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Moduli of stable pairs

By

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Introduction

Let S be a scheme of finite type over a universally Japanese ring Ξ and let $f: X \to S$ be a smooth, projective, geometrically integral morphism. We shall fix an f-very ample invertible sheaf $\mathcal{O}_X(1)$ and a locally free $\mathcal{O}_X$-module E of finite rank. An E-pair is a pair (F, φ) of a coherent sheaf F on a geometric fiber of f and an $\mathcal{O}_X$-homomorphism φ of F to $F \otimes_{\mathcal{O}_X} E$ such that φ induces a canonical structure of $S^*(E^\vee)$-module on F. An E-pair (F, φ) is said to be stable (or, semi-stable) if F is torsion free and if it satisfies the stability (or, semi-stability, resp.) inequality for all φ-invariant subsheaves of F (see § 1). Stable pairs were first introduced by N. J. Hitchin [3] in the case where $S = \operatorname{Spec}(k)$ with k an algebraically closed field and where X is a curve and E is a line bundle. In this case, the moduli spaces of stable E-pairs were constructed by N. Nitsure [10], and W. M. Oxbury studied some properties of the moduli spaces [11]. In higher dimensional cases, C. T. Simpson constructed the moduli spaces of semi-stable E-pairs over an algebraically closed field of characteristic zero [13]. In the method of C. T. Simpson, an E-pair (F, φ) were considered as a sheaf on $Y = \operatorname{Proj}(S^*(E^\vee) \oplus \mathcal{O}_X)$ and the problem was reduced to the study of stable points on $Q = \operatorname{Quot}^H_{\mathcal{O}_Y(-N)^{\oplus m}/Y/S}$ for large integers N, where $\mathcal{O}_Y(1)$ is a very ample invertible sheaf on Y and H is the Hilbert polynomial of F with respect to $\mathcal{O}_Y(1)$. To handle this problem he embedded Q into the Grassmann variety Grass $(H^0(\mathcal{O}_Y(l-N)^{\oplus m}), H(l))$ with l a sufficiently large integer. His proof depends, in essential way, on the boundedness theorem of M. Maruyama (Theorem 4.6 of [8]) which fails to hold in positive characteristic cases. The aim of this article is to construct a moduli scheme of semi-stable E-pairs along the method by D. Gieseker [2], M. Maruyama [6] and [7] and then our results hold good without assuming characteristic zero. The main idea is to find a space which seems as the "Gieseker space" in [2], [6] and [7]. It is the projective space $\mathbf{P}(\operatorname{Hom}_{\mathcal{O}_X}(V \otimes_\Xi (\bigoplus_{i=0}^{r-1} S^i(E^\vee)), L)^\vee)$, where L is a line bundle on X and r is the rank of F. On the other hand, to parametrize E-pairs we have to use a scheme Γ constructed in § 4 instead of Quot-scheme in the case of usual stable sheaves and to study stable points of Γ we have to introduce a morphism of Γ to a projective bundle on $\operatorname{Pic}_{X/S}$ whose fibers are

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new Gieseker spaces.

§1 is devoted to several definitions, a boundedness theorem and its corollaries. The moduli functor $\bar{\Sigma}^H_{E/X/S}$ is defined in §2. In §3, we shall extend the results of D. Gieseker on semi-stable points of Gieseker spaces to our new Gieseker spaces. In §4, we shall construct the scheme Γ and a morphism to a projective bundle on $\mathrm{Pic}_{X/S}$ whose fibers are new Gieseker spaces. And in §5, we shall construct the coarse moduli scheme of the functor $\bar{\Sigma}^H_{E/X/S}$.

M. Maruyama suggested trying this problem to me. W. M. Oxbury informed me the results of N. Nitsure and C. T. Simpson. I wish to thank Professors M. Maruyama, W. M. Oxbury and T. Sugie for their encoragement and valuable suggestions.

Notation and Convention

For an $\mathcal{O}_X$-module E on a scheme X, we denote by $S^i(E)$ the i-th symmetric product, by $S^*(E)$ the symmetric $\mathcal{O}_X$-algebra and by $S^*_r(E)$ the $\mathcal{O}_X$-module $\bigoplus_{i=0}^{r-1} S^i(E)$ for each positive integer r.

Let $f: X \to S$ be a smooth, projective, geometrically integral morphism of locally noetherian schemes and let $\mathcal{O}_X(1)$ be an f-very ample invertivle $\mathcal{O}_X$-module. If s is a geometric point of S, then X_s means the geometric fibre of X over s. For a coherent $\mathcal{O}_{X_s}$-module F, the degree of F with respect to $\mathcal{O}_X(1)$ is that of the first Chern class of F with respect to $\mathcal{O}_{X_s}(1) = \mathcal{O}_X(1) \otimes \mathcal{O}_{X_s}$ and it is denoted by $\deg_{\mathcal{O}_X(1)} F$ or simply $\deg F$. Moreover the rank of F is denoted by $\mathrm{rk}(F)$ and we denote by $\mu(F)$ (or, $P_F(m)$) the number $\deg(F)/\mathrm{rk}(F)$ (or, the polynomial $\chi(F \otimes \mathcal{O}_X(m))/\mathrm{rk}(F)$, resp.) when $\mathrm{rk}(F) \neq 0$. When X and Y are S-schemes and E (or, F) is an $\mathcal{O}_X$-module (or, $\mathcal{O}_Y$-module, resp.), $E \otimes_S F$ denotes the sheaf $p_X^*(E) \otimes_{\mathcal{O}_{X \times_S Y}} p_Y^*(F)$, where p_X (or, p_Y) is the projection of $X \times_S Y$ to X (or, Y, resp.). For an $\mathcal{O}_S$-module E and a morphism $f: X \to S$, we shall use the notation E_X instead of $f^*(E)$. In particular, if E and F are $\mathcal{O}_X$-modules, the $E \otimes_X F$ means $E \otimes_{\mathcal{O}_X} F$.

§1. Boundedness of the family of semi-stable pairs

Let $f: X \to S$ be a smooth, projective, geometrically integral morphism of noetherian schemes and let $\mathcal{O}_X(1)$ be an f-very ample invertible sheaf. Fix a locally free $\mathcal{O}_X$-module E of finite rank.

Definition 1.1. Let F be a coherent sheaf on X and φ be an $\mathcal{O}_X$-homomorphism of F to $F \otimes_X E$. φ induces a natural homomorphism φ' of $E^\vee$ to $\mathcal{E}nd_{\mathcal{O}_X}(F)$. A pair (F, φ) is said to be an E-pair if φ' can be extended to the natural homomorphism of $S^*(E^\vee)$ to $\mathcal{E}nd_{\mathcal{O}_X}(F)$ as $\mathcal{O}_X$-algebras. For an E-pair (F, φ), a subsheaf F' of F is said to be φ-invariant when $\varphi(F')$ is contained in $F' \otimes_X E$ and a quotient sheaf F'' of F is said to be φ-invariant when the kernel of the quotient map of F to F'' is φ-invariant. The numerical polynomial $\chi(F(m))$ is called the Hilbert polynomial of the E-pair (F, φ).

For an E-pair (F, φ), we obtain the following $\mathcal{O}_X$-homomorphism:

$$\tilde{\varphi}: F \otimes_X S^*(E^\vee) \longrightarrow F. \tag{1.1.1}$$

For a coherent subsheaf F' of F, we put

$$\bar{F}' = \tilde{\varphi}(F' \otimes_X S^*(E^\vee)). \tag{1.1.2}$$

It is easy to see that the sheaf $\bar{F}'$ is the minimal φ-invariant subsheaf of F containing F'. Now let (F, φ) be an E-pair on a geometric fiber X_s of f and let r be the rank of F as an $\mathcal{O}_{X_s}$-module. For a coherent subsheaf F' of F, we put

$$\bar{F}'_0 = \tilde{\varphi}(F' \otimes_X S^*_r(E^\vee)). \tag{1.1.3}$$

Lemma 1.2. *Under the above situation, suppose that F is torsion free on X_s. Then the degree of $\bar{F}'$ equals that of $\bar{F}'_0$.*

Proof. Let U be the maximal open subscheme of X_s where F is locally free then we have $\operatorname{codim}(X_s, X_s - U) \geq 2$. It is sufficient to prove that $\bar{F}'$ is equal to $\bar{F}'_0$ on each open subset $V = \operatorname{Spec}(A)$ of U where $E^\vee$ is a free A-module with a basis $x_1, \dots, x_m$. Then $\bar{F}'$ (or, $\bar{F}'_0$) is generated by the set

$$\{\varphi'(x_1)^{i_1} \cdots \varphi'(x_m)^{i_m}(f) \mid f \in F',\ 0 \leq i_1, \dots, i_m\}$$

$$(\text{or, } \{\varphi'(x_1)^{i_1} \cdots \varphi'(x_m)^{i_m}(f) \mid f \in F',\ 0 \leq i_1, \dots, i_m \leq r-1\}, \text{ resp.}),$$

where φ' is the induced homomorphism of $S^*(E^\vee)$ to $\mathcal{E}nd_{\mathcal{O}_X}(F)$ by φ. On the other hand, by Hamilton-Caylay's Theorem, each $\varphi'(x_i)$ satisfies a monic polynomial of degree r. Thus we see that $\bar{F}' = \bar{F}'_0$. Q.E.D.

