J. Math. Kyoto Univ. (JMKYAZ)
30-4 (1990) 575-584

On the mean curvature of surface with boundary

By

Keiichi SHIBATA

0. Introduction

The present study is concerned with the mean curvature of surfaces located in the Euclidean space $\boldsymbol{R}^3$. A great deal of interests has long been focussed upon the question in the scope of global analysis, under what condition such surfaces can possess an everywhere constant mean curvature at all.

The case of compact surface, namely, of the surface without boundary was treated mainly from the standpoint of isoperimetric problem, and the satisfactory results seem to have been achieved about in the last decade (cf. Barbosa-do Carmo [1], Osserman [5]). As for the case of surface with boundary, on the other hand, the theory remains still in its infancy and has much room to be investigated.

Our contribution in this paper will be, above all, (a) completion of a criterion for constancy of mean curvature with respect to volume and surface-area in the conditioned variation arguments up to its dual; and (b) presentation of a new convexity theorem valid for every surface with boundary that has a constant mean curvature.

To be precise, § 1 explains our situation, setting the problem and prepares the tools used. In § 2 we derive the first variations of volume- and area-functionals in our own terms. We state and prove, in § 3, our main theorems in this paper, among which are included a duality theorem for a conditioned critical point problem in calculus of variations, as well as the convexity theorem, which answers a question raised by Mrs. M. Koiso (cf. Koiso [4]). Partial reason for our restriction of the ambient space to $\boldsymbol{R}^3$, not to $\boldsymbol{R}^n$ $(n\geqq 3)$ lies in an attempt to adopt a few ideas together with their wordings from classical physics, relevant to the *soap bubble experiment* by blowing the tube. We expose such point of view in § 4, by virtue of which we have been able to obtain somewhat exacter information for the convexity than in § 3.

1. Preliminaries

Let $\Omega=\{(u, v)\in\boldsymbol{R}^2 \mid u^2+v^2<1\}$ denote the unit open disk in the $w=u+\sqrt{-1}\,v$-plane. Given a Jordan curve γ embedded in $\boldsymbol{R}^3$ we can define the differential-geometric orientable C^2-smooth surface $X=\boldsymbol{x}(w)$ of disk type which spans γ, as the mapping, subject to the three requirements:

1° $\boldsymbol{x}(w)\in C^0(\bar{\Omega}, \boldsymbol{R}^3)\cap C^2(\Omega, \boldsymbol{R}^3)$;

2° $\boldsymbol{x}(w)$ is an immersion of Ω into $\boldsymbol{R}^3$;

Communicated by Prof. Y. Kusunoki, November 25, 1988

3° $x(w)|_{\partial\Omega}$ is a homeomorphism $\partial\Omega \mapsto \gamma$.

Under these assumptions the surface $x(u, v)$ in question carries at every point its Gaussian frame consisting of the tangent vectors x_u, x_v and the unit normal vector e to it in the right-handed system in this order, which allows us to define the positively oriented unit exterior normal n to be identical with e.

In the following we shall commit ourselves to mean sometimes the above defined surface $x(u, v)$ under a briefer naming *surface with boundary* γ without particular reference to its smoothness in regard to the parametrization. The *locus* $\ll x(w) \gg = \{x(w) | w \in \bar{\Omega}\}$ is comprised, of course, in some open ball centred at the origin $X=0$ with sufficiently large radius, say R. A significant theorem is known to us in terms of this quantity R:

EXISTENCE THEOREM (Hildebrandt [3]). *For any real number H satisfying $|H|R<1$ there exists at least one surface with boundary γ, which has the constant mean curvature H.*

Beside this central proposition there is another useful lemma which lends itself to production of many surfaces admissible to comparison as its fit neighbourhood.

DISTORTION THEOREM (Böhme-Tomi [1]). *Any sufficiently smooth interior distortion of the surface $x(w)$ with boundary γ is expressed uniquely in the form*

$$x(w\,;\,t)=x(w)+t\lambda(w)n(x(w)) \tag{1}$$

in terms of a real-valued function $\lambda(w)$ with a support on a closed disk $\bar{D}=\{w \,|\, |w-w_0| \leqq a\}$ comprised in Ω, where t is a real parameter ranging over some open interval $(-\varepsilon, \varepsilon)$ with the origin in its interior.

Henceforth we take $\lambda(w)$ in the class $C_0^2(\bar{D}, \boldsymbol{R})$ with variable closed subdomain $\bar{D}$ of Ω and will call

$$x(w\,;\,\Delta t)=x(w)+\Delta t\lambda(w)n(x(w)), \qquad (|\Delta t| \ll 1) \tag{1'}$$

especially, to be a *small distortion* of $x(w)$. Of course, $x(w\,;\,0)$ is identical with $x(w)$ itself.

