J. Math. Kyoto Univ. (JMKYAZ)
31-1 (1991) 219–258

Spectral and scattering theory for the Schrödinger operators with penetrable wall potentials

Dedicated to Professor Tosio Kato on his 70th birthday

By

Teruo IKEBE and Shin-ichi SHIMADA

§ 0. Introduction

In this paper we shall consider the Schrödinger operator with a penetrable wall potential in $\boldsymbol{R}^3$ formally of the form

$$H_{formal} = -\Delta + q(x)\delta(|x| - a),$$

where $q(x)$ is real and smooth on $S_a = \{x; |x| = a\}$ $(a > 0)$ and δ denotes the one-dimensional delta function. This operator is said to provide a simple model for the α-decay (Petzold [15]). Other applications may be found in the references cited in Antoine-Gesztesy-Shabani [3]. Dolph-McLeod-Thoe [5] treated this operator ($q(x) \equiv$ const.) with concern for the analytic continuation of the scattering matrix, yet at the formal level.

The first problem one meets is to define properly H_{formal} as a selfadjoint opertor in $L_2(\boldsymbol{R}^3)$. For this purpose, let us consider the quadratic form h (which is associated with H_{formal})

$$h[u, v] = (H_{formal}\, u, v) = (\nabla u, \nabla v) + (q\gamma u, \gamma v)_a,$$

$$Dom[h] = H^1(\boldsymbol{R}^3).$$

Here γ is the trace operator from $H^1(\boldsymbol{R}^3)$ to $L_2(S_a)$, $Dom[h]$ denotes the form domain of h, (,) means the $L_2(\boldsymbol{R}^3)$ inner product, $(\, , \,)_a$ the $L_2(S_a)$ inner product, and $H^m(G)$ the Sobolev space of order m over G. h is shown to be a lower semibounded closed form, and thus determines a lower semibounded selfadjoint operator H. More precisely, H is seen to be the negative Laplacian with the boundary condition

$$q(x)(\gamma u)(x) - \left\{\frac{\partial u}{\partial n_+}(x) + \frac{\partial u}{\partial n_-}(x)\right\}\Big|_{S_a} = 0,$$

where $n_+ (n_-)$ denotes the outward (inward) normal to S_a. We should note here that while h is a "small" perturbation of h_0, which is defined by

$$h_0[u, v] = (\nabla u, \nabla v), \; Dom[h_0] = H^1(\boldsymbol{R}^3),$$

Received June 28, 1989

via an infinitesimally h_0-bounded form, $H - H_0$ is not H_0-bounded, where $H_0 = -\Delta$, $Dom(H_0) = H^2(\boldsymbol{R}^3)$, is the selfadjoint operator associated with h_0. We shall adopt this operator H as the rigorous selfadjoint realization of the formal expression H_{formal}. Antoine et al. [3] defined the Hamiltonians corresponding to H_{formal} as the selfadjoint extensions of $(-\Delta|_{C_0^\infty(\boldsymbol{R}^3\setminus S_a)})^\sim$ making use of the decomposition of $L_2(\boldsymbol{R}^3)$ with respect to angular momenta. Here $C_0^\infty(G)$ denotes the set of all infinitely continuously differentiable functions with compact support in G and $\sim$ means the closure.

After having determined the proper selfadjoint operator H corresponding to H_{formal}, we take interest in the spectral structure of H. It can be seen that the nagative part of the spectrum of H consists of a finite number of eigenvalues of finite multiplicity (Theorem 6.5). Further, we can show the difference of the resolvents of H and H_0 is a compact operator, which implies that the essential spectrum of H coincides with the interval $[0, \infty)$. A most interesting problem in the spectral theory for H is that of absolute continuity. Namely, let $E(\cdot)$ be the spectral measure associated with H. Then the problem is: Is H restricted to $E((0, \infty))L_2(\boldsymbol{R}^3)$ an absolutely continuous operator? This problem is affirmatively answered by making use of the so-called limiting absorption principle. Our limiting absorption principle for H states that the resolvent $(H - z)^{-1}$ can be extended to a $\boldsymbol{B}(L_2^s(\boldsymbol{R}^3), L_2^{-s}(\boldsymbol{R}^3))$-valued continuous function of z on $\Pi\setminus(\sigma_p(H)\cup\{0\})$ when $s > 1/2$. Here Π is the complex plane with the upper and lower edges of $(0, \infty)$ distinguished such that the upper (lower) edge is the boundary points from above (below) (see Kuroda [11, Appendix to Chap. IV]), and $\sigma_p(H)$ denotes the point spectrum of H, $\boldsymbol{B}(X, Y)$ the Banach space of bounded linear operators on X to Y, and $L_2^s(\boldsymbol{R}^3)$ the weighted L_2 space defined by

$$L_2^s(\boldsymbol{R}^3) = \{u(x); (1 + |x|^2)^{s/2}u(x) \in L_2(\boldsymbol{R}^3)\}$$

with the norm $\|u\|_{0,s} = \|(1 + |\cdot|^2)^{s/2}u\|$ ($\|u\| = \|u\|_{0,0}$ is the usual L_2-norm).

Let us recall some notions from scattering theory. In the situation described above the wave operators $W_\pm$ interwining the pair (H, H_0), defined as

$$W_\pm = \text{strong} \lim_{t\to\pm\infty} e^{itH}e^{-itH_0},$$

are shown to exist and to be complete. Thus, let us define the generalized Fourier transform $\mathscr{F}_\pm$ by

$$\mathscr{F}_\pm = \mathscr{F}W_\pm^*,$$

where $\mathscr{F}$ is the ordinary Fourier Transform defined by

$$(\mathscr{F}u)(\xi) = (2\pi)^{-3/2}\int_{\boldsymbol{R}^3} e^{-i\xi\cdot x}u(x)dx,$$

and * means adjoint. Then, with the aid of the limiting absorption principle for H we can construct the distorted plane waves $\varphi_\pm(x, \xi)$ which are the integral kernels of $\mathscr{F}_\pm$ and satisfy the following Lippmann-Schwinger equation

$$\varphi_{\pm}(x, \xi) = e^{i\xi\cdot x} - \frac{1}{4\pi}\int_{S_a} \frac{e^{\mp i|\xi||x-y|}}{|x-y|} q(y)\varphi_{\pm}(y, \xi)dS_y.$$

On the other hand, let $\lambda_1, \lambda_2, \cdots$ be the nonpositive eigenvalues of H (counting multiplicity) and $\varphi_1(x), \varphi_2(x), \cdots$ the corresponding normalized eigenfunctions of H. Then we have the following eigenfunction expansion formula

$$u(x) = \sum (u, \varphi_n)\varphi_n(x) + \text{l.i.m.}\,(2\pi)^{-3/2}\int_{\boldsymbol{R}^3} d\xi(\mathscr{F}_{\pm}u)(\xi)\varphi_{\pm}(x, \xi),$$

where l.i.m. means limit in the mean.

We shall outline here the contents of the present paper. In § 1 we shall define the proper selfadjoint operator H corresponding to H_{formal} and characterize the domain of H. § 2 will be devoted to studying some integral operators connected with the resolvent of H. The second resolvent equation for H and H_0 will be discussed in § 3. The existense and completeness of the wave operators will be shown in § 4. In § 5 we shall investigate the spectrum of H. An upper bound on the total number of the bound states of H will be given in § 6. In § 7 we shall show the limiting absorption principle for H, and in § 8 the eigenfunction expansion theorem concerning H.

Part of the results obtained here has been announced in LNM 1285, 211–214 (ed. I. W. Knowles and Y. Saitō). Also, a detailed discussion of the scattering matrices will be given elsewhere by one of the authors (S. S.).

§ 1. The Schrödinger operator H

Throughout the paper we shall make the following assumption.

Assumption 1.1. $q(x)$ is a real-valued, smooth function on S_a.

For a rigorous definition of the Schödinger operator H_{formal}, we need some lemmas concerning the trace operators.

Lemma 1.2. *Let γ_+ and γ_- be the trace operators from $H^1(\{x; |x| > a\})$ and $H^1(\{x; |x| < a\})$, respectively, to $L_2(S_a)$. Let $u \in H^1(\boldsymbol{R}^3)$. Then $\gamma_+ u = \gamma_- u$.*

Proof. Since $u \in H^1(\boldsymbol{R}^3)$ and $C_0^\infty(\boldsymbol{R}^3)$ is dense in $H^1(\boldsymbol{R}^3)$ we can choose a sequence $\{u_n\} \subset C_0^\infty(\boldsymbol{R}^3)$ such that $u_n \to u$ in $H^1(\boldsymbol{R}^3)$ as $n \to \infty$. Since $\gamma_\pm$ are bounded operators from $H^1(\boldsymbol{R}^3)$ to $L_2(S_a)$ (see, e.g. Mizohata [13, Chap. III]), respectively, there exists a constant C such that

(1.1) $$\|\gamma_\pm f\|_a \le C\|f\|_{H^1(\{x; |x| \gtrless a\})} \quad \text{for } f \in H^1(\{x; |x| \gtrless a\}),$$

where $\|u\|_a = \sqrt{(u, u)_a}$. In view of $(\gamma_+ u_n)(x) = (\gamma_- u_n)(x) = (u_n|_{S_a})(x)$ for each n, we have by (1.1)

$$(1.2)\qquad \|\gamma_+ u - \gamma_- u\|_a \le C\|u - u_n\|_{H^1(\{x;|x|>a\})} + C\|u - u_n\|_{H^1(\{x;|x|<a\})} \le 2C\|u - u_n\|_{H^1(\boldsymbol{R}^3)}.$$

Letting n tend to ∞ in (1.2), we obtain that $\gamma_+ u = \gamma_- u$. Q.E.D.

By the above lemma, we can define the trace operator γ from $H^1(\boldsymbol{R}^3)$ to $L_2(S_a)$ by $\gamma u = \gamma_+ u (= \gamma_- u)$ for $u \in H^1(\boldsymbol{R}^3)$.

Lemma 1.3. *Let u belong to $H^1(\boldsymbol{R}^3)$. Then we have for any $\varepsilon > 0$*

$$(1.3)\qquad \|\gamma u\|_a^2 \le \varepsilon\|\nabla u\|^2 + \frac{1}{\varepsilon}\|u\|^2,$$

$$(1.4)\qquad \|\gamma u\|_a \le \sqrt{a}\,\|\nabla u\|.$$

Proof. Since $C_0^\infty(\boldsymbol{R}^3)$ is dense in $H^1(\boldsymbol{R}^3)$ and γ is a bounded operator from $H^1(\boldsymbol{R}^3)$ to $L_2(S_a)$, it suffices to prove the lemma for $u \in C_0^\infty(\boldsymbol{R}^3)$. Let $u \in C_0^\infty(\boldsymbol{R}^3)$ and $\varepsilon > 0$. Using the inequality $2|p \cdot q| \le \varepsilon|p|^2 + \varepsilon^{-1}|q|^2$, we have for any $\omega \in S^2$ (the unit sphere of $\boldsymbol{R}^3$)

$$(1.5)\qquad |u(a\omega)|^2 = -2\,\mathrm{Re}\int_a^\infty \frac{\partial u}{\partial r}(r\omega)\,\overline{u(r\omega)}\,dr \le \varepsilon\int_a^\infty \left|\frac{\partial u}{\partial r}(r\omega)\right|^2 dr + \varepsilon^{-1}\int_a^\infty |u(r\omega)|^2\,dr \le \varepsilon\int_a^\infty \frac{r^2}{a^2}\left|\frac{\partial u}{\partial r}(r\omega)\right|^2 dr + \varepsilon^{-1}\int_a^\infty \frac{r^2}{a^2}|u(r\omega)|^2\,dr.$$

Multiplying both sides of (1.5) by a^2 and integrating with respect to ω over the unit sphere S^2 yield

$$(1.6)\qquad \int_{S_a} |u(x)|^2\,dS_x \le \varepsilon\int_{|x|\ge a}\left|\frac{\partial u}{\partial r}(x)\right|^2 dx + \varepsilon^{-1}\int_{|x|\ge a}|u(x)|^2\,dx \le \varepsilon\left\|\frac{\partial u}{\partial r}\right\|^2 + \varepsilon^{-1}\|u\|^2.$$

(1.3) follows from (1.6) and $\left|\frac{\partial u}{\partial r}(x)\right| \le |\nabla u(x)|$. To prove (1.4), we have by Schwarz' inequality

$$(1.7)\qquad |u(a\omega)|^2 = \left|-\int_a^\infty \frac{\partial u}{\partial r}(r\omega)\,dr\right|^2 \le \int_a^\infty \frac{dr}{r^2}\int_a^\infty r^2\left|\frac{\partial u}{\partial r}(r\omega)\right|^2 dr = \frac{1}{a}\int_a^\infty r^2\left|\frac{\partial u}{\partial r}(r\omega)\right|^2 dr.$$

Thus we have

(1.8) $$\int_{S_a} |u(x)|^2 \, dS_x \le a \int_{|x| \ge a} \left| \frac{\partial u}{\partial r}(x) \right|^2 dx \le a \left\| \frac{\partial u}{\partial r} \right\|^2 \le a \| \nabla u \|^2 .$$

Q.E.D.

Now we are in a position to define a selfadjoint operator corresponding to H_{formal} in a rigorous way. Consider the quadratic form

(1.9) $$h[u, v] = (\nabla u, \nabla v) + (q\gamma u, \gamma v)_a, \ Dom[h] = H^1(\boldsymbol{R}^3).$$

Since q is bounded on S_a by Assumption 1.1, it follows from Lemma 1.1 that h is a symmetric, lower semibounded, closed form. Therefore, by Kato [9, Chap. VI, Theorem 2.1] we have the following

Theorem 1.4. *Let h be the quadratic form defined by* (1.9). *Then there exists a unique selfadjoint operator H such that*

(1.10) $Dom(H) \subset Dom[h]$, $(Hu, v) = h[u, v]$ *for* $u \in Dom(H)$ *and* $v \in Dom[h]$.

We adopt this operator H as the Schrödinger operator corresponding to H_{formal} stated in the Introduction.

Theorem 1.5. *Let* $A = \min_{x \in S_a} q(x)$. *Then*

(1.11) $$H \ge -A^2.$$

Moreover, we have

(1.12) $$H \ge 0 \quad \text{for} \ -\frac{1}{a} \le A$$

and

(1.13) $$H \ge \frac{4}{a^2}(aA + 1) \quad \text{for} \ -\frac{2}{a} \le A \le -\frac{1}{a}.$$

Proof. By Theorem 1.4 we have for any $u \in Dom(H)$

(1.14) $$(Hu, u) = \| \nabla u \|^2 + \int_{S_a} q(x) |\gamma u(x)|^2 \, dS_x$$
$$\ge \| \nabla u \|^2 + A \| \gamma u \|_a^2 .$$

If $A \ge -\frac{1}{a}$, (1.12) follows immediately from (1.14) and (1.4) of Lemma 1.3. Let us assume that $-\frac{2}{a} \le A \le -\frac{1}{a}$. Rewriting (1.14), we have

(1.15) $$(Hu, u) \ge \| \nabla u \|^2 - A\left(\frac{2}{Aa} + 1 \right) \| \gamma u \|_a^2 + A\left(2 + \frac{2}{Aa} \right) \| \gamma u \|_a^2 .$$

By Lemma 1.3 $\left(\text{putting } \varepsilon = \frac{a}{2} \text{ in (1.3)}\right)$, we have

$$(1.16)\qquad (Hu, u) \geq \|\nabla u\|^2 - A\left(\frac{2}{Aa} + 1\right) a \|\nabla u\|^2 + A\left(2 + \frac{2}{Aa}\right)\left(\frac{a}{2}\|\nabla u\|^2 + \frac{2}{a}\|u\|^2\right) = \frac{4}{a^2}(aA + 1)\|u\|^2.$$

This implies (1.13). To complete the proof, we have only to show that (1.11) holds when $A < 0$. In this case, (1.11) follows from (1.14) and Lemma 1.3 with $\varepsilon = -\frac{1}{A}$.

Remark 1.6. The above theorem implies that H has no negative eigenvalues if $A \geq -\frac{1}{a}$. On the other hand, if $A < -\frac{1}{a}$, H can have negative eigenvalues. In fact, let $q(x) = V_0$ (constant) such that $V_0 < -\frac{1}{a}$. Then it is seen that H has a negative eigenvalue $-\lambda^2$ $(\lambda > 0)$, where λ is the unique solution of the equation $\frac{1 - e^{-2a\lambda}}{\lambda} = -\frac{2}{V_0}$, and a corresponding eigenfunction is $\frac{1}{|x|}(e^{-\lambda||x|-a|} - e^{-\lambda(|x|+a)})$ (see Dolph et al. [5, pp. 326–327], and cf. Theorem 5.3 below).

Now, we shall characterize the domain of H.

