|^2dx\leqq C_2(\|u\|^2_{B(R+2)}+\|(L-z)u\|^2_{B(R+2)})$$
$$\leqq C_3(\|u\|^2_{V_0;\,s-1-\delta}+\|(L-z)u\|^2_s)$$

where C_2 and C_3 are independent of u and z. Now gathering (4.15) and (4.16) and noting $s>(1/2)$ one can find a positive constant C_4 independent of u and z such that

$$(4.17)\qquad \|\mathcal{D}u\|^2_{s-1,E(R)}\leqq C_4(\|u\|^2_{V_0;\,s-1-\delta}+\|(L-z)u\|^2_s+\|\mathcal{D}u\|_{s-1,E(R)}\|u\|_{V_0;\,s-1-\delta}$$
$$\leqq C_4(\|u\|^2_{V_0;\,s-1-\delta}+\|(L-z)u\|^2_s+\varepsilon\|\mathcal{D}u\|^2_{s-1,E(R)}+\frac{1}{\varepsilon}\|u\|^2_{V_0;\,s-1-\delta})$$

for any $\varepsilon>0$. Therefore one can show (4.7) by taking ε in (4.17) so small that $\varepsilon C_4<1$.

Q. E. D.

Lemma 4.4. *For any $K^{\pm}(a, b)$ and $s>\frac{1}{2}$ there exist positive constants C and R such that*

$$\|u\|^2_{V_0;-s,E(\rho)}\leqq C\rho^{1-2s}(\|u\|^2_{V_0;-s}+\|(L-z)u\|^2_s+\|\mathcal{D}u\|^2_{s-1,E(R)})$$

for every $u\in C_0^\infty(\boldsymbol{R}^n)$, $z\in K^{\pm}(a, b)$ and $\rho>R$.

Proof. Recalling the definition (3.4) of $\mathcal{D}_r$ and putting

$$W_1=\mathrm{Re}\sqrt{z-V_0},\quad W_2=\mathrm{Im}\sqrt{z-V_0}$$

one obtains

$$(4.18)\qquad |\mathcal{D}_r u|^2=\Big|\Big(D_r-iW_1+W_2+\frac{n-1}{2r}+h\Big)u\Big|^2$$
$$=\Big|\Big(D_r+W_2+\frac{n-1}{2r}+\mathrm{Re}\,h\Big)u\Big|^2+(W_1-\mathrm{Im}\,h)^2|u|^2-2(W_1-\mathrm{Im}\,h)\mathrm{Im}[(D_r u)\bar{u}]$$
$$\geqq(W_1-\mathrm{Im}\,h)^2|u|^2-2(W_1-\mathrm{Im}\,h)\mathrm{Im}[(D_r u)\bar{u}].$$

The condition (A.1) and (A.5.4) show

$$h=\frac{-\partial_r V_0}{4(z-V_0)}\longrightarrow 0 \quad \text{as} \quad |x|\longrightarrow\infty$$

uniformly in $z\in K^{\pm}(a, b)$. Choose R sufficiently large so that

$$|h(x, z)|<\frac{1}{2}, \qquad a-\frac{V_0(x)}{2}\geqq\frac{1}{2} \tag{4.19}$$

for $x\in E(R)$. Then we have from (A.2) and (4.19)

$$\begin{aligned} W_1^2\geqq W_1^2-W_2^2&=\mathrm{Re}[z-V_0(x)]\geqq a-V_0(x) \\ &=a-\frac{V_0(x)}{2}-\frac{V_0(x)}{2}\geqq\frac{1}{2}(1-V_0(x))\geqq 1 \end{aligned} \tag{4.20}$$

and from (3.3)

$$2|W_1|\geqq\pm(W_1-\mathrm{Im}\,h)\geqq\frac{1}{2}|W_1| \qquad \text{for} \quad \pm\mathrm{Im}\,z>0. \tag{4.21}$$

Therefore it follows from (4.18), (4.20) and (4.21) that

$$\begin{aligned} 2|\mathcal{D}_r u^2|&\geqq\frac{2}{|W_1|}|\mathcal{D}_r u|^2\geqq\pm\frac{1}{(W_1-\mathrm{Im}\,h)}|\mathcal{D}_r u|^2 \\ &\geqq\pm(W_1-\mathrm{Im}\,h)|u|^2\mp 2\,\mathrm{Im}[(D_r u)\bar{u}] \\ &\geqq\frac{\sqrt{-V_0}}{4}|u|^2\mp 2\,\mathrm{Im}[(D_r u)\bar{u}] \qquad \text{for} \quad \pm\mathrm{Im}\,z>0. \end{aligned} \tag{4.22}$$

Taking the imaginary part of an integral

$$J=\int_{B(R)}(L-z)u\cdot\bar{u}\,dx$$

we have

$$\mathrm{Im}\,J=-\mathrm{Im}\int_{S(R)}(D_r u)\bar{u}\,dS-(\mathrm{Im}\,z)\int_{B(R)}|u|^2 dx.$$

The above inequality and

$$\|u\|_{-s}\leqq\|u\|_{V_0;-s}, \tag{4.23}$$

which is a consequence of (A.2), yield

$$\pm\int_{S(R)}(D_r u)\bar{u}\,dS\leqq\|(L-z)u\|_s\|u\|_{-s}\leqq\|(L-z)u\|_s\|u\|_{V_0;-s} \tag{4.24}$$

for $\pm\mathrm{Im}\,z>0$. (4.22) and (4.24) are gathered to obtain

$$\frac{1}{4}\int_{S(t)}\sqrt{-V_0}\,|u|^2 dS\leqq 2\int_{S(t)}|\mathcal{D}_r u|^2 dS+2\|(L-z)u\|_s\|u\|_{V_0;-s} \tag{4.25}$$

for any $t>R$. Multiplying (4.25) by $(1+t)^{-2s}$ and integrating with respect to t over $[\rho, \infty)(\rho>R)$ we have

$$\frac{1}{4}\|u\|^2_{V_0;-s,E(\rho)}$$

$$\leqq 2\|\mathcal{D}_r u\|^2_{-s,E(\rho)}+2\|(L-z)u\|_s\|u\|_{V_0;-s}\int_\rho^\infty \frac{dt}{(1+t)^{2s}}$$

$$\leqq 2(1+\rho)^{1-2s}\|\mathcal{D}_r u\|^2_{s-1,E(\rho)}+\frac{(1+\rho)^{1-2s}}{2s-1}(\|(L-z)u\|_s^2+\|u\|^2_{V_0;-s}),$$

which completes the proof. Q.E.D.

Lemma 4.5. *Let z be a non-real number and τ be a real number. Then there exist no non-trivial solutions $u(x)$ of*

$$Lu-zu=0$$

such that $u\in H_{2,\mathrm{loc}}\cap L^2_\tau$.

Proof. Let $u(x)$ be an arbitrary solution of

$$Lu-zu=0 \tag{4.26}$$

belonging to $H_{2,\mathrm{loc}}\cap L^2_\tau$. We shall show that if u is an $H_{2,\mathrm{loc}}$ solution of (4.26) satisfying

$$u\in L^2_s \quad \text{for a real number } t, \tag{4.27}$$

then we have

$$|Du|\in L^2_{t-(\alpha/2)} \tag{4.28}$$

(α is the number appearing in (A.2)) and

$$u\in L^2_{t+(1/2)-(\alpha/4)}. \tag{4.29}$$

Since $\alpha<2$ in consequence of (A.2), the repeated use of the above result gives finally

$$u\in L^2$$

and, therefore,

$$u\in D(H) \quad \text{and} \quad Hu=zu$$

in view of (2.1). Since z is non-real, the self-adjointness of H shows $u=0$.

We shall prove first (4.28). Integrating

$$(L-z)u\cdot(1+r)^{2t-\alpha}\bar{u}=0$$

over $B(R,\rho)$ and taking the real part we have

$$\begin{aligned}\int_{B(R,\rho)}&(1+r)^{2t-\alpha}|Du|^2dx\\ &=\left[\int_{S(\rho)}-\int_{S(R)}\right](1+r)^{2t-\alpha}\,\mathrm{Re}[(D_r u)\bar{u}]dS\\ &\quad-(2t-\alpha)\int_{B(R,\rho)}(1+r)^{2t-\alpha-1}\,\mathrm{Re}[(D_r u)\bar{u}]dx\\ &\quad+\int_{B(R,\rho)}(\mathrm{Re}\,z-V(x))(1+r)^{2t-\alpha}|u|^2dx,\end{aligned} \tag{4.30}$$

where

$$\left[\int_{S(\rho)}-\int_{S(R)}\right]f\,dS=\int_{S(\rho)}f\,dS-\int_{S(R)}f\,dS.$$

Let R be so large that (4.9) is valid. Then the condition (A.2) and the fact $u\in L_t^2$ give

(4.31) $$\int_{E(R)}(1+|V(x)|)(1+r)^{2t-\alpha}|u|^2dx<\infty.$$

The second integral of the right-hand side of (4.30) is estimated by

$$\int_{B(R,\rho)}\left\{\varepsilon(1+r)^{2t-\alpha}|Du|^2+\frac{1}{\varepsilon}(1+r)^{2t}|u|^2\right\}dx$$

for every $\varepsilon>0$. This fact and (4.30), (4.31) imply that we can obtain

(4.32) $$\int_{E(R)}(1+r)^{2t-\alpha}|Du|^2dx<\infty$$

if we show

(4.33) $$\liminf_{\rho\to\infty}\int_{S(\rho)}(1+r)^{2t-\alpha}\,\mathrm{Re}[(D_ru)\bar{u}]dS\leqq 0,$$

where

$$\liminf_{\rho\to\infty}f(\rho)=\lim_{\rho\to\infty}[\inf_{r\geqq\rho}f(r)].$$

Suppose that (4.33) is not true. Then there would exist positive constants δ and ρ_0 such that

(4.34) $$\int_{S(\rho)}(1+r)^{2t-\alpha}\,\mathrm{Re}[(D_ru)\bar{u}]dS\geqq\delta>0$$

for $\rho>\rho_0$. Integrating (4.34) over $[\rho_1,\rho_2](\rho_0<\rho_1)$ with respect to ρ gives

(4.35) $$(\rho_2-\rho_1)\delta\leqq\int_{B(\rho_1,\rho_2)}(1+r)^{2t-\alpha}\,\mathrm{Re}[(D_ru)\bar{u}]dS$$
$$=\frac{1}{2}\left[\int_{S(\rho_2)}-\int_{S(\rho_1)}\right](1+r)^{2t-\alpha}|u|^2dS-\frac{1}{2}\int_{B(\rho_1,\rho_2)}\left(\frac{n-1}{r}+\frac{2t-\alpha}{1+r}\right)(1+r)^{2t-\alpha}|u|^2dx.$$

The assumption $u\in L_t^2$ yields

$$\liminf_{\rho_2\to\infty}\rho_2\int_{S(\rho_2)}(1+r)^{2t}|u|^2dS=0$$

and that the right-hand side of (4.35) has a finite inferior limit as $\rho_2\to\infty$, while the left-hand side of (4.35) diverges to infinity. This is a contradiction. Thus, (4.28) has been proved.