Let w be any value of $\bar{D}$. Taking an arbitrary point Q on the ray $X=tn(x(w))$ $(0<|t|<+\infty)$, we introduce a spatial polar coordinates (r, ϕ, θ) with the pole at Q. Then there exists a neighbourhood $U=U(w)$ of w, such that the ray $\overrightarrow{Q, x(\omega)}$ meets the locus $\ll x(w\,;\,t) \gg$ just at a single point, so far as ω belongs to $U(w)$. The pole Q may alter with the choice of w. Letting w vary all over $\bar{D}$, we have a neighbourhood system $\{U(w)\}_{w\in\bar{D}}$, from which we can extract a finite sub-collection $\{U_j\}_{j=1,2,\cdots,m}$ convering $\bar{D}$.

Proposition 1. *Suppose a single-valued continuous function $f_j(p)$ is assigned to every point p of $\mathfrak{U}_j=x(U_j)$ $(j=1, 2, \cdots, m)$, so that the surface integral*

$$\iint f_j(p) r^2 \sin\theta \, d\phi \, d\theta$$

over $\mathfrak{U}_j$ may be well defined. If $d\sigma$ denotes the area element of the surface $\boldsymbol{x}(w)$, then there exists a continuous function $f(\boldsymbol{x}(w))$ defined on the whole closed disk $\bar{D}$ satisfying

1° $f(\boldsymbol{x}(w))|_{w\in U_j}=f_j(p)$

2° $\int_{U_j} f(\boldsymbol{x}(w))d\sigma(\boldsymbol{x}(w))=\int_{\mathfrak{U}_j}\int f_j(p)r^2\sin\theta d\phi d\theta\,.$ $\quad (j=1, 2, \cdots, m)$

Proof. With the aid of the technique of *Partition of Unity* we readily see the conclusion to hold.

2. Variational formulae for volume and area functionals

This section is devoted to derive the first variations corresponding to the distortion (1′) in terms of $\lambda(w)$, whose counterparts in surfaces without boudary are just the variations of volume and area respectively. According to the author's opinion it is quite unnatural to assign anyway a reaosnable volume functional $V[\boldsymbol{x}]$ to the surface with boundary, unlike for a compact surface. On the other hand one will be free to measure, however, the variation of volume for a small distortion of the original surface. As for the area functional $A[\boldsymbol{x}]$, lack of the boundary rectifiability causes the unmeasurability of $A[\boldsymbol{x}]$ itself, but the areal increment ΔA in the small is defined as we show in what follows.

Definition 1. (Increment of area functional)

$$\Delta A=\int_{\bar{D}}\int|\boldsymbol{x}_u(w\,;\,\Delta t)\times\boldsymbol{x}_v(w\,;\,\Delta t)|\,du\wedge dv-\int_{\bar{D}}\int|\boldsymbol{x}_u(w\,;\,0)\times\boldsymbol{x}_v(w\,;\,0)|\,du\wedge dv$$

Definition 2. (Increment of volume functional)

The volume increment ΔV shall be the signed sum of volumes for each component, into which the spatial open set bounded by the loci of $\boldsymbol{x}(w\,;\,0)$ and of $\boldsymbol{x}(w\,;\,\Delta t)$ is decomposed. Here the sign to be attached to the volume of each component is determined according as that of $\lambda(w)$ on the component in question.

Proposition 2. *The first variation of volume with respect to the distortion* (1′) *is*

$$\delta V=\delta t\int_{\Omega}\lambda(w)d\sigma(\boldsymbol{x}(w)) \tag{2}$$

at $t=0$.

Proof. Letting $w_1\in\bar{D}$ be arbitrary, we write $P_1=\boldsymbol{x}(w_1\,;\,0)$ for brevity. Take a pole Q_1 on the normal $X=t\boldsymbol{n}(P_1)$ $(0<|t|<+\infty)$ to set up a polar condinate system in the small. There is a neighbourhood U_1 of w_1, such that for every point P on $\mathfrak{U}_1=\boldsymbol{x}(U_1\,;\,0)$ the radius vector $\overrightarrow{Q_1,P}$ meets the small distortion $\boldsymbol{x}(w\,;\,\Delta t)$ only at one point $\tilde{P}$. To the increment $\Delta r=\overrightarrow{P,\tilde{P}}=\overrightarrow{Q_1,\tilde{P}}-\overrightarrow{Q_1,P}$ of radius vector r there corresponds a unique continuous function $\tilde{\lambda}(w)$, such that $\tilde{\lambda}(w)\Delta r=\lambda(w)\Delta t\,|\boldsymbol{n}(\boldsymbol{x}(w))|$.

