J. Math. Kyoto Univ. (JMKYAZ)
31-1 (1991) 265–279

Algebraically independent generators of invariant differential operators on a bounded symmetric domain

Dedicated to Professor Nobuhiko Tatsuuma on his 60th birthday

By

Takaaki NOMURA

Introduction

Let $\mathscr{D}$ be a bounded symmetric domain in a finite dimensional complex vector space V. We denote by G the identity component of the Lie group of holomorphic automorphisms of $\mathscr{D}$. Then, G is semisimple and acts transitively on $\mathscr{D}$. Regarding $\mathscr{D}$ as a Riemannian symmetric space, one knows that the algebra $\boldsymbol{D}(\mathscr{D})^G$ of G-invariant differential operators on $\mathscr{D}$ is isomorphic to a polynomial algebra of r indeterminates, where r is the real rank of G, cf. [1, p. 277].

On the other hand, it is now widely known that the category of (circled) bounded symmetric domains is equivalent to a certain category of Jordan triple systems, [2], [4], [7, p. 85] and [10, §2]. An advantage of the shift from Lie theoretic methods to Jordan theoretic ones is a transparent and an elementary description of the structure of bounded symmetric domains. This motivates an investigation of $\boldsymbol{D}(\mathscr{D})^G$ by making use of Jordan triple systems and the purpose of this paper is, in the same spirit as [6], to give explicitly a set of algebraically independent generators of $\boldsymbol{D}(\mathscr{D})^G$ without using the classification of $\mathscr{D}$.

Let us explain the content of this paper. A Jordan triple system (JTS for short) over $\mathbf{R}$ is a real vector space V equipped with a trilinear mapping $\{\cdot,\cdot,\cdot\}$: $V\times V\times V\to V$ such that

$$(0.1)\qquad \{u, v, w\} = \{w, v, u\},$$

$$(0.2)\qquad \{a, b, \{u, v, w\}\} - \{u, v, \{a, b, w\}\} = \{\{a, b, u\}, v, w\} - \{u, \{b, a, v\}, w\}$$

hold for all $u, v, w, a, b\in V$. A real JTS V is said to be hermitian if

(1) V is a complex vector space,

(2) $\{u, v, w\}$ is $\mathbf{C}$-linear in u, w and $\mathbf{C}$-antilinear in v.

Let V be a hermitian JTS. We define $\mathbf{C}$-linear operators $u\Box v$ $(u, v\in V)$ and $\mathbf{C}$-antilinear operators $Q(z)$ $(z\in V)$ on V by

$$(0.3)\qquad (u\Box v)w = \{u, v, w\},$$

Received July 18, 1989

(0.4) $$Q(z)w = \{z, w, z\}.$$

We note here that (0.2) is rewritten as

(0.5) $$[a \,\square\, b, u \,\square\, v] = ((a \,\square\, b)u) \,\square\, v - u \,\square\, ((b \,\square\, a)v),$$

where $[A, B] = AB - BA$ for two operators A, B. Now, V is said to be positive if $(u, v) := \mathrm{tr}(u \,\square\, v)$ defines a (positive definite) hermitian inner product on V. Let V be positive. We know that $z \,\square\, z$ $(z \in V)$ is a positive semi-definite hermitian operator. Letting $(z \,\square\, z)^{1/2}$ be the positive semi-definite square root of $z \,\square\, z$, we set

(0.6) $$\|z\|_\infty := \|(z \,\square\, z)^{1/2}\|.$$

Then, as the notation indicates, $\|\cdot\|_\infty$ becomes a norm on V (cf. [4, 3.17]), called the *spectral norm.* Let

$$\mathscr{D} := \{z \in V;\ \|z\|_\infty < 1\}.$$

Then, $\mathscr{D}$ is a (circled) bounded symmetric domain and every bounded symmetric domain arises in this way. Let G be the identity component of the Lie group of holomorphic automorphisms of $\mathscr{D}$ and we denote by K the stabilizer of G at 0.

Let $B(z, w)$ $(z, w \in V)$ be the Bergman operator:

(0.7) $$B(z, w) := I - 2z \,\square\, w + Q(z)Q(w).$$

One knows that if $z \in \mathscr{D}$, then $B(z, z)$ is a positive definite hermitian operator. Assume further that V is simple and of rank r. Let $\mathrm{Pol}(V^{\mathbf{R}})^K$ be the algebra of K-invariant polynomial functions on the underlying real vector space $V^{\mathbf{R}}$ of V and consider the polynomial functions f_j $(1 \leqq j \leqq r)$ on $V^{\mathbf{R}}$ defined by

$$f_j(v) := \frac{2}{q}(v^{(2j-1)}, v),$$

where q is the genus of V defined by (1.7) and $v^{(2j-1)}$ are the odd powers of v defined by (3.2). These f_j are shown to be real valued. Then,

Theorem 1. *The r polynomial functions $f_1, \dots, f_r$ are algebraically independent generators of* $\mathrm{Pol}(V^{\mathbf{R}})^K$.

Theorem 2. (1) *For each j $(1 \leqq j \leqq r)$, there is a polynomial function p_j on $V^{\mathbf{R}} \times V^{\mathbf{R}}$ such that*

$$f_j(B(z, z)^{1/2}v) = p_j(z, v) \quad (z \in \mathscr{D},\ v \in V).$$

(2) *These p_j are G-invariant functions on the cotangent bundle $T^*(\mathscr{D}) \approx \mathscr{D} \times V^{\mathbf{R}}$.*

(3) *The r differential operators $p_j(x, \partial/\partial x)$ form algebraically independent generators of $\boldsymbol{D}(\mathscr{D})^G$.*

An explicit expression for p_j is also given (see Proposition 3.5). Let $\mathrm{Pol}(V^{\mathbf{R}} \times V^{\mathbf{R}})^G$ be the algebra of polynomial functions on $V^{\mathbf{R}} \times V^{\mathbf{R}}$ whose restrictions to $\mathscr{D} \times V^{\mathbf{R}} \approx T^*(\mathscr{D})$ are G-invariant. Then,

Theorem 3. *The mapping* $\mathrm{Pol}(V^{\mathbf{R}})^K \ni f \mapsto \Phi f$, *where*

$$\Phi f(z, w) := f(B(z, z)^{1/2} w) \quad (z \in \mathscr{D},\ w \in V),$$

defines an algebra isomorphism of $\mathrm{Pol}(V^{\mathbf{R}})^K$ *onto* $\mathrm{Pol}(V^{\mathbf{R}} \times V^{\mathbf{R}})^G$.