Finally, we prove (4.29). Consider the imaginary part of

$$\int_{B(R,\rho)}(Lu-zu)(1+r)^{2t+1-(\alpha/2)}\bar{u}\,dx.$$

Then we have by integration by parts

(4.36) $$(\mathrm{Im}\,z)\int_{B(R,\rho)}(1+r)^{2t+1-(\alpha/2)}|u|^2dx$$

$$=-\left[\int_{S(\rho)}-\int_{S(R)}\right](1+r)^{2t+1-(\alpha/2)}\,\mathrm{Im}[(D_r u)\bar{u}]dS$$

$$+\int_{B(R,\rho)}\left(2t+1-\frac{\alpha}{2}\right)(1+r)^{2t-(\alpha/2)}\,\mathrm{Im}[(D_r u)\bar{u}]dx\,.$$

In view of $u\in L_t^2$ and $|Du|\in L_{t-(\alpha/2)}^2$ we have

$$(1+r)^{2t-(\alpha/2)}|u||Du|$$

$$=(1+r)^t|u|(1+r)^{t-(\alpha/2)}|Du|\in L^1$$

and

$$\liminf \rho\int_{S(\rho)}(1+r)^{2t-(\alpha/2)}|u||Du|dS=0\,.$$

Taking the inferior limit as $\rho\to\infty$ in (4.36) we have from

(3.37)
$$|\mathrm{Im}\,z|\int_{E(R)}(1+r)^{2t+1-(\alpha/2)}|u|^2dx$$

$$\leq\int_{S(R)}(1+r)^{2t+1-(\alpha/2)}|u||Du|dS+\left|2t+1-\frac{\alpha}{2}\right|\int_{E(R)}(1+r)^{2t-(\alpha/2)}|u||Du|dx$$

which shows (4.29). Q.E.D.

Remark. In the above proof we have used the condition $\alpha<2$ in (A.2). Without this condition we can not expect the assertion in general. In fact, there is a simple counterexample. Consider the case $n=1$. Then, $u=\exp(-ix^2/2)$ satisfies

$$-u''-x^2u=iu$$

and

$$u\in L_{-s}^2$$

for any $s>1/2$.

Proposition 4.6. *Let R be a positive number and $f(x)$ a C^1 function near $S(R)$. Then there exists a positive constant C independent of R such that*

$$|f(x')-f(x'')|\leq C\frac{|x'-x''|}{R}\sup_{x\in S(R)}\sum_{j=1}^{n}|(L_jf)(x)|$$

for any x', $x''\in S(R)$.

Proof. For any row vectors x', $x''\in S(R)$ consider an orthogonal matrix $U=[u_{ij}]$ such that

$$x'U=y'=(R,0,\cdots,0),$$

$$x''U=y''=(R\cos\theta,R\sin\theta,0,\cdots,0),$$

where $0\leq\theta\leq\pi$, and define λ

$$g(x)=f(xU^{-1})\,.$$

Then we have from the orthogonality of U

$$\frac{\partial g}{\partial x_k}(x)=\sum_{j=1}^{n} u_{jk}\frac{\partial f}{\partial x_j}(xU^{-1}),$$

and

$$\frac{\partial g}{\partial r}(x)=\sum_{j=1}^{n}\hat{x}_k\frac{\partial g}{\partial x_k}(x)=\sum_{j,k=1}^{n}\hat{x}_k u_{jk}\frac{\partial f}{\partial x_j}(xU^{-1})$$
$$=\frac{\partial f}{\partial r}(xU^{-1}).$$

Put $y=xU$. Then it is seen also from the orthonality of U

$$|(L_k g)(y)|=\left||y|\frac{\partial g}{\partial x_k}(y)-y_k\frac{\partial g}{\partial r}\right| \tag{4.38}$$
$$=\left|\sum_{j=1}^{n}\left[|y|u_{jk}\frac{\partial f}{\partial x_j}(x)-y_k\frac{\partial f}{\partial r}(x)\right]\right|$$
$$=\left|\sum_{j=1}^{n}\left[|x|u_{jk}\frac{\partial f}{\partial x_j}(x)-x_j u_{jk}\frac{\partial f}{\partial r}(x)\right]\right|$$
$$\leqq\left(\sum_{j=1}^{n}\left|r\frac{\partial f}{\partial x_j}(x)-x_j\frac{\partial f}{\partial r}(x)\right|^2\right)^{1/2}=\left(\sum_{j=1}^{n}|(L_j f)(x)|^2\right)^{1/2}.$$

Hence we obtain

$$|f(x')-f(x'')|=|g(y')-g(y'')| \tag{4.39}$$
$$=\left|\int_0^\theta \frac{d}{dt}g(R\cos t,\ R\sin t,\ 0,\ \cdots,\ 0)dt\right|$$
$$=\left|\int_0^\theta\left(-R\frac{\partial g}{\partial x_1}\sin t+R\frac{\partial g}{\partial x_2}\cos t\right)dt\right|$$
$$=\left|\int_0^\theta R\left\{-\left(\frac{\partial g}{\partial x_1}-\frac{\partial g}{\partial r}\cos t\right)\sin t+\left(\frac{\partial g}{\partial x_2}-\frac{\partial g}{\partial r}\sin t\right)\cos t\right\}dt\right|$$
$$\leqq\theta\max_{y\in S(R)}\{|(L_1 g)(y)|+|(L_2 g)(y)|\}.$$

An elementary inequality

$$R\cdot\theta=2R\frac{\theta}{2}\leqq 2R\frac{\pi}{2}\sin\frac{\theta}{2}=\frac{\pi}{2}|y'-y''|$$
$$=\frac{\pi}{2}|x'-x''|\qquad(0\leqq\theta\leqq\pi)$$

together with (4.38) and (4.39) yields the required inequality. Q. E. D.

Proposition 4.7. *Put*

$$g(R)=\frac{1}{|S(R)|}\int_{S(R)}\sqrt[4]{-V_0}\,dS\ \left(|S(R)|=\int_{S(R)}dS\right).$$

Then for each real λ

$$\sup_{x\in S(R)}|\sqrt[4]{\lambda-V_0(x)}-g(R)|\longrightarrow 0\qquad(R\longrightarrow\infty)$$

Proof. Since

$$\sqrt[4]{\lambda-V_0}-\sqrt[4]{-V_0}$$

$$=\frac{\lambda}{(\sqrt[4]{\lambda-V_0}+\sqrt[4]{-V_0})(\sqrt{\lambda-V_0}+\sqrt{-V_0})}\longrightarrow 0 \qquad (|x|\longrightarrow\infty)$$

in view of (A.1), it suffices to handle the case $\lambda=0$. The condition (A.5.3) implies

$$L_j\sqrt[4]{-V_0}=\frac{-1}{4}(L_jV_0)/(-V_0)^{3/4}=O(r^{-\delta})$$

as $r=|x|\to\infty$. Therefore setting $f(x)=\sqrt[4]{-V_0}$ in Proposition 4.6 we have

$$|\sqrt[4]{-V_0(x)}-g(R)|=\frac{1}{|S(R)|}\left|\int_{S(R)}(\sqrt[4]{-V_0(x)}-\sqrt[4]{-V_0(y)})dS_y\right|$$

$$\leqq\frac{C'}{|S(R)|}\int_{S(R)}(1+R)^{-\delta}dS=O(R^{-\delta}),$$

as $R\to\infty$, where the constant C' is independent of R. Q.E.D.

Lemma 4.8. *Let λ be a real number. There exist no non-trivial solutions $u\in H_{2,\mathrm{loc}}$ of*

$$Lu=\lambda u$$

satisfying the outgoing (incoming) radiation condition.

Proof. Let $u(x)$ be an $H_{2,\mathrm{loc}}$ solution of $Lu=\lambda u$ and satisfy the outgoing radiation condition. Then we shall show $u=0$. In the case that $u(x)$ satisfies the incoming radiation condition one can show $u=0$ similarly. A simple calculation gives

$$(4.40)\qquad |\mathcal{D}_r^+u|^2=\left|\left(D_r-i\sqrt{\lambda-V_0}+\frac{n-1}{2r}+h\right)u\right|^2$$

$$=|D_ru|^2+(\lambda-V_0)|u|^2+\left(\frac{n-1}{2r}+h\right)^2|u|^2$$

$$+2\left(\frac{n-1}{2r}+h\right)\mathrm{Re}[(D_ru)\bar{u}]-2\sqrt{\lambda-V_0}\,\mathrm{Im}[(D_ru)\bar{u}].$$

If we take R_0 so large that

$$(4.41)\qquad \lambda-V_0(x)\geqq 1,$$

$$\left|\frac{n-1}{2r}+h\right|=\left|\frac{n-1}{2r}-\frac{\partial_rV_0}{4(\lambda-V_0)}\right|\leqq\frac{1}{2}\qquad (|x|\geqq R_0)$$

by means of (A.1) and (A.5.4), we obtain from (4.40)

$$(4.42)\qquad |\mathcal{D}_r^+u|^2\geqq\frac{1}{2}\{|D_ru|^2+(\lambda-V_0)|u|^2\}-2\sqrt{\lambda-V_0}\,\mathrm{Im}[(D_ru)\bar{u}]\qquad (|x|\geqq R_0)$$

It follows from Proposition 4.7 that

$$\varepsilon(x)\equiv\sqrt[4]{\lambda-V_0(x)}-g(|x|)\longrightarrow 0,\qquad |x|\longrightarrow\infty.$$

Take a sufficiently large number $R_1(R_1>R_0)$ so that

$$|\varepsilon(x)|\leqq\frac{1}{8}\qquad(|x|\geqq R_1)$$

Then (4.42) implies

$$|(\mathcal{D}_r^+u)(x)|^2\geqq\frac{1}{2}\{|D_ru|^2+(\lambda-V_0)|u|^2\}$$

$$-2\{g(r)^2+2\varepsilon(x)\sqrt[4]{\lambda-V_0(x)}-\varepsilon(x)^2\}\,\mathrm{Im}[(D_ru)\bar{u}]$$

$$\geqq\frac{1}{2}\{|D_ru|^2+(\lambda-V_0)|u|^2\}-2g(r)^2\,\mathrm{Im}[(D_ru)\bar{u}]-\frac{1}{2}\left(\sqrt[4]{\lambda-V_0(x)}+\frac{1}{16}\right)|u||D_ru|,$$

from which and (4.41) we obtain

$$(4.43)\qquad |\mathcal{D}_r^+u|^2\geqq\frac{1}{8}\{|D_ru|^2+(\lambda-V_0)|u|^2\}-2g(r)^2\,\mathrm{Im}[(D_ru)\bar{u}]\quad(|x|\geqq R_1).$$

Taking the imaginary part of

$$\int_{B(R)}(Lu-\lambda u)\bar{u}\,dx=0,$$

implies

$$\mathrm{Im}\int_{S(R)}(D_ru)\bar{u}\,dS=0,$$

which and (4.43) give

$$(4.44)\qquad \int_{S(R)}|\mathcal{D}_r^+u|^2dS\geqq\frac{1}{8}\int_{S(R)}\{|D_ru|^2+(\lambda-V_0)|u|^2\}dS$$

for $R>R_1$. On the other hand we see

$$\liminf_{R\to\infty}R^{2s-1}\int_{S(R)}|\mathcal{D}_r^+u|^2dS=0$$

in view of $\mathcal{D}_r^+u\in L_{s-1}^2$. Hence it follows from (4.44)

$$(4.45)\qquad \liminf_{R\to\infty}R^{2s-1}\int_{S(R)}\{|Du|^2+(\lambda-V_0)|u|^2\}dS=0.$$

It is well known under a weaker condition that every eigenfunction satisfying (4.45) vanishes identically in an exterior domain $E(R)$ (see, e.g., Uchiyama [24]). Thus we have $u=0$ by the unique continuation property (A.6). Q.E.D.

§5. Proof of Theorems in §3.

We start with the following Lemma.

Lemma 5.1. *Let z be a non-real number. Then the set*

$$\{Lu-zu\,;\ u\in C_0^\infty(\boldsymbol{R}^n)\}$$

is dense in L_t^2 for every $t\in\boldsymbol{R}$.

The above Lemma is a consequence of Lemma 4.5 (cf. Lemma 1.10 in Ikebe-Saitō [12], where a detailed proof is given).