In reference to the above polar coordinate system we consider an appropriate curvilinear quadrangular cone with vertex Q_1 bounded by a pair of contiguous azimuthal planes and a pair of contiguous zenithal cones, which intersects a pair of spheres $\{P\,|\,\overline{Q_1, P}=\rho\}$ and $\{P\,|\,\overline{Q_1, P}=\rho+\varDelta\rho\}$ $(r, r+\varDelta r\in[\rho, \rho+\varDelta\rho])$, in order to produce a curvilinear hexahedron Σ. If the cone is taken, so that the ray $\overrightarrow{Q_1, P_1}$ may be its central axis and that the base surface-portion of $\partial\Sigma$ on the smaller circumference may have the area $\rho^2\sin\theta\,\varDelta\phi\varDelta\theta$, then the volume of Σ is $\varDelta\rho\rho^2\sin\theta\,\varDelta\phi\varDelta\theta$. Therefore the contribution of the subdomain bounded by $\ll\boldsymbol{x}(w\,;\,0)\gg$ and $\ll(\boldsymbol{x}\,;\,\varDelta t)\gg$ to $\varDelta V$ within Σ must be equal to $\tilde{\lambda}(w)\varDelta r\, r^2\sin\theta\,\varDelta\phi\varDelta\theta$. So we can calculate the full contribution on $\mathfrak{U}_1$ to the volume change at

$$\int_{\mathfrak{U}_1}\!\!\int\tilde{\lambda}\varDelta r\, r\sin\theta\, d\phi d\theta=\int_{U_1}\lambda(w)\varDelta t\, d\sigma(\boldsymbol{x}(w)),$$

from which follows

$$\varDelta V=\varDelta t\int_{\Omega}\lambda(w)\, d\sigma(\boldsymbol{x}(w))$$

by Proposition 1. q. e. d.

Concerning the sign of the principal curvature $1/R_j$ $(j=1, 2)$ at one point of a surface (1), let us agree as follows:

Definition 3. (sign of principal curvatures)
When the centre of curvature for R_j lies on the interior normal $X=\tau\boldsymbol{n}(\boldsymbol{x}(w\,;\,t)$ $(\tau<0)$, then $R_j>0$: otherwise, $R_j<0$ $(j=1, 2)$.[1)]

Proposition 3. *The first variation of the surface-area with respect to the distortion* (1) *is equal to*

$$\delta A=\delta t\int_{\Omega}\lambda(w)\Big(\frac{1}{R_1(\boldsymbol{x}(w))}+\frac{1}{R_2(\boldsymbol{x}(w))}\Big)d\sigma(\boldsymbol{x}(w))$$

at $t=0$.

Proof. Letting $w_0\in\bar{D}$ be at will, we set $\tilde{P}_0=\boldsymbol{x}(w_0\,;\,\varDelta t)$, at which we draw the normal $\mathfrak{N}$: $\boldsymbol{x}=\tau\boldsymbol{n}(P_0)$ $(-\infty<\tau<+\infty)$ to the surface $\ll\boldsymbol{x}(w\,;\,\varDelta t)\gg$. Let $\mathfrak{N}$ meet $\ll\boldsymbol{x}(w\,;\,0)\gg$ at a point P_0 and let a plane Π containing $\mathfrak{N}$ cut the two surfaces $\ll\boldsymbol{x}(w\,;\,\varDelta t)\gg$ and $\ll\boldsymbol{x}(w\,;\,0)\gg$ along the curves $\tilde{C}$ and C respectively, the former of which is a what we call "normal section" of this surface through $\tilde{P}_0$.

We are now interested in the case, in which the cutting plane Π produces a specific intersection with $\ll\boldsymbol{x}(w\,;\,\varDelta t)\gg$, namely, where $\tilde{C}$ accords with the lines of principal curvature. In this case Π and $\tilde{C}$ are denoted by Π_j and $\tilde{C}_j$ respectively: correspondingly the intersection of Π_j with $\ll\boldsymbol{x}(w\,;\,0)\gg$ is written as C_j, which may not necessarily coincide with the lines of principal curvature on $\ll\boldsymbol{x}(w\,;\,0)\gg$: here $\tilde{C}_1$ and $\tilde{C}_2$ can be taken for the parametric u- and v-curve of the surface respectively, this circumstance being easily realized at request via appropriate parameter transformation.

To the given $\lambda(w)$ there corresponds a C^2-function $\tilde{\lambda}(w)$, such that $\overrightarrow{\tilde{P}_0, P_0}=-\tilde{\lambda}(w_0)\varDelta t$.

1) On account of the immersedness assumption, the radii of principal curvature nowhere vanish.

Denote by Q_j the centre of curvature for $\tilde{C}_j$, which finds itself, of course, on the normal $\mathfrak{N}$. At an adequate choice of $\lambda(w)$ we may assume from the outset that the radii of principal curvature for $\ll x(w\,;\,\varDelta t)\gg$ at $w=w_0$ are both finite, i. e., $-\infty<\tilde{R}_j=\tilde{R}_j(\tilde{P}_0)<+\infty$, without loss of generality. Then the point $\tilde{P}_0$ (resp. P_0) stands at the distance $|\tilde{R}_j|$ (resp. $|\tilde{R}_j-\tilde{\lambda}\varDelta t|$) from Q_j.