Theorem 1.7. *$u \in Dom(H)$ if and only if*

$$(1.17)\qquad \begin{gathered} u \in H^1(\mathbf{R}^3),\ u \in H^2(\{x; |x| < a\}),\ u \in H^2(\{x; |x| > a\}) \text{ and} \\ q(x)(\gamma u)(x) - \left\{\frac{\partial u}{\partial n_+}(x) + \frac{\partial u}{\partial n_-}(x)\right\}|_{S_a} = 0. \end{gathered}$$

In this case, $Hu = -\Delta u$ in the distribution sense and u is continuous on $\mathbf{R}^3$.

Remark 1.8. Strictly speaking, $\frac{\partial u}{\partial n_\pm}|_{S_a}(x)$ denotes $\sum_{j=1}^{3} \langle n_\pm, e_j \rangle \gamma_\pm\left(\frac{\partial u}{\partial x_j}\right)(x)$, where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ and $\langle x, y \rangle$ means the scalar product of the vectors x and y.

Proof of the theorem. First let $u \in Dom(H)$. $u \in H^1(\mathbf{R}^3)$ is deirect from Theorem 1.4. Now, by Theorem 1.4 we have for any $v \in C_0^\infty(\{x; |x| < a\})$

$$(1.18)\qquad \int_{|x|<a} (Hu)(x)\overline{v(x)}\,dx = h[u, v] = \int_{|x|<a} (\nabla u)(x)\overline{(\nabla v)(x)}\,dx$$

$$= -\int_{|x|<a} u(x)\overline{(\Delta v)(x)}\,dx,$$

where and in the sequel it is understood that all derivatives are taken in the distribution sense. (1.18) implies that $Hu = -\Delta u$ in $\{x; |x| < a\}$ and $u \in H^2(\{x; |x| < a\})$ (see the note added in proof). Similarly, $Hu = -\Delta u$ in $\{x; |x| > a\}$ and $u \in H^2(\{x; |x| > a\})$. Therefore, it makes sense to speak of $\frac{\partial u}{\partial n_\pm}|_{S_a}$. Thus, by Theorem 1.4 and Green's Theorem (see e.g. Mizohata [13, Chap. III, §8]) we obtain for any $v \in C_0^\infty(\boldsymbol{R}^3)$

$$(1.19) \qquad (Hu, v) = h[u, v] = \int_{|x|<a} (\nabla u)(x)\overline{(\nabla v)(x)}\,dx + \int_{|x|>a} (\nabla u)(x)\overline{(\nabla v)(x)}\,dx + (q\gamma u, \gamma v)_a$$

$$= -\int_{|x|<a} (\Delta u)(x)\,\overline{v(x)}\,dx - \left(\frac{\partial u}{\partial n_-}|_{S_a}, \gamma v\right)_a - \int_{|x|>a} (\Delta u)(x)\,\overline{v(x)}\,dx - \left(\frac{\partial u}{\partial n_+}|_{S_a}, \gamma v\right)_a + (q\gamma u, \gamma v)_a$$

$$= (-\Delta u, v) + \left(q\gamma u - \left\{\frac{\partial u}{\partial n_+} + \frac{\partial u}{\partial n_-}\right\}|_{S_a}, \gamma v\right)_a,$$

and, since $u \in H^2(\{x; |x| \neq a\})$ and $Hu = -\Delta u$ as shown above,

$$\left(q\gamma u - \left\{\frac{\partial u}{\partial n_+} + \frac{\partial u}{\partial n_-}\right\}|_{S_a}, \gamma v\right)_a = 0 \quad \text{for any } v \in C_0^\infty(\boldsymbol{R}^3).$$

Since $\{\gamma v = v|_{S_a}; v \in C_0^\infty(\boldsymbol{R}^3)\}$ is dense in $L_2(S_a)$, we have

$$q\gamma u - \left\{\frac{\partial u}{\partial n_+} + \frac{\partial u}{\partial n_-}\right\}|_{S_a} = 0.$$

We have thus shown (1.17).

Conversely, let u verify (1.17). Define $w \in L_2(\boldsymbol{R}^3)$ by $w = -\Delta u$ (except on S_a). Then, for any $v \in Dom(H)$, we have, as we got (1.19),

$$(1.20) \qquad (Hv, u) = h[v, u] = (\nabla v, \nabla u) + (q\gamma v, \gamma u)_a$$

$$= (v, -\Delta u) + \left(\gamma v, q\gamma u - \left\{\frac{\partial u}{\partial n_+} + \frac{\partial u}{\partial n_-}\right\}|_{S_a}\right)_a$$

$$= (v, w)$$

This implies that $u \in Dom(H^*) = Dom(H)$.

Finally, let $u \in Dom(H)$. By what has been shown above, we have $u \in H^2(\{x; |x| < a\})$, $u \in H^2(\{x; |x| > a\})$ and $u \in H^1(\boldsymbol{R}^3)$. Thus, according to

Calderón's extension theorem (e.g. Agmon [1, p. 171, Theorem 11.12]), there exist u_1, $u_2 \in H^2(\boldsymbol{R}^3)$ such that

$$(1.21)\qquad \begin{aligned} (u_1|_{\{x;|x|<a\}})(x) &= u(x) \quad \text{for a.e. } x \text{ in } \{x;\, |x| < a\}, \\ (u_2|_{\{x;|x|>a\}})(x) &= u(x) \quad \text{for a.e. } x \text{ in } \{x;\, |x| > a). \end{aligned}$$

Since $u \in H^1(\boldsymbol{R}^3)$, we have in view of Lemma 1.2

$$(1.22)\qquad \gamma_-(u_1|_{\{x;|x|<a\}}) = \gamma u = \gamma_+(u_2|_{\{x;|x|>a\}}).$$

On the other hand, Sobolev's lemma (e.g. Reed-Simon [18, p. 32, Theorem 3.9]) implies that u_1 and u_2 are continuous on $\boldsymbol{R}^3$. Hence we have by (1.22)

$$(1.23)\qquad \begin{aligned} (u_1|_{\{x;|x|=a\}})(x) &= \gamma_-(u_1|_{\{x;|x|<a\}})(x) \\ &= \gamma_+(u_2|_{\{x;|x|>a\}})(x) = (u_2|_{\{x;|x|=a\}})(x) \quad \text{on } S_a. \end{aligned}$$

From (1.21) and (1.23), it follows that u is continuous on $\boldsymbol{R}^3$. Q.E.D.

§ 2. Preliminary lemmas

We shall introduce the following integral operators T_κ and $\tilde{T}_\kappa$ depending on a complex parameter κ defined by

$$(T_\kappa f)(x) = -\frac{1}{4\pi}\int_{S_a} \frac{e^{i\kappa|x-y|}}{|x-y|}\, q(y) f(y)\, dS_y \quad (x \in \boldsymbol{R}^3)$$

and

$$(\tilde{T}_\kappa f)(x) = -\frac{1}{4\pi}\int_{S_a} \frac{e^{i\kappa|x-y|}}{|x-y|}\, q(y) f(y)\, dS_y \quad (x \in S_a).$$

Before studying the properties of T_κ and $\tilde{T}_\kappa$, we shall state some lemmas. First, by direct computation using polar coordinates, we have

Lemma 2.1. *Let $\zeta \in \boldsymbol{C}$. Then we have for any $x \in \boldsymbol{R}^3$*

$$(2.1)\qquad \begin{aligned} \int_{S_a} \frac{e^{\zeta|x-y|}}{|x-y|}\, dS_y &= \frac{2\pi a}{\zeta|x|}\left(e^{\zeta(a+|x|)} - e^{\zeta|a-|x||}\right) \quad (\zeta \neq 0), \\ \int_{S_a} \frac{1}{|x-y|}\, dS_y &= \frac{2\pi a}{|x|}(a + |x| - |a - |x||) \quad (\zeta = 0). \end{aligned}$$

Lemma 2.2. *There exists a constant C such that for any x, $y \in S_a$,*

$$(2.2)\qquad \int_{S_a} \frac{1}{|x-z||z-y|}\, dS_z \leq C(1 + |\log|x-y||),$$

$$(2.3)\qquad \int_{S_a} \frac{1}{|x-z|}\, |\log|z-y||\, dS_z \leq C,$$

and for any $x \in \boldsymbol{R}^3$, $0 < r < 3$ and $r + s > 3$

$$\text{(2.4)} \qquad \int_{\boldsymbol{R}^3} \frac{dy}{|x-y|^r (1+|y|^2)^{s/2}} \leq \begin{cases} \dfrac{C}{(1+|x|)^{r+s-3}} & (s < 3) \\ \dfrac{C \log(1+|x|)}{(1+|x|)^r} & (s = 3) \\ \dfrac{C}{(1+|x|)^r} & (s > 3). \end{cases}$$

For the proof, see e.g. Kellogg [10, pp. 301–303] or Kuroda [12, p. 162].

Lemma 2.3. *Let* $\operatorname{Im}\kappa > 0$. *Then* T_κ *is a Hilbert-Schmidt operator from* $L_2(S_a)$ *to* $L_2(\boldsymbol{R}^3)$.

Proof. Put $b = \operatorname{Im}\kappa$. We compute the Hilbert-Schmidt norm of T_κ.

$$\| 4\pi T_\kappa \|_{H.S.}^2 = \int_{S_a} dS_y \int_{\boldsymbol{R}^3} dx |q(y)|^2 \frac{e^{-2b|x-y|}}{|x-y|^2} = \frac{2\pi}{b} \| q \|_a^2 < +\infty,$$

from which follows the assertion. Q.E.D.

Lemma 2.4. *Let* $\kappa \in \boldsymbol{C}$. *Then* $\tilde{T}_\kappa$ *is a compact operator from* $L_2(S_a)$ *to itself.*

Proof. Define the integral operator $G_\kappa^{(\varepsilon)}$ by

$$(G_\kappa^{(\varepsilon)} f)(x) = -\int_{S_a} \chi_{\{y \in S_a; |x-y| > \varepsilon\}}(y) \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} q(y) f(y) dS_y \quad (x \in S_a,\ \varepsilon > 0),$$

where $\chi_A(x)$ denotes the characteristic function of the set A. Since we have

$$\int_{S_a \times S_a} dS_x dS_y \left| \chi_{\{y \in S_a; |x-y| > \varepsilon\}}(y) \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} q(y) \right|^2$$
$$\leq \left(\frac{e^{2|\operatorname{Im}\kappa|a}}{4\pi\varepsilon} \max_{y \in S_a} |q(y)| \right)^2 (4\pi a^2)^2 < +\infty,$$

$G_\kappa^{(\varepsilon)}$ is a Hilbert-Schmidt, and a fortiori, compact operator from $L_2(S_a)$ to itself for each $\varepsilon > 0$. To prove the lemma, we have only to show that $G_\kappa^{(\varepsilon)}$ converges to $\tilde{T}_\kappa$ in the operator norm topology when $\varepsilon \downarrow 0$. In fact, using Schwarz' inequality we have for any $f \in L_2(S_a)$ and $x \in S_a$

$$\text{(2.5)} \qquad |(G_\kappa^{(\varepsilon)} f)(x) - (\tilde{T}_\kappa f)(x)|^2 \leq \left(\int_{S_a \cap \{y; |x-y| \leq \varepsilon\}} \frac{e^{|\operatorname{Im}\kappa||x-y|}}{4\pi|x-y|} |q(y)||f(y)| dS_y \right)^2$$
$$\leq (\max_{y \in S_a} |q(y)|)^2 \left(\int_{S_a \cap \{y; |x-y| \leq \varepsilon\}} \frac{e^{|\operatorname{Im}\kappa||x-y|}}{4\pi|x-y|} dS_y \right) \left(\int_{S_a} \frac{e^{|\operatorname{Im}\kappa||x-y|}}{4\pi|x-y|} |f(y)|^2 dS_y \right)$$
$$\leq (\max_{y \in S_a} |q(y)|)^2 e^{\varepsilon|\operatorname{Im}\kappa|} \frac{\varepsilon}{2} \int_{S_a} \frac{e^{|\operatorname{Im}\kappa||x-y|}}{4\pi|x-y|} |f(y)|^2 dS_y \quad (\text{if } \varepsilon \leq a/2),$$

where we have used the equality

$$\int_{S_a \cap \{y; |x-y| \leq \varepsilon\}} \frac{1}{4\pi|x-y|} dS_y = \frac{a}{2|x|}(\varepsilon - |a - |x||) \tag{2.6}$$

if $x \in \boldsymbol{R}^3$, $|a - |x|| < \varepsilon \leq a/2$. Integrating the both sides of (2.5) over S_a yields by Lemma 2.1 and Fubini's theorem

$$\| G_\kappa^{(\varepsilon)} f - \tilde{T}_\kappa f \|_a^2 \leq (\max_{y \in S_a} |q(y)|)^2 e^{\varepsilon|\mathrm{Im}\kappa|} \frac{\varepsilon}{2} e^{2a|\mathrm{Im}\kappa|} a \| f \|_a^2 . \tag{2.7}$$

From (2.7), the claim follows immediately. Q.E.D.

Define the Fourier transform $\mathscr{F}_{S_a}$ on $L_2(S_a)$ by

$$(\mathscr{F}_{S_a} f)(\xi) = (2\pi)^{-3/2} \int_{S_a} e^{-i\xi \cdot x} f(x)\, dS_x \quad (\xi \in \boldsymbol{R}^3). \tag{2.8}$$

Then, as is well known (e.g. Mochizuki [14, p. 16]), we have

Proposition 2.5. *Let $s > 1/2$. Then $\mathscr{F}_{S_a}$ is a bounded operator from $L_2(S_a)$ to $L_2^{-s}(\boldsymbol{R}^3)$, i.e. there exists a constant C such that*

$$\| \mathscr{F}_{S_a} f \|_{0,-s} \leq C \| f \|_a \quad \textit{for any } f \in L_2(S_a). \tag{2.9}$$

Lemma 2.6. *Let* $\mathrm{Im}\, \kappa > 0$. *Then T_κ is a bounded operator from $L_2(S_a)$ to $H^1(\boldsymbol{R}^3)$.*

Proof. For any $f \in L_2(S_a)$ we have by Fubini's theorem

$$\begin{aligned}(\mathscr{F} T_\kappa f)(\xi) &= - \int_{S_a} dS_y q(y) f(y) (2\pi)^{-3/2} \int_{\boldsymbol{R}^3} dx\, e^{-i\xi \cdot x} \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} \\ &= -(2\pi)^{-3/2} \int_{S_a} dS_y \frac{e^{-i\xi \cdot y}}{|\xi|^2 - \kappa^2} q(y) f(y) \\ &= - \frac{1}{|\xi|^2 - \kappa^2} (\mathscr{F}_{S_a}(qf))(\xi),\end{aligned} \tag{2.10}$$

where we used (2.8) and the fact that

$$\mathscr{F}\left(\frac{e^{i\kappa|\cdot - y|}}{4\pi|\cdot - y|} \right)(\xi) = (2\pi)^{-3/2} \frac{e^{-i\xi \cdot y}}{|\xi|^2 - \kappa^2}. \tag{2.11}$$

Take s such that $1/2 < s < 1$. Then, by Proposition 2.5 we can estimate the H^1-norm $\| T_\kappa f \|_{H^1}$ of $T_\kappa f$ as follows.

$$(2.12) \qquad \| T_\kappa f \|_{H^1}^2 = \int_{\mathbf{R}^3} d\xi (1 + |\xi|^2) |(\mathscr{F} T_\kappa f)(\xi)|^2$$

$$= \int_{\mathbf{R}^3} d\xi (1 + |\xi|^2) \left| \frac{-1}{|\xi|^2 - \kappa^2} (\mathscr{F}_{S_a}(qf))(\xi) \right|^2$$

$$\leq \sup_{\xi \in \mathbf{R}^3} \left\{ \frac{(1 + |\xi|^2)^{1+s}}{||\xi|^2 - \kappa^2|^2} \right\} \int_{\mathbf{R}^3} d\xi (1 + |\xi|^2)^{-s} |(\mathscr{F}_{S_a}(qf))(\xi)|^2$$

$$= \sup_{\xi \in \mathbf{R}^3} \left\{ \frac{(1 + |\xi|^2)^{1+s}}{||\xi|^2 - \kappa^2|^2} \right\} \cdot \| \mathscr{F}_{S_a}(qf) \|_{0,-s}^2$$

$$\leq \sup_{\xi \in \mathbf{R}^3} \left\{ \frac{(1 + |\xi|^2)^{1+s}}{||\xi|^2 - \kappa^2|^2} \right\} C^2 \{ \max_{x \in S_a} |q(x)| \}^2 \| f \|_a^2,$$

which implies the required result. Q.E.D.

By the above lemma, γT_κ ($\operatorname{Im} \kappa > 0$) is a well-defined bounded operator from $L_2(S_a)$ to itself. Furthermore, we have

Lemma 2.7. *Let* $\operatorname{Im} \kappa > 0$. *Then* $\gamma T_\kappa = \tilde{T}_\kappa$.