Lemma 5.2. *Let* $\frac{1}{2}<s\leqq\min\left(1, \frac{1+\delta}{2}\right)$ *and* $f\in L_s^2$. *Suppose that* $\{u_m\}\subset C_0^\infty(\boldsymbol{R}^n)$ *and* $\{z_m\}\subset \boldsymbol{C}$ *satisfy*

(5.1) $$\{u_m\} \textit{ is bounded in } L^2_{V_0;-s},$$

(5.2) $$(L-z_m)u_m \longrightarrow f \quad \textit{strongly in} \quad L_s^2,$$

(5.3) $$\operatorname{Im} z_m>0, \qquad m=1, 2, \cdots$$
$$(\operatorname{Im} z_m<0,\ m=1, 2, \cdots),$$

(5.4) $$z_m \longrightarrow z_0, \qquad m\longrightarrow \infty.$$

Then there exist a subsequence $\{u_{m_k}\}$ *of* $\{u_m\}$ *and* $u_0\in H_{2,\mathrm{loc}}\cap L^2_{V_0;-s}$ *such that*

(5.5) $$u_{m_k} \longrightarrow u_0 \quad \textit{strongly in} \quad L^2_{V_0;-s},$$

(5.6) $$u_0 \textit{ is a solution of}$$
$$(L-z_0)u_0=f$$

and satisfies the outgoing (incoming) radiation condition.

Proof. The assumptions (5.1), (5.2) and Lemma 4.1 imply the numerical sequence

$$\left\{\|u_m\|_{L^2(B(R))}+\sum_{j=1}^n\left\|\frac{\partial}{\partial x_j}u_m\right\|_{L^2(B(R))}\right\}_{m=1,2,\cdots}$$

is bounded for each $R>0$. In view of Rellich's theorem (see, e.g., Mizohata [19], Theorem 3.3) we can choose a subsequence $\{u_{m_k}\}$ of $\{u_m\}$ and $u_0\in L^2_{\mathrm{loc}}$ such that

(5.7) $$u_{m_k} \longrightarrow u_0 \quad \text{in} \quad L^2_{\mathrm{loc}}.$$

The assumptions (5.2), (5.4) and Lemma 4.1 are again used to show that

(5.8) $$u_0\in H_{1,\mathrm{loc}}, \qquad u_{m_k} \longrightarrow u_0 \quad \text{in} \quad H_{1,\mathrm{loc}}$$

and that the limit function u_0 satisfies

(5.9) $$Lu_0-z_0u_0=f\in L_s^2$$

in the distribution sense. Making use of Lemma 3 in Ikebe-Kato [11] in view of (A.1), (A.2) and (5.9), we obtain that u_0 belongs to $H_{2,\mathrm{loc}}$. Noting $s\leqq(1+\delta)/2$ and, therefore,

$$\|u\|_{V_0;s-1-\delta}\leqq\|u\|_{V_0;-s},$$

we have from Lemma 4.3, (5.1) and (5.2)

$$\{\mathcal{D}(z_m)u_m\} \quad \text{is bounded in} \quad L^2_{s-1,E(R_0)}$$

for a sufficiently large R_0, which together with Lemma 4.4 and (5.7) shows

(5.10) $$u_{m_k} \longrightarrow u_0 \quad \text{strongly in} \quad L^2_{V_0;-s}.$$

Lemma 4.3, (5.2) and (5.10) are used to show that

$$\{\mathcal{D}(z_{m_k})u_{m_k}\} \text{ is a Cauchy sequence in } L^2_{s-1,E(R_0)},$$

which, (5.4) and (5.8) give

$$\mathcal{D}(z_{m_k})u_{m_k} \longrightarrow \mathcal{D}^+(z_0)u_0 \qquad (\mathcal{D}^-(z_0)u_0)$$

strongly in $L^2_{s-1,E(R_0)}$. Thus u_0 satisfies the outgoing (incoming) radiation condition, which was to be shown. Q.E.D.

Before showing Theorem 3.1 we prove the following Lemma.

Lemma 5.3. *Let* $s>\frac{1}{2}$. *Then for any real* a, b *there exists a positive constant* C *such that*

$$\|u\|_{V_0;-s} \leqq C\|(L-z)u\|_s$$

for any $z\in K^\pm(a, b)$ *and* $u\in C_0^\infty(\boldsymbol{R}^n)$.

Proof. Since it suffices to prove the assertion for a sufficiently small s, we may assume

$$\frac{1}{2}<s\leqq\min\left(1, \frac{1+\delta}{2}\right)$$

so that we can use Lemma 5.2. Suppose that the assertion is false. Then there would be a sequence $\{u_m\}\subset C_0^\infty(\boldsymbol{R}^n)$ and $\{z_m\}\subset K^+(a, b)$(or $\{z_m\}\subset K^-(a, b)$) such that

$$\|u_m\|_{V_0;-s}=1, \qquad (L-z)u_m \dashrightarrow 0 \quad \text{in} \quad L^2_s.$$

We may assume, without loss of generality, that z_m coverges to a complex number z_0. Therefore Lemma 5.2 enables us to choose a strongly convergent subsequnce $\{u_{m_k}\}$ in $L^2_{V_0;-s}$ such that the limit function u_0 satisfies

$$Lu_0=z_0u_0$$

and the outgoing (or incoming) radiation condition. In view of Lemma 4.5 (if z_0 is non-real) and Lemma 4.8 (if z_0 is real) we have $u_0=0$. This is a contradiction, since

$$1=\lim_{k\to\infty}\|u_{m_k}\|_{V_0;-s}=\|u_0\|_{V_0;-s}=0,$$

which completes the proof. Q.E.D.

Now we prove our Theoroms.

Proof of Theorem 3.1. Let $f\in L^2_s$. By virtue of Lemma 5.1 we can take a sequence $\{u_m\}\subset C_0^\infty(\boldsymbol{R}^n)$ such that

$$(L-z)u_m \longrightarrow f \quad \text{in} \quad L^2_s. \tag{5.11}$$

Lemma 5.3 shows that $\{u_m\}$ is a Cauchy sequence in $L^2_{V_0;-s}$ Let u be the limit function. Then u satisfies

$$(L-z)u=f$$

in the distribution sense. Therefore Lemma 3 of Ikebe-Kato [11] gives $u\in H_{2,\text{loc}}$ and, by (2.1) and $f\in L^2_s\subset L^2(R^n)$,

$$u\in D(H), \qquad (H-z)u=f.$$

Since z is non-real, we have

$$u=(H-z)^{-1}f.$$

Hence we obtain from Lemma 5.3

(5.12)
$$\begin{aligned}\|(H-z)^{-1}f\|_{V_0;-s}&=\|u\|_{V_0;-s}\\&=\lim_{m\to\infty}\|u_m\|_{V_0;-s}\leqq C\lim_{m\to\infty}\|(L-z)u_m\|_s\\&=C\|f\|_s.\end{aligned}$$

Since $u_m\to u$ in $L^2_{V_0;-s}$ and $(L-z)u_m\to(L-z)u$ in L^2_s, Lemma 4.1 yields that

(5.13) u_m converges in $H_{1,\mathrm{loc}}$ to u.

Lemma 4.3 and 5.3 are combined to obtain

(5.14)
$$\begin{aligned}\|\mathcal{D}u_m\|_{s-1,B(R,\rho)}&\leqq C\{\|u_m\|_{V_0;s-1-\delta}+\|(L-z)u_m\|_s\}\\&\leqq C\{\|u_m\|_{V_0;-s}+\|(L-z)u_m\|_s\}\\&\leqq C'\|(L-z)u_m\|_s\end{aligned}$$

for some positive constants C, C', R and any $\rho>R$. Making m tend in (5.14), we have from (5.11) and (5.13)

$$\|\mathcal{D}u\|_{s-1:B(R,\rho)}\leqq C'\|f\|_s$$

and, by making ρ tend to infinity,

$$\|\mathcal{D}u\|_{s-1:E(R)}\leqq C'\|f\|_s,$$

which and (5.12) complete the proof. Q.E.D.

Proof of Theorem 3.2. Let f be as in Theorem 3.2 and $\{z_m\}$ be a sequence such that $z_m\to\lambda$ and $\mathrm{Im}\,z_m>0\,(m=1,2,\cdots)$. Then put $v_m=(H-z_m)^{-1}f$. One can take $\{u_m\}\subset C_0^\infty(\boldsymbol{R}^n)$, by means of Lemma 5.1, such that

(5.15)
$$\|(L-z_m)u_m-f\|_s\leqq\frac{1}{m}.$$

In view of Theorem 3.1 and (5.15) we have

(5.16)
$$\begin{aligned}\|v_m-u_m\|_{V_0:-s}&=\|(H-z_m)^{-1}(f-(L-z_m)u_m)\|_{V_0:-s}\\&\leqq C\|f-(L-z_m)u_m\|_s\leqq\frac{C}{m}.\end{aligned}$$

Since $\{v_m\}$ is bounded in $L^2_{V_0:-s}$ from Theorem 3.1, $\{u_m\}$ is also bounded in $L^2_{V_0:-s}$ from (5.16). Therefore we use Lemma 5.2 to choose a convergent sequence $\{u_{m_k}\}$ in $L^2_{V_0:-s}$ with the limit u^+ satisfying the outgoing radiation condition and

$$(L-\lambda)u^+=f.$$

From (5.16) we see

$$v_{m_k}\longrightarrow u^+\quad\text{in}\quad L^2_{V_0:-s}.$$

It remains to show that $v_m \to u^+$ in $L^2_{V_0:-s}$ as $m\to\infty$. Otherwise, one would have a subsequence $\{v_{m'_k}\}$ of $\{v_m\}$ and a positive constant δ such that

$$\|v_{m'_k} - u^+\|_{V_0:-s} \geqq \delta. \tag{5.17}$$

Applying the same argument as shown we can choose a subsequence $\{v_{m''_k}\}$ of $\{v_{m'_k}\}$ and $v^+\in L^2_{V_0:-s}$ such that

$$v_{m''_k} \longrightarrow v^+ \quad \text{in} \quad L^2_{V_0:-s}, \qquad (L-\lambda)v^+=f \tag{5.18}$$

and v^+ satisfies the outgoing radiation condition. Since, by Lemma 4.8, the solution of $(L-\lambda)u=f$ satisfying the outgoing radiation condition is unique, v^+ must be equal to u^+. Hence, (5.18) contradicts to (5.17).

For another sequence $\{z'_m\}$ such that $z'_m\to\lambda(\operatorname{Im} z'_m>0,\ m=1, 2, \cdots)$, we can choose, by the same argument as above, $w^+\in L^2_{V_0:-s}$ satisfying $(L-\lambda)u=f$ and the outgoing the radiation condition such that

$$R(z'_m)f \longrightarrow w^+ \qquad \text{strongly in} \quad L^2_{V_0:-s}$$

Lemma 4.8 is again used to get $u^+=w^+$. Therefore we have

$$\lim_{m\to\infty} R(z_m)f=u^+=v^+=\lim_{m\to\infty} R(z'_m)f$$

and see that the limit is independent of the choice of $\{z_m\}$ such that $z_m\to\infty$.

Q. E. D.

Proof of Theorem 3.3. Theorem 3.2 shows that for every $f\in L^2_s$ and every closed interval $[a, b]$

$$R\Big(\lambda+\frac{i}{m}\Big)f \longrightarrow R(\lambda+i0)f \tag{5.19}$$

and

$$R\Big(\lambda-\frac{i}{m}\Big)f \longrightarrow R(\lambda-i0)f \tag{5.20}$$

in $L^2_{V_0:-s}$ as $m\to\infty$, uniformly for $\lambda\in[a, b]$. We shall show (5.19)(as (5.20) can be shown similarly). Otherwise, there would be $f\in L^2_s$, an interval $[a, b]$, a positive number ε_0 and numerical sequences $\{m_k\}$, $\{n_k\}$, $\{\lambda_k\}$ such that

$$\Big\|R\Big(\lambda_k+\frac{i}{m_k}\Big)f-R\Big(\lambda_k+\frac{i}{n_k}\Big)f\Big\|_{V_0:-s}\geqq\varepsilon_0, \tag{5.21}$$

$m_k\to\infty$, $m_k\leqq n_k$ and $\lambda_k\in[a, b]$. In view of the compactness of $[a, b]$ we may assume that λ_k converges to a real number λ_0. Then Theorem 3.2 gives

$$\lim_{k\to\infty} R\Big(\lambda_k+\frac{i}{m_k}\Big)f=\lim_{k\to\infty} R\Big(\lambda_k+\frac{i}{n_k}\Big)f=R(\lambda_0+i0)f,$$

which contradicts to (5.21). The uniform convergence of (5.19) and (5.20) shows the continuity of $R(\lambda\pm i0)f$ in $L^2_{V_0:-s}$ with respect to λ. Q. E. D.