Let us work as follows: $\tilde{P}_j$ is a point on $\tilde{C}_j$ sufficiently near $\tilde{P}_0$, such that the parameter values (u, v) for them may fall on $\bar{D}$. Let the straight line $Q_j, \tilde{P}_j$ meet C_j at the point P_j. In this way we have a pair of narrow sectors on each Π_j, one of which has the radii $\overline{Q_j, \tilde{P}_j}$, $\overline{Q_j, \tilde{P}_j}$ and the arc $\overset{\frown}{\tilde{P}_0, \tilde{P}_j}$, the other the radii $\overline{Q_j, P_0}$, $\overline{Q_j, P_j}$ and the arc $\overset{\frown}{P_0, P_j}$. If the central angle $\varDelta\theta$ is sufficiently small, the ratio $\overset{\frown}{P_0, P_j} : \overset{\frown}{\tilde{P}_0, \tilde{P}_j}$ is approximately equal to the ratio $\overline{Q_j, P_0} : \overline{Q_j, \tilde{P}_0}$ up to the infinitesimals of higher order than $\varDelta\theta$. Hence we have

$$|x_u(w_0\,;\,0)\varDelta u|=(|x_u(w_0\,;\,\varDelta t)\varDelta u|\,|\tilde{R}_1(w_0)-\tilde{\lambda}(w_0)\varDelta t|/|\tilde{R}_1(w_0)|)+o(\varDelta u),$$

$$|x_v(w_0\,;\,0)\varDelta v|=(|x_v(w_0\,;\,\varDelta t)\varDelta v|\,|\tilde{R}_2(w_0)-\tilde{\lambda}(w_0)\varDelta t|/|\tilde{R}_2(w_0)|)+o(\varDelta v).$$

So the area element of $x(w\,;\,0)$ at $w=w_0$ reads

$$\begin{aligned}&|x_u(w_0\,;\,0)\times x_v(w_0\,;\,0)|\,du\wedge dv\\&=|1-(\tilde{\lambda}(w_0)\varDelta t/\tilde{R}_1(w_0))|\,|1-(\tilde{\lambda}(w_0)\varDelta t/\tilde{R}_2(w_0))|\,|x_u(w_0\,;\,\varDelta t)\times x_v(w_0\,;\,\varDelta t)|\,du\wedge dv\\&=|\tilde{R}_1(w_0)-\tilde{\lambda}(w_0)\varDelta t|\,|\tilde{R}_2(w_0)-\tilde{\lambda}(w_0)\varDelta t|\,|\tilde{e}_u(w_0)\times\tilde{e}_v(w_0)|\,du\wedge dv,\end{aligned}$$

where $\tilde{e}$ stands for the exterior unit normal vector of $x(w\,;\,\varDelta t)$. Therefore

$$\begin{aligned}\varDelta A[x]&=\int_\Omega\!\!\int(|x_u(w\,;\,\varDelta t)\times x_v(w\,;\,\varDelta t)|-|x_u(w\,;\,0)\times x_v(w\,;\,0)|)du\wedge dv\\&=\int_\Omega\!\!\int(|\tilde{R}_1(w)\tilde{R}_2(w)|-|\tilde{R}_1(w)-\tilde{\lambda}(w)\varDelta t|\,|\tilde{R}_2(w)-\tilde{\lambda}(w)\varDelta t|)|\tilde{e}_u\times\tilde{e}_v|\,du\wedge dv\\&=\pm\varDelta t\int_\Omega\!\!\int(|\tilde{R}_1(w)+\tilde{R}_2(w))\tilde{\lambda}(w)|\tilde{e}_u\times\tilde{e}_v|\,du\wedge dv+o(\varDelta t)^{2)}\\&=\pm\varDelta t\int_\Omega\!\!\int\tilde{\lambda}(w)\Big(\frac{1}{\tilde{R}_1(w)}+\frac{1}{\tilde{R}_2(w)}\Big)|x_u(w)\times x_v(w)|\,du\wedge dv+o(\varDelta t).\end{aligned}$$

But $\tilde{R}_j(w)\rightrightarrows R_j(x(w))$, $\tilde{\lambda}(w)\rightrightarrows\lambda(w)$ $(j=1, 2)$ as $\varDelta t\to 0$, so we conclude

$$\delta A=\delta t\int_\Omega\Big(\frac{1}{R_1(x(w))}+\frac{1}{R_2(x(w))}\Big)\lambda(w(d\sigma(x(w)).$$

q. e. d.