Notation. Let V be a finite dimensional complex vector space. Its underlying real vector space will be written as $V^{\mathbf{R}}$. We identify naturally the tangent spaces $T_v(V^{\mathbf{R}})$ at $v \in V$ with $V^{\mathbf{R}}$. Let W be another finite dimensional complex vector space. Let $\mathscr{D}$ be an open subset in V. If f is a W-valued C^∞-function defined on $\mathscr{D}$, then its tangent mapping $d_z f\colon V^{\mathbf{R}} \to W^{\mathbf{R}}$ at $z \in \mathscr{D}$ is defined by

$$d_z f(v) := \frac{d}{dt} f(z + tv)|_{t=0}.$$

We set

$$\partial_z f(v) := \frac{1}{2}[(d_z f)(v) - i(d_z f)(iv)],$$

$$\bar{\partial}_z f(v) := \frac{1}{2}[(d_z f)(v) + i(d_z f)(iv)].$$

Then, $d_z f = \partial_z f + \bar{\partial}_z f$ and it is clear that $v \mapsto \partial_z f(v)$ (resp. $v \mapsto \bar{\partial}_z f(v)$) is $\mathbf{C}$-linear (resp. $\mathbf{C}$-antilinear). Moreover, by Cauchy-Riemann equations, f is holomorphic if and only if $\bar{\partial}_z f = 0$ for all $z \in \mathscr{D}$, the latter being equivalent to saying that $v \mapsto d_z f(v)$ is $\mathbf{C}$-linear.

§ 1. Preliminaries

We summarize here fundamental facts of JTS. Their proofs can be found in [4], [8], for example.

1.1. Let V be a simple positive hermitian JTS. Then,

(1.1) $$(u, v) := \mathrm{tr}(u \,\Box\, v)$$

defines a hermitian inner product on V. For every linear operator T on V, we denote by T^* its adjoint operator: $(Tu, v) = (u, T^*v)$. Then, by (0.5) we have

(1.2) $$(z \,\Box\, w)^* = w \,\Box\, z \quad (z, w \in V),$$

and this gives

(1.3) $$(Q(z)u, v) = \overline{(u, Q(z)v)}.$$

Moreover, by [8, 18.2], we have

(1.4) $$Q(v)(u \square v) = (v \square u)Q(v).$$

An element $c \in V$ is called a tripotent if $\{c, c, c\} = c$. Every tripotent c gives an orthogonal direct sum decomposition of V (the *Peirce decomposition* of V relative to c):

$$V = V_0(c) \oplus V_{1/2}(c) \oplus V_1(c),$$
$$V_j(c) := \{v \in V;\ (c \square c)v = jv\} \quad (j = 0, 1/2, 1).$$

Two tripotents c_1, c_2 are said to be orthogonal if $c \square c = 0$. Every element $v \in V$ has the spectral decomposition:

(1.5) $$v = \lambda_1 c_1 + \cdots + \lambda_n c_n \quad (0 < \lambda_1 < \cdots < \lambda_n),$$

with $\{c_1, \dots, c_n\}$ a family of orthogonal tripotents. Let $\{c_1, \dots, c_n\}$ be a family of orthogonal tripotents. By (0.1), (0.5) and (1.2), $\{c_i \square c_i;\ 1 \leqq i \leqq n\}$ is a commutative family of selfadjoint operators, so that we have a simultaneous eigenspace decomposition (called the *Peirce decomposition* relative to $\{c_1, \dots, c_n\}$):

(1.6) $$V = \bigoplus_{0 \leqq i \leqq j \leqq n} V_{ij} \quad \text{(orthogonal direct sum)},$$

where

$$V_{ij} := \left\{ v \in V;\ (c_k \square c_k)v = \frac{1}{2}(\delta_{ik} + \delta_{jk})v \quad \text{for} \quad 1 \leqq k \leqq n \right\}.$$

A tripotent c is said to be primitive if c cannot be written as a sum of two non-zero orthogonal tripotents. A maximal orthogonal system of primitive tripotents is called a *frame*. Then, if $\{c_1, \dots, c_r\}$ is a frame, r equals, by definition, the *rank* of V. We assume from now on that V is of rank r. Let $V = \bigoplus_{0 \leqq i \leqq j \leqq r} V_{ij}$ be the Peirce decomposition (1.6) relative to a frame $\{c_1, \dots, c_r\}$. Then, $V_{00} = \{0\}$ and $V_{ii} = \mathbf{C}c_i$. Furthermore, $V_{ij}\,(1 \leqq i < j \leqq r)$ all have the same dimension and we put

$$a := \dim V_{ij} \quad (1 \leqq i < j \leqq r).$$

Since $\dim V_{0i}\,(1 \leqq i \leqq r)$ are also all equal, we set

$$b := \dim V_{0i} \quad (1 \leqq i \leqq r).$$

The number

(1.7) $$q := 2 + a(r-1) + b$$

is called the *genus* of V.

Let $GL(V)$ be the complex Lie group of complex linear automorphisms of V, and we denote by Aut V the automorphism group of V:

$$\operatorname{Aut} V := \{g \in GL(V);\ g\{u, v, w\} = \{gu, gv, gw\} \quad \text{for all } u, v, w \in V\}.$$

Then , Aut $V \subset U(V)$, where $U(V)$ is the unitary group of V relative to the inner product (1.1). Thus Aut V is a compact Lie group.