§6. Spectral representation of H

We shall assume hereafter the following conditions stronger than (A.4) and (A.5);

(A.4)′ Each $b_j(x)$ is a real-valued C^2 function.
(A.5)′ There exists a positive constant δ such that

$$\frac{1}{2}<\delta\leqq 1 \quad \text{and}$$

(A.5.1)′ $V_1(x)/(-V_0(x))^{1/4}=O(|x|^{-1-\delta})$,

(A.5.2)′ $B_{jk}(x)=\partial_j b_k(x)-\partial_k b_j(x)=O(|x|^{-1-\delta})$,

$$\sum_{j=1}^{n}\partial_j B_{jk}(x)=O(|x|^{-(3/2)-\delta}),$$

(A.5.3)′ $(L_jV_0)/(-V_0(x))^{1/2}$

$$=(r\partial_j V_0-x_j\partial_r V_0)/(-V_0(x))^{1/2}=O(|x|^{-\delta}),$$

$$(L_j^2V_0)/(-V_0(x))^{1/2}=O(|x|^{(1/2)-\delta}),$$

(A.5.4)′ $$\frac{(\partial/\partial x)^\gamma V_0(x)}{V_0(x)}=O(|x|^{-|\gamma|}) \qquad (|\gamma|\leqq 2),$$

as $|x|\to\infty$, where $1\leqq j,\ k\leqq n$.

Lemma 6.1. *Assume the condition* (A) *with* (A.4)′ *and* (A.5)′ *Let* $\lambda\in \boldsymbol{R}$, $f\in L_1^2$ *and* $u=R(\lambda+i0)f$. *Then we have*

$$|\mathcal{D}^+u|\in L^2(E(R)),$$

where R is so large that $\mathcal{D}^+$ can be defined as seen in §3.

Proof. It follows from Lemma 4.3 (by putting $s=1$) that

$$\|\mathcal{D}v\|_{E(R)}\leqq C\{\|v\|_{V_0:-\delta}+\|(L-\lambda-i\varepsilon)v\|_1\} \tag{6.1}$$

for every $v\in C_0^\infty(\boldsymbol{R})$ and $0<\varepsilon\leqq 1$. It should be remarked that the condition (A.5.4) is used in Lemma 4.3 only to estimate

$$(4.14)=\frac{(n-1)(n-3)}{4r^2}-\frac{\partial_r^2 V_0}{4(z-V_0)}-\frac{5}{16}\Big(\frac{\partial_r V_0}{z-V_0}\Big)^2=O(r^{-1-\delta}), \quad \text{as} \quad r\to\infty.$$

The condition (A.5.4)′ shows that $(4.14)=O(r^{-2})$ and, therefore, $(4.14)=O(r^{-1-\delta})$ from $\delta\leqq 1$. Noting $\delta>\frac{1}{2}$ we have from Lemma 5.3

$$\|v\|_{V_0:-\delta}\leqq C\|(L-\lambda-i\varepsilon)v\|_1,$$

which and (6.1) imply

$$\|\mathcal{D}v\|_{E(R)}\leqq C'\|(L-\lambda-i\varepsilon)v\|_1$$

for every $v\in C_0^\infty(\boldsymbol{R}^n)$ and $0<\varepsilon\leqq 1$. Since the set

$$\{Lu-zu\,;\ u\in C_0^\infty(\boldsymbol{R}^n)\}\qquad (\mathrm{Im}\,z\neq 0)$$

is dense in L_1^2 from Lemma 5.1, it is seen, by the similar arguement to the proof of Theorem 3.1, that

$$\|\mathcal{D}R(\lambda+i\varepsilon)f\|_{E(R)}\leqq C'\|f\|_1. \tag{6.2}$$

for $0<\varepsilon\leqq 1$. Put $u_m=R\left(\lambda+\frac{i}{m}\right)f$. Then Theorem 3.2 shows

$$u_m\longrightarrow u$$

and

$$Lu_m-\lambda u_m=f+\frac{i}{m}u_m\longrightarrow f$$

strongly in $L^2_{V_0:-s}$. Then we have from Lemma 4.1

$$u_m\longrightarrow u\quad\text{in}\quad H_{1,\mathrm{loc}} \tag{6.3}$$

The inequality (6.2) yields

$$\|\mathcal{D}u_m\|_{B(R,\rho)}\leqq C'\|f\|_1 \tag{6.4}$$

for any $\rho>R$. Taking the limit as $m\rightarrow\infty$ in (6.4), we have from (6.3)

$$\|\mathcal{D}^+u\|_{B(R,\rho)}\leqq C'\|f\|_1$$

and, by making tend to infinity,

$$\|\mathcal{D}^+u\|_{E(R)}\leqq C'\|f\|_1.$$

Q. E. D.

Lemma 6.2. *Let* $\lambda\in\boldsymbol{R}$, $\frac{1}{2}<s\leqq 1$ *and* $f\in L_1^2$. *Put* $u=R(\lambda+i0)f$. *Then there exists a sequence* $\{R_m\}$ *such that* $R_m\rightarrow\infty$ *and*

$$\lim_{m\to\infty}\int_{S(R_m)}\{R_m^{1-2s}\sqrt{-V_0}\,|u|^2+R_m^{2s-1}|\mathcal{D}^+u|^2\}dS=0. \tag{6.5}$$

For any sequence $\{R_m\}$ *satisfying* (6.5) *the following limit*

$$\lim_{m\to\infty}\int_{S(R_m)}\sqrt{\lambda-V_0}\,|u|^2dS \tag{6.6}$$

exists and is equal to

$$\frac{1}{2i}\{\langle u,f\rangle-\overline{\langle u,f\rangle}\}=\mathrm{Im}[\langle u,f\rangle]$$

where

$$\langle u,v\rangle=\int_{R^n}u(x)\overline{v(x)}dx$$

for $u\in L^2_{-s}$ *and* $v\in L^2_s$.

Proof. Since $u\in L^2_{V_0:-s}$ from Theorem 3.2, and $|\mathcal{D}^+u|\in L^2_{s-1,E(R_0)}$ from the assumption, we have

$$\liminf_{R\to\infty} R\int_{S(R)}\{R^{-2s}\sqrt{-V_0}\,|u|^2+R^{2s-2}|\mathcal{D}^+u|^2\}dS=0,$$

Therefore, the existtence of $\{R_m\}$ satisfying (6.5) is insured. Green's formula and the definition (3.5) of $\mathcal{D}_r^+$ yield

$$\begin{aligned}(6.7)\qquad &\int_{B(R_m)}(u\bar{f}-\bar{u}f)dx\\ &=\int_{B(R_m)}\{u(-\overline{D^2u}+V\bar{u}-\lambda\bar{u})-(D^2u+Vu-\lambda u)\bar{u}\}dx\\ &=\int_{S(R_m)}\{(D_r u)\bar{u}-\overline{(D_r u)}\}dS\\ &=2i\int_{S(R_m)}\sqrt{\lambda-V_0}\,|u|^2dS+\int_{S(R_m)}\{(\mathcal{D}_r^+u)\bar{u}-\overline{(\mathcal{D}_r^+u)}u\}dS\end{aligned}$$

for a sufficiently large R_m. The condition (A.3) and (6.5) imply

$$(6.8)\qquad \lim_{m\to\infty}\int_{S(R_m)}|\mathcal{D}_r^+u|\cdot|u|dS\leqq\lim_{m\to\infty}\int_{S(R_m)}\{R_m^{1-2s}\sqrt{-V_0}\,|u|^2+R_m^{2s-1}|\mathcal{D}^+u|^2\}dS=0.$$

Thus, Lemma 6.8 follows from (6.7) and (6.8). Q.E.D.

Proposition 6.3. *Put*

$$A(x)=\sum_{j=1}^n\int_0^1 b_j(tx)x_j dt$$

and

$$\beta_j(x)=b_j(x)-\partial_j A(x).$$

Then we have

$$(6.9)\qquad \beta_j(x)=O(r^{-\delta}),$$

$$(6.10)\qquad \sum_{j=1}^n\partial_j\beta_j(x)=O(r^{-(1/2)-\delta})$$

as $r\to\infty$, *and*

$$(6.11)\qquad \sum_{j=1}^n x_j\beta_j(x)=0.$$

Proof. (6.9) and (6.10) follow from Iwatsuka [14], Proposition 2.1 by using (A.5.2)′ and

$$\partial_j A(x)=b_j(x)+\int_0^1\sum_{k=1}^n x_k B_{jk}(tx)t\,dt,$$

which and the skew-symmetricity of the matrix $\{B_{jk}\}$ give

$$\sum_{j=1}^n x_j\beta_j(x)=-\int_0^1\sum_{j,k=1}^n x_jx_kB_{jk}(tx)t\,dt=0.$$

Thus, (6.11) is proved. Q.E.D.

Definition 6.4. Let $\zeta(t)$ be a C^∞ function on $\boldsymbol{R}$ such that

$$\zeta(t)=\begin{cases}1, & t\geqq1,\\ 0, & t\leqq0.\end{cases}$$

Then we define

$$\Phi(x, \lambda)=\int_0^{|x|}\sqrt{\lambda-V_0(t\hat{x})}\,\zeta(\lambda-V_0(t\hat{x}))dt$$

and

$$v(x, \lambda, \phi)=\frac{1}{\sqrt{\pi}}r^{-(n-1)/2}e^{i\Phi(x,\lambda)}e^{-iA(x)}(\lambda-V_0(x))^{-1/4}\zeta(\lambda-V_0(x)-1)\zeta(r-R_0^*)\phi(\hat{x})$$

for a C^∞ function ϕ defined near the sphere S^{n-1}, where R_0^* is so large that $V_1(x)$ is bounded in $E(R_0^*)$.

Proposition 6.5. *Let* $\lambda\in \boldsymbol{R}$, $\frac{1}{2}<s<\delta$ *and* ϕ *be as in Definition* 6.4.