Definition 1.3. An E-pair (F, φ) on a geometric fiber X_s of f is said to be semi-stable (or, stable) (with respect to $\mathcal{O}_X(1)$) if F is torsion free and for all non-trivial φ-invariant coherent subsheaves F' of F, we have

$$P_{F'}(m) \leq P_F(m) \quad (\text{or, } P_{F'}(m) < P_F(m), \text{ resp.})$$

for all large integers m.

Definition 1.4. An E-pair (F, φ) on a geometric fiber X_s of f is said to be μ-semi-stable (or, μ-stable) if F is torsion free and for all non-trivial φ-invariant coherent subsheaves F' of F,

$$\mu(F') \leq \mu(F) \quad (\text{or, } \mu(F') < \mu(F), \text{ resp.}).$$

As in the case of torsion free sheaves, we have the following relations:

$$\begin{array}{ccc} \mu\text{-stable} & \Longrightarrow & \text{stable} \\ \Downarrow & & \Downarrow \\ \mu\text{-semi-stable} & \Longleftarrow & \text{semi-stable} \end{array}$$

Definition 1.5. Let α be a rational number. An E-pair (F, φ) (or, a coherent sheaf F) on a geometric fiber X_s of f is said to be of type α, if F is torsion free and for all non-trivial φ-invariant coherent subsheaves (or, for all non-trivial coherent subsheaves, resp.) F' of F, the following holds

$$\mu(F') \leq \mu(F) + \alpha.$$

Now let us consider on the boundedness of the family of classes of E-pairs of type α with a fixed Hilbert polynomial.

Proposition 1.6. *Let α be a rational number. There is a rational number β which depends only on α, r and E such that if an E-pair (F, φ) is of type α, then F is of type β.*

Proof. Let F' be a μ-semi-stable subsheaf of F such that $\mu(F')$ is maximal among all coherent subsheaves of F. We can take a positive integer l so that $S_r^*(E^\vee) \otimes_X \mathcal{O}_X(l)$ is generated by global sections. Then for some positive integer m. $\bar{F}'_0$ is a quotient sheaf of $F' \otimes_X \mathcal{O}_X(-l)^{\oplus m}$. Since $F' \otimes_X \mathcal{O}_X(-l)^{\oplus m}$ is μ-semi-stable, we have $\mu(F' \otimes_X \mathcal{O}_X(-l)^{\oplus m}) = \mu(F') - l \cdot d \leq \mu(\bar{F}'_0)$, where d is the degree of X with respect to $\mathcal{O}_X(1)$. By Lemma 1.2 and our hypothesis, we have $\mu(\bar{F}'_0) = \mu(\bar{F}') \leq \max(\mu(F), \mu(F) + \alpha)$. Hence F is of type $\max(0, \alpha) + l \cdot d$. Q.E.D.

By the result of M. Maruyama [8], we have

Corollary 1.7. *Suppose that one of the following conditions is satisfied:*

(*a*) *S is a noetherian scheme over a field of characteristic zero.*

(*b*) *The rank is not greater than* 3.

(*c*) *The dimension of X over S is not greater than* 2.

Then the family of classes of E-pairs of type α with a fixed Hilbert polynomial is bounded. In particular, the family of μ-semi-stable pairs with a fixed Hilbert polynomial is bounded.

Definition 1.8. Let e be a non-negative integer and let (F, φ) be an E-pair on a geometric fiber X_s of X over S.

1) (F, φ) is said to be e-semi-stable (or, e-stable) (with respect to $\mathcal{O}_X(1)$) if it is semi-stable (or, stable) (with respect to $\mathcal{O}_X(1)$) and if for general non-singular curves $C = D_1 \cdot \cdots \cdot D_{n-1}$, $D_1, \ldots, D_{n-1} \in |\mathcal{O}_{X_s}(1)|$, $(F|_C, \varphi|_C)$ is of type e, where n is the dimension of X_s.

2) (F, φ) is said to be strictly e-semi-stable if it is e-semi-stable and if for every φ-invariant coherent quotient sheaf F' of F with $P_{F'}(m) = P_F(m)$, the E-pair (F', φ') induced by (F, φ) is e-semi-stable.

Let $\mathfrak{S}_{E/X/S}(e, H)$ be the family of classes of E-pairs on the fibers of X over S such that (F, φ) is contained in $\mathfrak{S}_{E/X/S}(e, H)$ if and only if (F, φ) is e-semi-stable and its Hilbert polynomial is H.

By Lemma 3.3 of [6] and Proposition 1.6, we have

Corollary 1.9. *For each e, H, $\mathfrak{S}_{E/X/S}(e, H)$ is bounded.*

By virtue of the fundamental lemma 2.2 in [6], Proposition 1.6 and the similar proof as in Proposition 3.6 in [6], we have

Proposition 1.10. *For each* $\mathfrak{S}_{E/X/S}(e, H)$, *there exists an integer* N *such that*

1) *for all* $(F, \varphi) \in \mathfrak{S}_{E/X/S}(e, H)$, $m \geq N$ *and* $i > 0$, $F(m)$ *is generated by its global sections and* $h^i(F(m)) = 0$,

2) *if* (F, φ) *is contained in* $\mathfrak{S}_{E/X/S}(e, H)$ *and if it is stable, then for all* $m \geq N$ *and all* φ*-invariant coherent subsheaves* F' *of* F *with* $0 \neq F' \subsetneqq F$,

$$h^0(F'(m))/\mathrm{rk}(F') < h^0(F(m))/rk(F)$$

3) *if* (F, φ) *is contained in* $\mathfrak{S}_{E/X/S}(e, H)$ *and if it is not stable, then for all* $m \geq N$ *and all* φ*-invariant coherent subsheaves* F' *of* F *with* $0 \neq F' \subsetneqq F$,

$$h^0(F'(m))/\mathrm{rk}(F') \leq h^0(F(m))/\mathrm{rk}(F)$$

and, moreover, there exists a non-trivial φ*-invariant coherent subsheaf* F_0 *of* F *such that* $h^0(F_0(m))/\mathrm{rk}(F_0) = h^0(F(m))/rk(F)$ *for all* $m \geq N$.

For the openness of the property "strictly e-semi-stability", we have

Proposition 1.11. *Let* $g: Y \to T$ *be a smooth, projective, geometrically integral morphism of locally noetherian schemes,* $\mathcal{O}_Y(1)$ *be a* g*-very ample invertible sheaf on* Y, E *be a locally free* $\mathcal{O}_Y$*-module and* (F, φ) *be an* E*-pair such that* F *is* T*-flat. If* $H^i(Y_t, \mathcal{O}_Y(1) \otimes k(t)) = 0$ *for all* $i > 0$ *and* $t \in T$, *then there exists an open set* U *of* T *such that for all algebraically closed field* k, $U(k) = \{t \in T(k) \mid (F, \varphi) \otimes k(t)$ *is strictly* e*-semi-stable with respect to* $\mathcal{O}_Y(1)\}$.

Proof. Let $\mathrm{Quot}_{(F,\varphi)/Y/T}$ be the subfunctor of $\mathrm{Quot}_{F/Y/T}$ defined in the following (1.11.1):

(1.11.1) $\mathrm{Quot}_{(F,\varphi)/Y/T}(S) = \{x \in \mathrm{Quot}_{F/Y/T}(S) \mid$ the quotient sheaf F' of F_S corresponding to x is φ-invariant$\}$.