Proof. Since γT_κ and $\tilde{T}_\kappa$ are bounded operators on $L_2(S_a)$ and the set of continuous functions on S_a is dense in $L_2(S_a)$, it suffices to prove that $\gamma T_\kappa = \tilde{T}_\kappa$ on this set. Assume that f is continuous on S_a. Then it follows in a standard way (e.g. Colton-Kress [4, p. 47, Theorem 2.12]) that $(T_\kappa f)(x)$ is continuous on $\mathbf{R}^3$. On the other hand, we have for a.e. $x \in S_a$

$$(\gamma T_\kappa f)(x) = \lim_{y \to x} (T_\kappa f)(y) \quad (y \text{ approaches } x \text{ along } n_\pm).$$

Therefore, $(\gamma T_\kappa f)(x) = (T_\kappa f)(x) = (\tilde{T}_\kappa f)(x)$ for a.e. $x \in S_a$. Thus the lemma has been proven. Q.E.D.

Lemma 2.8. *Let* $\kappa \in \mathbf{C}$. *Then* $(\tilde{T}_\kappa)^2$ *is a Hilbert-Schmidt operator from* $L_2(S_a)$ *to itself.*

Proof. The kernel of $(\tilde{T}_\kappa)^2$ is

$$\left(\frac{1}{4\pi} \right)^2 \int_{S_a} dS_z \frac{e^{i\kappa(|x-z| + |z-y|)}}{|x-z||z-y|} q(z) q(y).$$

Introducing polar coordinates, we have by Lemma 2.2

$$\int_{S_a \times S_a} dS_x dS_y \left| \int_{S_a} dS_z \frac{e^{i\kappa(|x-z| + |z-y|)}}{|x-z||z-y|} q(z) q(y) \right|^2$$

$$\leq e^{8a|\operatorname{Im}\kappa|} (\max_{z \in S_a} |q(z)|)^4 \int_{S_a \times S_a} dS_x dS_y \left(\int_{S_a} dS_z \frac{1}{|x-z||z-y|} \right)^2$$

$$\leq e^{8a|\mathrm{Im}\kappa|}(\max_{z\in S_a}|q(z)|)^4\int_{S_a\times S_a} dS_x dS_y C^2(1+|\log|x-y||)^2<+\infty,$$

which proves the lemma. Q.E.D.

Lemma 2.9. *Let* $s>1/2$. *Then* T_κ *is a* $\boldsymbol{B}(L_2(S_a),\ L_2^{-s}(\boldsymbol{R}^3))$*-valued continuous function of* κ *for* $\mathrm{Im}\,\kappa\geq 0$.

Proof. For any $f\in L_2(S_a)$, we consider the difference

$$(T_\kappa f)(x)-(T_{\kappa'}f)(x)=\frac{-1}{4\pi}\int_{S_a}dS_y\frac{e^{i\kappa|x-y|}-e^{i\kappa'|x-y|}}{|x-y|}q(y)f(y). \tag{2.13}$$

In view of the inequality

$$\begin{aligned}&|e^{i\kappa|x-y|}-e^{i\kappa'|x-y|}|\\ &\leq|\kappa-\kappa'|^\mu|x-y|^\mu e^{-\mu(\mathrm{Im}\kappa+\mathrm{Im}\kappa')|x-y|}\times\\ &\times(e^{-\mathrm{Im}\kappa|x-y|}+e^{-\mathrm{Im}\kappa'|x-y|})^{1-\mu}\quad(0\leq\mu\leq 1),\end{aligned} \tag{2.14}$$

we have

$$|(T_\kappa f)(x)-(T_{\kappa'}f)(x)|\leq\frac{2^{1-\mu}}{4\pi}|\kappa-\kappa'|^\mu\max_{y\in S_a}|q(y)|\int_{S_a}dS_y\frac{|f(y)|}{|x-y|^{1-\mu}}. \tag{2.15}$$

Taking μ such that $0<\mu<\min(s-1/2,1)$, we get by Schwarz' inequality and Fubini's theorem

$$\begin{aligned}&\|T_\kappa f-T_{\kappa'}f\|_{0,-s}^2\\ &\leq\int_{\boldsymbol{R}^3}dx(1+|x|^2)^{-s}\left(\frac{2^{1-\mu}}{4\pi}|\kappa-\kappa'|^\mu\max_{y\in S_a}|q(y)|\right)^2\left(\int_{S_a}dS_y\frac{|f(y)|}{|x-y|^{1-\mu}}\right)^2\\ &\leq\left(\frac{2^{1-\mu}}{4\pi}|\kappa-\kappa'|^\mu\max_{y\in S_a}|q(y)|\right)^2\int_{\boldsymbol{R}^3}dx(1+|x|^2)^{-s}\times\\ &\qquad\times\int_{S_a}dS_y\frac{1}{|x-y|^{2-2\mu}}\int_{S_a}dS_y|f(y)|^2\\ &=\left(\frac{2^{1-\mu}}{4\pi}|\kappa-\kappa'|^\mu\max_{y\in S_a}|q(y)|\right)^2\times\\ &\qquad\times\int_{S_a}dS_y\int_{\boldsymbol{R}^3}dx\frac{1}{|x-y|^{2-2\mu}(1+|x|^2)^s}\|f\|_a^2.\end{aligned} \tag{2.16}$$

(2.16) together with Lemma 2.2, (2.4) yields the required result. Q.E.D.

Lemma 2.10. $\tilde{T}_\kappa$ *is a* $\boldsymbol{B}(L_2(S_a))$*-valued continuous function of* κ *in* $\boldsymbol{C}$.

Proof. Using (2.15) ($\mu=1$), we have for $f\in L_2(S_a)$ and $x\in S_a$.

$$(2.18)\qquad |(\tilde{T}_\kappa f)(x) - (\tilde{T}_{\kappa'} f)(x)|^2 \le \left(\frac{|\kappa - \kappa'|e^{2a(|\mathrm{Im}\kappa| + |\mathrm{Im}\kappa'|)}}{4\pi} \max_{y\in S_a} |q(y)|\right)^2 \left(\int_{S_a} dS_y |f(y)|\right)^2.$$

Integrating the both sides of (2.18) over S_a and making use of Schwarz' inequality, we obtain

$$(2.19)\qquad \|\tilde{T}_\kappa f - \tilde{T}_{\kappa'} f\|_a^2 \le \left(\frac{|\kappa - \kappa'|e^{2a(|\mathrm{Im}\kappa| + |\mathrm{Im}\kappa'|)}}{4\pi} \max_{y\in S_a} |q(y)|\right)^2 (4\pi a^2)^2 \|f\|_a^2,$$

which completes the proof. Q.E.D.

Lemma 2.11. *Let $\kappa\in \boldsymbol{C}$ and let $u\in L_2(S_a)$. Then, for any $w\in C_0^\infty(\boldsymbol{R}^3)$ we have*

$$(2.20)\qquad \int_{\boldsymbol{R}^3} (T_\kappa u)(x)(-\Delta - \kappa^2)w(x)dx = -\int_{S_a} q(x)u(x)w(x)\,dS_x.$$

If $\mathrm{Im}\,\kappa \ge 0$, (2.20) *holds for any* $w\in\mathscr{S}$, *where* $\mathscr{S} = \mathscr{S}(\boldsymbol{R}^3)$ *denotes the set of functions which together with all their derivatives fall off faster than the inverse of any polynomial.*

Proof. By Fubini's theorem we have for $w\in C_0^\infty(\boldsymbol{R}^3)$

$$(2.21)\qquad \int_{\boldsymbol{R}^3} (T_\kappa u)(x)(-\Delta - \kappa^2)w(x)dx = -\int_{S_a} dS_y q(y)u(y)\int_{\boldsymbol{R}^3} dx\, \frac{e^{i\kappa|x-y|}}{4\pi|x-y|}(-\Delta - \kappa^2)w(x).$$

On the other hand, we have by Green's theorem

$$(2.22)\qquad \int_{\boldsymbol{R}^3} dx\, \frac{e^{i\kappa|x-y|}}{|x-y|}(-\Delta - \kappa)^2 w(x) = 4\pi w(y)$$

for $w\in C_0^\infty(\boldsymbol{R}^3)$. The first part of the lemma follows immediately from (2.21) and (2.22). The proof of the second half is similar. Q.E.D.

Lemma 2.12. *Let* $\mathrm{Im}\,\kappa > 0$. *Suppose that u is a non-trivial solution of the homogeneous equation $u = \tilde{T}_\kappa u$ in $L_2(S_a)$. Then $v \equiv T_\kappa u$ is a non-trivial eigenvector of H corresponding to eigenvalue κ^2. Conversely, if v is a non-zero eigenvector of H corresponding to eigenvalue κ^2, γv is a non-zero vector in $L_2(S_a)$ and satisfies the equation $\gamma v = \tilde{T}_\kappa(\gamma v)$.*

Proof. Assume that $u\in L_2(S_a)$, $u\ne 0$ and $u = \tilde{T}_\kappa u$. By Lemma 2.11, we have for any $w\in C_0^\infty(\boldsymbol{R}^3)$

$$(2.23)\qquad \int_{\boldsymbol{R}^3}(T_\kappa u)(x)(-\Delta-\kappa^2)\overline{w(x)}\,dx=-\int_{S_a}q(x)u(x)\overline{w(x)}\,dS_x.$$

Since $v\equiv T_\kappa u$ belongs to $H^1(\boldsymbol{R}^3)$ by Lemma 2.6 and hence $\gamma v=\gamma T_\kappa u=\tilde{T}_\kappa u=u$ by Lemma 2.7, we have on integration by parts

$$(2.24)\qquad (\nabla v,\nabla w)-\kappa^2(v,w)+(q\gamma v,\gamma w)_a=0\quad \text{for any } w\in C_0^\infty(\boldsymbol{R}^3)$$

Since $C_0^\infty(\boldsymbol{R}^3)$ is dense in $H^1(\boldsymbol{R}^3)$, $C_0^\infty(\boldsymbol{R}^3)$ is a form core of h by Kato [9, Chap. VI, Theorem 1.21] and Lemma 1.3. So we have by (2.24)

$$h[v,w]=\kappa^2(v,w)\quad \text{for any } w\in H^1(\boldsymbol{R}^3).$$

Therefore, we obtain by Theorem 1.4

$$(v,(H-\bar{\kappa}^2)w)=0\quad \text{for any } w\in Dom(H),$$

which implies that $v\in Dom(H)$ and $(H-\kappa^2)v=0$. If $v=0$, we have by Lemma 2.7 $u=\tilde{T}_\kappa u=\gamma T_\kappa u=\gamma v=0$, which is a contradiction. Thus v is non-trivial, and is an eigenvector with eigenvalue κ^2.

Conversely, let v verify that $v\in Dom(H)$, $v\neq 0$ and $(H-\kappa^2)v=0$. Since $v\in H^1(\boldsymbol{R}^3)$ by Theorem 1.7 and hence $\gamma v\in L_2(S_a)$, we have by Theorem 1.4.

$$\begin{aligned}(2.25)\qquad &(\nabla v,\nabla w)-\kappa^2(v,w)+(q\gamma v,\gamma w)_a\\ &=h[v,w]-\kappa^2(v,w)=((H-\kappa^2)v,w)\\ &=0\qquad \text{for any } w\in H^1(\boldsymbol{R}^3).\end{aligned}$$

On the other hand, as we got (2.24) from (2.23), we obtain by Lemma 2.11 and in view of $\gamma v\in L_2(S_a)$ for any $w\in\mathscr{S}$

$$(2.26)\qquad (\nabla(T_\kappa\gamma v),\nabla w)-\kappa^2(T_\kappa\gamma v,w)+(q\gamma v,\gamma w)_a=0,$$

(note that $T_\kappa(\gamma v)\in H^1(\boldsymbol{R}^3)$ by Lemma 2.6). Therefore, from (2.25) and (2.26) it follws that

$$(2.27)\qquad (\nabla(T_\kappa\gamma v-v),\nabla w)-\kappa^2(T_\kappa\gamma v-v,w)=0\quad \text{for any } w\in\mathscr{S}.$$

By Parseval's identity we can rewrite (2.27) as

$$(2.28)\qquad (\mathscr{F}(T_\kappa\gamma v-v),(|\cdot|^2-\bar{\kappa}^2)\mathscr{F}w)=0\quad \text{for any } w\in\mathscr{S}.$$

Put $w(x)=\mathscr{F}^{-1}\left(\dfrac{h}{|\cdot|-\bar{\kappa}^2}\right)(x)$ for $h\in\mathscr{S}$. Since w belongs to $\mathscr{S}$, we obtain by (2.28)

$$(\mathscr{F}(T_\kappa\gamma v-v),h)=0\quad \text{for any } h\in\mathscr{S},$$

and hence

$$(2.29)\qquad T_\kappa\gamma v-v=0 \text{ in } L_2(\boldsymbol{R}^3).$$

If $\gamma v = 0$, $v = 0$ by (2.29), which is a contradiction. Thus γv is a non-zero vector and $\gamma v = \gamma T_\kappa(\gamma v) = \tilde{T}_\kappa(\gamma v)$ by (2.29). We have thus completed the proof of the lemma. Q.E.D.

Lemma 2.13. *Let* $\operatorname{Im}\kappa > 0$. *Then*

$$T_\kappa^* = -q\gamma R_0(\bar{\kappa}^2), \tag{2.30}$$

which maps from $L_2(\boldsymbol{R}^3)$ *to* $L_2(S_a)$.

Proof. By Fubini's theorem we have for $u \in L_2(S_a)$ and $v \in L_2(\boldsymbol{R}^3)$

$$\begin{aligned}(T_\kappa u, v) &= \int_{\boldsymbol{R}^3} dx \left(\frac{-1}{4\pi} \int_{S_a} dS_y \frac{e^{i\kappa}|x-y|}{|x-y|} q(y)u(y) \right) \overline{v(x)} \\ &= \int_{S_a} dS_y u(y) \overline{\left(\frac{-q(y)}{4\pi} \int_{\boldsymbol{R}^3} dx \frac{e^{i(-\bar{\kappa})|x-y|}}{|x-y|} v(x) \right)} \\ &= (u, -q\gamma R_0(\bar{\kappa}^2)v)_a,\end{aligned} \tag{2.31}$$

where we have used the reality and boundedness of q and $(\gamma R_0(z)v)(x) = (R_0(z)v)|_{S_a}(x) \in L_2(S_a)$ for $z \notin [0, \infty)$ as is seen by Sobolev's lemma in view of $Ran(R_0(z)) = H^2(\boldsymbol{R}^3)$. The lemma follows from (2.31) immediately. Q.E.D.

Define the integral operators $T_\kappa^{(1)}$ and $\tilde{T}_\kappa^{(1)}$ with a complex parameter κ by

$$(T_\kappa^{(1)} f)(x) = \frac{-1}{4\pi} \int_{S_a} dS_y \frac{e^{i\kappa|x-y|}}{|x-y|} f(y) \quad (x \in \boldsymbol{R}^3)$$

and

$$(\tilde{T}_\kappa^{(1)} f)(x) = \frac{-1}{4\pi} \int_{S_a} dS_y \frac{e^{i\kappa|x-y|}}{|x-y|} f(y) \quad (x \in S_a).$$

We remark that if $q(x) \equiv 1$, then $T_\kappa = T_\kappa^{(1)}$ and $\tilde{T}_\kappa = \tilde{T}_\kappa^{(1)}$, respectively.

Lemma 2.14. *Let* $\kappa \in \boldsymbol{C}$. *Then*

$$(\tilde{T}_\kappa)^* = q\tilde{T}_{-\bar{\kappa}}^{(1)}, \tag{2.32}$$

which maps from $L_2(S_a)$ *to itself.*

Proof. By Fubini's theorem we have for $u, v \in L_2(S_a)$

$$\begin{aligned}(\tilde{T}_k u, v)_a &= \int_{S_a} dS_x \left(\frac{-1}{4\pi} \int_{S_a} dS_y \frac{e^{i\kappa|x-y|}}{|x-y|} q(y)u(y) \right) \overline{v(x)} \\ &= \int_{S_a} dS_y u(y) \overline{\left(\frac{-q(y)}{4\pi} \int_{S_a} dS_x \frac{e^{i(-\bar{\kappa})|x-y|}}{|x-y|} v(x) \right)} \\ &= (u, q\tilde{T}_{-\bar{\kappa}}^{(1)} v)_a,\end{aligned}$$

from which follows the assertion. Q.E.D.

§3. The resolvent equation

In this section, we shall study the resolvent $R(z)$ of H. As remarked in the proof of Lemma 2.13, $\gamma R_0(z)$ is a bounded operator from $L_2(\boldsymbol{R}^3)$ to $L_2(S_a)$. More precisely, combining Lemmas 2.3 and 2.13 ($q(x) \equiv 1$), we have

Lemma 3.1. *Let $z \notin [0, \infty)$. Then $\gamma R_0(z)$ is a Hilbert-Schmidt operator from $L_2(\boldsymbol{R}^3)$ to $L_2(S_a)$.*

Theorem 3.2. *Let $z \in \rho(H) \cap \rho(H_0)$, where ρ denotes the resolvent set. Then $\gamma R(z)$ is a bounded operator from $L_2(\boldsymbol{R}^3)$ to $L_2(S_a)$ and the following resolvent equation holds:*

$$R(z) - R_0(z) = T_{\sqrt{z}}\gamma R(z), \tag{3.1}$$

where and in the sequel, by $\sqrt{z}$ is meant the branch of the square root of z with $\operatorname{Im}\sqrt{z} \geq 0$.