(6.12) $$v(x, \lambda, \phi)\in L^2_{V_0:-s},$$

(6.13) $$\mathscr{D}_j^+v(x, \lambda, \phi)=i[\beta_j(x)+\Psi_j(x, \lambda)]v(x, \lambda, \phi)+\frac{1}{r}v(x, \lambda, L_j\phi),$$

and

(6.14) $$\mathscr{D}_r^+v(x, \lambda, \phi)=0$$

for $|x|\geqq R_0^*+1$ *such that* $\lambda-V_0(x)\geqq 2$, *where*

$$L_j=r\partial_j-x_j\partial_r$$

and

$$\Psi_j(x, \lambda)=-\int_0^{|x|}\frac{1}{|x|}(L_jV_0)(t\hat{x})\left[\frac{\zeta(\lambda-V_0(t\hat{x}))}{2\sqrt{\lambda-V_0(t\hat{x})}}+\sqrt{\lambda-V_0(t\hat{x})}\,\zeta'(\lambda-V_0(t\hat{x}))\right]dt.$$

Moreover, we have

(6.15) $$\Psi_j(x, \lambda)=O(r^{-\delta}),\quad \partial_j\Psi_j(x, \lambda)=O(r^{-(1/2)-\delta})\quad \text{as } r\longrightarrow\infty\ (1\leqq j\leqq n).$$

(6.16) $$(L-\lambda)v(x, \lambda, \phi)\in L^2_s,$$
$$|\mathscr{D}^+v(x, \lambda, \phi)|\in L^2_{-s+\delta}.$$

Proof. (6.12) can be shown from (A.1), (A.2) as follows

$$\begin{aligned}|v(x, \lambda, \phi)|&=\frac{1}{\sqrt{\pi}}r^{-(n-1)/2}(\lambda-V_0(x))^{-1/4}\zeta(\lambda-V_0(x)-1)\zeta(r-R_0^*)|\phi(\hat{x})|\\ &\leqq C(\lambda)(-V_0(x))^{-1/4}(1+r)^{-(n-1)}|\phi(\hat{x})|\\ &\in L^2_{V_0:-s}\end{aligned}$$

for any $s>(1/2)$, where $C(\lambda)$ is independent of x and ϕ. Noting

$$\frac{\partial}{\partial x_j}\phi(\hat{x})=\frac{1}{r^2}(L_j\phi)(\hat{x}),$$

we have, by an elementary calculation

(6.17) $$\begin{aligned}D_jv&=(\partial_j+ib_j(x))v(x, \lambda, \phi)\\ &=\left[-\frac{n-1}{2r}\hat{x}_j+i\Psi_j(x, \lambda)+i\hat{x}_j\sqrt{\lambda-V_0(x)}-i\partial_jA(x)+ib_j(x)+\frac{\partial_jV_0(x)}{4(\lambda-V_0(x))}\right]v(x, \lambda, \phi)\\ &\quad+\frac{1}{r}v(x, \lambda, L_j\phi)\end{aligned}$$

for any x satisfying $|x|\geqq R_0^*+1$ and $\lambda-V_0(x)\geqq 2$. So, (6.13) follows from the definition (3.4) of $\mathscr{D}_j^+$ and Proposition 6.3. (6.14) is seen from

$$\sum_{j=1}^n \hat{x}_j\beta_j(x)=0$$

by (6.11), and

(6.18) $$\sum_{j=1}^n \hat{x}_j L_j=\sum_{j=1}^n (r\hat{x}_j\partial_{x_j}-\hat{x}_j x_j\partial_r)=r\partial_r-r\partial_r=0,$$

which gives

(6.19) $$\sum_{j=1}^n \hat{x}_j\Psi_j(x, \lambda)=0.$$

Put $\xi(t)=\sqrt{t}$. Then the choice of $\zeta(t)$ implies

$$\left|\frac{\zeta(t)}{2\xi(t)}+\zeta'(t)\xi(t)\right|\leqq\begin{cases} C, & t\leqq 1, \\ \dfrac{C}{\sqrt{t}}, & t\geqq 1, \end{cases}$$

for a positive constant C. Hence, we have, making use of (A.5.3)′ and (A.3),

$$\begin{aligned}|\Psi_j(x, \lambda)|&\leqq\int_0^R \frac{C}{|x|}|(L_jV_0)(t\hat{x})|\,dt+\int_R^{|x|}\frac{C}{|x|}|(L_jV_0)(t\hat{x})|(\lambda-V_0(t\hat{x}))^{-1/2}dt \\ &\leqq\frac{C'}{|x|}+\int_R^{|x|}\frac{C'}{|x|}(1+t)^{-\delta}dt=O(r^{-\delta})\end{aligned}$$

as $r\to\infty$, where C' is a positive constant, and R is a sufficiently large number such that

(6.20) $$\lambda-V_0(x)\geqq 2, \quad\text{and}\quad |x|\geqq R_0^*+1.$$

Similarly, one can estimate

$$\begin{aligned}\partial_j\Psi_j(x, \lambda)=&-\frac{x_j}{|x|^2}(L_jV_0)(x)\frac{1}{2\sqrt{\lambda-V_0(x)}}+\frac{x_j}{|x|^2}\Psi_j(x, \lambda) \\ &-\int_0^{|x|}\frac{1}{|x|^2}(L_j^2V_0)(t\hat{x})\left[\frac{\zeta}{2\xi}+\zeta'\xi\right](\lambda-V_0(t\hat{x}))dt \\ &+\int_0^{|x|}\frac{1}{|x|^2}[(L_jV_0)(t\hat{x})]^2\left[\frac{\zeta'}{2\xi}-\frac{\xi'\zeta}{2\xi^2}+\zeta''\xi+\zeta'\xi'\right](\lambda-V_0(t\hat{x}))dt \\ &\qquad (|x|\geqq R),\end{aligned}$$

using (A.5.3)′ and (A.3), as follows,

$$\begin{aligned}|\partial_j\Psi_j(x, \lambda)|&\leqq\frac{|(L_jV_0)(x)}{2|x|\sqrt{\lambda-V_0(x)}}+\frac{1}{|x|}|\Psi_j(x, \lambda)|+\frac{C''}{|x|^2}+\frac{C''}{|x|^2}\int_R^{|x|}[(1+t)^{(1/2)-\delta}+(1+t)^{-2\delta}]dt, \\ &=O(r^{-1-\delta})+O(r^{-1-\delta})+O(r^{-2})+O(r^{-(1/2)-\delta})+O(r^{-2\delta}) \quad \text{as} \quad r\to\infty,\end{aligned}$$

where C'' is a positive constant, which shows

$$\partial_j\Psi_j(x, \lambda)=O(r^{-(1/2)-\delta}) \quad \text{as} \quad r\to\infty$$

because of $\delta>(1/2)$. Finally, we shall prove (6.16). It follows from (4.2), (4.14), (6.13), (6.14) and Proposition 6.3 that

$$(L-\lambda)v(x,\lambda,\phi)=-\sum_{j=1}^{n}D_j\Big\{i[\beta_j(x)+\Psi_j(x,\lambda)]v(x,\lambda,\phi)+\frac{1}{r}v(x,\lambda,L_j\phi)\Big\}$$
$$+\Big[V_1(x)+\frac{(n-1)(n-3)}{4r^2}-\frac{\partial_r^2V_0}{4(\lambda-V_0)}-\frac{5}{16}\Big(\frac{\partial_rV_0}{\lambda-V_0}\Big)^2\Big]v(x,\lambda,\phi)$$

and, by noting (6.11), (6.17) and (6.18),

$$(6.21)\qquad (L-\lambda)v(x,\lambda,\phi)$$
$$=-\sum_{j=1}^{n}\Big\{\Big[i(\partial_j\Psi_j+\partial_j\beta_j)+i(\Psi_j+\beta_j)\Big(i\Psi_j+i\beta_j+\frac{\partial_jV_0}{4(\lambda-V_0)}\Big)\Big]v(x,\lambda,\phi)$$
$$+\Big[\frac{2i}{r}(\beta_j+\Psi_j)-\frac{\partial_jV_0}{4r(\lambda-V_0)}\Big]v(x,\lambda,L_j\phi)+\frac{1}{r^2}v(x,\lambda,L_j^2\phi)\Big\}$$
$$+\Big[V_1(x)+\frac{(n-1)(n-3)}{4r^2}-\frac{\partial_r^2V_0}{4(\lambda-V_0)}-\frac{5}{16}\Big(\frac{\partial_rV_0}{\lambda-V_0}\Big)^2\Big]v(x,\lambda,\phi)$$

for $|x|\geqq R$. The condition (A.5)′, (6.9), (6.10) and (6.15) show

$$\beta_j(x)=O(r^{-\delta}),\qquad \sum_{j=1}^{n}\partial_j\beta_j(x)=O(r^{-(1/2)-\delta}),$$
$$\Psi_j(x,\lambda)=O(r^{-\delta}),\qquad \partial_j\Psi_j(x,\lambda)=O(r^{-(1/2)-\delta}),$$
$$\frac{\partial_jV_0}{\lambda-V_0}=O(r^{-1}),\qquad V_1(x)/(-V_0(x))^{1/4}=O(r^{-1-\delta})$$

as $r\to\infty$, which suggests that all the coefficients of $v(x,\lambda,\phi)$, $v(x,\lambda,L_j\phi)$ and $v(x,\lambda,L_j^2,\phi)$ in (6.21) are

$$O(r^{-(1/2)-\delta}\sqrt[4]{-V_0(x)}),\qquad r\to\infty.$$

According to Proposition 6.5, $v(x,\lambda,\phi)$, $v(x,\lambda,L_j\phi)$ and $v(x,\lambda,L_j^2\phi)$ belong to $L^2_{V_0;-s}$ for every $s>(1/2)$. Hence, we obtain

$$(L-\lambda)v(x,\lambda,\phi)\in L_s^2$$

for every $s<\delta$, which shows the first assertion of (6.16). The second assertion of (6.16) follows from (6.10), (6.13) and (6.15). Q.E.D.

In view of Lemma 6.2 we obtain that for each $f\in L_1^2$

$$\{R_m^{(n-1)/2}\sqrt[4]{\lambda-V_0(R_m\cdot)}[R(\lambda+i0)f](R_m\cdot)\}_{m=1,2,\cdots}$$

is a bounded sequence in the Hilbert space $h=L^2(S^{n-1})$.

We shall show below that

$$\{R_m^{(n-1)/2}e^{-i\Phi(R_m\cdot,\lambda)}e^{iA(R_m\cdot)}\sqrt[4]{\lambda-V_0(R_m\cdot)}[R(\lambda+i0)f](R_m\cdot)\}_{m=1,2,\cdots}$$

is a Cauchy sequence in h.

Lemma 6.6. *Let* $\frac{1}{2}<s\leqq\frac{1+\delta}{2}$, $\lambda\in\boldsymbol{R}$, $f\in L_1^2$ *and* $u=R(\lambda+i0)f$. *Take a sequence* $\{R_m\}$ *as in* (6.5) *in Lemma* 6.2 *and put*

$$w_m=\frac{1}{\sqrt{\pi}}R_m^{(n-1)/2}e^{-i\Phi(R_m\cdot,\lambda)}e^{iA(R_m\cdot)}\sqrt[4]{\lambda-V_0(R_m\cdot)}u(R_m\cdot).$$

Then there exists a positive constant C such that

$$|(w_m-w_l,\phi)_h|\leqq C\|\phi\|_h\Big[\|f\|_{1,B(R_m,R_l)}+\|u\|_{V_0;-s,B(R_m,R_l)}+\|\mathcal{D}^+u\|_{B(R_m,R_l)}$$
$$+\sum_{j=m,l}\Big(\int_{S(R_j)}|\mathcal{D}_r^+u|^2dS\Big)^{1/2}\Big]+C\|\mathcal{D}^+u\|_{B(R_m,R_l)}$$
$$\times\Big[\sum_{j=1}^{n}R_m^{-1/2}\|(\lambda-V_0(R_m\cdot))^{-1/4}L_j\phi\|_h\Big],$$

for $\phi\in H_2(\Omega)$ and $R_l>R_m>R$, where Ω is an open neighborhood of S^{n-1} and R is the number satisfying (6.20).