$\mathrm{Quot}_{(F,\varphi)/X/S}$ is represented by a closed subscheme of $\mathrm{Quot}_{F/X/S}$ (see Lemma 4.3). We omit the rest of the proof, since it is same as the proof of Proposition 3.6 in [7] if we use the scheme $\mathrm{Quot}_{(F,\varphi)/X/S}$ instead of $\mathrm{Quot}_{F/X/S}$. □

§2. Definition of moduli functors

Let X be a non-singular projective variety over an algebraically closed field k, with a very ample invertible sheaf $\mathcal{O}_X(1)$ and let E be a locally free sheaf of finite rank on X.

Definition 2.1. Let (F, φ) be a semi-stable E-pair. A filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_t = F$ by φ-invariant coherent subsheaves is called a Jordan-Hölder filtration if $(F_i/F_{i-1}, \varphi_i)$ is stable and $P_{F_i}(m) = P_F(m)$ $(1 \leq i \leq t)$, where φ_i is a homomorphism induced by φ. For a Jordan-Hölder filtration $0 = F_0 \subset F_1 \subset \cdots$

$\subset F_t = F$, define $\mathrm{gr}(F, \varphi)$ to be $(\bigoplus_{i=0}^{t} F_i/F_{i-1}, \bigoplus_{i=0}^{t} \varphi_i)$.

By the same argument as in Proposition 1.2 of [7], we have the following.

Proposition 2.2. *Every semi-stable E-pair (F, φ) has a Jordan-Hölder filtration. If $0 = F_0 \subset F_1 \subset \cdots \subset F_t = F$ and $0 = F'_0 \subset F'_1 \subset \cdots \subset F'_s = F$ are two Jordan-Hölder filtrations for (F, φ), then $t = s$ and there exists a permutation σ of $\{1, 2, \ldots, t\}$ such that $(F_i/F_{i-1}, \varphi_i)$ is isomorphic to $(F'_{\sigma(i)}/F'_{\sigma(i-1)}, \varphi'_{\sigma(i)})$.*

Now we define the moduli functor of semi-stable E-pairs. Let $f: X \to S$ be a smooth, projective, geometrically integral morphism of noetherian schemes with an f-very ample invertible sheaf $\mathcal{O}_X(1)$. We denote by (Sch/S) the category of locally noetherian schemes over S. Let E be a locally free $\mathcal{O}_X$-module of finite rank and $H(m)$ be a numerical polynomial. The functor $\bar{\Sigma}^H_{E/X/S}$ of (Sch/S) to the category of sets is defined as follows.

For an object T of (Sch/S),

$\bar{\Sigma}^H_{E/X/S}(T) = \{(F, \varphi) | F$ is a T-flat, coherent $\mathcal{O}_{X \times_S T}$-module and φ is an $\mathcal{O}_{X \times_S T}$-homomorphism of F to $F \otimes_X E$ with the property (2.3.1)$\}/\sim$, where $\sim$ is the equivalence relation defined in (2.3.2).

(2.3.1) For every geometric point t of T, $(F \otimes_T k(t), \varphi \otimes_T k(t))$ is a semi-stable $E \otimes_S k(t)$-pair and the Hilbert polynomial of $F \otimes_T k(t)$ is $H(m)$.

(2.3.2) $(F, \varphi) \sim (F', \varphi')$ is and only if (1) $(F, \varphi) \simeq (F' \otimes_T L, \varphi \otimes_T id_L)$ or (2) there exist filtrations $0 = F_0 \subset F_1 \subset \cdots \subset F_u = F$ and $0 = F'_0 \subset F'_1 \subset \cdots \subset F'_u = F'$ by φ (or, φ') invariant coherent $\mathcal{O}_{X \times_S T}$-modules such that for every geometric point t of T, their restrictions to $X \times_T \mathrm{Spec}\, k(t)$ provide us with Jordan-Hölder filtrations of $(F \otimes_T k(t), \varphi \otimes_T k(t))$ and $(F' \otimes_T k(t), \varphi' \otimes_T k(t))$, respectively, $\bigoplus_{i=0}^{u} F_i/F_{i-1}$ is T-flat and that $(\bigoplus_{i=0}^{u} F_i/F_{i-1}, \bigoplus_{i=0}^{u} \varphi_i) \simeq ((\bigoplus_{i=0}^{u} F'_i/F'_{i-1}) \otimes_T L, \bigoplus_{i=0}^{u} \varphi'_i \otimes id_L)$, for some invertible sheaf L on T. The equivalence class of (F, φ) is denoted by $[(F, \varphi)]$.

For a morphism $g: T' \to T$ in (Sch/S), g^* defines a map of $\bar{\Sigma}^H_{E/X/S}(T)$ to $\bar{\Sigma}^H_{E/X/S}(T')$. It is obvious that $\bar{\Sigma}^H_{E/X/S}$ is a contravariant functor of (Sch/S) to $(Sets)$.

Moreover, we need to define a subfunctor of $\bar{\Sigma}^H_{E/X/S}$. Let e be a non-negative integer. For an object T of (Sch/S),

$$\bar{\Sigma}^{H,e}_{E/X/S}(T) = \{[(F, \varphi)] \in \bar{\Sigma}^H_{E/X/S}(T) | (F, \varphi) \text{ satisfies the property } (2.4)^e\}.$$

$(2.4)^e$ For every geometric point t of T, $(F \otimes_T k(t), \varphi \otimes_T k(t))$ is strictly e-semi-stable.

If $(F, \varphi) \sim (F', \varphi')$ and (F, φ) satisfies the property $(2.4)^e$, then (F', φ') has the same property (see § 3 of [7]). Hence the above definition is well-defined. By virtue of Proposition 1.11, if $H^i(X_s, \mathcal{O}_X(1) \otimes \mathcal{O}_{X_s}) = 0$ for all $i > 0$, $s \in S$, then $\bar{\Sigma}^{H,e}_{E/X/S}$ is an open subfunctor of $\bar{\Sigma}^H_{E/X/S}$.

§3. Semi-stable points of extended Gieseker spaces

Let X be a smooth, projective variety over a field k and $\mathcal{O}_X(1)$ be a very ample invertible sheaf. Take an N-dimensional vector space V over k. Let E and F be locally free $\mathcal{O}_X$-modules of rank l and m, respectively. Fix a non-negative integer r. The algebraic group $G = GL(V) \simeq GL(k, N)$ acts naturally on the vector space $W = \mathrm{Hom}_{\mathcal{O}_X}(\overset{r}{\wedge}(V \otimes_k E), F)$. Hence we have an action of G on the projective space $\mathbf{P}(W^\vee)$ and a G-linearized invertible sheaf $\mathcal{O}(1)$ on $\mathbf{P}(W^\vee)$. If $E = \mathcal{O}_X$, then $W = \mathrm{Hom}_k(\overset{r}{\wedge} V, H^0(X, F))$ and $\mathbf{P}(W^\vee)$ is the Gieseker space $P(V, r, H^0(X, F))$ which has been exploited to construct a moduli of semi-stable sheaves (see [2], [6], [7]). We denote $\mathbf{P}(W^\vee)$ with the action of G and the G-linearized invertible sheaf $\mathcal{O}(1)$ defined as above by $P_E(V, r, F)$. It is called also a Gieseker space. From now on, we assume that F is an invertible sheaf.

For a field K containing k, a non-zero element T of $\mathrm{Hom}_{\mathcal{O}_X}(\overset{r}{\wedge}(V \otimes_k E), F) \otimes_k K = \mathrm{Hom}_{\mathcal{O}_{X_K}}(\overset{r}{\wedge}(V_K \otimes_K E_K), F_K)$ gives rise to a K-rational point of $P_E(V, r, F)$, which is denoted by T, too. For vector subspaces $V_1, \dots, V_r$ of $V \otimes_k K$, the image of $(V_1 \otimes_K E_K) \otimes \cdots \otimes (V_r \otimes_K E_K)$ by the canonical homomorphism $(V_K \otimes_K E_K)^{\otimes r} \to \overset{r}{\wedge}(V_K \otimes_K E_K)$ is denoted by $[V_1, \dots, V_r]$ and if V_i is a one-dimensional subspace generated by x_i, we use the notation $[V_1, \dots, V_{i-1}, x_i, V_{i+1}, \dots, V_r]$ for $[V_1, \dots, V_r]$.

We shall extend the notion "T-independence" to our new Gieseker spaces.