Proof. To prove the first part of the theorem, we have only to show that $R(z)$ is a bounded operator from $L_2(\boldsymbol{R}^3)$ to $H^1(\boldsymbol{R}^3)$. From Theorem 1.4, it follows that $Ran\ R(z) = Dom(H) \subset H^1(\boldsymbol{R}^3)$ and $Dom(R(z)) = L_2(\boldsymbol{R}^3)$. Let $\{u_n\}$ be such that for some $u \in L_2(\boldsymbol{R}^3)$ and $v \in H^1(\boldsymbol{R}^3)$, $u_n \to u$ in $L_2(\boldsymbol{R}^3)$ and $R(z)u_n \to v$ in $H^1(\boldsymbol{R}^3)$ as $n \to \infty$. Then, since $R(z)$ is a bounded operator from $L_2(\boldsymbol{R}^3)$ to itself, we have

$$R(z)u = \lim_{n\to\infty} R(z)u_n = v \quad \text{in } L_2(\boldsymbol{R}^3),$$

and hence $R(z)$ is a closed operator from $L_2(\boldsymbol{R}^3)$ to $H^1(\boldsymbol{R}^3)$. Therefore, from the closed graph theorem it follows that $R(z)$ belongs to $\boldsymbol{B}(L_2(\boldsymbol{R}^3), H^1(\boldsymbol{R}^3))$.

Finally, let us show the resolvent equation. Let $u \in Dom(H)$ and $v \in Dom(H_0)$. In view of Theorem 1.4 and $Dom(H_0) = H^2(\boldsymbol{R}^3)$, we have

$$((H - z)u, v) = h[u, v] - (u, \bar{z}v) = (u, (H_0 - \bar{z})v) + (q\gamma u, \gamma v)_a, \tag{3.2}$$

and hence, on putting $u = R(z)\varphi$ and $v = R_0(\bar{z})\psi = R_0(z)^*\psi$, we obtain

$$\begin{aligned}(R_0(z)\varphi, \psi) &= (R(z)\varphi, \psi) + (q\gamma R(z)\varphi, \gamma R_0(\bar{z})\psi)_a \\ &= (R(z)\varphi - T_{\sqrt{z}}\gamma R(z)\varphi, \psi),\end{aligned} \tag{3.3}$$

where we have used Lemma 2.13. The required resolvent equation follows from (3.3) immediately. Q.E.D.

§4. The wave operators

The wave operators $W_\pm$ which intertwine H and H_0 are defined as

$$W_{\pm} = \text{strong} \lim_{t \to \pm\infty} e^{itH} e^{-itH_0},$$

if they exist. In this section we shall prove the following

Theorem 4.1. *$W_{\pm}$ exist and are complete.*

The proof of the above theorem will be given after proving the next

Lemma 4.2. *$\gamma R(-b^2)$ is a Hilbert-Schmidt operator from $L_2(\boldsymbol{R}^3)$ to $L_2(S_a)$ for a sufficiently large $b > 0$.*

Proof. On operating γ from left on the resolvent equation (3.1) ($z = -b^2$), we have, using Lemma 2.7,

$$(4.1) \qquad (1 - \tilde{T}_{ib})\gamma R(-b^2) = \gamma R_0(-b^2).$$

If we show that $1 - \tilde{T}_{ib}$ has a bounded inverse for a suitable $b > 0$, the the lemma follows, for $\gamma R_0(-b^2)$ is a Hilbert-Schmidt operator by Lemma 3.1. Using Schwarz' inequality, Fubini's theorem and Lemma 2.1, we have for any $u \in L_2(S_a)$

$$(4.2) \qquad \| \tilde{T}_{ib} u \|_a^2 = \int_{S_a} dS_x \left| \int_{S_a} dS_y \frac{e^{-b|x-y|}}{-4\pi|x-y|} q(y)u(y) \right|^2$$

$$\leq \left(\frac{1}{4\pi} \max_{y \in S_a} |q(y)| \right)^2 \int_{S_a} dS_x \int_{S_a} dS_y \frac{e^{-2b|x-y|}}{|x-y|} \int_{S_a} dS_y \frac{|u(y)|^2}{|x-y|}$$

$$= \left(\frac{1}{4\pi} \max_{y \in S_a} |q(y)| \right)^2 \int_{S_a} dS_x \frac{\pi}{b}(1 - e^{-4ba}) \int_{S_a} dS_y \frac{|u(y)|^2}{|x-y|}$$

$$= \left(\frac{1}{4\pi} \max_{y \in S_a} |q(y)| \right)^2 \frac{\pi}{b}(1 - e^{-4ab}) \int_{S_a} dS_y |u(y)|^2 \int_{S_a} dS_x \frac{1}{|x-y|}$$

$$= (\max_{y \in S_a} |q(y)|)^2 \frac{a}{4b}(1 - e^{-4ab}) \| u \|_a^2.$$

Therefore, we obtain

$$(4.3) \qquad \| \tilde{T}_{ib} \| \leq (\max_{y \in S_a} |q(y)|) \left\{ \frac{a}{4b}(1 - e^{-4ab}) \right\}^{1/2},$$

and hence, the operator norm of $\tilde{T}_{ib}$ is less than unity for sufficiently large $b > 0$, which makes possible the Neumann series inversion of $1 - \tilde{T}_{ib}$. Q.E.D.

Proof of Theorem 4.1. It is known that the wave operators exist and are complete if the difference of the resolvents is a trace-class operator (Kato [9, Chap. X, Theorem 4.8]). On the other hand, as is well known, an operator is in the trace-class if and only if it is a product of two Hilbert-Schmidt operators (e.g. Kato [9, p. 521]). Thus, from Lemma 2.3, Theorem 3.2 and Lemma 4.2 it follows that $R(z) - R_0(z)$ is in the trace-class. The proof is now complete. Q.E.D.

§5. The spectrum of H

As is mentioned in the previous section, the difference of the resolvents of H and H_0 is a trace-class operator (see the proof of Theorem 4.1). Thus, concerning the essential spectrum $\sigma_{ess}(H)$ of H, we have by Weyl's theorem (e.g. Reed-Simon [19, p.112, Theorem XIII. 14])

Theorem 5.1. $\sigma_{ess}(H) = \sigma_{ess}(H_0) = [0, \infty)$.

As to the point sectrum of H, we get the following result.

Theorem 5.2. $\sigma_p(H) \cap (0, \infty) = \phi$.

Proof. Assume that $\lambda > 0$, $(H - \lambda)u = 0$ and $u \in Dom(H)$. By Theorem 1.7 u satisfies

$$(5.1) \qquad (\Delta + \lambda)u(x) = 0 \quad \text{in} \quad \{x; |x| < a\} \cup \{x; |x| > a\}.$$

In view of Mizohata [13, Chap. VIII, Lemma 8.4], we have

$$(5.2) \qquad u(x) = 0 \quad \text{in } \{x; |x| > a\}.$$

Thus it follows from (5.2) and Theorem 1.7 that

$$(5.3) \qquad \frac{\partial u}{\partial n_-}|_{S_a}(x) = u|_{S_a}(x) = 0.$$

Now let us define $\tilde{u}(x)$ by

$$(5.4) \qquad \tilde{u}(x) = \begin{cases} u(x) & |x| \leq a \\ 0 & |x| > a. \end{cases}$$

Then, for any $\varphi \in C_0^\infty(\boldsymbol{R}^3)$, we have by (5.1), (5.3) and Green's theorem

$$\int_{\boldsymbol{R}^3} \tilde{u}(x)(\Delta + \lambda)\varphi(x)dx = \int_{|x|<a} u(x)\Delta\varphi(x)dx + \lambda\int_{|x|<a} u(x)\varphi(x)dx$$
$$= \int_{|x|<a} (\Delta + \lambda)u(x)\varphi(x)dx = 0,$$

which implies

$$(5.5) \qquad (\Delta + \lambda)\tilde{u}(x) = 0 \quad \text{in } \boldsymbol{R}^3.$$

Operating the Fourier transform on the both sides of (5.5), we have

$$(\lambda - |\xi|^2)(\mathscr{F}\tilde{u})(\xi) = 0.$$

Since $\mathscr{F}\tilde{u} \in L_2(\boldsymbol{R}^3)$, we obtain $\mathscr{F}\tilde{u} = 0$, and hence

$$(5.6) \qquad \tilde{u}(x) = 0 \quad \text{for a.e. } x \in \boldsymbol{R}^3.$$

Therefore, from (5.2), (5.4) and (5.6) it follows that

$$u(x) = 0 \quad \text{in } L_2(\boldsymbol{R}^3). \qquad \text{Q.E.D.}$$

In contrast to the above theorem the point 0 may or may not belong to $\sigma_p(H)$. If q is constant on S_a, however, we get the following criterion.

Theorem 5.3. *Let $q(x) = V_0$ (constant). Then $0 \in \sigma_p(H)$ if and only if there exists a positive integer n such that $aV_0 + 2n + 1 = 0$. In this case, the corresponding eigenspace is spanned by the vecotors $v(|x|)Y_n^m$ $(m = -n, -n + 1, \ldots, n)$, where*

$$v(r) = \begin{cases} r^n & r \leq a \\ a^{2n+1} r^{-n-1} & r \geq a, \end{cases}$$

and $Y_n^m (n = 0, 1, \ldots, m = -n, -n+1, \ldots, n)$ denote the spherical harmonics which provide a basis for $L_2(S^2)$ (S^2 the unit sphere in $\boldsymbol{R}^3$).

Proof. (cf. Colton-Kress [4, pp. 78–79])

Suppose that $Hu = 0$ and $u \in Dom(H)$. By Theorem 1.7 we have

(5.7) $$\Delta u(x) = 0 \quad \text{in } \{x; |x| < a\} \cup \{x; |x| > a\}.$$

Thus $u(x)$ is a C^∞-function in the above region by Weyl's lemma (e.g. Reed-Simon [18, p. 53]). Let (r, θ, φ) denote the spherical coordinates with $r = |x|$. For each fixed r we can expand u in a uniformly convergent series

$$u(x) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} v_{km}(r) Y_k^m(\theta, \varphi),$$

where

$$v_{km}(r) = \int_0^{2\pi} \int_0^{\pi} u(r, \theta, \varphi) \overline{Y_k^m(\theta, \varphi)} \sin\theta \, d\theta d\varphi.$$

Since $u \in C^\infty (\{x; |x| \neq a\})$, we can differentiate under the integral and integrate by parts using $\Delta u = 0$ to conclude that v_{km} is a solution of the following equation

$$\frac{d^2}{dr^2} v_{km} + \frac{2}{r}\frac{d}{dr} v_{km} - \frac{k(k+1)}{r^2} v_{km} = 0,$$

which has a fundamental system of solutions r^k and r^{-k-1}. Since v_{km} is bounded near zero and belongs to $L_2(0, \infty); r^2 dr)$ by Theorem 1.7, v_{km} has the form

$$v_{00}(r) = \alpha_{00} \; (r < a), \; = 0 \quad (r > a),$$

$$v_{km}(r) = \begin{cases} \alpha_{km} r^k & (r < a) \\ \beta_{km} r^{-k-1} & (r > a) \end{cases} \quad (k \geq 1),$$

where α_{km} and β_{km} are constants. In view of Theorem 1.7, v_{km} is continuous at $r = a$ and satisfies the boundary condition

$$V_0 v_{km}(a) + \left(\frac{d}{dr} v_{km}\right)(a - 0) - \left(\frac{d}{dr} v_{km}\right)(a + 0) = 0,$$

where $f(a \pm 0)$ denotes $\lim_{\varepsilon \downarrow 0} f(a \pm \varepsilon)$. Therefore, α_{km} and β_{km} satisfy the following equations

$$\alpha_{00} = 0,$$

$$\alpha_{km} a^{2k+1} = \beta_{km}, \ (aV_0 + 2k + 1)\alpha_{km} = 0 \quad (k \geq 1),$$

from which the required result follows immediately. Q.E.D.

§6. Bound states of H

Let us define the quadratic form h_t depending on a real parameter t by

$$h_t[u, v] = (\nabla u, \nabla v) + t(q\gamma u, \gamma v)_a, \quad (6.1)$$

$$Dom[h_t] = H^1(\boldsymbol{R}^3).$$

The form h_t can be seen to be lower semibounded and closed in exactly the same way as for h ($t = 1$). Therefore, Theorem 1.4 applies to h_t. We denote the corresponding unique selfadjoint operator by H_t ($H_1 = H$ (see § 1)). Put for $n = 1, 2, \cdots$ and $t \in \boldsymbol{R}$,

$$\mu_n(t) = \sup_{\substack{\varphi_1,\dots,\varphi_{n-1} \\ \varphi_j \in L_2(\boldsymbol{R}^3)}} \ \inf_{\substack{u \in H^1(\boldsymbol{R}^3) \cap [\varphi_1,\dots,\varphi_{n-1}]^{\perp} \\ \|u\| = 1}} \min(h_t[u], 0), \quad (6.2)$$

where $h_t[u] = h_t[u, u]$ and $[\varphi_1, \varphi_2, \dots, \varphi_{n-1}]^{\perp}$ is short hand for $\{u; (u, \varphi_j) = 0, j = 1, 2, \dots, n-1\}$. Then our min-max principle will read as follows:

Lemma 6.1. *Let n and $t \in \boldsymbol{R}$ be fixed. Then, either (a) $0 = \mu_n(t) = \mu_{n+1}(t) = \mu_{n+2}(t) = \cdots$ and there are at most $n - 1$ eigenvalues of H_t (counting multiplicity), or (b) there are n eigenvalues of H_t (counting multiplicity) and $\mu_n(t)$ is the n-th negative eigenvalue of H_t (counting multiplicity) from below.*

Proof. (cf. Reed-Simon [19, p. 76, Theorem XIII. 1])

Let $E_t(\cdot)$ be the spectral measure for H_t. First let us show

$$\dim[\mathrm{Ran}(E_t((-\infty, \alpha)))] < n \qquad \text{if } \alpha < \mu_n(t) \quad (6.3)$$

$$\dim[\mathrm{Ran}(E_t((-\infty, \alpha)))] \geq n \qquad \text{if } \alpha > \mu_n(t) \quad (6.4)$$

Here we remark that $\mu_n(t)$ is finite for each $t \in \boldsymbol{R}$ and

$$Ran(E_t((-\infty, \alpha))) \subset Dom(H_t) \quad (\subset H^1(\boldsymbol{R}^3)) \text{ if } \alpha < +\infty, \quad (6.5)$$

because of the fact that H_t is bounded from below by Theorem 1.5.

Suppose that (6.3) is false. Then, for any $\varphi_1, \varphi_2, \dots, \varphi_{n-1}$ we can find u such that $u \in Ran(E_t((-\infty, \alpha))) \cap [\varphi_1, \varphi_2, \dots, \varphi_{n-1}]^{\perp}$ and, by (6.5), $(H_t u, u) \leq \alpha \|u\|^2$. By Theorem 1.4, this implies that

$$\inf_{\substack{u \in H^1(\boldsymbol{R}^3) \cap [\varphi_1, \varphi_2, \ldots, \varphi_{n-1}]^\perp \\ \|u\|=1}} \min(h_t[u], 0) \le \alpha$$

for any $\varphi_1, \varphi_2, \ldots, \varphi_{n-1} \in L_2(\boldsymbol{R}^3)$, and hence $\mu_n(t) \le \alpha$, which is a contradiction. This proves (6.3).

Since $\mu_n(t) \le 0$ and Theorrem 5.1 holds, we have only to prove (6.4) when $\mu_n(t) < \alpha \le 0$. Thus, suppose that (6.4) is false when $\mu_n(t) < \alpha \le 0$. Then we can find $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}$ such that L. h. $\{\varphi_1, \varphi_2, \ldots, \varphi_{n-1}\} = Ran(E_t((-\infty, \alpha)))$, where L. h. A denotes the subspace spanned by A. Since any $u \in [\varphi_1, \varphi_2, \ldots, \varphi_{n-1}]^\perp \cap Dom(H_t)$ is in $Ran(E_t([\alpha, \infty)))$, we have by Theorem 1.4, $h_t[u] = (H_t u, u) \ge \alpha \|u\|^2$. Since $Dom(H_t)$ is a form core for h_t (e.g. Reed-Simon [17, p.281]), it follows that

$$h_t[u] \ge \alpha \|u\|^2 \quad \text{for any } u \in [\varphi_1, \varphi_2, \ldots, \varphi_{n-1}]^\perp \cap H^1(\boldsymbol{R}^3).$$

Therefore, noting that $\alpha \le 0$, we obtain

$$\inf_{\substack{u \in [\varphi_1, \varphi_2, \ldots, \varphi_{n-1}]^\perp \cap H^1(\boldsymbol{R}^3) \\ \|u\|=1}} \min(h_t[u], 0) = \inf_{\substack{u \in [\varphi_1, \varphi_2, \ldots, \varphi_{n-1}]^\perp \cap H^1(\boldsymbol{R}^3) \\ \|u\|=1}} h_t[u] \ge \alpha.$$

and hence $\mu_n(t) \ge \alpha$, which is a contradiction. This proves (6.4).