3. Main theorems

Suppose, there exists a surface $x(w\,;\,0)$ with boundary, whose mean curvatures satisfy

2) Note that $\tilde{R}_1\tilde{R}_2$ has a definite sign by virtue of 1). The signs $\pm$ take place according as $\tilde{R}_1\tilde{R}_2$ is positive or negative respectively.

$$\int_{\Omega}\lambda(w)\Big(\frac{1}{R_1(\boldsymbol{x}(w))}+\frac{1}{R_2(\boldsymbol{x}(w))}\Big)d\sigma(\boldsymbol{x}(w))=0 \tag{3}$$

under all distortions of type (1′) such that

$$\int_{\Omega}\lambda(w)d\sigma(\boldsymbol{x}(w))=0\,. \tag{4}$$

Then, if $(1/R_1(\boldsymbol{x}(w)))+(1/R_2(\boldsymbol{x}(w)))\not\equiv$const. on Ω, it is possible to choose some $\lambda^*(w)$ with the normalization (4) satisfying

$$\int_{\Omega}\lambda^*(w)\Big(\frac{1}{R_1(\boldsymbol{x}(w))}+\frac{1}{R_2(\boldsymbol{x}(w))}\Big)d\sigma(\boldsymbol{x}(w))>0\,,$$

which contradicts the assumption.

Conversely, the condition $(1/R_1(\boldsymbol{x}(w)))+(1/R_2(\boldsymbol{x}(w)))\equiv$const. everywhere on Ω obviously implies (3) for all $\lambda(w)$ satisfying (4). Thus, in view of Propositions 2 and 3 we have proved

Theorem 1. *The surface of class $C^0(\bar{\Omega}, \boldsymbol{R}^3)\cap C^2(\Omega, \boldsymbol{R}^3)$ with boundary has a constant mean curvature if and only if its surface-area is critical among all surfaces of this class that bears no change of volume with respect to the original one.*

Remark 1. Apart from some kinds of known proofs leading to the similar conclusion as above theorem (cf. Hildebrandt [3], Barbosa-do Carmo [1], Koiso [5]), the present line of argument just exposed makes it possible to complement the statement to its dual in the conditioned variational problem as

Theorem 2. *The surface $\boldsymbol{x}(w\,;\,0)$ with boundary γ has a constant mean curvature if and only if its volume is critical compared with all surfaces with the same area.*

Proof. On closer inspection into the arguments so far one will notice that the reasoning that has derived Theorem 1 has only to be repeated almost verbatim.

Theorem 3. *If the surface with boundary has a constant mean curvature $H\neq 0$ everywhere, it is convex.*

Proof. Under the condition that such a surface $\boldsymbol{x}(w\,;\,0)$ has the constant mean curvature H, we shall show $(1/R_1)(1/R_2)>0$ everywhere on it.

($\boldsymbol{a}$) Suppose on the contrary that $\ll\boldsymbol{x}(w\,;\,0)\gg$ contains a point $\boldsymbol{x}_0$, where $R_1(\boldsymbol{x}_0)R_2(\boldsymbol{x}_0)<0$. Then $\boldsymbol{x}_0$ is a hyperbolic point of this surface, namely, $\boldsymbol{x}(w\,;\,0)$ behaves in a neighbourhood of $\boldsymbol{x}_0$ approximately like near the saddle point of the hyperbolic paraboloid. Because we have only to rotate the $\boldsymbol{x}$-space $\boldsymbol{R}^3=\{(x^1, x^2, x^3)\}$ if necessary, we may assume from the outset that the outer normal $\boldsymbol{e}$ at $\boldsymbol{x}_0$ points to the positive x^3-axis. Then $\ll\boldsymbol{x}(w\,;\,0)\gg$ contains a closed subdomain $\boldsymbol{B}$, which enjoys the followinf properties:

1° $\partial\boldsymbol{B}$ is a smooth Jordan curve enclosing the point $\boldsymbol{x}_0$ in its interior, such that both $\operatorname{Max} x^3$ and $\operatorname{Min} x^3$ on $\boldsymbol{B}$ are taken on $\partial\boldsymbol{B}$;

2° for any given $\varepsilon>0$, $\operatorname{Osc} x^3=\operatorname{Max} x^3-\operatorname{Min} x^3$ on $\boldsymbol{B}$ is bounded above by ε;

3° there exists a pair of cylinders $\overline{\boldsymbol{K}}$ and $\underline{\boldsymbol{K}}$, both of which intersect $\ll \boldsymbol{x}(w; 0)\gg$ along $\partial\boldsymbol{B}$;

4° $\overline{\boldsymbol{K}}$ (resp. $\underline{\boldsymbol{K}}$) is higher (resp. lower) than $\boldsymbol{B}$ to the effiect that for any points $P\in\overline{\boldsymbol{K}}$ (resp. $\underline{\boldsymbol{K}}$), $P'\in\boldsymbol{B}$, the third component x^3 of P is not less (resp. greater) than that of P'; further the former is actually greater (resp. less) than the latter except on $\partial\boldsymbol{B}$;