Definition 3.1. Let K be an algebraically closed field containing k and let T be a non zero element of $\mathrm{Hom}_{\mathcal{O}_{X_K}}(\overset{r}{\wedge}(V_K \otimes_K E_K), F_K)$ or a K-rational point of $P_E(V, r, F)$. Vectors $x_1, \dots, x_d$ in V_K are said to be T-independent if the restriction of T to the subspace $[x_1, \dots, x_d, V, \dots, V]$ is not zero. A vector x is said to be T-dependent on $x_1, \dots, x_d$ if the restriction of T to the subspace $[x_1, \dots, x_d, x, V, \dots, V]$ is zero. For a vector subspace V' of V_K, vectors $x_1, \dots, x_d$ in V' is called a T-base of V' if $x_1, \dots, x_d$ are T-independent and if all vectors in V' are T-dependent on $x_1, \dots, x_d$. For a T-base $x_1, \dots, x_d$, the number d is called its length and the maximal (or, minimal) length among all T-bases of V' is called the maximal (or, minimal) T-dimension of V' and denoted by $\overline{\dim}_T V'$ (or, $\underline{\dim}_T V'$, resp.).

By a similar proof as in Proposition 2.2 and Proposition 2.3 of [2], we hae

Proposition 3.2. *Let K be an algebraically closed field containing k.*

1) *A point T in $P_E(V, r, F)(K)$ is properly stable (or, semi-stable) with respect to the action $\bar{\sigma}$ of $PGL(V)$ if for all vector subspaces V' of V_K, the following inequalities hold*

$$\dim_K V' < (N/r) \cdot \underline{\dim}_T V'$$

$$(or,\ \dim_K V' \leq (N/r) \cdot \underline{\dim}_T V',\ resp).$$

2) *If a point T in $P_E(V, r, F)(K)$ stable (or, semi-stable), then for all vector subspaces V' of V_K, the following inequalities hold*

$$\dim_K V' < (N/r) \cdot \overline{\dim}_T V'$$

$$(or,\ \dim_K V' \leq (N/r) \cdot \overline{\dim}_T V',\ resp).$$

Corollary 3.3. *Let T be a K-valued geometric point of $P_E(V, r, F)$ with the following property* (3.3.1).

(3.3.1) *For all vector subspaces V' of V_K, $\overline{\dim}_T V' = \underline{\dim}_T V'$.*

Then T is semi-stable (or, stable) if and only if for all vector subspaces V' of V_K(or, for all vector ubspaces V' of V_K such that $0 < \dim_T V' < r$),

$$\dim_K V' < (N/r) \cdot \dim_T V'$$

$$(or,\ \dim_K V' \leq (N/r) \cdot \dim_T V',\ resp).$$

Next we must analyze orbit spaces of $P_E(V, r, F)$.

Definition 3.4. Let T, T' and T'' be K-valued geometric points of $P_E(V, r, F)$, $P_E(V', r', F')$ and $P_E(V'', r'', F'')$, respectively. Let $\phi: F' \otimes F'' \to F$ be an injective homomorphism. T is said to be a ϕ-extention or, simply an extention of T'' by T' if the following conditions are satisfied;

1) $r = r' + r''$,
2) there exists an exact sequence

$$0 \longrightarrow V' \otimes_k K \xrightarrow{f} V \otimes_k K \xrightarrow{g} V'' \otimes_k K \longrightarrow 0$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \overset{r'}{\wedge}(V'_K \otimes_K E_K) \otimes_{\mathcal{O}_{X_K}} \overset{r''}{\wedge}(V_K \otimes_K E_K) & \longrightarrow & \overset{r}{\wedge}(V_K \otimes_K E_K) \\ {\scriptstyle T' \otimes (\overset{r''}{\Lambda}(g \otimes id_E))} \Big\downarrow & & \Big\downarrow {\scriptstyle T} \\ F'_K \otimes_{\mathcal{O}_{X_K}} F''_K & \xrightarrow[\phi_K]{} & F_K . \end{array}$$

In this case T' (or, T'') is said to be a subpoint (or, quotient point, resp.) of T.

Definition 3.5. Let T be a K-valued geometric point of $P_E(V, r, F)$. T is said to be excellent if it has the property (3.3.1) and the following (3.5.1).

(3.5.1) For every subpoint T' of T, if $x_1, \dots, x_d$ is a T'-base of a subspace V'_0

of V', then $f(x_1), \dots, f(x_d)$ is a T-base of V'_0.

(3.5.1) implies the following (3.5.1)$'$.

(3.5.1)$'$ For every subpoint T' of T and every subspace V'_0 of V'_K,

$$\underline{\dim}_T V'_0 \leq \underline{\dim}_{T'} V'_0 \leq \overline{\dim}_{T'} V'_0 \leq \overline{\dim}_T V'_0 .$$

Definition 3.6. Let T, T' and T'' be K-valued geometric points of $P_E(V, r, F)$, $P_E(V', r', F')$ and $P_E(V'', r'', F'')$, respectively and let $\phi: F' \otimes F'' \to F$ be an injective homomorphism. Assume T is a ϕ-extention of T'' by T' and let

$$0 \longrightarrow V' \otimes_k K \xrightarrow{f} V \otimes_k K \xrightarrow{g} V'' \otimes_k K \longrightarrow 0$$

be the underlying exact sequence of the extention. T is said to be a ϕ-direct sum of T' and T'' if there exists a linear map $i: V'' \otimes_k K \to V \otimes_k K$ such that $g \circ i = id_{V'' \otimes K}$ and $T|_{[i(y_1),\dots,i(y_s),w_{s+1},\dots,w_r]} = 0$ for all $y_1, \dots, y_s$ in $V'' \otimes_k K$ and for all $w_{s+1}, \dots, w_r$ in $V \otimes_k K$ whenever $s > r''$.

If T_1 and T_2 are two ϕ-direct sums of T' and T'', then $T_1 \simeq T_2$ (see Lemma 2.16 of [7]). Thus a direct sum of T' and T'' can be denoted by $T' \oplus T''$. Moreover let T'_i be a K-valued geometric point of $P_E(V'_i, l_i, F'_i)$ $(1 \leq i \leq t)$ and put $r_i = l_1 + \cdots + l_i$ and $V_i = V'_1 \oplus \cdots \oplus V'_i$. Let $\phi_i: F_{i-1} \otimes F'_i \to F_i$ be a sequence of injective homomorphisms $(1 \leq i \leq t,\ F_0 = \mathcal{O}_X)$. We can define ϕ_i-direct sum of T_{i-1} and T'_i inductively. Each T_i is a K-valued geometric point of $P_E(V_i, r_i, F_i)$ and it is denoted by $(\cdots((T'_1 \oplus T'_2) \oplus T'_3) \oplus \cdots) \oplus T'_i)$. By a similar argument as in Lemma 2.19 and corollary 2.19.1 of [7] we can denote T_i by $T'_1 \oplus \cdots \oplus T'_i$.

Now the main result in §2 of [7] can be extended to our case. Since the proof is similar to that of Theorem 2.13 and 2.22 of [7] and it is not difficult to rewrite so as to suit our case, we omit the proof.

Theorem 3.7. *Let $\phi_i: F_{i-1} \otimes F'_i \to F_i$ be injective homomorphisms $(1 \leq i \leq t$, $F_0 = \mathcal{O}_X)$, $0 < r_1 < \cdots < r_t = r$ be a sequence of integers and let D_i be a $GL(V_i)$-invariant closed set of $P_E(V_i, r_i, F_i)$ $(1 \leq i \leq t)$. Assume that for every algebraically closed field K containing k, all the points of $D_i(K)$ are excellent and that $\dim_k V_1/r_1 = \cdots = \dim_k V_t/r_t$. Let S_i be a stable, excellent point in $P_E(V'_i, l_i, F'_i)(\bar{k})$ which is k-rational, where $l_i = r_i - r_{i-1}$ and $\bar{k}$ is the algebraic closure of k. Then there exists a $GL(V_t)$-invariant closed set $Z_t = Z(S_1, \dots, S_t)$ of $D_t^{ss} = D_t^{ss}(\mathcal{O}(1) \otimes \mathcal{O}_{D_t})$ such that for every algebraically closed field K containing k,*

$$Z_t(K) = \{T \in D_t(K) \mid T \textit{ has the following property } (*)_t\}.$$

$(*)_t$: *There exists a K-valued geometric point T_i in each $D_i^{ss} = D_i^{ss}(\mathcal{O}(1) \otimes \mathcal{O}_{D_i})$ such that $T_1 = S_1$, T_i is a ϕ_i-extention of S_i by T_{i-1} $(2 \leq i \leq t)$ and $T = T_t$.*

Moreover if $Z(S_1, \dots, S_t)$ is not empty, then $GL(V_t)$-orbit $o(S_1, \dots, S_t)$ of $S_1 \oplus \cdots \oplus S_t$ is a unique closed orbit in $Z(S_1, \cdots S_t)$.