Proof. It suffices to prove Lemma 6.6 for any $\phi\in C^\infty(S^{n-1})$. In view of Definition 6.4 we have

$$(w_m,\phi)_h=\int_{S(R_m)}\sqrt{\lambda-V_0}u\bar{v}\,dS,$$

for $R_m\geqq R_0$, where $v=v(x,\lambda,\phi)$ So, the same calculation as seen in (6.7) gives

$$2i(w_m,\phi)_h$$
$$=2i\int_{S(R_m)}\sqrt{\lambda-V_0}u\bar{v}\,dS=\int_{S(R_m)}\{u(\overline{\mathcal{D}_r^+v})-(\mathcal{D}_r^+u)\bar{v}\}dS+\int_{S(R_m)}\{(D_ru)\bar{v}-u(\overline{D_rv})\}dS$$
$$=\int_{S(R_m)}\{u(\overline{\mathcal{D}_r^+v})-(\mathcal{D}_r^+u)\bar{v}\}dS-\int_{B(R_m)}\{\bar{v}(L-\lambda)u-u(\overline{L-\lambda)v}\}dx.$$

(4.2) and (4.14) give us to obtain

$$(L-\lambda)v=-\sum_{j=1}^{n}D_j\mathcal{D}_j^+v+k^+(x,\lambda)\mathcal{D}_r^+v+\Big[V_1(x)+\frac{(n-1)(n-3)}{4r^2}-\frac{\partial_r^2V_0}{4(\lambda-V_0)}-\frac{5}{16}\Big(\frac{\partial_rV_0}{\lambda-V_0}\Big)^2\Big]v.$$

Therefore we have, noting (6.14) in Proposition 6.5,

(6.22)

$$2i(w_m,\phi)_h=-\int_{S(R_m)}(\mathcal{D}_r^+u)\bar{v}\,dS-\int_{B(R_m)}[-f\bar{v}+u(\overline{L-\lambda)v}]dx=-\int_{S(R_m)}(\mathcal{D}_r^+u)\bar{v}\,dS$$
$$-\int_{B(R_m)}\Big\{\Big[f\bar{v}+u\Big(\sum_{j=1}^{n}\overline{D_j\mathcal{D}_j^+v}\Big)\Big]-u\Big[V_1(x)+\frac{(n-1)(n-3)}{4r^2}$$
$$-\frac{\partial_r^2V_0}{4(\lambda-V_0)}-\frac{5}{16}\Big(\frac{\partial_rV_0}{\lambda-V_0}\Big)^2\Big]\bar{v}\Big\}dx.$$

Thus we have, by integration parts and by using $(L-\lambda)u=f$, the definition (3.5) of $\mathcal{D}_j^+$ and (6.14),

(6.23) $2i(w_m-w_l,\phi)_h$

$$=-\Big[\int_{S(R_l)}-\int_{S(R_m)}\Big](\mathcal{D}_r^+u)\bar{v}\,dS-\int_{B(R_m,R_l)}f\bar{v}\,dx+\sum_{j=1}^{n}\int_{B(R_m,R_l)}(\mathcal{D}_j^+u)(\overline{\mathcal{D}_j^+v})dx$$
$$+\int_{B(R_m,R_l)}u\Big[V_1+\frac{(n-1)(n-3)}{4r^2}-\frac{\partial_r^2V_0}{4(\lambda-V_0)}-\frac{5}{16}\Big(\frac{\partial_rV_0}{\lambda-V_0}\Big)^2\Big]\bar{v}\,dx$$

$$=-I_1-I_2+I_3+I_4\,.$$

These terms are estimated as follows. It follows from Definition 6.4 that

$$|v|=|v(x,\lambda,\phi)|\leqq\frac{1}{\sqrt{\pi}}r^{-(n-1)/2}|\phi(\hat{x})|$$

for $|x|\geqq R_0$, which yields

$$\int_{S(R)}|v|^2dS\leqq\frac{1}{\pi}\|\phi\|_h^2\quad(R\geqq R_0+1)$$

and, by Schwarz inequality,

$$|I_1|\leqq\frac{1}{\sqrt{\pi}}\sum_{j=m,l}\left(\int_{S(R_j)}|\mathcal{D}_r^+u|^2dS\right)^{1/2}\|\phi\|_h\,. \tag{6.24}$$

Proposition 6.5 and the proof show

$$\|v\|_{V_0;-s}\leqq C\|\phi\|_h \tag{6.25}$$

from which and (3.8) we have, by noting $\frac{1}{2}<s\leqq\frac{1+\delta}{2}\leqq1$

$$|I_2|=\left|\int_{B(R_m,R_l)}f\bar{v}\,dx\right|\leqq\|f\|_{s,B(R_m,R_l)}\|v\|_{V_0;-s}\leqq C\|f\|_{1,B(R_m,R_l)}\|\phi\|_h\,. \tag{6.26}$$

It follows from (6.9), (6.13), (6.15), (A.3) and (A.5.3)′ that

$$\begin{aligned}|I_3|&\leqq\sum_{j=1}^{n}\|\mathcal{D}_j^+v\|_{B(R_m,R_l)}\|\mathcal{D}_j^+u\|_{B(R_m,R_l)}\\&\leqq\|\mathcal{D}^+u\|_{B(R_m,R_l)}\left(C\|v\|_{V_0;-\delta}+\sum_{j=1}^{n}\left\|\frac{v(x,\lambda,L_j\phi)}{r}\right\|_h\right)\\&\leqq\|\mathcal{D}^+u\|_{B(R_m,R_l)}\left\{C'\|\phi\|_h+\sum_{j=1}^{n}\left(\int_{B(R_m,R_l)}\frac{|(L_j\phi)(x)|^2dx}{\pi r^{n+1}(\lambda-V_0(x))^{1/2}}\right)^{1/2}\right\}\\&\leqq C''\|\mathcal{D}^+u\|_{B(R_m,R_l)}\left\{\|\phi\|_h+R_m^{-1/2}\sum_{j=1}^{n}\|(\lambda-V_0(R_m\cdot))^{-1/4}(L_j\phi)\|_h\right\}\end{aligned} \tag{6.27}$$

where C, C' and C'' are some positive constants. Since the conditions (A.5.1)′ and (A.5.4)′

$$\begin{aligned}&\frac{(n-1)(n-3)}{4r^2}-\frac{\partial_r^2V_0}{4(\lambda-V_0)}-\frac{5}{16}\left(\frac{\partial_rV_0}{\lambda-V_0}\right)^2=O(r^{-2}),\\&\frac{V_1(x)}{\sqrt[4]{-V_0(x)}}=O(r^{-1-\delta})\end{aligned} \tag{6.28}$$

as $r\to\infty$, we obtain from (3.8), (6.25) and $s\leqq\frac{1+\delta}{2}$ that

$$\begin{aligned}|I_4|&\leqq\|u\|_{V_0;-(1+\delta)/2,B(R_m,R_l)}\|v\|_{-(1+\delta)/2}\\&\leqq C\|u\|_{V_0;-s,B(R_m,R_l)}\|\phi\|_h\end{aligned} \tag{6.29}$$

for $R_l>R_m\geqq R_0$.

Gathering (6.23), (6.24), (6.26), (6.27) and (6.29) completes the proof of Lemma 6.6.

The following Lemma assures the strong convergence of $\{w_m\}$ in $h=L^2(S^{n-1})$.

Lemma 6.7. *Let* $f\in L_1^2$ *and Let* $\lambda, s, u, \{R_m\}$ *and* $\{w_m\}$ *be as in Lemma* 6.2 *and* 6.6. *Then the sequence* $\{w_m\}$ *converges strongly in* h *to an element* $w_\infty\in h$. *The limit* w_∞ *is determined uniquely for each* $f\in L_1^2$ *and does not depend on the choice of* $\{R_m\}$ *satisfying* (6.7).

Proof. Since Lemma 6.2 and 6.6 imply that $\{w_m\}$ is bounded in h and $\{(w_m, \phi)_h\}$ is a Cauchy sequence for any $\phi\in C^\infty(S^{n-1})$ which is dense in h. So $\{(w_m, \psi)_h\}$ is also a Cauchy sequence for every $\psi\in h$. Therefore $\{w_m\}$ has a weak limit w_∞ in h (e.g., Kato [17], III-1-6). We shall see below the strong convergence of $\{w_m\}$ by making use of Lemma 6.6.

The condition (A.5.4)$'$, (6.9) and (6.15) show, by a similar calculation to the proof of (6.13),

$$
\begin{aligned}
&|(L_j w_m)(\omega)|\\
&=\left|\frac{1}{\sqrt{\pi}}R_m^{(n-1)/2}L_j\{e^{-i\Phi(R_m\cdot,\lambda)}e^{iA(R_m\cdot)}\sqrt[4]{\lambda-V_0(R_m\cdot)}u(R_m\cdot)\}(\omega)\right|\\
&=\left|R_m\left[-i\Psi_j(R_m\omega,\lambda)+i(\partial_j A)(R_m\omega)-i\omega_j(\partial_r A)(R_m\omega)\right.\right.\\
&\qquad\left.-\frac{1}{4}\frac{(\partial_j V_0)(R_m\omega)-\omega_j(\partial_r V_0)(R_m\omega)}{\lambda-V_0(R_m\omega)}\right]w_m\\
&\quad\left.+\frac{1}{\sqrt{\pi}}R_m^{(n+1)/2}e^{-i\Phi(R_m\omega,\lambda)}e^{iA(R_m\omega)}\sqrt[4]{\lambda-V_0(R_m\omega)}[(\partial_j u)(R_m\omega)-\omega_j(\partial_r u)(R_m\omega)]\right|\\
&\leqq C R_m\sqrt[4]{\lambda-V_0(R_m\omega)}[R_m^{-\delta}|w_m|+R_m^{(n-1)/2}|(D_j u)(R_m\omega)-\omega_j(D_r u)(R_m\omega)|]\quad(\omega\in S^{n-1}),
\end{aligned}
$$

for $R_m\geqq R_0$, where C is independent of m. The above inequality yields

$$
\begin{aligned}
(6.30)\quad&\sum_{j=1}^n R_m^{-1/2}\|(\lambda-V_0(R_m))^{-1/4}L_j w_m\|_h\\
&\leqq C' R_m^{(1/2)-\delta}\left(\int_{S(R_m)}\sqrt{\lambda-V_0}|u|^2dS\right)^{1/2}+\left(\sum_{j=1}^n\int_{S(R_m)}r|(D_j u)-\hat{x}_j(D_r u)|^2dS\right)^{1/2},
\end{aligned}
$$

where C' is some constant independent of m. Noting

$$
\begin{aligned}
&\sum_{j-1}^n|(D_j-x_jD_r)u|^2\\
&\leqq\sum_{j=1}^n|(D_j-x_jD_r)u|^2+|\hat{x}_j\mathcal{D}_r^+u|^2=\sum_{j=1}^n|(D_j-\hat{x}_jD_r+\hat{x}_j\mathcal{D}_j^+u|^2=|\mathcal{D}^+u|^2
\end{aligned}
$$

by means of $\sum_{j=1}^n\hat{x}_j(D_j-x_jD_r)=0$ and (3.5), we have from Lemma 6.2, 6.6 and (6.30) we have

$$
(6.31)\qquad |(w_m-w_l, w_m)_h|\leqq\varepsilon_{ml}
$$

for $m\to\infty$, where

$$
\begin{aligned}
\varepsilon_{ml}=C\|w_m\|_h\Big[&\|f\|_{1,B(R_m,R_l)}+\|u\|_{V_0;-s,B(R_m,R_l)}+\|\mathcal{D}^+u\|_{B(R_m,R_l)}\\
&+\left(\int_{S(R_m)}|\mathcal{D}_r^+u|^2dS\right)^{1/2}+\left(\int_{S(R_l)}|\mathcal{D}_r^+u|^2dS\right)^{1/2}\Big]
\end{aligned}
$$

$$+\|\mathscr{D}^+u\|_{B(R_m,R_l)}\Big[R_m^{(1/2)-\delta}\Big(\int_{S(R_m)}\sqrt{\lambda-V_0}|u|^2dS\Big)^{1/2}+\Big(\int_{S(R_m)}r|\mathscr{D}_r^+u|^2dS\Big)^{1/2}\Big],$$