First, suppose that

$$\text{(6.6)} \qquad \dim[Ran(E_t((-\infty, \mu_n(t) + \varepsilon)))] = \infty \quad \text{for all } \varepsilon > 0.$$

Then the situation (a) holds. In fact, by (6.3) we have

$$\dim[Ran(E_t((-\infty, \mu_n(t) - \varepsilon)))] < n \quad \text{for all } \varepsilon > 0,$$

and hence

$$\dim[Ran(E_t([\mu_n(t) - \varepsilon, \mu_n(t) + \varepsilon)))] = \infty \quad \text{for all } \varepsilon > 0.$$

This implies that

$$\text{(6.7)} \qquad \mu_n(t) \in \sigma_{ess}(H_t).$$

Since $\mu_n(t) \le 0$ and $\sigma_{ess}(H_t) = [0, \infty)$ by Theorem 5.1, it follows that $\mu_n(t) = 0$. If $\mu_{n+1}(t) > \mu_n(t)$, we have by putting $\alpha = \frac{1}{2}(\mu_{n+1}(t) + \mu_n(t))$ $(< \mu_{n+1}(t))$ in (6.3)

$$\dim[Ran(E_t((-\infty, \frac{1}{2}(\mu_{n+1}(t) + \mu_n(t)))] < n + 1,$$

which contradicts (6.6). Thus, noting that $\mu_{n+1}(t) \ge \mu_n(t)$, we obtain $\mu_n(t) = \mu_{n+1}(t) \cdots$. Finally, if there are n eigenvalues strictly below $\mu_n(t)$ and λ is the n-th eigenvalue, we have

$$\dim [Ran(E_t((-\infty, \frac{1}{2}(\mu_n(t)+\lambda))))] \geq n,$$

which contradicts (6.3) ($\alpha = \frac{1}{2}(\mu_n(t)+\lambda) < \mu_n(t)$). Thus it is seen that there are at most $n-1$ eigenvalues of H_t.

Next, assume that (6.6) fails, i.e., for some $\varepsilon_0 > 0$

$$\dim [Ran(E_t((-\infty, \mu_n(t)+\varepsilon_0)))] < +\infty. \tag{6.8}$$

Then the situation (b) arises. In fact, we have by (6.3) an (6.4)

$$\dim [Ran(E_t((\mu_n(t)-\varepsilon, \mu_n(t)+\varepsilon)))] \geq 1 \quad \text{for any } \varepsilon > 0. \tag{6.9}$$

On the other hand, (6.8) implies

$$\dim [Ran(E_t((\mu_n(t)-\varepsilon_0, \mu_n(t)+\varepsilon_0)))] < +\infty. \tag{6.10}$$

Thus it follows from (6.9) an (6.10) that $\mu_n(t)$ is a discrete eigenvalue of H_t. Take $\delta > 0$ such that $(\mu_n(t)-\delta, \mu_n(t)+\delta) \cap \sigma(H_t) = \{\mu_n(t)\}$. Then we have by (6.4)

$$\begin{aligned} &\dim [Ran(E_t((-\infty, \mu_n(t)])))] \\ &= \dim [Ran(E_t((-\infty, \mu_n(t)+\delta)))] \geq n. \end{aligned}$$

Thus there exist at least n eigenvalues of H_t: $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \mu_n(t)$. If $\lambda_n < \mu_n(t)$, we have by putting $\alpha = \frac{1}{2}(\mu_n(t)+\lambda_n)$ $(<\mu_n(t))$ in (6.3)

$$\begin{aligned} n &\leq \dim [Ran(E_t((-\infty, \lambda_n])))] \\ &\leq \dim [Ran(E_t((-\infty, \alpha)))] < n, \end{aligned}$$

which is a contradiction. Therefore, $\lambda_n = \mu_n(t)$, i.e. $\mu_n(t)$ is the n-th eigenvalue of H_t. The lemma has now been proven. Q.E.D.

Lemma 6.2. *For each n, $\mu_n(t)$ is monotone nonincreasing in t on* $[0, \infty)$.

Proof. Since $\min(h_t[u], 0)$ is monotone nonincreasing in t on $[0, \infty)$, the required result follows immediately. Q.E.D.

Lemma 6.3. *For each n, $\mu_n(t)$ is continuous in t on $\boldsymbol{R}$.*

Proof. (cf. Simon [21, p. 71, Theorem II. 33])

For each $u \in H^1(\boldsymbol{R}^3)$ with $\|u\| = 1$, we put

$$f(t; u) = \min(h_t[u], 0).$$

If we show that $\{f(\cdot; u); u \in H^1(\boldsymbol{R}^3), \|u\| = 1\}$ is equicontinuous, the conclusion follows.

Given $t_0 \in \boldsymbol{R}^1$, we have by Lemma 1.3 ($\varepsilon = [2(\max_{x \in S_a}|q(x)| + 1)(|t_0| + 1)]^{-1}$)

$$|(q\gamma u, \gamma u)_a| \leq \max_{x \in S_a}|q(x)| \cdot \|\gamma u\|_a^2 \leq \frac{\max|q(x)|}{2(\max|q(x)|+1)(|t_0|+1)} \|\nabla u\|^2 + \tag{6.11}$$

$$+ 2(\max|q(x)|)(\max|q(x)| + 1)(|t_0| + 1)\|u\|^2 \leq \frac{1}{2(|t_0| + 1)}\|\nabla u\|^2 + b_{t_0},$$

where we put $b_{t_0} = 2(\max|q(x)| + 1)^2(|t_0| + 1)$. Suppose that $h_t[u] \leq 0$ for some t such that $|t - t_0| < 1$. Then we have

$$f(t; u) = h_t[u] = \|\nabla u\|^2 + t(q\gamma u, \gamma u)_a \leq 0,$$

which implies

$$\|\nabla u\|^2 \leq -t(q\gamma u, \gamma u)_a \leq |t||(q\gamma u, \gamma u)_a| \leq (|t_0| + 1)|(q\gamma u, \gamma u)_a|. \tag{6.12}$$

Therefore, from (6.11) and (6.12) it follows that

$$|(q\gamma u, \gamma u)_a| \leq \frac{1}{2}|(q\gamma u, \gamma u)_a| + b_{t_0},$$

and hence

$$|(q\gamma u, \gamma u)_a| \leq 2b_{t_0} \tag{6.13}$$

if $h_t[u] \leq 0$ for some t such that $|t - t_0| < 1$.

Now, for given $\varepsilon > 0$, let $\delta = \min\left(\frac{\varepsilon}{2b_{t_0}}, 1\right)$. Let $|t - t_0| < \delta$. If $h_t[u] \leq 0$ or $h_{t_0}[u] \leq 0$, then we have by (6.13)

$$|f(t; u) - f(t_0; u)| \leq |h_t[u] - h_{t_0}[u]| \leq |t - t_0||(q\gamma u, \gamma u)_a| < \delta \cdot 2b_{t_0} \leq \varepsilon.$$

The above inequality is trivially satisfied if $h_t[u] > 0$ and $h_{t_0}[u] > 0$. We have thus obtained the required equicontinuity. Q.E.D.

Lemma 6.4. *For each n, $\mu_n(t)$ is strictly monotone decreasing on $[t_1, +\infty)$ once $\mu_n(t_1) < 0$ for some $t_1 \geq 0$.*

Proof. Let t_1 be such that $E \equiv \mu_n(t_1) < 0$ and $t_1 \geq 0$. Assume that there exists t_2 such that $t_1 \leq t_2$ and $\mu_n(t_1) = \mu_n(t_2) = E < 0$. Then, for any $t \in [t_1, t_2]$, $\mu_n(t) = E$ holds by Lemma 6.2. Therefore, by Lemma 6.1 we can find u_t for each $t \in [t_1, t_2]$ which satisfies

$$u_t \in Dom(H_t),\ u_t \neq 0 \text{ and } (H_t - E)u_t = 0. \tag{6.14}$$

In view of Lemma 2.12, we have

$$\gamma u_t = t \cdot \tilde{T}_{i\sqrt{-E}}(\gamma u_t) \quad \text{and} \quad \gamma u_t \neq 0 \text{ in } L_2(S_a). \tag{6.15}$$

This implies that for every $t \in [t_1, t_2]$ t^{-1} is an eigenvalue of $\tilde{T}_{i\sqrt{-E}}$, which is a contradiction, for $\tilde{T}_{i\sqrt{-E}}$ is a compact operator by Lemma 2.4. Therefore, we must have the lemma in view of Lemma 6.2. Q.E.D.

Now, as an analogue of the Birman-Schwinger bound (e.g. Reed-Simon [19, p.98, Theorem XIII.10]), we shall give a bound on the total numbr of bound states of H. Let $E < 0$ and define $N(E)$ by

$$N(E) = \#\{n;\ \mu_n(1) < E\},$$

where $\#A$ denotes the cardinality of the set A. Then we have the following

Theorem 6.5. *Let $E < 0$. Then*

$$N(E) \le \|(\tilde{T}_{i\sqrt{-E}})^2\|_{H.S.}^2 \le M < +\infty, \tag{6.16}$$

where M is a constant independent of $E < 0$. In particular, the total number of negative eigenvalues of H is finite.

Proof. Since $\mu_n(0) = 0$ for every n and $\mu_n(t)$ is continuous by Lemma 6.3, it follows from the intermediate value theorem and Lemma 6.4 that $\mu_n(1) < E$ if and only if $\mu_n(t) = E$ for exactly one $t \in (0, 1)$. Using Lemma 2.12 repeatedly, it is seen that t^{-2} satisfying the equation $\mu_n(t) = E$ is an eigenvalue of $(\tilde{T}_{i\sqrt{-E}})^2$. Further, since $(\tilde{T}_{i\sqrt{-E}})^2$ is a Hilbert-Schmidt operator by Lemma 2.8, we have

$$\begin{aligned}
N(E) &= \#\{n;\ \mu_n(t) = E \quad \text{for some } t \in (0, 1)\} \\
&\le \sum_{\{t\in(0,1);\mu_k(t)=E,k=1,2,\dots,N(E)\}} t^{-4} \\
&\le \sum_{\{t\in(0,1);\mu_k(t)=E,k=1,2,\dots\}} t^{-4} \\
&\le \sum_{\{t\in(0,1);t^{-2}\text{ is an eigenvalue of }(\tilde{T}_{i\sqrt{-E}})^2\}} t^{-4} \\
&\le \|(\tilde{T}_{i\sqrt{-E}})^2\|_{H.S.}^2 \\
&\le C^2(\max_{z\in S_a}|q(z)|)^4 \int_{S_a\times S_a} dS_x dS_y (1 + |\log|x-y||)^2 \equiv M < +\infty,
\end{aligned}$$

where C is a constant which is independent of E (see the proof of Lemma 2.8). The above inequality shows the theorem. Q.E.D.

§7. The limiting absorption principle for H

In this section we shall prove the limiting absorption principle for H.

Theorem 7.1. *Let $s > \frac{1}{2}$. Then $R(z)$ can be extended to a $\boldsymbol{B}(L_2^s(\boldsymbol{R}^3), L_2^{-s}(\boldsymbol{R}^3))$-valued continuous function of z on $\Pi \backslash (\sigma_p(H) \cup \{0\})$.*

Proof. Let us recall the resolvent equation

$$R(z) - R_0(z) = T_{\sqrt{z}}\gamma R(z). \tag{7.1}$$

If we assume that $(1 - \tilde{T}_{\sqrt{z}})^{-1}$ exists, we have on operating γ from left on the both sides of (7.1) and solving for $R(z)$,

$$R(z) = R_0(z) + T_{\sqrt{z}}(1 - \tilde{T}_{\sqrt{z}})^{-1}\gamma R_0(z) \tag{7.2}$$

for $z \in \rho(H) \cap \rho(H_0)$. Here we have used Lemma 2.7. By Lemma 2.9 $T_{\sqrt{z}}$ is a $\boldsymbol{B}(L_2(S_a), L_2^{-s}(\boldsymbol{R}^3))$-valued continuous function of z on $\mathrm{Im}\sqrt{z} \geq 0$ if $s > 1/2$. Thus $(T_{\sqrt{z}})^*$ is a $\boldsymbol{B}(L_2^s(\boldsymbol{R}^3), L_2(S_a))$-valued continuous function of z on $\mathrm{Im}\sqrt{z} \geq 0$ if $s > 1/2$. On the other hand, we have by Lemma 2.13 ($q(x) \equiv 1$)

$$\gamma R_0(z) = -(T^{(1)}_{\sqrt{\bar{z}}})^* \quad \text{if } \mathrm{Im}\sqrt{\bar{z}} > 0.$$

Thus, since $T_\kappa = T^{(1)}_\kappa$ if $q(x) \equiv 1$, $\gamma R_0(z)$ can be extended to a $\boldsymbol{B}(L_2^s(\boldsymbol{R}^3), L_2(S_a))$-valued continuous function of z on Π if $s > 1/2$. Therefore, in view of the well-known limiting absorption principle for H_0 (see e.g. Agmon [2]), the proof of the above theorem is reduced to the next

Lemma 7.2. *Let $z \in \Pi \backslash (\sigma_p(H) \cup \{0\})$. Then $(1 - \tilde{T}_{\sqrt{z}})^{-1}$ exists and belongs to $\boldsymbol{B}(L_2(S_a))$. In this case, $(1 - \tilde{T}_{\sqrt{z}})^{-1}$ is a $\boldsymbol{B}(L_2(S_a))$-valued continuous function of z on $\Pi \backslash (\sigma_p(H) \cup \{0\})$, where $\boldsymbol{B}(X)$ denotes $\boldsymbol{B}(X, X)$.*

We will show this lemma after proving a series of lemmas. First, we have by Lemma 2.12

Lemma 7.3. *Let $\mathrm{Im}\sqrt{z} > 0$. Then $1 \in \sigma_p(\tilde{T}_{\sqrt{z}})$ if and only if $z \in \sigma_p(H)$.*

Lemma 7.4. *Let $\zeta \in \boldsymbol{C}$ and let $u \in L_2(S_a)$ satisfy the homogeneous equation $u = \tilde{T}_\zeta u$ in $L_2(S_a)$. Then u is bounded on S_a.*

Proof. Let $k(x, y)$ be the integral kernel of $(\tilde{T}_\zeta)^3$. It follows from Lemma 2.2 that $k(x, y)$ is bounded on $S_a \times S_a$. Thus we have by Schwarz' inequality

$$\begin{aligned} |u(x)| = |(\tilde{T}_\zeta)^3 u(x)| &= \left| \int_{S_a} dS_y k(x, y) u(y) \right| \\ &\leq \sup_{(x,y) \in S_a \times S_a} |k(x, y)| \int_{S_a} dS_y |u(y)| \\ &\leq \sup_{(x,y) \in S_a \times S_a} |k(x, y)| (4\pi a^2)^{1/2} \|u\|_a \quad < +\infty, \end{aligned}$$

which proves the lemma. Q.E.D.

Lemma 7.5. *Under the conditions of Lemma 7.4 $u(x)$ is Hölder continuous on S_a.*

Proof. We consider the difference

$$\begin{aligned} u(x) - u(x') &= \frac{-1}{4\pi} \int_{S_a} \frac{e^{i\zeta|x-y|} - e^{i\zeta|x'-y|}}{|x-y|} q(y) u(y) dS_y \\ &\quad + \frac{-1}{4\pi} \int_{S_a} \left(\frac{1}{|x-y|} - \frac{1}{|x'-y|} \right) e^{i\zeta|x'-y|} q(y) u(y) dS_y \\ &= J_1 + J_2. \end{aligned} \tag{7.3}$$

We shall estimate J_1 and J_2 as follows. Considering the inequality

$$|e^{i\zeta|x-y|} - e^{i\zeta|x'-y|}| \leq |\zeta||x - x'|e^{|\mathrm{Im}\zeta||x-x'|},$$

we have for $x, x' \in S_a$

$$(7.4) \qquad |J_1| \leq \frac{1}{4\pi} A|\zeta||x - x'|e^{2a|\mathrm{Im}\zeta|} \int_{S_a} dS_y \frac{1}{|x-y|}$$
$$= Aa|\zeta|e^{2a|\mathrm{Im}\zeta|}|x - x'|,$$

where $A = \sup_{y \in S_a}|q(y)u(y)| < +\infty$ by Assumption 1.1, Lemma 7.4 and Lemma 2.1. We proceed to estimate J_2. In view of the inequality

$$\left|\frac{1}{|x-y|} - \frac{1}{|x'-y|}\right| \leq \frac{|x-x'|}{|x-y||x'-y|},$$

we have for $x, x' \in S_a$

$$(7.5) \qquad |J_2| \leq \frac{1}{4\pi} Ae^{2a|\mathrm{Im}\zeta|}|x - x'| \int_{S_a} dS_y \frac{1}{|x-y||x'-y|}$$
$$\leq \frac{1}{4\pi} Ae^{2a|\mathrm{Im}\zeta|} C|x - x'|(1 + |\log|x - x'||),$$

where we used Lemma 2.2. The conclusion follows from (7.3), (7.4) and (7.5).