§4. Morphism to Gieseker spaces

To construct a moduli scheme of semi-stable sheaves, D. Gieseker [2] and M. Maruyama [6], [7] constructed a morphism μ of a Quot-scheme to a projective bundle in the étale topology on a finite union of connected components of $\mathrm{Pic}_{X/S}$. Our aim in this section is to construct a scheme which is an analogy of Quot-schemes for our problem and which plays the same role as the above μ.

From now on, we shall fix the following situation:

(4.1) Let S be a scheme of finite type over a universally Japanese ring Ξ and let $f: X \to S$ be a smooth, projective, geometrically integral morphism such that the dimension of each fiber of X over S is n. Let $\mathcal{O}_X(1)$ be an f-very ample invertible sheaf such that for all points s in S and for all positive integers i, $H^i(X_s, \mathcal{O}_X(1) \otimes \mathcal{O}_{X_s}) = 0$ and let E be a locally free $\mathcal{O}_X$-module of finite rank.

Let V be a free Ξ-module of rank N and let G be the Ξ-group scheme $GL(V)$. Fix a numerical polynomial $H(m)$ which is the Hilbert polynomial of a coherent sheaf of rank r on a geometric fiber of f. Take $\tilde{Q}$ a union of some of connected components of $\mathrm{Quot}^H_{V \otimes_\Xi S^*_r(E^\vee)/X/S}$ and the universal quotient sheaf $\tilde{\phi}: V \otimes_\Xi S^*_r(E^\vee)_{X_{\tilde{Q}}} \to \tilde{F}$ on $X_{\tilde{Q}}$. We denote by $\tilde{\phi}^i$ the restriction of $\tilde{\phi}$ to $V \otimes_\Xi S^i(E^\vee)_{X_{\tilde{Q}}}$. Let $\tilde{Q}$ be the subset of $\tilde{Q}$ such that a point x of $\tilde{Q}$ is contained in $\tilde{Q}$ if and only if $\tilde{\phi}^0 \otimes_{\tilde{Q}} k(x)$ is surjective. By the properness of the projection of $X_{\tilde{Q}}$ to $\tilde{Q}$, $\tilde{Q}^0$ is an open set of $\tilde{Q}$ and clearly it is G-stable. Since the restriction of $\tilde{\phi}^0$ to $X_{\tilde{Q}^0}$ is surjective, it defines a morphism of $\tilde{Q}$ to $\mathrm{Quot}^H_{V \otimes_\Xi \mathcal{O}_X/X/S}$. Clearly it is a G-morphism. Let Q be a union of connected components with a non-empty intersection with the image of $\tilde{Q}^0$. Then we obtain a G-morphism of $\tilde{Q}^0$ to Q.

We shall need the following proposition (cf. EGA III (7.7.8), (7.7.9) or [1]).

Proposition 4.2. *Let $f: X \to S$ be a proper morphism of noetherian schemes, and let I and F be two coherent $\mathcal{O}_X$-modules with F flat over S. Then there exist a coherent $\mathcal{O}_S$-module $H(I, F)$ and an element $h(I, F)$ of* $\mathrm{Hom}_X(I, F \otimes_S H(I, F))$ *which represents the functor*

$$M \longmapsto \mathrm{Hom}_X(I, F \otimes_S M)$$

defined on the category of quasi-coherent $\mathcal{O}_S$-modules M, and the formation of the pair commutes with base change; in other words, the Yoneda map defined by $h(I, F)$

$$y: \mathrm{Hom}_T(H(I, F)_T, M) \longrightarrow \mathrm{Hom}_{X_T}(I_T, F \otimes_S M) \tag{4.2.1.}$$

is an isomorphism for every S-scheme T and every quasi-coherent $\mathcal{O}_T$-module M. Moreover if I is flat over S and if $\mathrm{Ext}^1_{X_s}(I \otimes k(s), F \otimes k(s)) = 0$ *for all points s of S, then $H(I, F)$ is locally free.*

Let $\phi: V \otimes_\Xi \mathcal{O}_{X_Q} \to F$ be the universal quotient sheaf on X_Q. Now let us apply Proposition 4.2 to the case $X = X_Q$, $S = Q$, $I = F$ and $F = F \otimes_X E$. Then we obtain a coherent $\mathcal{O}_Q$-module $H(F, F \otimes_X E)$. By virtue of Proposition 4.2, we know that the scheme $\Gamma' = \mathbf{V}(H(F, F \otimes_X E))$ represents the functor,

$$T \longmapsto \mathrm{Hom}_{X_T}(F_{X_T}, F_{X_T} \otimes_X E)$$

defined on the category of Q-schemes, moreover we have the universal homomorphism $\Phi\colon F_{X_{\Gamma'}} \to F_{X_{\Gamma'}} \otimes_X E$.

Lemma 4.3. *Let $f\colon X \to S$ be a proper morphism of noetherian schemes and let $\varphi\colon I \to F$ be an $\mathcal{O}_X$-homomorphism of coherent $\mathcal{O}_X$-modules with F flat over S. Then there exists a unique closed subscheme Z of S such that for all morphism $g\colon T \to S$, $g^*(\varphi) = 0$ if and only if g factors through Z.*

Proof. By the isomorphism (4.2.1), φ corresponds to an $\mathcal{O}_S$-homomorphism $\psi\colon H(I, F) \to \mathcal{O}_S$. The closed subscheme Z of S defined by the ideal sheaf $\mathrm{Image}(\psi)$ is the desired one. □

By virtue of Lemma 4.3, there exists a closed subscheme Γ of Γ' such that for all morphism $g\colon T \to \Gamma'$, $g^*(\Phi)$ can be extended to the homomorphism $F_{X_T} \otimes_X S^*(E^\vee) \to F_{X_T}$ defined as in (1.1.1) if and only if g factors through Γ. We have also the universal homomorphism $\tilde{\Phi}\colon F_{X_\Gamma} \otimes_X S^*(E^\vee) \to F_{X_\Gamma}$. Let $\pi\colon \Gamma \to Q$ be the structure morphism. The surjective homomorphism $\tilde{\Phi} \circ (\mathrm{id}_{X_Q} \times \pi)^*(\phi \otimes \mathrm{id}_{S^*(E^\vee)})\colon V \otimes_\Xi S^*(E^\vee)_{X_\Gamma} \to F_{X_\Gamma}$ defines a Q-morphism λ of Γ to $\tilde{Q}^0$ and clearly λ is a G-morphism. It is easy to see that λ is a closed immersion if we use Lemma 4.3 repeatedly.

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\lambda} & \tilde{Q}^0 & \longrightarrow & \tilde{Q} \\ & {\scriptstyle\pi}\searrow & \downarrow & & \\ & & Q & & \end{array}$$

From now on, we assume

(4.4) if an invertible sheaf L on a geometric fiber X_s of $X_{\tilde{Q}}$ has the same Hilbert polynomial as $(\det \tilde{F}) \otimes_{\tilde{Q}} k(s)$, then

$$\mathrm{Ext}^j_{\mathcal{O}_{X_s}}(\overset{r}{\wedge}(V \otimes_\Xi S^*_r(E^\vee)) \otimes_S k(s), L) = 0$$

for all positive integers j.

Remark 4.5. $\det \tilde{F}$ is the sheaf defined in Lemma 4.2 of [6] which is a G-linearized sheaf and we have a natural G-homomorphism γ of $\overset{r}{\wedge}\tilde{F}$ to $\det \tilde{F}$.