1.2. Let $\mathscr{D}$ be the open unit ball of V relative to the spectral norm $\|\cdot\|_\infty$ defined by (0.6). We remark here that the spectral norm is invariant under Aut V. The domain $\mathscr{D}$ is a (circled) bounded symmetric domain in V and the symmetry at $0 \in \mathscr{D}$ is given by $z \mapsto -z$. Let $B(z, w)$ be the Bergman operator defined by (0.7). Then, by (1.2) and (1.3), we have

$$B(z, w)^* = B(w, z). \tag{1.8}$$

Let G be the identity component of the Lie group of holomorphic automorphisms of $\mathscr{D}$. Then, G is semisimple (and simple) and acts transitively on $\mathscr{D}$. We have

$$B(gz, gw) = (d_z g) B(z, w)(d_w g)^* \quad (g \in G,\ z, w \in \mathscr{D}). \tag{1.9}$$

We note also the following important identity (see [3, p. 21] or [8, 21.8]):

$$Q(B(z, w)u) = B(z, w) Q(u) B(w, z) \quad (z, w, u \in V). \tag{1.10}$$

Since $B(z, z)$ is positive definite hermitian for $z \in \mathscr{D}$,

$$h_z(u, v) := (B(z, z)^{-1} u, v) \quad (z \in \mathscr{D},\ u, v \in V)$$

defines a G-invariant hermitian structure on $\mathscr{D}$. Thus

$$b_z(u, v) := \operatorname{Re} h_z(u, v) \tag{1.11}$$

defines a G-invariant Riemannian structure on $\mathscr{D}$.

1.3. Let $\mathfrak{X}_{\mathrm{hol}}(\mathscr{D})$ denote the set of holomorphic vector fields on $\mathscr{D}$. Every orthonormal basis $e_1, \ldots, e_n$ ($n = \dim V$) of V relative to the inner product (1.1) gives rise to a coordinate system $(z_1, \ldots, z_n)$ in V by $z = \sum z_i e_i$. Then, each $X \in \mathfrak{X}_{\mathrm{hol}}(\mathscr{D})$ is written as

$$X = \sum_{i=1}^{n} h_i(z) \frac{\partial}{\partial z_i}, \tag{1.12}$$

where h_i are holomorphic functions on $\mathscr{D}$. Let $\mathcal{O}(\mathscr{D}, V)$ be the space of V-valued holomorphic functions on $\mathscr{D}$. Putting $h(z) = \sum h_i(z) e_i$ in (1.12), we get $h \in \mathcal{O}(\mathscr{D}, V)$ and this h is independent of the choice of orthonormal basis of V. It is clear that the correspondence $\mathfrak{X}_{\mathrm{hol}}(\mathscr{D}) \ni X \mapsto h \in \mathcal{O}(\mathscr{D}, V)$ is bijective. This being so, we write the vector field (1.12) as $X = h(z)\partial/\partial z$. Thus, if f is a holomorphic function on $\mathscr{D}$ into another complex vector space W, we have

$$Xf(z) = d_z f(h(z)). \tag{1.13}$$

The space $\mathfrak{X}_{\mathrm{hol}}(\mathscr{D})$ is a Lie algebra by the Poisson bracket:

$$\left[h(z)\frac{\partial}{\partial z}, k(z)\frac{\partial}{\partial z} \right] := (d_z k(h(z)) - d_z h(k(z))) \frac{\partial}{\partial z}.$$

Let $\mathfrak{P}(V)$ be the algebra of holomorphic polynomial mappings of V into V. We denote by $\mathfrak{P}_\nu(V)$ the subspace of $\mathfrak{P}(V)$ of homogeneous polynomial mappings of degree ν. Then, $\mathfrak{P}(V)$ is an algebraic direct sum $\mathfrak{P}(V) = \bigoplus_{\nu=0}^{\infty} \mathfrak{P}_\nu(V)$. Clearly, we have $\mathfrak{P}(V) \subset \mathcal{O}(\mathcal{D}, V)$.

Now let $\mathfrak{g}$ be the Lie algebra of G. Every $X \in \mathfrak{g}$ is considered as an element of $\mathfrak{X}_{\text{hol}}(\mathcal{D})$ by

$$Xf(z) = \frac{d}{dt} f(\exp(-tX)\cdot z)|_{t=0},$$

where exp is the exponential mapping $\mathfrak{g} \to G$ and f a holomorphic function on $\mathcal{D}$. For every $v \in V$, we define $q_v \in \mathfrak{P}_2(V)$ and $\xi_v \in \mathfrak{P}_0(V) + \mathfrak{P}_2(V)$ by

$$q_v(z) := Q(z)v, \tag{1.14}$$

$$\xi_v(z) := v - q_v(z). \tag{1.15}$$

Then the following proposition is known, [4, §4] or [10, §2].

Proposition 1.1. (1) *If* $X = h(z)\partial/\partial z \in \mathfrak{g}$, *then* $h \in \sum_{\nu=0}^{2} \mathfrak{P}_\nu(V)$.

(2) *Let K be the stabilizer of G at $0 \in \mathcal{D}$. Then, K is the identity component of* Aut V.

(3) *Put*

$$\mathfrak{k} := \{T(z)\partial/\partial z;\ T \text{ is a derivation of } V\},$$

$$\mathfrak{p} := \{\xi_v(z)\partial/\partial z;\ v \in V\}.$$

Then, $\mathfrak{k} = \text{Lie } K$ *and* $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ *is a Cartan decomposition of* $\mathfrak{g}$.