Q.E.D.

Lemma 7.6. *Let $\mu \in \boldsymbol{R}$ and $u \in L_2(S_a)$. Put $U(x) \equiv (T_\mu u)(x)$. Then $U(x)$ has the following asymptotic behavior*

$$(7.6) \qquad U(x) = \frac{-1}{4\pi} \frac{e^{i\mu|x|}}{|x|} \int_{S_a} e^{i\mu\omega_x \cdot y} q(y)u(y)dS_y + O\left(\frac{1}{|x|^2}\right)$$

as $|x| \to \infty$, where ω_x denotes the unit vector with the direction of x. Further, $U(x)$ satisfies the following radiation condition

$$(7.7) \qquad \frac{\partial U}{\partial|x|}(x) - i\mu U(x) = O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to +\infty.$$

Proof. In view of the relation

$$\frac{e^{i\mu|x-y|}}{|x-y|} = \frac{1}{|x|} e^{i\mu|x| - i\mu\omega_x \cdot y + i\mu|x|\eta_1} + \frac{\eta_2}{|x|} e^{i\mu|x-y|}$$
$$+ \frac{1}{|x|^2} \omega_x \cdot y e^{i\mu|x-y|} \quad (x \in \boldsymbol{R}^3, \ |y| < R < +\infty),$$

where η_1 and η_2 are real valued functions satisfying $\eta_1 = O\left(\frac{|y|}{|x|^2}\right)$, and η_2

$= O\left(\frac{|y|}{|x|^2}\right)$ when $|x| \to +\infty$ (see e.g. Ikebe [6, p. 11]), we have

$$U(x) = \frac{-1}{4\pi}\frac{e^{i\mu|x|}}{|x|}\int_{S_a} e^{-i\mu\omega_x\cdot y}\, q(y)u(y)dS_y$$

$$-\frac{1}{4\pi}\frac{e^{i\mu|x|}}{|x|}\int_{S_a} e^{-i\mu\omega_x\cdot y}\,(e^{-i\mu|x|\eta_1}-1)q(y)u(y)dS_y$$

$$-\frac{1}{4\pi}\frac{1}{|x|}\int_{S_a} \eta_2\, e^{i\mu|x-y|}\, q(y)u(y)\,dS_y$$

$$-\frac{1}{4\pi}\frac{1}{|x|^2}\int_{S_a} \omega_x\cdot y\; e^{i\mu|x-y|} q(y)u(y)\,dS_y$$

$$= \frac{-1}{4\pi}\frac{e^{i\mu|x|}}{|x|}\int_{S_a} e^{-i\mu\omega_x\cdot y}\, q(y)u(y)dS_y + I_1 + I_2 + I_3.$$

I_i $(i = 1, 2, 3)$ are estimated as follows:

$$|I_1| \le \frac{1}{4\pi}\max_{y\in S_a}|q(y)|\frac{1}{|x|}\int_{S_a} |e^{-i\mu|x|\eta_1}-1|\,|u(y)|dS_y$$

$$\le \frac{1}{4\pi}\max_{y\in S_a}|q(y)|\frac{1}{|x|}\int_{S_a} C|\mu|\,|x|\frac{a}{|x|^2}|u(y)|\,dS_y$$

$$\le \frac{const.}{|x|^2}\int_{S_a} |u(y)|\,dS_y$$

$$\le \frac{const.}{|x|^2}\|u\|_a,$$

$$|I_2| \le \frac{const.}{|x|^3}\|u\|_a, \quad |I_3| \le \frac{const.}{|x|^2}\|u\|_a.$$

These estimates prove (7.6).

Let us show (7.7). By differentiation under the integral sign, we have

(7.8) $$\frac{\partial U}{\partial|x|}(x) - i\mu U(x)$$

$$= -\frac{i\mu}{4\pi}\int_{S_a}\frac{e^{i\mu|x-y|}}{|x-y|}\left(\frac{|x|^2 - x\cdot y}{|x|\,|x-y|} - 1\right)q(y)u(y)dS_y$$

$$+\frac{1}{4\pi}\int_{S_a}\frac{|x|^2 - x\cdot y}{|x|\,|x-y|^3}\, e^{i\mu|x-y|}\, q(y)u(y)dS_y$$

$$= J_1 + J_2.$$

Considering $|y| = a$, we have

$$|x-y|^{-1}=|x|^{-1}\left(1+\omega_x\cdot\left(\frac{y}{|x|}\right)+O\left(\frac{1}{|x|^2}\right)\right)$$

as $|x|\to+\infty$, and hence

$$\frac{1}{|x-y|}\left(\frac{|x|^2-x\cdot y}{|x||x-y|}-1\right)=O\left(\frac{1}{|x|^3}\right),$$

$$\frac{|x|^2-x\cdot y}{|x||x-y|^3}=O\left(\frac{1}{|x|^2}\right)\quad \text{as } |x|\to+\infty.$$

Therefore, we have

$$|J_1|\le\frac{const.}{|x|^3}\int_{S_a}|u(y)|\,dS_y\le\frac{const.}{|x|^3}\|u\|_a, \tag{7.9}$$

$$|J_2|\le\frac{const.}{|x|^2}\|u\|_a.$$

Thus (7.7) follows from (7.8) and (7.9) immediately. Q.E.D.

Lemma 7.7. *Let $\mu\in\boldsymbol{R}\backslash\{0\}$ and let $u\in L_2(S_a)$ satisfy $u=\tilde{T}_\mu u$ in $L_2(S_a)$. Then, for an aribitrary unit vector ω we have*

$$\int_{S_a}e^{-i\mu\omega\cdot y}q(y)u(y)\,dS_y=0. \tag{7.10}$$

Proof. Put $U(x)\equiv(T_\mu u)(x)$. Since $u(x)$ is continuous on S_a by Lemma 7.5, $U(x)$ is continuous on $\boldsymbol{R}^3$ (see e.g. Colton-Kress [4, p. 47, Theorem 2.12]), and hence

$$(U|_{S_a})(x)=u(x). \tag{7.11}$$

On the other hand, by Lemma 2.11 ($(T_\mu u)(x)\equiv U(x)$), $U(x)$ satisfies the reduced wave equation

$$(\Delta+\mu^2)U(x)=0\quad \text{on } \{x;|x|<a\}\cup\{x;|x|>a\}. \tag{7.12}$$

Further, $\left(\dfrac{\partial U}{\partial n_+}\right)(x)$ can be continuously extended from $\{x;|x|<a\}$ to $\{x;|x|\le a\}$ and from $\{x;|x|>a\}$ to $\{x;|x|\ge a\}$ with the limiting values

$$\left(\frac{\partial U}{\partial n_+}\right)^{(\pm)}(x)=\pm\frac{1}{2}q(x)u(x)+W(x)\quad(x\in S_a), \tag{7.13}$$

respectively. Here $W(x)=\dfrac{-1}{4\pi}\displaystyle\int_{S_a}\left(\frac{\partial}{\partial n_+}\right)_y\left(\frac{e^{i\mu|x-y|}}{|x-y|}\right)q(y)u(y)dS_y$ (the integral exists as an improper integral) and $\left(\dfrac{\partial U}{\partial n_+}\right)^{(\pm)}(x)$ are the limits of $\left(\dfrac{\partial U}{\partial n_+}\right)(x)$ obtained by approaching S_a from $\{x;|x|>a\}$ and $\{x;|x|<a\}$, respectively, that is,

$$\left(\frac{\partial U}{\partial n_+}\right)^{(+)}(x) = \lim_{\substack{y \to x \\ |y| > a}} \left(\frac{\partial U}{\partial n_+}\right)(y),$$

$$\left(\frac{\partial U}{\partial n_+}\right)^{(-)}(x) = \lim_{\substack{y \to x \\ |y| < a}} \left(\frac{\partial U}{\partial n_+}\right)(y), \quad x \in S_a$$

(see e.g. Colton-Kress [4, p.47]). Using (7.12), (7.13) and Green's theorem, we have

$$(7.14) \qquad 0 = \int_{|x|<a} \{(\Delta + \mu^2)U(x) \cdot \overline{U(x)} - \overline{(\Delta + \mu^2)U(x)} \cdot U(x)\} dx$$

$$= \int_{|x|<a} \{(\Delta U)(x)\overline{U(x)} - \overline{(\Delta U)(x)}\, U(x)\} dx$$

$$= \int_{S_a} \left\{\left(\frac{\partial U}{\partial n_+}\right)^{(-)}(x)\overline{U(x)} - \overline{\left(\frac{\partial U}{\partial n_+}\right)^{(-)}(x)} U(x)\right\} dS_x$$

$$= \int_{S_a} (W(x)\overline{U(x)} - \overline{W(x)}\, U(x)) dS_x,$$

where we have used the fact that μ and $q(x)$ are real-valued. Similarly, for any b such that $b > a$ we have

$$(7.15) \qquad 0 = \int_{a<|x|<b} \{(\Delta + \mu^2)U(x) \cdot \overline{U(x)} - \overline{(\Delta + \mu^2)U(x)} \cdot U(x)\} dx$$

$$= -\int_{S_a} \left\{\left(\frac{\partial U}{\partial n_+}\right)^{(+)}(x)\overline{U(x)} - \overline{\left(\frac{\partial U}{\partial n_+}\right)^{(+)}(x)}\, U(x)\right\} dS_x$$

$$+ \int_{S_b} \left\{\left(\frac{\partial U}{\partial n_+}\right)(x)\overline{U(x)} - \overline{\left(\frac{\partial U}{\partial n_+}\right)(x)}\, U(x)\right\} dS_x$$

$$= -\int_{S_a} (W(x)\overline{U(x)} - \overline{W(x)}\, U(x)) dS_x$$

$$+ \int_{S_b} \left\{\left(\frac{\partial U}{\partial n_+}\right)(x)\overline{U(x)} - \overline{\left(\frac{\partial U}{\partial n_+}\right)(x)}\, U(x)\right\} dS_x.$$

Thus we obtain by (7.14) and (7.15)

$$(7.16) \qquad \int_{|x|=b} \left\{\left(\frac{\partial U}{\partial n_+}\right)(x)\overline{U(x)} - \overline{\left(\frac{\partial U}{\partial n_+}\right)(x)}\; U(x)\right\} dS_x = 0,$$

for any b such that $b > a$. Once Lemma 7.6 and (7.16) are shown, an argument similar to Povzner [16, Chap. II, Lemma 5] gives

$$\int_{S_a} e^{-i\mu\omega\cdot y} q(y)(U|_{S_a})(y)dS_y = 0 \quad (\omega \in S^2), \tag{7.17}$$

which implies (7.10) by (7.11). Q.E.D.

Lemma 7.8. *Let* $\lambda > 0$ *and let* $u \in L_2(S_a)$ *satisfy* $u = \tilde{T}_{\sqrt{\lambda+i0}} u$ *(or* $u = \tilde{T}_{\sqrt{\lambda-i0}} u$*) in* $L_2(S_a)$. *Then* $u = 0$ *in* $L_2(S_a)$.

Proof. Put $U(x) \equiv (T_{\sqrt{\lambda+i0}} u)(x)$ ($= T_{\sqrt{\lambda}} u)(x)$). Then, by Lemmas 7.6, 7.7 and 2.11, we have

$$(\Delta + \lambda) U(x) = 0 \text{ on } \{x; |x| > a\}, \ U(x) = O\left(\frac{1}{|x|^2}\right) \text{ as } |x| \to +\infty.$$

Thus, in view of Mizohata [13, Chap. VIII §5, Lemma 8.4], we have

$$U(x) \equiv 0 \quad \text{on } \{x; |x| > a\}.$$

Since $U(x)$ is continuous on $\boldsymbol{R}^3$ as mentioned in the proof of Lemma 7.7, we obtain

$$U(x) \equiv 0 \quad \text{on } \{x; |x| \geq a\},$$

and hence

$$u(x) = (U|_{S_a})(x) = 0.$$

Similarly, the case that $u = \tilde{T}_{\sqrt{\lambda-i0}} u$ can be proven. Q.E.D.

We are now in a position to make use of the Fredholm-Riesz theory of compact operators in a Hilbert space, according to which, if T is a compact operator in a Hilbert space X, $1 - T$ is injective if and only if $(1 - T)^{-1}$ exists and belongs to $\boldsymbol{B}(X)$ (see e.g. Riesz-Nagy [20, Chap. IV]). Thus, by Lemmas 2.4, 7.3 and 7.8 we have the following

Lemma 7.9. *Let* $z \in \Pi \backslash (\sigma_p(H) \cup \{0\})$. *Then* $(1 - \tilde{T}_{\sqrt{z}})^{-1}$ *exists and belongs to* $\boldsymbol{B}(L_2(S_a))$.

Lemma 7.10. $(1 - \tilde{T}_{\sqrt{z}})^{-1}$ *is a* $\boldsymbol{B}(L_2(S_a))$*-valued continuous function of* z *on* $\Pi \backslash (\sigma_p(H) \cup \{0\})$.

Proof. The conclusion follows from Lemma 2.10 and the standard estimate

$$\|(1 - \tilde{T}_{\sqrt{z}})^{-1} - (1 - \tilde{T}_{\sqrt{z'}})^{-1}\|$$
$$\leq \frac{\|\tilde{T}_{\sqrt{z}} - \tilde{T}_{\sqrt{z'}}\| \, \|(1 - \tilde{T}_{\sqrt{z}})^{-1}\|}{1 - \|\tilde{T}_{\sqrt{z}} - \tilde{T}_{\sqrt{z'}}\| \, \|(1 - \tilde{T}_{\sqrt{z}})^{-1}\|}.$$

Q.E.D.

The above two lemmas imply Lemma 7.2. Therefore, Theorem 7.1 has now been proven.

Once the limiting absorption principle for H is established, the absolute continuity of H on $(0, \infty)$ readily follows from the same argument as Ikebe-Saitō [8]. Thus we have the following

Theorem 7.11. *$E((0, \infty))H$ is an absolutely continuous operator, where $E(\cdot)$ is the spectral measure associated with H.*

§ 8. Eigenfunction expansions

We shall proceed to show the eigenfunction expansion theorem. Our method is based on Kuroda [12] and Ikebe [6, 7].

We shall start with a well-known formula.

Lemma 8.1. *Let $s > \frac{1}{2}$. Suppose that $u \in L_2^s(\boldsymbol{R}^3)$ and $\mathscr{F} v \in C_0^\infty(\boldsymbol{R}^3 \backslash \{0\})$. Then we have*

$$(u, W_\pm v) = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_{-\infty}^{+\infty} (R(\lambda \pm i\varepsilon)u, R_0(\lambda \pm i\varepsilon)v) d\lambda. \tag{8.1}$$

For the proof, see e.g. Kuroda [11, § 5.4].

Lemma 8.2. *Let $s > \frac{1}{2}$. Suppose that $u \in L_2^s(\boldsymbol{R}^3)$ and $\mathscr{F} v \in C_0^\infty(\boldsymbol{R}^3 \backslash \{0\})$ such that $\operatorname{supp} \mathscr{F} v \subset \{\xi ; \alpha < |\xi|^2 < \beta\}$ $(0 < \alpha < \beta)$. Then we have*

$$(u, W_\pm v) = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_\alpha^\beta (R(\lambda \pm i\varepsilon)u, R_0(\lambda \pm i\varepsilon)v) d\lambda. \tag{8.2}$$

Here supp means support.