(**b**) Note that

$$\text{area}\,\overline{\boldsymbol{K}}<\text{area}\,\boldsymbol{B}, \qquad \text{area}\,\underline{\boldsymbol{K}}<\text{area}\,\boldsymbol{B}. \tag{5}$$

Let B denote the parameter subdomain of Ω, such that $B=\{w\,|\,\boldsymbol{x}(w; 0)=\boldsymbol{B}\}$. Then it is possible to find a nice parametrization $\overline{\boldsymbol{y}}(w; 0)$ of $\overline{\boldsymbol{K}}$ on B, such that the mapping is continuous on B, smooth (of class C^∞) in the interior to B. Define the modification $\bar{z}(w)=\bar{z}(w; 0)$ of $\boldsymbol{x}(w; 0)$ as follows:

$$\bar{z}(w; 0)=\begin{cases} \boldsymbol{x}(w; 0) & \text{on } \Omega\setminus B, \\ \overline{\boldsymbol{y}}(w; 0) & \text{on } B. \end{cases} \tag{6}$$

Taking a compact subdomain A of Ω, such that $\operatorname{Clo}B\subset\operatorname{Int}A\subset\operatorname{Clo}A\subset\Omega$, we introduce a function $\bar{\mu}(w)\in C_0^\infty(\Omega, \boldsymbol{R}^+)$ supported by A to define a distortion of $\bar{z}(w; 0)$ of type (1):

$$\bar{z}(w; \eta)=\bar{z}(w; 0)+\eta\bar{\mu}(w)\boldsymbol{n}(\bar{z}(w)), \tag{7}$$

where η is a positive real number.

(**c**) In view of (5), (6) and (7) there is a suitable choice of η for any $\bar{\mu}(w)$, such that

$$A[\bar{z}(\Omega; \eta)]=A[\boldsymbol{x}(\Omega; 0)]. \tag{8}$$

(**d**) With (8) in our eyes we combine (6), (7) with 4° of (**a**). We see that for any $\bar{\mu}$ and η, $\bar{z}(w; \eta)$ is an admissible surface in the variational problem in Theorem 2, while $\ll\bar{z}(w; \eta)\gg$ is located higher than the original one $\ll\boldsymbol{x}(w; 0)\gg$.

(**e**) We might work on $\underline{\boldsymbol{K}}$ in entirely the same way as done earlier with $\overline{\boldsymbol{K}}$, namely, the parametrization $\underline{\boldsymbol{y}}(w; 0)$ of $\underline{\boldsymbol{K}}$, the modification $\underline{z}(w; 0)$, a real-valued smooth function $\underline{\mu}(w)$ of class $C_0(\Omega, \boldsymbol{R}^-)$ with support $A(B\subset A\subset\Omega)$ and the distortion $\underline{z}(w; \eta)$ with a similar real positive parameter η enter into consideration in place of $\overline{\boldsymbol{y}}(w; 0)$, $\bar{z}(w; 0)$, $\bar{\mu}(w)$ and $\bar{z}(w; \eta)$ aimed at finds itself in its turn, at the position lower than $\ll\boldsymbol{x}(w; 0)\gg$.

(**f**) We conclude from (**d**), (**e**) and 2° of (**a**) that the admissible surfaces, both higher and lower, can be found as close to $\boldsymbol{x}(w; 0)$ as one pleases. In other words, the volume $V[\boldsymbol{x}(w; 0)]$ cannot be critical among all the admissible surfaces satisfying (3), which is a contradiction on account of Theorem 2. q.e.d.

4. Principle of Virtual Work

The present author now wishes to propose a plan associating the purely mathematical situation as well as the results so far investigated with a physical interpretation by regarding the parameter $t\in(-\varepsilon, \varepsilon)$ as a time variable. Such circle of ideas seems neither dogmatic nor novel, but can often be found in significant mathematical works, especially in the ones on global analysis. For example, in Eells-Sampson [2] the parameter t of time enters the energy functional and the harmonic maps have successfully been dealt with in connexion with the heat equation.

Nevertheless our intention cannot surpass further the mere adoption of physical terminology: so, what we call *surface* here shall be of vanishing thickness, an everywhere constant density and a constant surface-tension coefficient as well. In addition, we commit ourselves to take no influence of gravity into consideration.