Recalling (6.5), $s\leqq\delta$, $f\in L_1^2$, $u\in L_{V;-s}^2$, $|\mathscr{D}^+u|\in L^2(E(R_0))$, (by Lemma 6.1) and the boundness of $\{w_m\}$ in h (by Lemma 6.2), one can see that the limit $\lim_{l\to\infty}\varepsilon_{ml}$ exists for every m, and

$$\lim_{l\to\infty}\lim_{l\to\infty}\varepsilon_{ml}=0.$$

Therefore, it follows from (6.31) that

$$|(w_m-w_\infty,\ w_m)_h|\leqq(\lim_{l\to\infty}\varepsilon_{ml})\longrightarrow 0\quad(\text{as } m\to\infty)$$

and, by the weak convergence of $\{w_m\}$,

$$\|w_m-w_\infty\|_h^2=(w_m-w_\infty,\ w_m-w_\infty)_h=(w_m-w_\infty,\ w_m)_h-(w_m-w_\infty,\ w_\infty)_h\longrightarrow 0,$$

which shows the strong convergence of $\{w_m\}$. Finally, we shall prove that the limit w_∞ does not depend on the choice of $\{R_m\}$ satisfying (6.5). (6.22) and (6.23) imply

$$2i(w_m,\ \phi)_h=-\int_{S(R_m)}(\mathscr{D}_r^+u)\bar{v}\,dS+\int_{B(R_m)}\Big\{-f\bar{v}+\sum_{j=1}^{n}(\mathscr{D}_j^+u)\overline{(\mathscr{D}_j^+v)}$$
$$+u\Big[V_1+\frac{(n-1)(n-3)}{4r^2}-\frac{\partial_r^2V_0}{4(\lambda-V_0)}-\frac{5}{16}\Big(\frac{\partial_rV_0}{\lambda-V_0}\Big)^2\Big]\bar{v}\Big\}dx,\tag{6.32}$$

where $v=v(x,\lambda,\phi)$. In the right-hand side of (6.32) the first term tends to 0 as $m\to\infty$, and the integrand of the second term is integrable over R^n, as seen in the proof of Lemma 6.6. Therefore, one can get

$$2i(w_\infty,\ \phi)_h=2i\lim_{m\to\infty}(w_m,\ \phi)_h=\int_{R^n}\Big\{-f\bar{v}+\sum_{j=1}^{n}(\mathscr{D}_j^+u)\overline{(\mathscr{D}_j^+v)}$$
$$+u\Big[V_1+\frac{(n-1)(n-3)}{4r^2}-\frac{\partial_r^2V_0}{4(\lambda-V_0)}-\frac{5}{16}\Big(\frac{\partial_rV_0}{\lambda-V_0}\Big)^2\Big]\bar{v}\,dx\tag{6.33}$$

for any ϕ in a dense set $C^\infty(S^{n-1})$ in h. The right-hand side of (6.33) is independent of the choice of $\{R_m\}$. So w_∞ is also independent of the choice of $\{R_m\}$. Q. E. D.

Definition 6.8. Let $\lambda\in R$ and $f\in L_1^2$.

$$F(\lambda)f=s-\lim_{m\to\infty}w_m$$
$$=s-\lim_{m\to\infty}\frac{1}{\sqrt{\pi}}R_m^{(n-1)/2}e^{-i\Phi(R_m\cdot,\lambda)}e^{iA(R_m\cdot)}\sqrt[4]{\lambda-V_0(R_m\cdot)}(R(\lambda+i0)f)(R_m\cdot),$$

where $s-\lim_{m\to\infty}$ denotes the limit in $h=L^2(S^{n-1})$.

Let E be the spectral measure of the self-adjoint operator H. Then we have

Lemma 6.9. *Let $f\in L_1^2$ and $I=(a,b)$. Then we have*

$$(E(I)f,\ f)_{L^2}=\int_a^b\|F(\lambda)f\|_h^2d\lambda$$

and

$$(F(\lambda)f,\ \phi)_{\boldsymbol{h}}=\frac{1}{2i}\{\langle u,\ (L-\lambda)v\rangle-\overline{\langle v,\ f\rangle}\}$$

for any $\phi\in C^{\infty}(S^{n-1})$, *where* $v=v(x,\ \lambda,\ \phi)$ *and* $u_{\lambda}=R(\lambda+i0)f$.

Proof. In view of Lemma 6.2 and Definition 6.8 we obtain

$$\begin{aligned}\text{(6.34)}\qquad \operatorname{Im}[\langle u_{\lambda},\ f\rangle]&=\lim_{m\to\infty}\int_{S(R_m)}\sqrt{\lambda-V_0}\,|u_{\lambda}|^2dS\\ &=\lim_{m\to\infty}\|w_m\|_{\boldsymbol{h}}^2=\|F(\lambda)f\|_{\boldsymbol{h}}^2.\end{aligned}$$

Therefore, Corollary 3.4 enables us to obtain

$$\begin{aligned}(E(I)f,\ f)_{L^2}&=\frac{1}{2\pi i}\int_a^b\{\langle u_{\lambda},\ f\rangle-\overline{\langle u_{\lambda},\ f\rangle}\}d\lambda\\ &=\int_a^b\|F(\lambda)f\|_{\boldsymbol{h}}^2d\lambda.\end{aligned}$$

The second identity follows from (6.22) in the proof of Lemma 6.6, by making m tend to infiuity. Q.E.D.

Definition 6.10. For each $f\in L_1^2$ define and h-valued function Ff on $\boldsymbol{R}$ such that

$$(Ff)(\lambda)=F(\lambda)f.$$

For an interval $I=(a,\ b)$, $L^2(I;\ \boldsymbol{h})$ denotes the Hilbert space of all h-valued square integrable functions on I.

Theorem 6.11. *For any* $f\in L_1^2$, Ff *is an* h-*valued strongly continuous function on* $\boldsymbol{R}$. *For any* $f\in C_0^{\infty}(\boldsymbol{R}^n)$ *we have*

$$(FHf)(\lambda)=\lambda(Ff)(\lambda).$$

Proof. Let $f\in L_1^2$ and $\lambda_0\in\boldsymbol{R}$ and take an arbitrary sequence $\{\lambda_m\}$ converging to λ_0. Since

$$\text{(6.35)}\qquad \|F(\lambda_m)f\|_{\boldsymbol{h}}^2=\frac{1}{2\pi i}\langle R(\lambda_m+i0)f,\ f\rangle-\overline{\langle R(\lambda_m+i0)f,\ f\rangle}$$

as seen in (6.34), the right-hand side of (6.35) tends to $\|F(\lambda_0)f\|_{\boldsymbol{h}}^2$ in view of Theorem 3.3. Using the identity

$$\|F(\lambda_m)f-F(\lambda_0)f\|_{\boldsymbol{h}}^2=\|F(\lambda_m)f\|_{\boldsymbol{h}}^2-2\operatorname{Re}(F(\lambda_m)f,\ F(\lambda_0)f)_{\boldsymbol{h}}+\|F(\lambda_0)f\|_{\boldsymbol{h}}^2,$$

in order to show $F(\lambda_m)f\to F(\lambda_0)f$ strongly in h, we have only to prove the weak convergence. Moreover, to this end it suffices to show for $\phi\in C^{\infty}(S^{n-1})$

$$\text{(6.36)}\qquad (F(\lambda_m)f,\ \phi)_{\boldsymbol{h}}\longrightarrow(F(\lambda_0)f,\ \phi)_{\boldsymbol{h}}.$$

because of the boundness of $\{F(\lambda_m)f\}$ in h. Set $u_m=R(\lambda_m+i0)f$, $v_m=v(x,\ \lambda_m,\ \phi)$ and $g_m(x)=[(L-\lambda_m)v_m](x)$ $(m=0,\ 1,\ \cdots)$. Then, Definition 6.4 and (6.21) imply that $\lim_{m\to\infty}v_m(x)=v_0(x)$ and $\lim_{m\to\infty}g_m(x)=g_0(x)$ for each $x\in\boldsymbol{R}^n$. It follows from Proposition 6.5 and the proof that there exists a positive constant $C=C(\phi)$ such that

$$|v_m(x)|\leqq C(1+r)^{-(n-1)/2}, \qquad |g_m(x)\leqq C(1+r)^{-(n/2)-1-\delta},$$

which shows, by Lebesgue's dominated convergence theorem,

$$v_m \longrightarrow v_0 \text{ in } L^2_{-s}, \qquad g_m \longrightarrow g_0 \text{ in } L^2_s \tag{6.37}$$

for any $s>(1/2)$ such that $s<\delta$. Lemma 6.9 shows

$$2i(F(\lambda_m)f, \phi)_h=\langle u_m, g_m\rangle-\overline{\langle v_m, f\rangle}$$

and, therefore,

$$2i(F(\lambda_m)f, \phi)_h=2i(F(\lambda_0)f, \phi)_h=\langle u_m, (g_m-g_0)\rangle+\langle(u_m-u_0), g_0\rangle-\overline{\langle(v_m-v_0), f\rangle}.$$

Since $u_m\rightarrow u_0$ in L^2_{-s} from Theorem 3.3, we have (6.36) from (6.37).

Let $f\in C_0^\infty(\boldsymbol{R}^n)$. Since Hf belongs to L^2 with compact support, we have $Hf\in L^2_1$. Put $g=(H-\lambda)f$ and $u=R(\lambda+i0)g$. Then, u is a solution of $(L-\lambda)u=g$ and satisfies the outgoing ratiation condition in view of Theorem 3.2. The compactness of the support of f implies that f satisfies the outgoing radiation condition. Therefore, u must coincide with f by virtue of Lemma 4.8. Then we have

$$R(\lambda+i0)(Hf-\lambda f)=f,$$

which shows

$$\begin{aligned}(FHf)(\lambda)&=s-\lim_{m\to\infty}\frac{1}{\sqrt{\pi}}R_m^{(n-1)/2}e^{-i\Phi(R_m\cdot,\lambda)}e^{iA(R_m\cdot)}\sqrt[4]{\lambda-V_0(R_m\cdot)}(R(\lambda+i0)Hf)(R_m\cdot)\\&=\lambda(Ff)(\lambda),\end{aligned}$$

since the support of f is compact. Q.E.D.

Theorem 6.12. *F defined in Definition 6.10 can be uniquely extended to an isometric operator $\mathcal{F}$ on $L^2(\boldsymbol{R}^n)$ to $\hat{H}=L^2(\boldsymbol{R}, h)$. For $f\in D(H)$ we have*

$$(\mathcal{F}Hf)(\lambda)=\lambda(\mathcal{F}f)(\lambda), \quad a.e.\ \lambda\in\boldsymbol{R}.$$

Proof. Lemma 6.9 and Theorem 6.11 show

$$Ff\in\hat{H}, \qquad \|Ff\|_{\hat{H}}=\|f\|_{L^2}$$

for any $f\in L^2_1$. Since L^2_1 is dense in $L^2(\boldsymbol{R}^n)$, F can be uniquely extended to an isometric operator on $L^2(\boldsymbol{R}^n)$ to H. According to Ikebe-Kato [11], L defined on $C_0^\infty(\boldsymbol{R}^n)$ is essentially self-adjoint under our condition. Therefore for any $f\in D(H)$ we can choose a sequence $\{f_m\}\subset C_0^\infty(R^n)$ such that $f_m\rightarrow f$, $Hf_m\rightarrow Hf$ in L^2. Then we have from 6.11 that

$$\mathcal{F}Hf_m\longrightarrow\mathcal{F}Hf \text{ in } \hat{H},$$

$$\mathcal{F}f_m\longrightarrow\mathcal{F}f \text{ in } \hat{H}$$

and

$$\begin{aligned}(\mathcal{F}Hf_m)(\lambda)&=(FHf_m)(\lambda)=\lambda(Ff_m)(\lambda)\\&=\lambda(\mathcal{F}f_m)(\lambda),\end{aligned}$$

Thus, one obtaius

$$(\mathcal{F}Hf)(\lambda)=\lambda(\mathcal{F}f)(\lambda) \quad \text{a.e. } \lambda\in\boldsymbol{R}.$$

Q.E.D.