§2. Compactification of V

To realize the complexification $G_{\mathbb{C}}$ of G, we will introduce a compactification of V defined by Loos [4, §7] (see also [5]). A pair (x, y) of elements of V is said to be *invertible* if the operator $B(x, y)$ is invertible, that is, if det $B(x, y) \neq 0$. Note that (x, y) is invertible if and only if so is (y, x) by (1.8). If (x, y) is invertible, we set

$$x^y := B(x, y)^{-1}(x - Q(x)y), \tag{2.1}$$

$$(x, y)^{-1} := (x^y, -y). \tag{2.2}$$

We call x^y the quasi-inverse of (x, y). We have

$$B(x, y)^{-1} = B((x, y)^{-1}). \tag{2.3}$$

Lemma 2.1. *If $z, w \in \mathcal{D}$, then (z, w) is invertible.*

Proof. Note that since $B(0, w) = I$, $(0, w)$ is invertible. Now, the transitivity of G on $\mathcal{D}$ together with (1.9) proves the lemma. ■

We define a relation $\sim$ in $V \times V$ by the following:

$$(x, y) \sim (x', y') \text{ if } (x, y - y') \text{ is invertible and } x' = x^{y-y'}.$$

Then, $\sim$ is an equivalence relation. Let X be the set of equivalence classes. The equivalence class of (x, y) will be denoted by $(x: y)$. V is considered as a subset of X by the injective mapping $x \mapsto (x: 0)$. Then, $(x: y) \in V$ if and only if (x, y) is invertible. For each $v \in V$, let $U_v := \{(x: v); x \in V\}$. Then, by [4, 7.7], there is a unique structure of smooth algebraic variety on X such that U_v is an open affine subvariety, which is isomorphic to V under $(x: y) \mapsto x$. Moreover, X is a projective variety by [4, 7.10].

Considering X as a compact complex manifold, the complexification $G_{\mathbf{C}}$ of G is now realized as the (Zariski-) connected component of the identity of the group of holomorphic automorphisms of X. The Lie algebra $\mathfrak{g}_{\mathbf{C}}$ of $G_{\mathbf{C}}$ is the set of all holomorphic vector fields on X. By restriction, $\mathfrak{g}_{\mathbf{C}}$ may be considered as a Lie algebra of vector fields on V.

For every $v \in V$, let

$$\tilde{t}_v(x: a) := (x: a + v) \quad ((x: a) \in X). \tag{2.4}$$

Then, $\tilde{t}_v$ is well defined and $\tilde{t}_v \in G_{\mathbf{C}}$. Clearly we have $\tilde{t}_{v_1}\tilde{t}_{v_2} = \tilde{t}_{v_1+v_2}$. Let Str V denote the structure group of V:

$$\text{Str } V := \{g \in GL(V); Q(gz) = gQ(z)g^* \quad \text{for all } z \in V\}.$$

We see easily that $g \in GL(V)$ belongs to Str V if and only if

$$g(z \,\square\, w)g^{-1} = (gz) \,\square\, (g^{*-1}w) \quad (z, w \in V).$$

This together with (1.2) shows that Str V is stable under $g \mapsto g^*$. Now, Str V acts on X by

$$h(x: a) := (hx: h^{*-1}a) \quad (h \in \text{Str } V, (x: a) \in X).$$

We denote by H the identity component of Str V. Thus H is considered as a subgroup of $G_{\mathbf{C}}$. It should be noted here that $K = H \cap U(V)$. Finally, for every $u \in V$, the translation t_u: $V \ni x \mapsto x + u \in V$ is uniquely extended to an element of $G_{\mathbf{C}}$. Let

$$U^+ := \{t_u; u \in V\}, \quad U^- := \{\tilde{t}_v; v \in V\}.$$

Then, $U^\pm$ are abelian subgroups of $G_{\mathbf{C}}$, and a computation shows

$$ht_uh^{-1} = t_{hu}, \quad h\tilde{t}_vh^{-1} = \tilde{t}_{h^{*-1}v},$$

for all $h \in H$ and $u, v \in V$. We note here that if (x, y) is invertible, then (1.8) and (1.10) imply that $B(x, y) \in \text{Str } V$. Further, the identity (as elements of $G_{\mathbf{C}}$)

$$B(x, y) = \tilde{t}_{y^x} t_{-x} \tilde{t}_{-y} t_{x^y} \quad (\text{cf. [4, 8.11]})$$

says that $B(x, y) \in H$.

The Lie algebras $\mathfrak{u}^{\pm}$ of $U^{\pm}$ are identified as

$$\mathfrak{u}^{+} = \{\text{constant vector fields on } V\},$$
$$\mathfrak{u}^{-} = \{q_v(z)\partial/\partial z;\ v \in V\}.$$

Let $\mathfrak{h} := \text{Lie } H$. Then, we have a vector space direct sum of Lie subalgebras: $\mathfrak{g}_{\mathbf{C}} = \mathfrak{u}^{+} + \mathfrak{h} + \mathfrak{u}^{-}$. Let

$$\Xi := \{g \in G_{\mathbf{C}};\ g(0:0) \in V\}.$$

Then, $\Xi \approx U^{+} \times H \times U^{-}$ and $G \subset \Xi$. For every $v \in V$, one has (cf. [4, 9.8])

$$\exp(\xi_v(z)\partial/\partial z) = t_{\tanh v} \cdot B(\tanh v,\ \tanh v)^{1/2} \cdot \tilde{t}_{-\tanh v}, \tag{2.5}$$

with $B(\tanh v, \tanh v)^{1/2} \in H$, where if $v = \sum \lambda_i c_i$ is the spectral decomposition (1.4) of v, then $\tanh v := \sum(\tanh \lambda_i)c_i$. Thus, $\tanh v \in \mathscr{D}$ for any $v \in V$. The following proposition will be needed later.

Proposition 2.2. *Put* $g_v := \exp(\xi_v(z)\partial/\partial z) \in \exp \mathfrak{p}$ *for each* $v \in V$. *Then,*

(1) $g_v \cdot 0 = \tanh v$,

(2) $d_z g_v = B(\tanh v, \tanh v)^{1/2} B(z, -\tanh v)^{-1} \in H$ *for every* $z \in \mathscr{D}$.

Proof. Put $u = \tanh v \in \mathscr{D}$ for simplicity. Lemma 2.1 implies that if $z \in \mathscr{D}$, then $(z, -u)$ is invertible. Hence

$$\tilde{t}_{-u}(z:0) = (z:-u) = (z^{-u}:0).$$

Thus, $\tilde{t}_{-u}$ induces a holomorphic mapping $\mathscr{D} \ni z \mapsto z^{-u} \in V$.