To any point $\boldsymbol{x}_0$ on the surface $S=\{\boldsymbol{x}(w;0)\mid w\in\Omega\}$ there corresponds an open ball neighbourhood $\mathfrak{V}(\boldsymbol{x}_0)$, such that the surface-portion $S_0=S\cap\mathfrak{V}(\boldsymbol{x}_0)$ decomposes the open set $\mathfrak{V}(\boldsymbol{x}_0)\setminus S_0$ into the union of just two connected components O^+ and O^-. We agree to regard a point $p\in S_0$ as two different boundary points p^+ and p^-, which are accessible from the interior of the spatial domains O^+ and O^- respectively. If we set $S_0^{\pm}=\{p^{\pm}\mid p\in S_0\}$, we may assume without losing the generality that the correspondence $S_0^+\ni p^+$ into Ω is orientation-preserving and $S_0^-\ni p^-$ into Ω orientation-reversing. The compact set $\{\boldsymbol{x}(w;0)\mid w\in\bar{D}\}$ can be covered by some finite sub-collection $\{\mathfrak{V}_j\}_{j=1,2,\dots,m}$ of the neighbourhood system $\bigcup_{x\in\bar{D}}\mathfrak{V}(\boldsymbol{x})$: each pair of boundaries $S_j^{\pm}$ of the two spatial domains corresponding to $\mathfrak{V}_j$, which are originated from a portion of S, are connected with one another with running index $1\leqq j\leqq m$. We may call $S^+=\bigcup_{j=1}^{m}S_j^+$ and $S^-=\bigcup_{j=1}^{m}S_j^-$ the top and the bottom of the surface-portion $S=\langle\!\langle\boldsymbol{x}(\bar{D};0)\rangle\!\rangle$ respectively.[3)]

We want here to quote some physical facts and laws, which we expose in our own locution. The first of them is the postulate that the atmospheric pressure is a 3-vector acting perpendicularly on the wall, which depends only on the space coordinates. So, if $\boldsymbol{p}^{\pm}(\boldsymbol{x})$ denote the atmospheric pressures acting on the sides $S^{\pm}$ respectively, then $\boldsymbol{p}^+(x)$ (resp. $\boldsymbol{p}^-(x)$) is an inward- (resp. outward-) pointing normal vector at the point $\boldsymbol{x}$ with the absolute value $|\boldsymbol{p}^+(\boldsymbol{x})|$ (resp. $|\boldsymbol{p}^-(\boldsymbol{x})|$).

Secondly we refer to a physical law known as 'Principle of Virtual Work', which is classical in mechanics. Namely, we consider the infinitesimal displacement $(\delta t)\boldsymbol{n}(x(w))\lambda(w)$ of one point $\boldsymbol{x}(w)$ on the surface $\boldsymbol{x}(w;0)$ has been caused by the external force $-\boldsymbol{p}^+(\boldsymbol{x}(w))+\boldsymbol{p}^-(\boldsymbol{x}(w))$ acting at this point. In analogy with the case of the particle-system in the elementary mechanics, the inner product

$$(9)\qquad \begin{aligned}&\langle -\boldsymbol{p}^+(\boldsymbol{x}(w))+\boldsymbol{p}^-(\boldsymbol{x}(w)),\ (\delta t)\boldsymbol{n}(\boldsymbol{x}(w))\lambda(w)\rangle\\ &\quad=(\delta t)(-|\boldsymbol{p}^+(\boldsymbol{x}(w))|+|\boldsymbol{p}^-(\boldsymbol{x}(w))|)\lambda(w)\end{aligned}$$

of these two vectors, parallel to each other, is the work for the infinitesimal displacement of this point. If we recall the fact that the product of the area element with the small displacement (9) to the normal direction makes up the volume element (Proposition 2), we are able to calculate the work to change the volume against the external force as follows:

Proposition 4. *The total work $W(\delta V)$ to be done for the infinitesimal volume change*

3) The index m does not necessarily indicate the same at each occurrence.

is equal to

$$W(\delta V)=\delta t\int_{\Omega}(-|\boldsymbol{p}^{+}(\boldsymbol{x}(w))|+|\boldsymbol{p}^{-}(\boldsymbol{x}(w))|)\lambda(w)d\sigma(\boldsymbol{x}(w)).$$

Legitimacy of the following definition, too, will readily be approved in view of (9) and of the classical 'Priciple of Virtual Work' for the particle system.

Definition 4. The surface $\boldsymbol{x}(w\,;\,0)$ is said to be *stable* under the distortion (1'), if the condition

$$\int_{\bar{D}}(-|\boldsymbol{p}^{+}(\boldsymbol{x}(w))|+|\boldsymbol{p}^{-}(\boldsymbol{x}(w))|)\lambda(w)d\sigma(\boldsymbol{x}(w))=0 \tag{10}$$

is satisfied for every $\lambda(w)\in C^{2}(\bar{D},\,\boldsymbol{R}),\,(\bar{D}\subset\Omega)$.

Remark 2. In accordance with the above Proposition 4, it is known to us that the work $W(\delta A)$ to be done for the infinitesimal change of surface area A is given in the form $W(\delta A)=\alpha\delta A$ by means of an absolute constant α, the coefficient of surface-tension, which is proper for the material constituting the surface.