§ 7. The unitarity of $\mathcal{F}$

In this section we shall prove that the isometric operator $\mathcal{F}$ defined in Theorem 6.12 is unitary, that is,

$$\mathcal{F} \text{ maps } L^2(\boldsymbol{R}^n) \text{ onto } \hat{H}=L^2(\boldsymbol{R}, \boldsymbol{h}) \tag{7.1}$$

Lemma 7.1. *Let $I=(a, b)$ be an open interval. Then we have*

$$(\mathcal{F} E(I) f)(\lambda)=\chi_I(\lambda)(\mathcal{F} f)(\lambda),$$

for any $f\in L^2$, where $\chi_I(\lambda)$ is the characteristic function of I.

Proof. Lemma 6.9 and Theorem 6.12 imply

$$\|\mathcal{F} E(I) f\|_{L^2}^2=\int_I \|(\mathcal{F} f)(\lambda)\|_{\boldsymbol{h}}^2 d\lambda \tag{7.2}$$

for any $f\in L^2$ and any open interval I, since L_1^2 is dense in L^2. Take any interval open interval $B\subset I$. Then (7.2) gives

$$\int_B \|(\mathcal{F} E(I) f)(\lambda)-(\mathcal{F} f)(\lambda)\|_{\boldsymbol{h}}^2 d\lambda=\|E(B)(E(I)-1) f\|_{L^2}^2=\|E(B) f-E(B) f\|_{L^2}=0. \tag{7.3}$$

On the other hand, for any open interval B' included in the complement I^c of I, we have

$$\int_{B'} \|(\mathcal{F} E(I) f)(\lambda)\|_{\boldsymbol{h}}^2 d\lambda=\|E(B') E(I) f\|_{L^2}^2=0. \tag{7.4}$$

Therefore, (7.3) and (7.4) give

$$(\mathcal{F} E(I) f)(\lambda)\begin{cases} =(\mathcal{F} f)(\lambda), & \lambda\in I, \\ =0, & \lambda\in I^c \end{cases}$$

Q. E. D.

Lemma 7.2. *Let $\hat{f}\in\hat{H}=L^2(\boldsymbol{R}, \boldsymbol{h})$ and $I=(a, b)$. Then we have*

$$((\mathcal{F} E(I))^* f, g)_{L^2}=\int_I \langle F(\lambda)^*\hat{f}(\lambda), g\rangle d\lambda$$

for any $g\in L_1^2$, where A^* denotes the adjoint operator of A ($F(\lambda)^*$ maps $\boldsymbol{h}=L^2(S^{n-1})$ into the dual space L_{-1}^2 of L_1^2).

Proof. It follows from Definition 6.10, Theorem 6.11 and Lemma 7.1 that

$$\begin{aligned} ((\mathcal{F} E(I))^* f, g)_{L^2} &= (f, \mathcal{F} E(I) g)_H \\ &=\int_I (\hat{f}(\lambda), F(\lambda) f)_{\boldsymbol{h}} d\lambda \\ &=\int_I \langle F(\lambda)^*\hat{f}(\lambda), g\rangle d\lambda. \end{aligned}$$

Q. E. D.

Lemma 7.3. *Take $s>\frac{1}{2}$ such that $s-\frac{1}{2}$ is so small as in Theorem* 3.1. *Then, for* *each* $\lambda\in\boldsymbol{R}$, $F(\lambda)$ *defined in Definition* 6.8 *can be uniquely extended to a bounded operator on* L_s^2 *to* $h=L^2(S^{n-1})$. *We denote the bounded operator by* $\tilde{F}(\lambda)$. *Moreover, if we choose* $\{R_m\}$ *satisfying* (6.5) *for each* $f\in L_s^2$, *we have*

$$(\tilde{F}(\lambda)f,\ \phi)_h=\lim_{m\to\infty}(w_m,\ \phi)_h,$$

for any $\phi\in C^\infty(S^{n-1})$, *where*

$$w_m=\frac{1}{\sqrt{\pi}}R_m^{(n-1)/2}e^{-i\Phi(R_m\cdot,\lambda)}e^{iA(R_m\cdot)}\sqrt[4]{\lambda-V_0}(R_m\cdot)(R(\lambda+i0)f)(R_m\cdot)$$

as in Lemma 6.6.

Proof. Let $f\in L_s^2$ and take $\{f_k\}\subset L_1^2$ such that $f_k\to f$ in L_s^2. For any $\phi\in C^\infty(S^{n-1})$, we have, in view of Lemma 6.9,

$$(F(\lambda)f_k,\ \phi)_h=\frac{1}{2i}\{\langle u_k,\ (L-\lambda)v\rangle-\overline{\langle v,\ f_k\rangle}\},$$

where $u_k=R(\lambda+i0)f_k$ and $v=v(x,\ \lambda,\ \phi)$ as in Definition 6.4. Set $u=R(\lambda+i0)f$. Since Theorems 3.1 and 3.2 imply

$$\|R(\lambda+i0)g\|_{V_0;-s}\leqq C\,\|g\|_s$$

for any $g\in L_s^2$, we have $u_k\to u$ strongly in $L_{V_0;-s}^2$. Noting $u\in L_{V_0;-s}^2$ and $(L-\lambda)v\in L_s^2$ by virtue of Proposition 6.5, we have

$$\lim_{k\to\infty}(F(\lambda)f_k,\ \phi)_h=\frac{1}{2i}\{\langle u,\ (L-\lambda)v\rangle-\overline{\langle v,\ f\rangle}\}. \tag{7.5}$$

Let us take $\{R_m\}$ satisfying (6.5). Then Lemma 6.2 and Theorem 3.1 show

$$\begin{aligned}\lim_{m\to\infty}\|w_m\|_h^2&=\lim_{m\to\infty}\int_{S(R_m)}\sqrt{\lambda-V_0}\,|u|^2\,dS\\&=\mathrm{Im}[\langle u,\ f\rangle]\leqq\|u\|_{-s}\|f\|_s\\&\leqq C\,\|f\|_s^2\end{aligned} \tag{7.6}$$

The same argument as in the beginning of the proof of Lemma 6.6 leads us to obtain

$$\begin{aligned}&\frac{1}{2i}\{\langle u,\ (L-\lambda)v\rangle-\overline{\langle v,\ f\rangle}\}\\&=\lim_{m\to\infty}\frac{1}{2i}\int_{B(R_m)}\{u(L-\lambda)v-f\bar{v}\}\,dx\\&=\lim_{m\to\infty}\int_{S(R_m)}\sqrt{\lambda-V_0}\,u\,\bar{v}\,dS\\&=\lim_{m\to\infty}(w_m,\ \phi)_h.\end{aligned} \tag{7.7}$$

Thus we obtain from (7.5), (7.6), (7.7) that

$$|\lim_{k\to\infty}(F(\lambda)f_k,\ \phi)_h|=|\lim_{m\to\infty}(w_m,\ \phi)_h|\leqq C\|\phi\|_h\|f\|_s, \tag{7.8}$$

for any $\phi\in C^\infty(S^{n-1})$. Since $C^\infty(S^{n-1})$ is dense in h, (7.8) shows the existence of the weak limit of $F(\lambda)f_k$ as $k\to\infty$. The weak limit is independent of the choice of $\{R_m\}$

in view of (7.5). We denote the weak limit by $\tilde{F}(\lambda)f$. Then, we have from (7.8)

$$\|\tilde{F}(\lambda)f\|_h \leq C\|f\|_s .$$

which shows the boundedness of $\tilde{F}(\lambda)$ on L_s^2 to h. In view of (7.5) we have

$$(\tilde{F}(\lambda)f, \phi)_h = \frac{1}{2i}\{\langle u, (L-\lambda)v\rangle - \overline{\langle v, f\rangle}\}$$

which and Lemma 6.9 give that $\tilde{F}(\lambda)$ coincide with $F(\lambda)$ on L_1^2. Thus we complete the proof. Q. E. D.

Theorem 7.4. *The isometric operator $\mathcal{F}$ is unitary on $L^2(\boldsymbol{R}^n)$ onto $\hat{H}=L^2(\boldsymbol{R}, h)$.*

Proof. In order to prove that $\mathcal{F}$ maps L^2 onto $\hat{H}$, it suffices to show that the null $N(\mathcal{F}^*)$ consists of zero vector 0 only, that is,

$$\mathcal{F}^*\hat{f}=0 \quad (\hat{f}\in\hat{H}) \quad \text{implies } \hat{f}=0 .$$

Let $\hat{f}\in N(\mathcal{F}^*)$ and I be an open interval in $\boldsymbol{R}$. Then, using Lemma 7.2 we have

$$\begin{aligned}(7.9) \qquad 0 &= (\mathcal{F}^*\hat{f}, E(I)g)_{L^2} = (\hat{f}, \mathcal{F}(E(I)g))_{\hat{H}} \\ &= ((\mathcal{F}E(I))^*\hat{f}, g)_{L^2} \\ &= \int_I \langle F(\lambda)^*\hat{f}(\lambda), g\rangle\, d\lambda\end{aligned}$$

for any $g\in L_1^2$. Since $L^2(\boldsymbol{R}^n)$ is a separable Hilbert space, L_1^2 is also separable. In fact, if we take a countable dense set $\{\varphi_m\}$ in $L^2(\boldsymbol{R}^n)$, then $\{(1+|x|)^{-1}\varphi_m\}$ is a countable dense set in L_1^2. Put $q_m=(1+|x|)^{-1}\varphi_m$. As a result of (7.9), for each q_m there exists a set N_k in $\boldsymbol{R}$ such that the Lebesgue measure of N_k is equal to zero and

$$(7.10) \qquad \langle F(\lambda)^*\hat{f}(\lambda), q_k\rangle = 0, \qquad \lambda\notin N_k .$$

Let $N=\bigcup_{k=1}^{\infty} N_k$. Then, N is of Lebesgue measure zero. From (7.10) we obtain

$$(7.11) \qquad (\hat{f}(\lambda), F(\lambda)g)_h = \langle F(\lambda)^*\hat{f}(\lambda), g\rangle = 0$$

for any $g\in L_1^2$ and $\lambda\notin N$. In view of Lemma 7.3 and (7.11)

$$(7.12) \qquad 0=(\hat{f}(\lambda), F(\lambda)f)_h = \lim_{m\to\infty}(\hat{f}(\lambda), w_m)_h$$

for any $f\in L_s^2$ $(s>(1/2))$. Proposition 6.5 yields

$$v=v(x, \lambda, \phi)\in L^2_{V_0; -s}, \qquad |\mathcal{D}^+v|\in L^2_{s-1, E(R)}$$

and

$$(L-\lambda)v\in L_s^2$$

for any $\phi\in C^\infty(S^{n-1})$, where $s>(1/2)$ and $s-(1/2)$ is a sufficiently small number, and R is a sufficiently large number. Theorem 3.2 shows

$$v=R(\lambda+i0)g ,$$

where $g=(L-\lambda)v$. Substituting $f=g$ in (7.12) and noting then

$$(7.13)\qquad w_m=\frac{1}{\sqrt{\pi}}R_m^{(n-1)/2}e^{-i\Phi(R_m\cdot,\lambda)}e^{iA(R_m\cdot)}\sqrt[4]{\lambda-V_0(R_m\cdot)}(R(\lambda+i0)g)(R_m\cdot)$$
$$=\frac{1}{\pi}\phi$$

for any sufficiently large R_m. Therefore, (7.12) and (7.13) give

$$(\hat{f}(\lambda),\ \phi)_h=0$$

for any $\phi\in C^\infty(S^{n-1})$ and $\lambda\notin N$. Thus we have

$$\hat{f}(\lambda)=0\quad \text{a.e.},$$

which completes the proof. Q. E. D.

Added in proof. Recently, it turns out that the condition $\alpha<2$ in (A.2), which is used only to prove Lemma 4.5, can be replaced by a weaker condition $V_0(x)=o(r^2)$ near infinity, since Lemma 4.5 can be proved under the later condition by M. Arai and O. Yamada, [27].

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