By (4.2.1), the homomorphism $\gamma \circ (\overset{r}{\wedge}\tilde{\phi})\colon \overset{r}{\wedge}(V \otimes_\Xi S^*_r(E^\vee)_{X_{\tilde{Q}}}) \to \det \tilde{F}$ defines the $\mathcal{O}_{\tilde{Q}}$-homomorphism δ of $H(\overset{r}{\wedge}(V \otimes_\Xi S^*_r(E^\vee)_{X_{\tilde{Q}}}), \det \tilde{F})$ to $\mathcal{O}_{\tilde{Q}}$. δ is surjective since for all points x of $\tilde{Q}$, $\delta \otimes k(x)$ corresponds to the non-zero homomorphism $(\gamma \circ (\overset{r}{\wedge}\tilde{\phi})) \otimes k(x)$ by (4.2.1). Hence δ defines a section $\sigma\colon \tilde{Q} \to \mathbf{P}(H(\overset{r}{\wedge}(V \otimes_\Xi S^*_r$

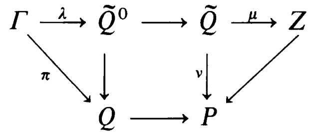
$(E^\vee)_{X_{\tilde{Q}}})$, det $\tilde{F})$). If f has a section, there exists a unique Poincaré sheaf L on $X \times_S \mathrm{Pic}_{X/S}$. det $\tilde{F}$ defines a G-morphism v of $\tilde{Q}$ to $\mathrm{Pic}_{X/S}$ with the trivial action of G on $\mathrm{Pic}_{X/S}$ (see Lemma 4.5 of [6]). Let P be a union of a finite number of connected components of $\mathrm{Pic}_{X/S}$ having non-empty intersection with $v(\tilde{Q})$. By virtue of Proposition 4.2 and the assumption (4.4) the $\mathcal{O}_P$-module $H(\overset{r}{\wedge}(V \otimes_{\Xi} S_r^*(E^\vee)_{X_{\tilde{Q}}}), L)$ is locally free. Set $Z = \mathbf{P}(H(\overset{r}{\wedge}(V \otimes_{\Xi} S_r^*(E^\vee)_{X_{\tilde{Q}}}), L))$. By the universality of L, we see that $(1_X \times v)^*(L) \simeq (\det \tilde{F}) \otimes_{\tilde{Q}} M$ for some invertible sheaf M on $\tilde{Q}$. By the universality of $H(-, -)$, we see that

$$v^*(H(\overset{r}{\wedge} \otimes V \otimes_{\Xi} S_r^*(E^\vee)_{X_P}), L) \simeq H(\overset{r}{\wedge}(V \otimes_{\Xi} S_r^*(E^\vee)_{X_{\tilde{Q}}}), (\det \tilde{F}) \otimes_{\tilde{Q}} M)$$

$$\simeq H(\overset{r}{\wedge}(V \otimes_{\Xi} S_r^*(E^\vee)_{X_{\tilde{Q}}}), \det \tilde{F}) \otimes_{\tilde{Q}} M^\vee.$$

Therefore we have $Z \times_P \tilde{Q} \simeq \mathbf{P}(H(\overset{r}{\wedge}(V \otimes_{\Xi} S_r^*(E^\vee)_{X_{\tilde{Q}}}), \det \tilde{F})$ and the section σ defines a P-morphism μ of $\tilde{Q}$ to Z which is also a G-morphism.

(4.6)



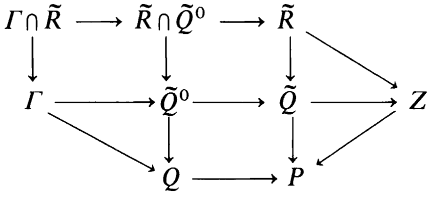
Let $\tilde{R}$ be the open set of $\tilde{Q}$ such that for every algebraically closed field K, $\tilde{R}(K) = \{x \in \tilde{Q}(K) | \tilde{F} \otimes k(x) \text{ is torsion free}\}$ (see [5]). $\tilde{Q}$ has a natural G-action and clearly $\tilde{R}$ is a G-stable open set of $\tilde{Q}$. By the similar argument as in [6], we have

Proposition 4.7. *Assume* (4.4) *holds for* $\tilde{Q}$ *and* $\tilde{F}$. *Then there exist an open and closed subscheme* P *of* $\mathrm{Pic}_{X/S}$ *of finite type over* S *and a* $\mathbf{P}^m$*-bundle* $p: Z \to P$ *in the étale topology on* P *such that*

1) *G acts on Z and there exists a p-ample G-linearized invertible sheaf H on Z,*
2) *there exists a G-morphism $\mu: \tilde{Q} \to Z$ with $\mu|_{\tilde{R}}$ an immersion.*
3) *if $u: S' \to S$ is an étale, surjective morphism such that $f' = f \times_S S'$ has a section, then $Z \times_S S'$ and $\mu \times_S S'$ are the same defined in* (4.6).

Consequently we obtain the following commutative diagram of G-morphism:

(4.8)



§5. Construction of moduli spaces

Let $f: X \to S$, $\mathcal{O}_X(1)$ and E be as in (4.1). We may assume that S is connected. Set $H^{(i)}(m) = i \cdot H(m)/r$ for $1 \le i \le r$, where $r = rk(F)$ for an (F, φ) with $[(F, \varphi)] \in \bar{\Sigma}^H_{E/X/S}(\operatorname{Spec} k(s))$. By an argument similar to Lemma 4.2 of [7] and Proposition 1.10, we have

Lemma 5.1. *For each non-nagative integer e, there exists an integer m_e such that if $m \ge m_e$, then for all geometric points s of S and for all strictly e-semi-stable pairs (F, φ) on X_s with $\operatorname{rk}(F) = i$ and $\chi(F(m)) = H^{(i)}(m)$,*

(5.1.1) *$F(m)$ is generated by its global sections and $h^j(X_s, F(m)) = 0$ if $j > 0$,*

(5.1.2) *for all φ-invariant coherent subsheaves F' of F with $F' \neq 0$, $h^0(F'(m)) \le \operatorname{rk}(F') \cdot h^0(F(m))/i$ and moreover, the equality holds if and only if $P_{F'}(m) = P_F(m) = H(m)/r$,*

(5.1.3.) *if an invertible sheaf L on X_s has the same Hilbert polynomial as* $\det(F(m))$, *then* $\operatorname{Ext}^j_{\mathcal{O}_{X_s}}(\overset{r}{\wedge}(V \otimes_{\Xi} S^*_r(E^\vee), L) = 0$ *for all positive integers j, where V is a free Ξ-module of rank r.*

Remark 5.1.4. If (5.1.3) holds, then for all invertible sheaf L on X_s with the same Hilbert polynomial as $\det(F(m))$ and for all free Ξ-module V, $\operatorname{Ext}^j_{\mathcal{O}_{X_s}}(\overset{r}{\wedge}(V \otimes_{\Xi} S^*_r(E^\vee), L) = 0 \ (j > 0)$.

We may assume that $m_e \ge m_{e'}$ if $e \ge e'$. Set $H^{(i,e)}(m) = H^{(i)}(m + m_e)$ and $N^{(i,e)} = H^{(i,e)}(0) = H^{(i)}(m_e)$. Let $V_{i,e}$ be a free Ξ-module of rank $N^{(i,e)}$ and let G_i be the Ξ-group scheme $GL(V_{i,e})$. Let us consider the scheme

$$\tilde{Q}_i = \operatorname{Quot}^{H^{(i,e)}}_{V_{i,e}\otimes_{\Xi} S^*_r(E^\vee)/X/S}$$

and its subscheme Γ_i constructed in §4. Let $\phi^e_i\colon V_{i,e} \otimes_{\Xi} \mathcal{O}_{X_{\Gamma_i}} \to F^e_i$ be the universal quotient and $\varphi^e_i\colon F^e_i \to F^e_i \otimes_X E$ be the universal homomorphism on X_{Γ_i}. By virtue of Proposition 1.11 and (5.1,1), there exists an open set $R^{e,e'}_i$ in Γ_i such that a geometric point y of Γ_i is contained in $R^{e,e'}_i$ if and only if

(5.2.1) $\Gamma(\phi^e_i \otimes k(y))\colon V_{i,e} \otimes_{\Xi} k(y) \to H^0(X_y, F^e_i \otimes_{\Gamma_i} k(y))$ is bijective and

(5.2.2) $(F^e_i \otimes_{\Gamma_i} k(y), \varphi^e_i \otimes_{\Gamma_i} k(y))$ is strictly e'-semi-stable.