(1) Since $0^{-u} = 0$ by (2.1), the assertion follows from (2.5).

(2) Let $z \in \mathscr{D}$, $x \in V$ and $s \in \mathbf{R}$. By [3, Theorem 3.7 (b)], we have

$$(z + sx)^{-u} = z^{-u} + B(z, -u)^{-1}(sx)^{(-u)^z},$$

provided $|s|$ is sufficiently small. We put $y = (-u)^z$ for brevity. Since

$$(sx)^y = B(sx, y)^{-1}(sx - Q(sx)y) = sB(sx, y)^{-1}(x - sQ(x)y)$$
$$= s(I + sO(1))^{-1}(x - sQ(x)y) = s(x + sO(1)),$$

we get

$$(z + sx)^{-u} = z^{-u} + sB(z, -u)^{-1}(x + sO(1)).$$

Hence, $d_z\tilde{t}_{-u} = B(z, -u)^{-1}$. Now, the proposition follows from (2.5) and the chain rule. ■

§3. Invariant polynomial functions

We begin with the following two lemmas which are more or less known to JTS specialists.

Lemma 3.1. *Let $\{c_1, \dots, c_r\}$ be a frame and put $c := c_1 + \cdots + c_r$. Then, for every pair (j, l), there is $x \in V$* such that *$T := B(x, c) \in H$ satisfies*

$$T^2 = I, \quad Tc_j = T^*c_j = c_l, \quad Tc_m = T^*c_m = c_m \quad \textit{for all } m \neq j, l.$$

Proof. We put

$$\mathfrak{A}(c) := V_1(c) \cap \{x \in V;\, Q(c)x = x\}.$$

Then, by [4, 3.13 (c)], $\mathfrak{A}(c)$ is a formally real Jordan algebra with the product $xy := \{x, c, y\}$. Note that $c_m \in \mathfrak{A}(c)$ for all m. The proof of [3, 17.1] shows that for every pair (j, l), there is $x \in \mathfrak{A}(c)$ such that $T := B(x, c)$ satisfies

$$T^2 = I,\ Tc_j = c_l,\ Tc_m = c_m \quad \text{for } m \neq j, l.$$

Since $x \in \mathfrak{A}(c)$, we have $c \square x = x \square c$ by [4, 9.13]. Moreover, it holds that

$$Q(c)Q(x)c_m = Q(x)c_m = Q(x)Q(c)c_m \qquad \text{for all } m,$$

where the first equality is the consequence of

$$\begin{aligned} Q(x)c_m &= \{x, \{c, c_m, c\}, x\} \\ &= -\{c_m, c, \{x, c, x\}\} + 2\{\{c_m, c, x\}, c, x\} \quad \text{(by (0.1), (0.2))} \\ &= -c_m x^2 + 2(c_m x)x \in \mathfrak{A}(c). \end{aligned}$$

The above observation yields

$$\begin{aligned} Tc_m &= B(x, c)c_m = c_m - 2(x \square c)c_m + Q(x)Q(c)c_m \\ &= c_m - 2(c \square x)c_m + Q(c)Q(x)c_m = B(c, x)c_m = T^*c_m \end{aligned}$$

for all m. This proves the lemma. ■

Corollary 3.2. *Let $\{c_1, \dots, c_r\}$ be a frame. Then, for each permutation σ of r letters, there is $k \in K$ such that $kc_m = c_{\sigma(m)}$ for all m.*

Proof. It suffices to show the corollary in the case where σ are transpositions. Let σ be the transposition of j, l and $T \in H$ be as in Lemma 3.1. Consider the polar decomposition $T = U|T|$ of the operator T, where U is unitary and $|T| = (T^*T)^{1/2}$ positive definite selfadjoint. Since $T^* \in H$, we have $|T| = \exp(\frac{1}{2}\log T^*T) \in H$, so that $U \in H \cap U(V) = K$. Since $|T|c_m = c_m$ for all m, this U is a required one. ■

Lemma 3.3. *Let $\{c_1, \dots, c_r\}$, $\{d_1, \dots, d_r\}$ be frames. Then, there is $k \in K$ such that $kc_m = d_m$ for all m.*

Proof. The real subspaces

$$C := \sum_{m=1}^{r} \mathbf{R}c_m, \qquad D := \sum_{m=1}^{r} \mathbf{R}d_m$$

are maximal flat subspaces in the sense of [4, 3.10]. By [4, 5.3 (a)], there is $k_1 \in K$ such that $k_1(C) = D$. Since $k_1 \in \mathrm{Aut}\, V$, $k_1 c_m$ are tripotents, so that the primitivity and the linear independence of c_m imply that there is a permutation σ of r letters such that $k_1 c_m = \pm d_{\sigma(m)}$ for all m. By Corollary 3.2, we get $k_2 \in K$ such that

$$k_2 c_m = \pm d_m \quad \text{for all } m. \tag{3.1}$$

On the other hand, (0.2) says that $i(c_j \square c_j)$ are derivations of V. Put $k^{(j)} := \exp(\pi i(c_j \square c_j)) \in K$. Then,

$$k^{(j)} c_j = -c_j, \quad k^{(j)} c_m = c_m \quad \text{for } m \neq j.$$

This together with (3.1) proves the lemma. ■

We will fix a frame $\{c_1, \dots, c_r\}$ of V throughout the rest of this section. We define odd powers $v^{(2j-1)}$ $(j = 1, 2, \dots)$ of an element $v \in V$ inductively by

$$v^{(1)} := v, \quad v^{(2j+1)} := Q(v)v^{(2j-1)} \quad (j = 1, 2, \dots). \tag{3.2}$$