Proof. (cf. Kuroda [12, p. 151, Proposition 5.12]) Let $J = \boldsymbol{R} \backslash [\alpha, \beta]$. By Lemma 8.1 we have only to show

$$\lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_J (R(\lambda \pm i\varepsilon)u, R_0(\lambda \pm i\varepsilon)v) d\lambda = 0. \tag{8.3}$$

By Schwarz' inequality we have

$$\begin{aligned} &\left| \frac{\varepsilon}{\pi} \int_J (R(\lambda \pm i\varepsilon)u, R_0(\lambda \pm i\varepsilon)v) d\lambda \right| \\ &\quad \leq \left(\frac{\varepsilon}{\pi} \int_J \| R(\lambda \pm i\varepsilon)u \|^2 d\lambda \right)^{1/2} \left(\frac{\varepsilon}{\pi} \int_J \| R_0(\lambda \pm i\varepsilon)v \|^2 d\lambda \right)^{1/2} \\ &\quad = I_1(\varepsilon)^{1/2} \cdot I_2(\varepsilon)^{1/2}. \end{aligned}$$

Thus, to prove (8.3) it is sufficient to show

$$\lim_{\varepsilon \downarrow 0} I_2(\varepsilon) = 0, \tag{8.4}$$

$$I_1(\varepsilon) \leq \| u \|^2 \quad \text{for all } \varepsilon > 0. \tag{8.5}$$

Using the spectral representation for H_0, we have

$$\frac{\varepsilon}{\pi}\|R_0(\lambda \pm i\varepsilon)v\|^2 = \int_{-\infty}^{+\infty} \frac{\varepsilon}{\pi}\cdot\frac{1}{(\mu-\lambda)^2+\varepsilon^2}\, d(E_0(\mu)v, v), \tag{8.6}$$

where $E_0(\cdot)$ denotes the spectral measure associated with H_0. Using the fact that H_0 is an absolutely continuous operator, we have

$$\frac{\varepsilon}{\pi}\|R_0(\lambda \pm i\varepsilon)v\|^2 = (P_\varepsilon * \rho)(\lambda),$$

where $P_\varepsilon(\mu) = \dfrac{\varepsilon}{\pi(\mu^2+\varepsilon^2)}$ (the Poisson kernel), $\rho(\mu) = \dfrac{d}{d\mu}(E_0(\mu)v, v)$ and $*$ means convolution. Further, $\rho(\mu)$ belongs to $L_1(\boldsymbol{R}^1)$ and $\rho(\mu) = 0$ for a.e.$\mu \in J$ since $E_0(J)v = 0$. Thus we obtain

$$\begin{aligned} I_2(\varepsilon) &= \int_J (P_\varepsilon * \rho)(\lambda)d\lambda = \int_J ((p_\varepsilon * \rho)(\lambda) - \rho(\lambda))d\lambda \\ &\leq \|P_\varepsilon * \rho - \rho\|_{L_1(\boldsymbol{R}^1)} \longrightarrow 0 \text{ as } \varepsilon \downarrow 0, \end{aligned}$$

which implies (8.4). Let us show (8.5). As we got (8.6), we have

$$\begin{aligned} I_1(\varepsilon) = \frac{\varepsilon}{\pi}\int_J \|R(\lambda \pm i\varepsilon)u\|^2 d\lambda &= \int_J d\lambda \int_{-\infty}^{+\infty} P_\varepsilon(\mu-\lambda)d(E(\mu)u, u) \\ &\leq \int_{-\infty}^{+\infty} d(E(\mu)u, u) \int_{-\infty}^{+\infty} P_\varepsilon(\mu-\lambda)d\lambda = \|u\|^2, \end{aligned}$$

where we used Fubini's theorem and the well-known properties of $P_\varepsilon(\mu)$ that

$$P_\varepsilon(\mu) > 0 \quad \text{for all } \mu \text{ and } \int_{-\infty}^{+\infty} P_\varepsilon(\mu)d\mu = 1.$$

This implies (8.5). Q.E.D.

Let us define the generalized Fourier transform $\mathscr{F}_\pm$ and the generalized eigenfunctions $\varphi_\pm(x, \xi)$ by

$$\mathscr{F}_\pm = \mathscr{F} W_\pm^*, \tag{8.7}$$

$$\varphi_\pm(x, \xi) = e^{i\xi\cdot x} + [T_{\mp|\xi|}^{(1)}(1 - q\tilde{T}_{\mp|\xi|}^{(1)})^{-1}(e^{i\xi\cdot}q)](x) \tag{8.8}$$

for $(x, \xi) \in \boldsymbol{R}^3 \times (\boldsymbol{R}^3\backslash\{0\})$, respectively. We should note here that by Lemmas 2.14 and 7.2 $(1 - q\tilde{T}_{\mp|\xi|}^{(1)})^{-1}$ exist and satisfy the relations

$$(1 - q\tilde{T}_{\mp|\xi|}^{(1)})^{-1} = [(1 - \tilde{T}_{\pm|\xi|})^{-1}]^* \quad \text{for } \xi \in \boldsymbol{R}^3\backslash\{0\}. \tag{8.9}$$

We also remark that $\varphi_\pm(x, \xi)$ are regarded as the generalized eigenfunctions of H in the sense stated in Theorem 8.6. Further, they are seen to be the integral kernels of $\mathscr{F}_\pm$ by the following theorem.

Theorem 8.3. *For any* $u \in L_2(\boldsymbol{R}^3)$, $\mathscr{F}_\pm$ *have the form*

$$(8.10)\qquad (\mathscr{F}_\pm u)(\xi) = \underset{R\to+\infty}{\text{l.i.m.}}\ (2\pi)^{-3/2} \int_{|x|\le R} \overline{\varphi_\pm(x, \xi)}\ u(x)dx,$$

where l.i.m. *means the limit in the mean.*

Proof. Let $u \in C_0^\infty(\boldsymbol{R}^3)$ and $v \in C_0^\infty(\boldsymbol{R}^3\backslash\{0\})$ such that $\operatorname{supp} v \subset \{x; \alpha < |x|^2 < \beta\}$ $(0 < \alpha < \beta)$. Using (8.7), Lemma 8.2 and (7.2), we have

$$(8.11)\qquad (\mathscr{F}_\pm u, v) = (u, W_\pm \mathscr{F}^* v)$$

$$= \lim_{\varepsilon\downarrow 0} \frac{\varepsilon}{\pi} \int_\alpha^\beta (R_0(\lambda \pm i\varepsilon)u, R_0(\lambda \pm i\varepsilon)\mathscr{F}^* v)d\lambda$$

$$+ \lim_{\varepsilon\downarrow 0} \frac{\varepsilon}{\pi} \int_\alpha^\beta (T_{\sqrt{\lambda\pm i\varepsilon}}(1 - \tilde{T}_{\sqrt{\lambda\pm i\varepsilon}})^{-1} \gamma R_0(\lambda \pm i\varepsilon)u, R_0(\lambda \pm i\varepsilon)\mathscr{F}^* v)d\lambda$$

$$= \lim_{\varepsilon\downarrow 0} J_1(\varepsilon) + \lim_{\varepsilon\downarrow 0} J_2(\varepsilon).$$

For the first term of the right hand side of (8.11), as is well known (see e.g. Kuroda [12, p.54]), we have

$$(8.12)\qquad \lim_{\varepsilon\downarrow 0} J_1(\varepsilon) = \int_{\alpha<|\xi|^2<\beta} d\xi\, (\mathscr{F} u)(\xi)\, \overline{v(x)} = (\mathscr{F} u, v).$$

We shall consider the second term. In view of Parseval's equality and (2.10), we have

$$(8.13)\qquad (T_{\sqrt{\lambda\pm i\varepsilon}}(1 - \tilde{T}_{\sqrt{\lambda\pm i\varepsilon}})^{-1} \gamma R_0(\lambda \pm i\varepsilon)u, R_0(\lambda \pm i\varepsilon)\mathscr{F}^* v)$$

$$= \left(- \frac{1}{|\cdot|^2 - (\lambda \pm i\varepsilon)} \mathscr{F}_{S_a} q(1 - \tilde{T}_{\sqrt{\lambda\pm i\varepsilon}})^{-1} \gamma R_0(\lambda \pm i\varepsilon)u, \frac{v}{|\cdot|^2 - (\lambda \pm i\varepsilon)} \right)$$

$$= \int_{\boldsymbol{R}^3} d\xi \left\{ - \frac{1}{|\xi|^2 - (\lambda \pm i\varepsilon)} (2\pi)^{-3/2} \int_{S_a} dS_y e^{-i\xi\cdot y} q(y) \times \right.$$

$$\left. \times [(1 - \tilde{T}_{\sqrt{\lambda\pm i\varepsilon}})^{-1} \gamma R_0(\lambda \pm i\varepsilon)u](y) \right\} \frac{\overline{v(\xi)}}{|\xi|^2 - (\lambda \mp i\varepsilon)}$$

$$= -(2\pi)^{-3/2} \int_{\boldsymbol{R}^3} d\xi \frac{\overline{v(\xi)}}{(\lambda - |\xi|^2)^2 + \varepsilon^2} \times$$

$$\times ((1 - \tilde{T}_{\sqrt{\lambda\pm i\varepsilon}})^{-1} \gamma R_0(\lambda \pm i\varepsilon)u, e^{i\xi\cdot} q)_a.$$

Since $((1 - \tilde{T}_{\sqrt{\lambda\pm i\varepsilon}})^{-1} \gamma R_0(\lambda \pm i\varepsilon)u, e^{i\xi\cdot} q)_a$ are continuous in λ and ε on $[\alpha, \beta] \times [0, 1]$ by Lemmas 7.2, 2.9 with $q(x) \equiv 1$ and the fact that $\gamma R_0(z) = -(T^{(1)}_{\sqrt{\bar{z}}})^*$, we have

(8.14) $$\lim_{\varepsilon\downarrow 0}\int_\alpha^\beta d\lambda\,\frac{\varepsilon}{\pi}\cdot\frac{1}{(\lambda-|\xi|^2)^2+\varepsilon^2}((1-\tilde{T}_{\sqrt{\lambda+i\varepsilon}})^{-1}\gamma R_0(\lambda\pm i\varepsilon)u,\, e^{i\xi\cdot}q)_a$$
$$=((1-\tilde{T}_{\pm|\xi|})^{-1}\gamma R_0(|\xi|^2\pm i0)u,\, e^{i\xi\cdot}q)_a,$$

where we have made use of the well-known relation

$$\lim_{\varepsilon\downarrow 0}\int_\alpha^\beta d\lambda\,\frac{\varepsilon}{\pi}\cdot\frac{1}{(\lambda-a)^2+\varepsilon^2}f(\lambda,\varepsilon)$$
$$=\begin{cases}0 & \text{if } a<\alpha \text{ or } \beta<a\\ f(a,0) & \text{if } \alpha<a<\beta.\end{cases}$$

in which $f(\lambda, \varepsilon)$ is a continuous function of (λ, ε) for $(\lambda, \varepsilon)\in[\alpha, \beta]\times[0, 1]$ (see e.g. Titchmarsh [22, p.31]). Further, from (8.9) and the fact that $\gamma R_0(z) = -(T^{(1)}_{\sqrt{\bar{z}}})^*$, it follws that

(8.15) $$((1-\tilde{T}_{\pm|\xi|})^{-1}\gamma R_0(|\xi|^2\pm i0)u,\, e^{i\xi\cdot}q)_a$$
$$=(\gamma R_0(|\xi|^2\pm i0)u,\,(1-q\tilde{T}^{(1)}_{\mp|\xi|})^{-1}(e^{i\xi\cdot}q))_a$$
$$=-\int_{\boldsymbol{R}^3}dx\,u(x)\overline{[T^{(1)}_{\mp|\xi|}(1-q\tilde{T}^{(1)}_{\mp|\xi|})^{-1}(e^{i\xi\cdot}q)](x)},$$

where we have used Fubini's theorem in the last equality. Thus, making use of Fubini's theorem and the dominated convergence theorem, we see from (8.13), (8.14) and (8.15) that

(8.16) $$\lim_{\varepsilon\downarrow 0}J_2(\varepsilon)$$
$$=(2\pi)^{-3/2}\int_{\boldsymbol{R}^3}d\xi\left(\int_{\boldsymbol{R}^3}dx\,u(x)\overline{[T^{(1)}_{\mp|\xi|}(1-q\tilde{T}^{(1)}_{\mp|\xi|})^{-1}(e^{i\xi\cdot}q)](x)}\right)\overline{v(\xi)}.$$

Now we have by (8.8), (8.11) (8.12) and (8.16)

$$(\mathscr{F}_\pm u)(\xi)=(2\pi)^{-3/2}\int_{\boldsymbol{R}^3}dx\,\overline{\varphi_\pm(x,\xi)}u(x)\qquad\text{for any } u\in C_0^\infty(\boldsymbol{R}^3).$$

Since $C_0^\infty(\boldsymbol{R}^3)$ is dense in $L_2(\boldsymbol{R}^3)$, the conclusion follows. Q.E.D.

To prove the continuity and boundedness of $\varphi_\pm(x, \xi)$, we need the following lemma.

Lemma 8.4. *Let K be a compact set in $\boldsymbol{R}^3$. Let $f(x, \xi)$ be a continuous function of $(x, \xi)\in S_a\times K$. Then $(T^{(1)}_{\pm|\xi|}f(\cdot, \xi))(x)$ is bounded and continuous in $(x, \xi)\in\boldsymbol{R}^3\times K$. In particular, $(\tilde{T}^{(1)}_{\pm|\xi|}f(\cdot, \xi))(x)$ is continuous in $(x, \xi)\in S_a\times K$.*

Proof. By Lemma 2.1 we have

$$|(T^{(1)}_{\pm|\xi|}f(\cdot,\xi))(x)| \le \frac{1}{4\pi}\max_{(y,\xi)\in S_a\times K}|f(y,\xi)|\int_{S_a}\frac{1}{|x-y|}dS_y$$

$$= \frac{a}{2}\max_{(y,\xi)\in S_a\times K}|f(y,\xi)|\frac{a+|x|-|a-|x||}{|x|} \le a\max_{(y,\xi)\in S_a\times K}|f(y,\xi)|,$$

which proves the boundeness of $(T^{(1)}_{\pm|\xi|}f(\cdot,\xi))(x)$.

Let us show the continuity. Let us introduce the functions $G^{(\varepsilon)}_{\pm}(x,\xi)$ with a real parameter ε by

$$G^{(\varepsilon)}_{\pm}(x,\xi) = -\frac{1}{4\pi}\int_{S_a\cap\{y;|x-y|>\varepsilon\}}\frac{e^{\pm i|\xi||x-y|}}{|x-y|}f(y,\xi)dS_y,$$

$(x,\xi)\in \boldsymbol{R}^3\times K$, $\varepsilon>0$. It is easily seen that for each ε, $G^{(\varepsilon)}_{\pm}(x,\xi)$ is continuous in (x,ξ) in $\boldsymbol{R}^3\times K$. Further, $G^{(\varepsilon)}_{\pm}(x,\xi)$ uniformly converges to $(T^{(1)}_{\pm|\xi|}f(\cdot,\xi))(x)$ when $\varepsilon\downarrow 0$. In fact, we have for a sufficiently small ε

$$|G^{(\varepsilon)}_{\pm}(x,\xi)-(T^{(1)}_{\pm|\xi|}f(\cdot,\xi))(x)|$$

$$\le \max_{(y,\xi)\in S_a\times K}|f(y,\xi)|\int_{S_a\cap\{y;|x-y|\le\varepsilon\}}\frac{1}{4\pi|x-y|}dS_y$$

$$=\begin{cases}\max_{(y,\xi)\in S_a\times K}|f(y,\xi)|\dfrac{a(\varepsilon-|a-|x||)}{2|x|} & \text{if } |a-|x||<\varepsilon\\ 0 & \text{if } |a-|x||\ge\varepsilon\end{cases}$$

$$\le \max_{(y,\xi)\in S_a\times K}|f(y,\xi)|\varepsilon \quad \text{if } \varepsilon\le\frac{a}{2},$$

where we have used (2.6). Thus the continuity of $(T^{(1)}_{\pm|\xi|}f(\cdot,\xi))(x)$ has been proven. Since $(\tilde{T}^{(1)}_{\pm|\xi|}f(\cdot,\xi))(x)=(T^{(1)}_{\pm|\xi|}f(\cdot,\xi))(x)$ $(x\in S_a)$, the assertion for $(\tilde{T}^{(1)}_{\pm|\xi|}f(\cdot,\xi))(x)$ holds. Q.E.D.