By virtue of (5.1.1) and the universality of Γ_i, for every geometric point s of S, we have the surjective map;

$$\begin{aligned}\xi^{e,e'}_i(s)\colon R^{e,e'}_i(k(s)) &\longrightarrow \bar{\Sigma}^{H^{(i)},e'}_{E/X/S}(m_e)(\operatorname{Spec} k(s)) \\ &= \{[(F(m_e), \varphi \otimes 1_{\mathcal{O}(m_e)})] \mid (F, \varphi) \in \bar{\Sigma}^{H^{(i)},e'}_{E/X/S}(\operatorname{Spec} k(s))\},\end{aligned}$$

where $\xi^{e,e'}_i(s)$ maps $k(s)$-valued point y of $R^{e,e'}_i$ to the pair $(F^e_i \otimes_{\Gamma_i} k(y), \varphi^e_i \otimes_{\Gamma_i} k(y))$. Moreover, $R^{e,e'}_i$ is G_i-invariant and K-valued geometric points y_1 and y_2 of $R^{e,e'}_i$ are in the same orbit of $G_i(K)$ if and only if $(F^e_i \otimes_{\Gamma_i} k(y_1), \varphi^e_i \otimes_{\Gamma_i} k(y_1)) \simeq (F^e_i \otimes_{\Gamma_i} k(y_2), \varphi^e_i \otimes_{\Gamma} k(y_2))$ (see §5 of [6]).

Let $\bar{R}^{e,e'}_i$ be the scheme theoretic closure of $R^{e,e'}_i$ in $\tilde{Q}_i$. Now we replace $\tilde{Q}_i$ by a union of connected components of $\tilde{Q}_i$ having a non-empty intersection with $R_i^{e,e'}$. Let ν_i be the morphism of $\tilde{Q}_i$ to $\mathrm{Pic}_{X/S}$ defined in §4 and let P_i be the union of connected components which intersect with $\nu_i(\tilde{Q}_i)$. Then by the condition (5.1.3) we obtain a G_i-morphism μ_i of $\tilde{Q}_i$ to Z_i defined in Proposition 4.7. Let Δ_i be the scheme theoretic image of $R_i^{e,e'}$ by μ_i. Then μ_i induces an open immersion of $R_i^{e,e'}$ to Δ_i. Consequently, we obtain the following commutative diagram of G_i-morphisms:

$$\begin{array}{ccccc} R_i^{e,e'} & \longrightarrow & \bar{R}_i^{e,e'} & \longrightarrow & \tilde{Q}_i & & \\ & \searrow & \downarrow & & \downarrow{\scriptstyle \mu_i} & \searrow{\scriptstyle \nu_i} & \\ & & \Delta_i & \longrightarrow & Z_i & \xrightarrow[p_i]{} & P_i. \end{array}$$

For all K-valued geometric points x of P_i, $(Z_i)_x$ is isomorphic to the Gieseker space $P_{S_i^*(E^\vee)}(V_{i,e} \otimes_\Xi K, i, L_x)$, where L_x is an invertible sheaf on X_K corresponding to x. By an argument similar to Lemma 4.4 of [7], we know that if T is a K'-valued geometric point of $(\Delta_i)_x$, then T is excellent in $(Z_i)_x = P_{S_i^*(E^\vee)}(V_{i,e} \otimes_\Xi K, i, L_x)$ and for every vector subspace V of $V_{i,e} \otimes_\Xi K'$,

$$(5.3.1) \qquad \overline{\dim}_T V = \underline{\dim}_T V = \mathrm{rk}(\Phi_i^e(V \otimes_K S_r^*(E^\vee))).$$

Let L_i be a G_i-linearized p_i-ample invertible sheaf on Z_i. Then there exist G_i-invariant open subshemes Δ_i^s and Δ_i^{ss} of Δ_i such that for all algebraically closed field K, $\Delta_i^s(K) = \{x \in \Delta_i(K) | x$ is a properly stable point of $(\Delta_i)_y$ with respect to the pull back of L_i to $(\Delta_i)_y$, where $y = p_i(K)(x)\}$ and $\Delta_i^{ss}(K) = \{x \in \Delta_i(K) | x$ is a semi-stable point of $(\Delta_i)_y$ with respect to the pull back of L_i to $(\Delta_i)_y$, where $y = p_i(K)(x)\}$. By virtue of Corollary 3.3, (5.1.2) and (5.3.1), the same argument as in Lemma 4.15 of [6] provides us with the following.

Lemma 5.4. *μ_i induces an open immersion of $R_i^{e,e'}$ to Δ_i^{ss}. Moreover, for a geometric point x of $R_i^{e,e'}$, if $(F_i^e \otimes k(x), \varphi_i^e \otimes k(x))$ is stable, then $\mu_i(x)$ is in Δ_i^s.*

By virtue of Theorem 4 of [12], there exists a good quotient $\pi: \Delta_r^{ss} \to Y$. Since S is of finite type over a universally Japanese ring, Y is projective over S. $\Delta_r^{ss} - \mu_r(R_r^{e,e'})$ is G_r-invariant closed set of Δ_r^{ss}. Set $\bar{M}_{e,e'} = Y - \pi(\Delta_r^{ss} - \mu_r(R_r^{e,e'}))$. $\bar{M}_{e,e'}$ is an open subscheme of Y. Hence $\bar{M}_{e,e'}$ is quasi-projective over S.

Let x be a k-valued geometric point of $R_i^{e,e'}$. Since $(F, \varphi) = (F_i^e \otimes k(x), \varphi_i^e \otimes k(x))$ is strictly e'-semi-stable, we can find a Jordan-Hölder filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_\alpha = F$. Set $r_i = rk(F_i)$ and $l_i = r_i - r_{i-1}$. Then $(F_{\alpha-1}, \varphi_{\alpha-1})$ and $(\bar{F}_{\alpha-1}, \varphi_{\alpha-1})$ and $(\bar{F}_\alpha, \bar{\varphi}_\alpha)$ are strictly e'-semi-stable (see lemma 3.5 of [7]) where $\bar{F}_\alpha = F/F_{\alpha-1}$. By virtue of (5.1.1), we get the following commutative diagrom;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(X_x, F_{\alpha-1}) & \longrightarrow & H^0(X_x, F) & \longrightarrow & H^0(X_x, F/F_{\alpha-1}) & \longrightarrow & 0 \\ & & {\scriptstyle \eta_{\alpha-1}}\uparrow\wr & & {\scriptstyle \eta_\alpha}\uparrow\wr & & {\scriptstyle \eta_{\bar{\alpha}}}\uparrow\wr & & \\ & & V_{r-1,e} \otimes_\Xi k & \longrightarrow & V_{r,e} \otimes_\Xi k & \longrightarrow & V_{l_\alpha,e} \otimes_\Xi k & & \end{array}$$

where $\eta_\alpha = \Gamma(\phi_r^e \otimes k(x))$. An isomorphism $\eta_{\alpha-1}$ (or, $\eta_{\bar{\alpha}}$) defines a k-rational point $x_{\alpha-1}$ (or $\bar{x}_\alpha$, resp.) of $R_{\alpha-1}^{e,e'}$ (or, $R_{l_\alpha}^{e,e'}$, resp.). If $T_\alpha = \mu_r(k)(x)$, $T_{\alpha-1} = \mu_{r_{\alpha-1}}(k)(x_{\alpha-1})$ and $\bar{T}_\alpha = \mu_{l_\alpha}(k)(\bar{x}_\alpha)$, then $T_\alpha \in P_{S_\Xi^*(E^\vee)}(V_{r,e} \otimes_\Xi k, r, \det F)$, $T_{\alpha-1} \in P_{S_\Xi^*(E^\vee)}(V_{r_{\alpha-1},e} \otimes_\Xi k, r_{\alpha-1}, \det F_{\alpha-1})$ and $\bar{T}_\alpha \in P_{S_\Xi^*(E^\vee)}(V_{l_\alpha,e} \otimes_\Xi k, l_\alpha, \det \bar{F}_\alpha)$. Let $\psi_\alpha\colon \det F_{\alpha-1} \otimes \det \bar{F}_\alpha \to \det F_\alpha$ be the canonical isomorphism. Then T_α is a ψ_α-extention of $\bar{T}_{\alpha-1}$ (see §4 of [7]). Let $\bar{F}_j = F_j/F_{j-1}$ and $\psi_j\colon \det F_{j-1} \otimes \det \bar{F}_j \to \det F_j$. Repeating the similar argument to the above, we get T_j in $P_{S_\Xi^*(E^\vee)}(V_{r_j,e} \otimes_\Xi k, r_j, \det F_j)$ $(1 \le j \le \alpha)$ and $\bar{T}_j$ in $P_{S_\Xi^*(E^\vee)}(V_{l_j,e} \otimes_\Xi k, l_j, \det \bar{F}_j)(1 \le j \le \alpha)$ such that

(5.4.1) $T_j = \mu_{r_j}(k)(x_j)$ for some x_j in $R_{r_j}^{e,e'}(k)$ and $\bar{T}_j = \mu_{l_j}(k)(\bar{x}_j)$ for some $\bar{x}$ in $R_{l_j}^{e,e'}(k)$. Moreover, $\bar{T}_j$ is in $\Delta_{l_j}^s(k)$.