Clearly we have

$$v^{(2j+1)} = Q(v)^j v \quad (j = 1, 2, \dots). \tag{3.3}$$

An easy induction argument using (1.4) and (3.3) shows

$$v^{(2j+1)} = (v \square v)^j v \quad (j = 1, 2, \dots). \tag{3.4}$$

Letting q be the genus of V defined by (1.7), we normalize the inner product (1.1) as

$$(u, v)_0 := \frac{2}{q}(u, v).$$

Then, since

$$\mathrm{tr}(c_m \square c_m) = 1 + \frac{b}{2} + \frac{1}{2}a(r-1) = \frac{q}{2} \quad \text{for all } m, \tag{3.5}$$

the normalization of the inner product is made so that the norm of c_m is one for every m. We put

$$f_j(v) := (v^{(2j-1)}, v)_0 \quad (j = 1, 2, \dots). \tag{3.6}$$

We note that by (3.4) and (1.2), each f_j is real valued. Since $Q(kv) = kQ(v)k^{-1}$ for all $k \in K$ and $v \in V$, it is clear from (3.3) that $f_j \in \mathrm{Pol}(V^{\mathbf{R}})^K$. We note also that if λ_m are all real, then

$$f_j\Big(\sum_{m=1}^{r} \lambda_m c_m\Big) = \frac{2}{q}\,\mathrm{tr} \sum_{m=1}^{r} \lambda_m^{2j} c_m \square c_m = \sum_{m=1}^{r} \lambda_m^{2j} \quad (\text{by } (3.5)). \tag{3.7}$$

Theorem 3.4. *The polynomial functions f_j $(1 \leqq j \leqq r)$ form algebraically independent generators of* $\mathrm{Pol}(V^{\mathbf{R}})^K$.

Proof. Let $f \in \mathrm{Pol}(V^{\mathbf{R}})^K$ and consider the polynomial

$$F(\lambda_1, \dots, \lambda_r) := f(\sum_{m=1}^{r} \lambda_m c_m) \quad (\lambda_m \in \mathbf{R}).$$

Take the frames $\{\varepsilon_1 c_1, \dots, \varepsilon_r c_r\}$ with $\varepsilon_j \in \{-1, 1\}$. Then, since f is K-invariant, Lemma 3.3 implies that F is of the form

$$F(\lambda_1, \dots, \lambda_r) = F_0(\lambda_1^2, \dots, \lambda_r^2)$$

for some polynomial F_0. Let σ be an arbitrary permutation of r letters. Then, Corollary 3.2 and the K-invariance of f say that the polynomial F_0 is invariant under σ. Hence, there is a polynomial P of r variables such that

$$f(\sum_{m=1}^{r} \lambda_m c_m) = P(\sum_{m=1}^{r} \lambda_m^2, \dots, \sum_{m=1}^{r} \lambda_m^{2j}, \dots, \sum_{m=1}^{r} \lambda_m^{2r}). \tag{3.8}$$

Now, consider the polynomial function

$$R(v) := f(v) - P(f_1(v), \dots, f_r(v)) \quad (v \in V^{\mathbf{R}}),$$

where f_j are defined by (3.6). We have $R \in \mathrm{Pol}(V^{\mathbf{R}})^K$. Let $v \in V$ be arbitrary and consider its spectral decomposition: $v = \sum_{j=1}^{n} \mu_j d_j$. If the tripotent $d := \sum_{j=1}^{n} d_j$ is not maximal, then choose a tripotent $d' \in V_0(d)$, the Peirce 0-space of d, so that $d^0 := d + d'$ is a maximal tripotent. Anyhow, the spectral decomposition is refined as $v = \sum_{m=1}^{r} \lambda_m d_m^0$ with a frame $\{d_1^0, \dots, d_r^0\}$ and $0 \leqq \lambda_1 \leqq \cdots \leqq \lambda_r$. Then, by Lemma 3.3, there is $k \in K$ such that $kd_m^0 = c_m$ for all m. Hence $kv = \sum_{j=1}^{r} \lambda_j c_j$. This together with (3.7) and (3.8) yields $R = 0$. Since the polynomials $\sum_m \lambda_m^j$ $(1 \leqq j \leqq r)$ are algebraically independent as is well known, so are $f_1, \dots, f_r$ by (3.7). ■

We now make G act on $\mathscr{D} \times V^{\mathbf{R}}$ by

$$g \cdot (z, w) := (gz, (d_z g)^{*-1} w) \quad (z \in \mathscr{D}, w \in V). \tag{3.9}$$

Let $\mathrm{Pol}(V^{\mathbf{R}} \times V^{\mathbf{R}})^G$ be the algebra of polynomial functions on $V^{\mathbf{R}} \times V^{\mathbf{R}}$ whose restrictions to $\mathscr{D} \times V^{\mathbf{R}}$ are invariant under the action of G defined by (3.9). We will define an injective mapping of $\mathrm{Pol}(V^{\mathbf{R}})^K$ into $\mathrm{Pol}(V^{\mathbf{R}} \times V^{\mathbf{R}})^G$.

Let $f_j \in \mathrm{Pol}(V^{\mathbf{R}})^K$ be as in (3.6) and recall that $B(z, z)$ is positive definite hermitian for $z \in \mathscr{D}$.

Proposition 3.5. *For each j $(1 \leqq j \leqq r)$, there is a unique polynomial function $p_j \in \mathrm{Pol}(V^{\mathbf{R}} \times V^{\mathbf{R}})^G$ such that*

$$f_j(B(z, z)^{1/2} v) = p_j(z, v) \quad (z \in \mathscr{D}, v \in V). \tag{3.10}$$

Moreover, these p_j are given as

$$p_j(z, v) = ((Q(v)B(z, z))^{j-1} v, B(z, z)v)_0 \quad (j = 1, 2, \dots). \tag{3.11}$$