Theorem 8.5. *$\varphi_{\pm}(x,\xi)$ is continuous in $(x,\xi)\in\boldsymbol{R}^3\times(\boldsymbol{R}^3\backslash\{0\})$ and bounded on $\boldsymbol{R}^3\times K$, where K is any compact set in $\boldsymbol{R}^3\backslash\{0\}$.*

Proof. Put $\psi_{\pm}(x,\xi)=[(1-q\tilde{T}^{(1)}_{\mp|\xi|})^{-1}(e^{i\xi\cdot}q)](x)$. Then $\psi_{\pm}(\cdot,\xi)$ is an $L_2(S_a)$-valued continuous function of $\xi\in\boldsymbol{R}^3\backslash\{0\}$ by Lemma 7.2 and (8.9). If we show that $\psi_{\pm}(x,\xi)$ is a continuous function of $(x,\xi)\in S_a\times K$, the conclusion follows from (8.8) and Lemma 8.4. Since $\psi_{\pm}(x,\xi)$ satisfy the equation

$$\psi_{\pm}(x,\xi)=e^{i\xi\cdot x}q(x)+(q\tilde{T}^{(1)}_{\mp|\xi|})\psi_{\pm}(x,\xi), \tag{8.17}$$

we have, using (8.17) repeatedly,

$$\begin{aligned}\psi_{\pm}(x,\xi)&=e^{i\xi\cdot x}q(x)+[(q\tilde{T}^{(1)}_{\mp|\xi|})(e^{i\xi\cdot}q)](x)\\&\quad+[(q\tilde{T}^{(1)}_{\mp|\xi|})^2(e^{i\xi\cdot}q)](x)+[(q\tilde{T}^{(1)}_{\mp|\xi|})^3(e^{i\xi\cdot}q)](x)\\&\quad+[(q\tilde{T}^{(1)}_{\mp|\xi|})^4\psi_{\pm}(\cdot,\xi)](x).\end{aligned} \tag{8.18}$$

It follows from Lemma 8.4 that the first four terms of the right hand side of (8.18) are continuous in $(x, \xi) \in S_a \times K$. Thus the proof of this theorem is reduced to showing the continuity of $[(q\tilde{T}^{(1)}_{\mp|\xi|})^4 \psi_{\pm}(\cdot, \xi)](x)$. Let $\mathscr{K}_{\mp}(x, y,; \xi)$ be the integral kernel of $(q\tilde{T}^{(1)}_{\mp|\xi|})^4$. Then, in the same way as we proved Lemma 7.5, we can show that $\mathscr{K}_{\mp}(x, y; \xi)$ is continuous in (x, y, ξ) on $S_a \times S_a \times K$. We consider the difference

(8.19)
$$\begin{aligned}
&[(q\tilde{T}^{(1)}_{\mp|\xi|})^4 \psi_{\pm}(\cdot, \xi)](x) - [(q\tilde{T}^{(1)}_{\mp|\xi_0|})^4 \psi_{\pm}(\cdot, \xi_0)](x_0) \\
&= \int_{S_a} (\mathscr{K}_{\mp}(x, y; \xi) - \mathscr{K}_{\mp}(x_0, y; \xi))\psi_{\pm}(x, \xi) dS_y \\
&+ \int_{S_a} (\mathscr{K}_{\mp}(x_0, y; \xi) - \mathscr{K}_{\mp}(x_0, y; \xi_0))\psi_{\pm}(x, \xi) dS_y \\
&+ \int_{S_a} \mathscr{K}_{\mp}(x_0, y; \xi_0)(\psi_{\pm}(x, \xi) - \psi_{\pm}(x, \xi_0)) dS_y \\
&= J_1 + J_2 + J_3.
\end{aligned}$$

$J_i (i = 1, 2, 3)$ are estimated as follows:

(8.20)
$$\begin{aligned}
|J_1| &\le \max_{y \in S_a} |\mathscr{K}_{\mp}(x, y; \xi) - \mathscr{K}_{\mp}(x_0, y; \xi)| \int_{S_a} |\psi_{\pm}(y, \xi)| dS_y \\
&\le \max_{y \in S_a} |\mathscr{K}_{\mp}(x, y; \xi) - \mathscr{K}_{\mp}(x_0, y; \xi)| (4\pi a^2)^{1/2} \, \|\psi_{\pm}(\cdot, \xi)\|_a, \\
|J_2| &\le \max_{y \in S_a} |\mathscr{K}_{\mp}(x_0, y; \xi) - \mathscr{K}_{\mp}(x_0, y; \xi_0)| (4\pi a^2)^{1/2} \, \|\psi_{\pm}(\cdot, \xi)\|_a, \\
|J_3| &\le \max_{y \in S_a} |\mathscr{K}_{\mp}(x_0, y; \xi_0)| (4\pi a^2)^{1/2} \, \|\psi_{\pm}(\cdot, \xi) - \psi_{\pm}(x, \xi_0)\|_a,
\end{aligned}$$

It follows from (8.19) and (8.20) that $[(q\tilde{T}^{(1)}_{\mp|\xi|})^4 \psi_{\pm}(\cdot, \xi)](x)$ is continuous in (x, ξ) on $S_a \times K$. Thus the theorem follows. Q.E.D.

Theorem 8.6. *Let $\xi \in \boldsymbol{R}^3 \backslash \{0\}$. Then $\varphi_{\pm}(x, \xi)$ satisfy the followng equations*

(8.21)
$$\varphi_{\pm}(x, \xi) = e^{i\xi \cdot x} - \frac{1}{4\pi} \int_{S_a} \frac{e^{\mp i|\xi||x-y|}}{|x-y|} q(y) \varphi_{\pm}(y, \xi) dS_y$$

(the Lippmann-Schwinger equation),

(8.22)
$$\begin{aligned}
&\int_{\boldsymbol{R}^3} \varphi_{\pm}(x, \xi)(-\Delta - |\xi|^2) v(x) dx + \\
&+ \int_{S_a} q(x) \varphi_{\pm}(x, \xi) v(x) dS_x = 0 \quad \text{for any } v \in C_0^{\infty}(\boldsymbol{R}^3).
\end{aligned}$$

Proof. By (8.8) we have

$$-\frac{1}{4\pi}\int_{S_a}\frac{e^{\mp i|\xi||x-y|}}{|x-y|}q(y)\varphi_\pm(y,\xi)dS_y = T^{(1)}_{\mp|\xi|}(q(\varphi_\pm|_{S_a})(\cdot,\xi))(x)$$
$$= T^{(1)}_{\mp|\xi|}(qe^{i\xi\cdot})(x) + T^{(1)}_{\mp|\xi|}(q\tilde{T}^{(1)}_{\mp|\xi|}(1-q\tilde{T}^{(1)}_{\mp|\xi|})^{-1}(qe^{i\xi\cdot}))(x)$$
$$= T^{(1)}_{\mp|\xi|}(qe^{i\xi\cdot})(x) + T^{(1)}_{\mp|\xi|}([-(1-q\tilde{T}^{(1)}_{\mp|\xi|})+1](1+q\tilde{T}^{(1)}_{\mp|\xi|})^{-1}(qe^{i\xi\cdot}))(x)$$
$$= T^{(1)}_{\mp|\xi|}((1-q\tilde{T}^{(1)}_{\mp|\xi|})^{-1}(qe^{i\xi\cdot}))(x)$$
$$= \varphi_\pm(x,\xi) - e^{i\xi\cdot x},$$

which implies (8.21). Let us show (8.22). In view of (8.21), $\varphi_\pm(x,\xi)$ can be written as

$$\varphi_\pm(x,\xi) = e^{i\xi\cdot x} + (T_{\mp|\xi|}(\varphi_\pm|_{S_a}(\cdot,\xi))(x),$$

Therefore, by Lemma 2.11 we have for any $v\in C_0^\infty(\boldsymbol{R}^3)$

$$\int_{\boldsymbol{R}^3}\varphi_\pm(x,\xi)(-\Delta-|\xi|^2)v(x)dx$$
$$= \int_{\boldsymbol{R}^3}(\varphi_\pm(x,\xi)-e^{i\xi\cdot x})(-\Delta-|\xi|^2)v(x)dx$$
$$= \int_{\boldsymbol{R}^3}(T_{\mp|\xi|}\varphi_\pm|_{S_a}(\cdot,\xi))(x)(-\Delta-|\xi|^2)v(x)dx$$
$$= -\int_{S_a}q(x)\varphi_\pm(x,\xi)v(x)dS_x,$$

which implies (8.22). Q.E.D.

Theorem 8.7. *$\mathscr{F}_\pm$ are partially isometric operators with the domain $E((0,\infty))$ $L_2(\boldsymbol{R}^3)$ and the range $L_2(\boldsymbol{R}^3)$. Further, $\mathscr{F}_\pm$ have the following properties: Let Λ be any Borel set on $\boldsymbol{R}$. Then,*

(8.23) $$\mathscr{F}_\pm E(\Lambda) = \chi_{\{\xi;|\xi|^2\in\Lambda\}}\mathscr{F}_\pm,$$

where χ_A denotes the operator of multiplication by the characteristic function of A. In particular, if $u\in L_2(\boldsymbol{R}^3)$, and α and β are such that $0<\alpha<\beta$, then

(8.24) $$\|E((\alpha,\beta))u\|^2 = \int_{\alpha<|\xi|^2<\beta}|(\mathscr{F}_\pm u)(\xi)|^2d\xi,$$

(8.25) $$E((\alpha,\beta))u(x) = (2\pi)^{-3/2}\int_{\alpha<|\xi|^2<\beta}(\mathscr{F}_\pm u)(\xi)\varphi_\pm(x,\xi)d\xi.$$

Proof. First, let us recall the well-known relations

(8.26) $$\mathscr{F}E_0(\Lambda)\mathscr{F}^* = \chi_{\{\xi;|\xi|^2\in\Lambda\}},$$

(8.27) $$E(\Lambda)W_\pm = W_\pm E_0(\Lambda),$$

where Λ is a Borel set on $\boldsymbol{R}$ (see e.g. Kuroda [12, §3.4, Theorem 2]). Putting $\Lambda = (0, \infty)$ in (8.27), we have

$$E((0, \infty))W_\pm = W_\pm E_0((0, \infty)) = W_\pm,$$

from which it follows that

$$Ran(W_\pm) \subset E((0, \infty))L_2(\boldsymbol{R}^3).$$

On the other hand, it follows from Theorem 7.11 that $E((0, \infty))L_2(\boldsymbol{R}^3)$ is included in the absolute continuous subspace $\mathscr{X}_{ac}(H)$ of $L_2(\boldsymbol{R}^3)$ relative to H. Therefore, we have by Theorem 4.1.

$$E((0, \infty))L_2(\boldsymbol{R}^3) \subset \mathfrak{X}_{ac}(H) = Ran(W_\pm) \subset E((0, \infty))L_2(\boldsymbol{R}^3),$$

and hence

$$Ran(W_\pm) = E((0, \infty))L_2(\boldsymbol{R}^3).$$

This implies that $W_\pm$ are partially isometric operators with the domain $L_2(\boldsymbol{R}^3)$ and the range $E((0, \infty))L_2(\boldsymbol{R}^3)$. Thus, it follows from (8.7) that $\mathscr{F}_\pm$ are partially isometric operators with the domain in $E((0, \infty))L_2(\boldsymbol{R}^3)$ and the range $L_2(\boldsymbol{R}^3)$. By (8.7), (8.26) and (8.27) we have

$$\mathscr{F}_\pm E(\Lambda) = \mathscr{F}W_\pm^* E(\Lambda) = \mathscr{F}E_0(\Lambda)W_\pm^* = \mathscr{F}E_0(\Lambda)\mathscr{F}^*\mathscr{F}W_\pm^* = \chi_{\{\xi;|\xi|^2 \in \Lambda\}}\mathscr{F}_\pm.$$

This proves (8.23). Let us show (8.24) and (8.25). Since $\mathscr{F}_\pm$ are partially isometric operators with domain $E((0, \infty))L_2(\boldsymbol{R}^3)$, it follows that

$$\mathscr{F}_\pm^*\mathscr{F}_\pm = E((0, \infty)).$$

Therefore, we have by (8.23) ($\Lambda = (\alpha, \beta) \subset (0, \infty)$)

$$E((\alpha, \beta)) = \mathscr{F}_\pm^* \chi_{\{\xi;\alpha<|\xi|^2<\beta\}}(\xi)\mathscr{F}_\pm,$$

from which (8.24) and (8.25) follow immediately. Q.E.D.

We shall now proceed to the eigenfunction expansion theorem.

Theorem 8.8. *Let $\lambda_1, \lambda_2, \cdots$ be the nonpositive eigenvalues of H (counting multiplicity) and $\{\varphi_1, \varphi_2, \cdots\}$ a corresponding orthonormal system of eigenfunctions of H, if any. Then, for any $u \in L_2(\boldsymbol{R}^3)$ we have the following expansion formula*

$$u(x) = \sum_n (u, \varphi_n)\varphi_n(x) + \underset{\alpha\downarrow 0, \beta\uparrow\infty}{\text{l.i.m.}} (2\pi)^{-3/2} \int_{\alpha<|\xi|<\beta} d\xi(\mathscr{F}_\pm u)(\xi)\varphi_\pm(x, \xi). \tag{8.28}$$

Further, $u \in Dom(H)$ if and only if $|\cdot|^2\mathscr{F}_\pm u \in L_2(\boldsymbol{R}^3)$. In this case, we have the following representation of H

$$Hu(x) = \sum_n \lambda_n(u, \varphi_n)\varphi_n(x) + \underset{\alpha\downarrow 0, \beta\uparrow\infty}{\text{l.i.m.}} (2\pi)^{-3/2} \int_{\alpha<|\xi|<\beta} d\xi(\mathscr{F}_\pm u)(\xi)\varphi_\pm(x, \xi). \tag{8.29}$$

for $u \in Dom(H)$.

Proof. According to Theorem 5.1 $E((-\infty, 0])L_2(\boldsymbol{R}^3)$ is spanned by

$\{\varphi_1, \varphi_2, \cdots\}$. Therefore, we have for any $u \in L_2(\boldsymbol{R}^3)$

(8.30) $$u(x) = \sum_n (u, \varphi_n)\varphi_n(x) + \operatorname*{l.i.m.}_{\alpha \downarrow 0, \beta \uparrow \infty} E((\alpha, \beta))u(x).$$

(8.28) follows from (8.30) and Theorem 8.7, (8.25). Using Theorem 8.7, (8.24), we have

(8.31) $$\int_0^\infty \lambda^2 d\, \| E(\lambda)u \|^2 = \int_{\boldsymbol{R}^3} |\xi|^4 |(\mathscr{F}_\pm u)(\xi)|^2 d\xi.$$

On the other hand, it follows from Theorem 6.5 that

(8.32) $$E((-\infty, 0])L_2(\boldsymbol{R}^3) \subset Dom(H).$$

Thus, we have by (8.31) and (8.32)

(8.33) $$Dom(H) = \{u;\, u \in L_2(\boldsymbol{R}^3),\ |\cdot|^2 \mathscr{F}_\pm u \in L_2(\boldsymbol{R}^3)\}.$$

Finally, let us show (8.29). From the "intertwining" relation $W_\pm H_0 \subset HW_\pm$ (see e.g. Kuroda [11, §3.4]) and (8.33), it follows that

(8.34) $$\mathscr{F}_\pm H = |\cdot|^2 \mathscr{F}_\pm .$$

Therefore, if we replace u by Hu in (8.28), (8.29) follows rom (8.34) and the fact that $(Hu, \varphi_n) = \lambda_n(u, \varphi_n)$. Thus the theorem has been proven. Q.E.D.

Added in proof. The proof of the assertion $u \in H^2(\boldsymbol{R}^3 \backslash S_a)$ is incomplete. But this can be proven by the standard argument for showing the global regularity for solutions to elliptic boundary-value problems (see e.g. Mizohata [13, Chap 3, §12]) if one takes into account the already known facts that $\Delta u \in L_2(\boldsymbol{R}^3)$, $u \in H^1(\boldsymbol{R}^3) = \mathrm{Dom}[h]$ and that $h[u, v] = (Hu, v)$ for any $v \in H^1(\boldsymbol{R}^3)$.

Department of Mathematics
Kyoto University (T. I.)
Fukui National College of Technology
(S. S.)

References

[1] S. Agmon, Lectures on Elliptic Boundary Value Propblems, Van Nostrand, 1966.
[2] S. Agmon, Spectral properties of Schrödinger operators and scattering theory, Ann. Scuola Norm. Sup. Pisa, Ser. IV, **2** (1975), 151–218.
[3] J. P. Antoine, F. Gesztesy and J. Shabani, Exactly solvable models of sphere interactions in quantum mechanics, J. Phys. A: Math. Gen., **20** (1987), 3687–3712.
[4] D. Colton and R. Kress, Integral Equation Methods in Scattering Theory, Wiley-Interscience, 1983.
[5] C. L. Dolph, J. B. McLeod and D. Thoe, The analytic continuation of the resolvent kernel and

scattering operator associated with the Schrödinger operator, J. Math. Anal. Appl., **16** (1966), 311–332.

[6] T. Ikebe, Eigenfunction expansions associated with the Schrödinger operators and their applications to scattering theory, Arch. Rational Mech. Anal., **5** (1960), 1–34.

[7] T. Ikebe, Remarks on the orthogonality of eigenfunctions for the Schrödinger operator in R^n, J. Fac. Sci. Univ. Tokyo, Sect. I, **17** (1970), 355–361.

[8] T. Ikebe and Y. Saitō, Limiting absorption method and absolute continuity for the Schrödinger operator, J. Math. Kyoto Univ., **12** (1972), 513–542.

[9] T. Kato, Perturbation Theory for Linear Operators, Springer, 1966.

[10] O.D. Kellogg, Foundations of Potential Theory, Springer, 1929.

[11] S. T. Kuroda, An Introduction to Scattering Theory, Lecture Notes Series No. 51, Aarhus Univ., 1978.

[12] S. T. Kuroda, Spectral Theory II, Iwanami, 1979, (in Japanese).

[13] S. Mizohata, The Theory of Partial Differential Equations, Cambridge, 1973.

[14] K. Mochizuki, Scattering Theory for Wave Equaiions, Kinokuniya, 1984 (in Japanese).

[15] J. Petzold, Wie gut gilt das Exponentialgesetz beim α-Zerfall?, Zeits. f. Phys., **155** (1959), 422–432.

[16] A. Ya. Povzner, The expansion of arbitrary functions in terms of eigenfunctions of the operator $-\Delta u + cu$, A.M.S. Translations, Series 2, **60** (1967), 1–49.

[17] M. Reed and B. Simon, Methods of Modern Mathematical Physics I: Functional Analysis, Academic Press, New York, 1972.

[18] M. Reed and B. Simon, Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness, Academic Press, New York, 1975.

[19] M. Reed and B. Simon, Methods of Modern Mathematical Physics IV: Analysis of Operators, Academic Press, New York, 1978.

[20] F Riesz and B. Von Sz-Nagy, Functional Analysis, New York, F. Ungar Publ. Co., 1955.

[21] B. Simon, Quantum Mechanics for Hamiltonians Defined as Quadratic Forms, Princeton Univ. Press, 1971.

[22] E. C. Titchmarsh, Introduction to the Theory of Fourier Integrals, Oxford, 1937.