(5.4.2) T_j is a ψ_j-extention of $\bar{T}_j$ by T_{j-1} and $T_1 \simeq \bar{T}_1$.

By a proof similar to lemma 4.7 of [7], we have

Lemma 5.5. $T_j \simeq T_{j-1} \oplus T_j$ *if and only if* $(F_j, \varphi_j) \simeq (F_{j-1}, \varphi_{j-1}) \oplus (\bar{F}_j, \bar{\varphi}_j)$.

Since $\mathrm{gr}(F, \varphi)$ is strictly e'-semi-stable (see Corollary 3.5.1 of [7]), $\mathrm{gr}(F, \varphi)$ corresponds to a point y in $R_r^{e,e'}(k)$.

Corollary 5.5.1. $\mu_r(k)(y) = \bar{T}_1 \oplus \cdots \oplus \bar{T}_\alpha$.

By virtue of Theorem 3.7 and a proof similar to Proposition 4.8, we obtain

Proposition 5.6. *Let y be a k-valued geometric point of P_r and let s be the image of y by the structure morphism $P_r \to S$. Let $(\bar{F}_1, \varphi_1), \dots, (\bar{F}_\alpha, \varphi_\alpha)$* be e'-stable E-pairs on X_s such that $l_i = rk(\bar{F}_i)$, $\chi(\bar{F}_i(m)) = H^{(l_i)}(m)$ *and* $l_1 + \cdots + l_\alpha = r$. *Then there exists a G_r-invariant closed subset $Z((\bar{F}_1, \varphi_1), \dots, (\bar{F}_\alpha, \varphi_\alpha))$ of $(R_r^{e,e'})_y = (v_r)^{-1}(y) \cap R_r^{e,e'}$ such that*

(5.6.1) *$\mu_r(Z((\bar{F}_1, \varphi_1), \dots, (\bar{F}_\alpha, \varphi_\alpha)))$ is closed in $(\Delta_r^{ss})_y$,*

(5.6.2) *for every algebraically closed field K containing k, $Z((\bar{F}_1, \varphi_1), \dots, (\bar{F}_\alpha, \varphi_\alpha))(K) = \{x \in (R_r^{e,e'}) | gr((F_r^e, \varphi_r^e) \otimes k(x)) \simeq (\oplus \bar{F}_i, \oplus \varphi_i)_K\}$*

(5.5.6) *the G_r-orbit of x_0 corresponding to $(\oplus \bar{F}_i, \oplus \varphi_i)$ is the unique closed orbit in $Z((\bar{F}_1, \varphi_1), \dots, (\bar{F}_\alpha, \varphi_\alpha))$.*

By Theorem 4 of [12], Proposition 5.6 and a proof similar to that of Proposition 4.9 and 4.10 of [7], we have

Proposition 5.7. *$\bar{M}_{e,e'}$ has the following properties*:

(5.7.1) *For each geometric point s of S, there exists a natural bijection* $\bar{\theta}_s\colon \bar{\Sigma}_{E/X/S}^{H,e'}(\mathrm{Spec}\,(k(s))) \to \bar{M}_{e,e'}(k(s))$.

(5.7.2) *For $T \in (Sch/S)$ and a pair (F, φ) of a T-flat coherent $\mathcal{O}_{X \times_S T}$-module F and an $\mathcal{O}_{X \times_S T}$-homomorphism of F to $F \otimes_X E$ with the property* (2.3.1) *and* (2.4)$^{e'}$, *there exists a morphism $\bar{f}_{(F,\varphi)}^{e,e'}$ of T to $\bar{M}_{e,e'}$ such that $\bar{f}_{(F,\varphi)}^{e,e'}(t) = \bar{\theta}([(F \otimes_T k(t), \varphi \otimes_T k(t))])$ for all points t in $T(k(s))$. Moreover, for a morphism $g\colon T' \to T$ in*

(Sch/S),

$$\bar{f}^{e,e'}_{(F,\varphi)} \circ g = \bar{f}^{e,e'}_{(1_X \times g)^*(F,\varphi)}.$$

(5.7.3) *If* $\bar{M}' \in (Sch/S)$ *and maps* $\bar{\theta}'_s$: $\bar{\Sigma}^{H,e'}_{E/X/S}(\mathrm{Spec}(k(s))) \to \bar{M}'(k(s))$ *have the above property* (5.7.2), *then there exists a unique S-morphism* $\bar{\Psi}$ *of* $\bar{M}_{e,e'}$ *to* $\bar{M}'$ *such that* $\bar{\Psi}(k(s)) \circ \bar{\theta}_s = \bar{\theta}'_s$ *and* $\bar{\Psi} \circ \bar{f}^{e,e'}_{(F,\varphi)} = \bar{f}'_{(F,\varphi)}$ *for all geometric points s of S and for all* (F, φ), *where* $\bar{f}'_{(F,\varphi)}$ *is the morphism given by the property* (5.7.2) *for* $\bar{M}'$ *and* $\bar{\theta}'_s$.

The construction of a moduli scheme of the functor $\bar{\Sigma}^H_{E/X/S}$ is completely same as in §4 of [7], that is, $\bar{M}_{E/X/S}(H) = \varinjlim_e \bar{M}_{e,e'}$.

Theorem 5.8. *In the situation of* (4.1), *there exists an S-scheme* $\bar{M}_{E/X/S}(H)$ *with the following properties*:

1) $\bar{M}_{E/X/S}(H)$ *is locally of finite type and separated over S.*

2) *There exists a coarse moduli scheme* $M_{E/X/S}(H)$ *of stable E-pairs with Hilbert polynimial H and it is contained in* $\bar{M}_{E/X/S}(H)$ *as an open subscheme.*

3) *For each geometric point s of S, there exists a natural bijection* $\bar{\theta}_s$: $\bar{\Sigma}^H_{E/X/S}(\mathrm{Spec}(k(s))) \to \bar{M}_{E/X/S}(H)(k(s))$.

4) *For* $T \in (Sch/S)$ *and a pair* (F, φ) *of a T-flat coherent* $\mathcal{O}_{X \times_S T}$*-module F and an* $\mathcal{O}_{X \times_S T}$*-homomorphism of F to* $F \otimes_X E$ *with the property* (2.3.1), *there exists a morphism* $\bar{f}_{(F,\varphi)}$ *of T to* $\bar{M}_{E/X/S}(H)$ *such that* $\bar{f}_{(F,\varphi)}(t) = \bar{\theta}_s([(F \otimes_T k(t), \varphi \otimes_T k(t))])$ *for all points t in* $T(k(s))$. *Moreover, for a morphism* g: $T' \to T$ *in* (Sch/S),

$$\bar{f}_{(F,\varphi)} \circ g = \bar{f}_{(1_X \times g)^*(F,\varphi)}.$$

5) *If* $\bar{M}' \in (Sch/S)$ *and maps* $\bar{\theta}'_s$: $\bar{\Sigma}^H_{E/X/S}(\mathrm{Spec}(k(s))) \to \bar{M}'(k(s))$ *have the above property* 4), *then there exists a unique S-morphism* $\bar{\Psi}$ *of* $\bar{M}_{E/X/S}(H)$ *to* $\bar{M}'$ *such that* $\bar{\Psi}(k(s)) \circ \bar{\theta}_s = \bar{\theta}'_s$ *and* $\bar{\Psi} \circ \bar{f}_{(F,\varphi)} = \bar{f}'_{(F,\varphi)}$ *for all geometric points s of S and for all* (F, φ), *where* $\bar{f}'_{(F,\varphi)}$ *is the morphism given by the property* 4) *for* $\bar{M}'$ *and* $\bar{\theta}'_s$.

Corollary 5.8.1. *If* $\mathfrak{S}_{E/X/S}(H)$ *is bounded, then* $\bar{M}_{E/X/S}(H)$ *is quasi-projective over S.*

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