Remark 3.6. It is interesting to show directly that the right hand side of (3.11) is real. By (1.3), this is clear if j is odd. To see the case $j = 2m$, we note

$$
\begin{aligned}
&B(z, z)(Q(v)B(z, z))^{2m-1}v \\
&\quad = (Q(B(z, z)v)Q(v))^{m-1}Q(B(z, z)v)v \qquad \text{(by (1.10))} \\
&\quad = (Q(B(z, z)v)Q(v))^{m-1}((B(z, z)v)\Box v)B(z, z)v \\
&\quad = ((B(z, z)v)\Box v)(Q(B(z, z)v)Q(v))^{m-1}B(z, z)v \quad \text{(by (1.4))}.
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
&((Q(v)B(z, z))^{2m-1}v, B(z, z)v)_0 \\
&\quad = ((Q(B(z, z)v)Q(v))^{m-1}B(z, z)v, (v\Box B(z, z)v)v)_0 \\
&\quad = ((Q(B(z, z)v)Q(v))^{m-1}B(z, z)v, Q(v)B(z, z)v)_0 \\
&\quad = \overline{((Q(v)B(z, z))^{2m-1}v, B(z, z)v)_0} \quad \text{(by (1.3), (1.10))}.
\end{aligned}
$$

Proof of Proposition 3.5. Since $\mathscr{D}$ is open in V, the uniqueness is clear. Let $z\in\mathscr{D}$ and $v\in V$. Since $B(z, z)^{1/2}\in \mathrm{Str}\, V$, we have

$$Q(B(z, z)^{1/2}v) = B(z, z)^{1/2}Q(v)B(z, z)^{1/2}. \tag{3.12}$$

Then, by (3.6), (3.3) and (3.12), we get

$$
\begin{aligned}
f_j(B(z, z)^{1/2}v) &= (Q(B(z, z)^{1/2}v)^{j-1}B(z, z)^{1/2}v, B(z, z)^{1/2}v)_0 \\
&= (B(z, z)^{1/2}(Q(v)B(z, z))^{j-1}v, B(z, z)^{1/2}v)_0 \\
&= p_j(z, v).
\end{aligned}
$$

Hence, it remains to prove that p_j are G-invariant. For this, we need the following lemma.

Lemma 3.7. *If $g\in G$, then $d_zg\in H$ for all $z\in\mathscr{D}$, where H is the identity component of* Str V.

Proof. Let us write g as $g = kg_v$ with $k\in K$, $v\in V$ and g_v as in Proposition 2.2. By chain rule, it suffices to show Lemma 3.7 for $g = k$ and g_v separately. Suppose $g = k\in K$. Then, k is $\mathbf{C}$-linear, so that $d_zk = k$ for all $z\in\mathscr{D}$. Therefore, $d_zk\in K\subset H$. Let $g = g_v\in \exp\mathfrak{p}$. Then, Proposition 2.2 (2) says that $d_zg_v\in H$. ∎

Let us return to the proof of Proposition 3.5. Let $g\in G$, $z\in\mathscr{D}$ and $v\in V$. Since H is stable under $T\mapsto T^*$, Lemma 3.7 yields

$$Q((d_zg)^{*-1}v) = (d_zg)^{*-1}Q(v)(d_zg)^{-1}.$$

This together with (1.9) completes the proof. ∎

Proposition 3.8. *The mapping* $\mathrm{Pol}(V^{\mathbf{R}})^K\ni f\mapsto \Phi f$, *where*

$$\Phi f(z, v) := f(B(z, z)^{1/2}v) \quad (z\in\mathscr{D}, v\in V)$$

defines an injection of $\mathrm{Pol}(V^{\mathbf{R}})$ *into* $\mathrm{Pol}(V^{\mathbf{R}} \times V^{\mathbf{R}})^G$.

Proof. The injectivity follows from $f(v) = \Phi f(0, v)$. If $f \in \mathrm{Pol}(V^{\mathbf{R}})^K$, then write f as a polynomial of $f_1, \dots, f_r$ by Theorem 3.4. Then, Φf is a polynomial of $p_1, \dots, p_r$ in Propositin 3.5. ■

We establish the surjectivity of $f \mapsto \Phi f$ in the next section.

§4. G-invariant differential operators on $\mathscr{D}$

Let $T^*(\mathscr{D}) \approx \mathscr{D} \times V^{\mathbf{R}}$ be the cotangent bundle of $\mathscr{D}$ with the natural G-action:

$$g \cdot (z, w) = (gz, (d_z g)^{*-1} w) \quad (g \in G,\ z \in \mathscr{D},\ w \in V). \tag{4.1}$$

If $L \in C^\infty(\mathscr{D} \times V^{\mathbf{R}})$ and if $V^{\mathbf{R}} \ni w \mapsto L(z, w)$ is polynomial for each fixed $z \in \mathscr{D}$, then we associate a differential operator $L(x, \partial/\partial x)$ with the property that

$$L(x, \partial/\partial x) e^{\langle x, y \rangle} = L(x, y) e^{\langle x, y \rangle} \quad (x \in \mathscr{D},\ y \in V^{\mathbf{R}}), \tag{4.2}$$

where $\langle x, y \rangle := \mathrm{Re}\,(x, y)_0$. In this way, one obtains every differential operator on $\mathscr{D}$ with coefficients in $C^\infty(\mathscr{D})$.

The differential operator $L(x, \partial/\partial x)$ on $\mathscr{D}$ is said to be *G-invariant*, if it commutes with the G-action:

$$gL(x, \partial/\partial x) g^{-1} = L(x, \partial/\partial x) \quad \text{for all } g \in G.$$

It is easy to see that the differential operator $L(x, \partial/\partial x)$ is G-invariant if and only if the corresponding function L is invariant under the G-action defined by (4.1):

$$L(gz, (d_z g)^{*-1} w) = L(z, w) \quad (z \in \mathscr{D},\ w \in V^{\mathbf{R}}). \tag{4.3}$$

Proposition 4.1. *Let $L(z, w)$ be C^∞ in $z \in \mathscr{D}$ and polynomial in $w \in V^{\mathbf{R}}$. If L satisfies* (4.3), *then there is a polynomial P of r variables such that*

$$L(z, w) = P(p_1(z, w), \dots, p_r(z, w)) \quad (z \in \mathscr{D},\ w \in V^{\mathbf{R}}),$$

where $p_j \in \mathrm{Pol}(V^{\mathbf{R}} \times V^{\mathbf{R}})^G$ are as in Proposition 3.5.