Theorem 4. *If a stable surface* $\boldsymbol{x}(w\,;\,0)$ *with boundary* γ *provides a critical value of the surface-area among all surfaces* $\boldsymbol{x}(w\,;\,t)\,(-\varepsilon<t<\varepsilon)$ *of class* $C^{0}(\bar{\Omega},\,\boldsymbol{R}^{3})\cap C^{2}(\Omega,\,\boldsymbol{R}^{3})$ *with the same boundary, then* $\boldsymbol{x}(w\,;\,0)$ *fulfills Laplace-Thomson's formula*

$$-|\boldsymbol{p}^{+}(\boldsymbol{x})|+|\boldsymbol{p}^{-}(\boldsymbol{x})|=\alpha\Big(\frac{1}{R_{1}(\boldsymbol{x})}+\frac{1}{R_{2}(\boldsymbol{x})}\Big).$$

Proof. We may assume that

$$\int_{\Omega}\Big(\frac{1}{R_{1}(\boldsymbol{x}(w))}+\frac{1}{R_{2}(\boldsymbol{x}(w))}\Big)\lambda(w)d\sigma(\boldsymbol{x}(w))=0 \tag{11}$$

is valid for all $\lambda(w)\in C^{2}(\Omega,\,\boldsymbol{R})$ satisfying

$$\int_{\Omega}(-|\boldsymbol{p}^{+}(\boldsymbol{x}(w))|+|\boldsymbol{p}^{-}(\boldsymbol{x}(w))|)\lambda(w)d\sigma(\boldsymbol{x}(w))=0. \tag{12}$$

Then we notice that

$$g(w)=-|\boldsymbol{p}^{+}(\boldsymbol{x}(w))|+|\boldsymbol{p}^{-}(\boldsymbol{x}(w))|-\alpha\Big(\frac{1}{R_{1}(\boldsymbol{x}(w))}+\frac{1}{R_{2}(\boldsymbol{x}(w))}\Big)$$

has a constant sign all over Ω.

In fact, suppose on the contrary that Ω contains non-void subsets $G^{\pm}$, for which

$$g(w)>0,\qquad (w\in G^{+}), \tag{13}$$

$$g(w)<0,\qquad (w\in G^{-}). \tag{14}$$

We can choose a $\lambda(w)$ such as to satisfy (12) and to be positive and negative on G^{+} and G^{-} respectively. Multiplying this $\lambda(w)$ on both sides of (13), (14) and adding them together, we see that the left-hand side in (11) must be negative, which is a contradiction. Hence we may assume without loss of generality that $g(w)\geqq 0$ on Ω.

In order to show that $g(w)=0$ throughout Ω, we suppose, contrary to the assertion that Ω contains a point w_{0} such $g(w)>0$ in a neighbourhood of w_{0}. It is possible to take a $\lambda(w)$ admitted to the consideration at hand, such that $\lambda(w_0)>0$,

$$\int_\Omega g(w)\lambda(w)d\sigma(\boldsymbol{x}(w))>0$$

$$\int_\Omega(-|\boldsymbol{p}^+(\boldsymbol{x}(w))|+|\boldsymbol{p}^-(\boldsymbol{x}(w))|)\lambda(w)d\sigma(\boldsymbol{x}(w))=0.$$

It contradicts the assumption at the beginning of this proof. q. e. d.

Proposition 5. *For any distortion of surface with boundary with vanishing volume variation, the work done to change the volume must trivially be zero.*

Hence combination of Theorems 1,3,4 with Proposition 5 yields the

Corollary. *The difference of the atomospheric pressures working on the two sides of a stable surface with boundary is constant at every point on it. The surface is convex towards the side, on which the storonger pressure is acting.*

Remark 3. The Laplace-Thomson formula has hitherto been derived under the assumption that the surface is in the state of thermodynamic equilibrium, i.e., $W(\delta V)+W(\delta A)=0$ for *all* distortions. Our result (Theorem 4) is proved under the weaker assumption of stability (10).

FACULTY OF SCIENCE
OKAYAMA UNIVERSITY OF SCIENCE

References

[1] J.L. Barbosa and M. do Carmo, Stability of hypersurfaces with constant mean curvature, Math. Z., **185** (1984), 339-353.

[2] R. Böhme und F. Tomi, Zur Struktur der Lösungsmenge des Plateau-problems, Math. Z., **133** (1973), 1-29.

[3] J. Eells, Jr. and J.H. Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. math., **86** (1964), 109-160.

[4] S. Hildebrandt, On the Plateau problem for surfaces of constant mean curvature, Comm. Pure Appl. Math., **23** (1970), 97-114.

[5] M. Koiso, Stability of surfaces with constant mean curvature in $\boldsymbol{R}^3$, MPI/87-17, Max-Planck-Institut für Mathematik, Bonn (1987), 1-29.

[6] R. Osserman, The isoperimetric inequality, Bull. Amer. Math. Soc., **84** (1978), 1182-1238.