Proof. Set $l(w) := L(0, w)$. Since every element of K is a unitary operator, we have $(d_z k)^{*-1} = k$ for all $z \in \mathscr{D}$ and $k \in K$. Hence (4.3) implies $l \in \mathrm{Pol}(V^{\mathbf{R}})^K$. By Theorem 3.4, there is a polynomial P of r variables such that

$$l(w) = P(f_1(w), \dots, f_r(w)) \quad (w \in V^{\mathbf{R}}).$$

Now, let $z \in \mathscr{D}$. Considering the spectral decomposition (1.5) of z, we see that there is $v \in V$ such that $z = \tanh v$. Let g_v be as in Proposition 2.2. Then, $g_v \cdot 0 = \tanh v = z$ and

$$d_0 g_v = B(z, z)^{1/2}. \tag{4.4}$$

By virtue of the G-invariance (4.3) of L, we get

$$\begin{aligned} L(z, w) &= L(g_v \cdot 0, (d_0 g_v)^{*-1}(d_0 g_v)^* w) \\ &= L(0, B(z, z)^{1/2} w) = l(B(z, z)^{1/2} w) \qquad \text{(by (4.4))} \\ &= P(f_1(B(z, z)^{1/2} w), \dots, f_r(B(z, z)^{1/2} w)) \\ &= P(p_1(z, w), \dots, p_r(z, w)). \quad \blacksquare \end{aligned}$$

Since $\mathscr{D}$ is symmetric, we know that the algebra $\boldsymbol{D}(\mathscr{D})^G$ of G-invariant differential operators on $\mathscr{D}$ is commutative. Put

$$D_j = p_j(x, \partial/\partial x) \quad (j = 1, 2, \dots, r).$$

By Proposition 3.5, we have $p_j \in \mathrm{Pol}(V^{\mathbf{R}} \times V^{\mathbf{R}})^G$. Hence

$$D_j \in \boldsymbol{D}(\mathscr{D})^G \quad (j = 1, 2, \dots, r).$$

Lemma 4.2. $D_1, \dots, D_r$ are algebraically independent.

Proof. For any polynomial $q(x_1, \dots, x_r)$ of r variables $x_1, \dots, x_r$, we call the degree of $q(x_1, x_2^2, \dots, x_r^r)$ the weight of q. Let now q be a polynomial of r variables such that $q(D_1, \dots, D_r) = 0$. Suppose the weight of q is m. We denote by q_μ the sum of the monomials in q of weight μ. Then, $q = \sum_{\mu=0}^m q_\mu$. Let $t \in \mathbf{R}$. Since $y \mapsto p_j(x, y)$ is homogeneous of degree $2j$, (4.2) yields

$$D_j e^{\langle x, ty \rangle} = t^{2j} p_j(x, y) e^{\langle x, ty \rangle}.$$

Then, we get

$$\begin{aligned} &q_\mu(D_1, \dots, D_r) e^{\langle x, ty \rangle} \\ &\qquad = e^{\langle x, ty \rangle} [t^{2\mu} q_\mu(p_1(x, y), \dots, p_r(x, y)) + \text{lower order terms in } t]. \end{aligned}$$

Hence

$$\begin{aligned} &q(D_1, \dots, D_r) e^{\langle x, ty \rangle} \\ &\qquad = e^{\langle x, ty \rangle} [t^{2m} q_m(p_1(x, y), \dots, p_r(x, y)) + \text{lower order terms in } t]. \end{aligned}$$

so that $q(D_1, \dots, D_r) = 0$ leads us to

$$q_m(p_1(x, y), \dots, p_r(x, y)) = \frac{1}{(2m)!} \frac{d^{2m}}{dt^{2m}} e^{-\langle x, ty \rangle} q(D_1, \dots, D_r) e^{\langle x, ty \rangle} = 0$$

for all x, y. Putting $x = 0$, we obtain

$$q_m(f_1(y), \dots, f_r(y)) = q_m(p_1(0, y), \dots, p_r(0, y)) = 0.$$

This implies $q_m = 0$ by Theorem 3.4, whence $q = 0$. $\blacksquare$

We now arrive at two main theorems by virtue of Propositions 4.1, 3.8 and Lemma 4.2.

Theorem 4.3. $D_1, \dots, D_r$ form algebraically independent generators of $\boldsymbol{D}(\mathscr{D})^G$.

Theorem 4.4. For every $f \in \mathrm{Pol}(V^{\mathbf{R}})^K$, put

$$\Phi f(x, y) := f(B(x, x)^{1/2} y) \quad (x \in \mathscr{D},\ y \in V).$$

Then, Φ defines an algebra isomorphism of $\mathrm{Pol}(V^{\mathbf{R}})$ onto $\mathrm{Pol}(V^{\mathbf{R}} \times V^{\mathbf{R}})^G$.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[1] S. Helgason, Groups and geometric analysis, Academic press, New York, 1984.
[2] M. Koecher, An elementary approach to bounded symmetric domains, Lecture Notes, Rice Univ., 1969.
[3] O. Loos, Jordan pairs, Lecture Notes in Math., **460**, Springer, Berlin, 1975.
[4] O. Loos, Bounded symmetric domains and Jordan pairs, Lecture Notes, Univ. California at Irvine, 1977.
[5] O. Loos, Homogeneous algebraic varieties defined by Jordan pairs, Monatsh. Math., **86** (1978), 107–129.
[6] T. Nomura, Algebraically independent generators of invariant differential operators on a symmetric cone, J. Reine Angew. Math., **400** (1989), 122–133.
[7] I. Satake, Algebraic structures of symmetric domains, Iwanami Shoten and Princeton Univ. Press, Tokyo and Princeton, 1980.
[8] H. Upmeier, Symmetric Banach manifolds and Jordan C^*-algebras, North-Holland, Amsterdam, 1985.
[9] H. Upmeier, Jordan algebras and harmonic analysis on symmetric spaces, Amer. J. Math., **108** (1986), 1–25.
[10] H. Upmeier, Jordan algebras in analysis, operator theory and quantum mechanics, Regional Conf. in Math., **67**, Amer. Math. Soc., Rhode Island, 1987.