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Homotopy operations in symplectic and orthogonal groups

By

Albert T. LUNDELL

Using the Bott periodicity maps, we define the structure of a right stable homotopy module on $\pi_*(G)$ for $G=SO$ or $G=Sp$. Because of the simple structure of $\pi_*(G)$, most operations of stable homotopy π_*^S are trivial.

Specializing to Sp, we compute some non-trivial Toda brackets in $\pi_*(Sp)$, obtaining some new non-trivial primary operations in $\pi_*(Sp)$.

Using these Toda bracket calculations and the non-stable Bott maps, we can transfer the calculations to $\pi_*(SO)$ and to certain non-stable homotopy of $SO(n)$. This results in the fact that the generators of $\pi_{8m+r}(SO)$, $r=0, 1$, originate in $\pi_{8m+r}(SO(6))$.

Throughout this work we use the notation of Toda [T] with the modifications of Mori [M] for generators of the various homotopy groups of spheres.

1. Generators of stable homotopy and primary operations

Since $Sp(1)=S^3$, we choose $\beta_{3,1}=\iota_3$, $\beta_{4,1}=\eta$, and $\beta_{5,1}=\eta\circ\eta=\eta^2$, in $\pi_k(S^3)$ for $k=3, 4, 5$. If $j\colon Sp(1)\to Sp$ is the inclusion, let $\beta_k=j_*(\beta_{k,1})$. Let $\alpha_{k,4}$ be the generator of $\pi_k(O(4))$ for $k=0, 1$, and if $h\colon Sp(1)=S^3\to O(4)$ is the inclusion, let $\alpha_{3,4}=h_*(\iota_3)\in\pi_3(O(4))$. If $i\colon O(4)\to O$ is the inclusion, let $\alpha_k=i_*(\alpha_{k,4})$ for $k=0, 1, 3$. Using the Cayley numbers, one can construct a cross-section $s\colon S^7\to SO(8)$ of the fibre bundle $SO(7)\to SO(8)\to S^7$ such that if $s_*(\iota_7)=\alpha_{7,8}\in\pi_7(SO(8))$, then $i_*(\alpha_{7,8})=\alpha_7\in\pi_7(SO)$ is a generator. If $k\colon SO(8)\to Sp(8)$ is the inclusion, set $\beta_{7,8}=k_*(\alpha_{7,8})$ and $\beta_7=j_*(\beta_{7,8})\in\pi_7(Sp)$, both of which are generators. Note that by following s or h by the inverse map, we can change the sign of $\alpha_{7,8}$ or $\beta_{7,8}$.

We have Bott [Bo1] maps $B\colon Sp\to\Omega^4 SO$ and $B'\colon O\to\Omega^4 Sp$, and the composites $B_{Sp}=(\Omega^4 B')\circ B\colon Sp\to\Omega^8 Sp$, and $B_O=(\Omega^4 B)\circ B'\colon O\to\Omega^8 SO$, all of which are homotopy equivalences and yield isomorphisms

$$\tilde{B}\colon \pi_k(Sp)\xrightarrow[\cong]{B_*}\pi_k(\Omega^4 O)\xrightarrow[\cong]{\partial^{-4}}\pi_{k+4}(O),$$

$$\tilde{B}'\colon \pi_k(O)\xrightarrow[\cong]{B'_*}\pi_k(\Omega^4 Sp)\xrightarrow[\cong]{\partial^{-4}}\pi_{k+4}(Sp),$$

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$$\tilde{B}_G\colon \pi_k(G) \xrightarrow[\cong]{B_{G*}} \pi_k(\Omega^8 G) \xrightarrow[\cong]{\partial^{-8}} \pi_{k+8}(G),$$

where $G = O$ or Sp, and ∂^{-n} is the inverse of the n-fold boundary operator isomorphism in the path-space fibration.

By composing with the inverse map if necessary we may insure that $\tilde{B}(\beta_3) = \alpha_7$ and $\tilde{B}'(\alpha_3) = \beta_7$.

Finally, set $\alpha_{8m+k} = \tilde{B}^m_O(\alpha_k)$ for $k \equiv 0, 1, 3, 7 \pmod 8$, and $\beta_{8m+k} = \tilde{B}^m_{Sp}(\beta_k)$ for $k \equiv 3, 4, 5, 7 \pmod 8$. We have $\tilde{B}(\beta_k) = \alpha_{k+4}$ and $\tilde{B}'(\alpha_k) = \beta_{k+4}$.

The following gives relations between these generators.

Lemma 1.1. *For* $m \geq 1$,

(1) $\alpha_{8m} = \alpha_{8m-1} \circ \eta$ *and* $\alpha_{8m+1} = \alpha_{8m} \circ \eta = \alpha_{8m-1} \circ \eta^2$;

(2) $\beta_{8m+4} = \beta_{8m+3} \circ \eta$ *and* $\beta_{8m+5} = \beta_{8m+4} \circ \eta = \beta_{8m+3} \circ \eta^2$.

Proof. Part (1) is due to Kervaire [K, Lemma 2]. For part (2), observe that $\beta_{8m+4} = \tilde{B}'(\alpha_{8m}) = \tilde{B}'(\alpha_{8m-1} \circ \eta) = \tilde{B}'(\alpha_{8m-1}) \circ \eta = \beta_{8m+3} \circ \eta$, and $\beta_{8m+5} = \tilde{B}'(\alpha_{8m+1}) = \tilde{B}'(\alpha_{8m} \circ \eta) = \tilde{B}'(\alpha_{8m}) \circ \eta = \beta_{8m+4} \circ \eta$, since η and η^2 are suspension elements in the homotopy of spheres. ■

We begin by describing $\pi_*(G)$ as a π^S_*-module. If $\theta \in \pi^S_k$ and $\gamma_m \in \pi_m(G)$, choose n large enough that $\theta \in \pi_{k+m+8n}(S^{m+8n}) = \pi^S_k$, and form $\tilde{B}_G^{-n}(\tilde{B}^n_G(\gamma_m) \circ \theta) \in \pi_{m+k}(G)$, where $\tilde{B}^n_G$ is the n-fold iterate of $\tilde{B}_G$. Of course this operation of π^S_* on $\pi_*(G)$ is often trivial. Non-trivial examples of this operation are provided by Lemma 1.1 above, and we give others below.

Of course one might ask about the operation of non-stable homotopy of spheres on $\pi_*(G)$. The answer is provided by the following proposition. Let $E\colon \pi_k(X) \to \pi_{k+1}(EX)$ be the suspension homomorphism.

Proposition 1.2. *If* $\theta \in \pi_n(S^k)$ *and* $\theta \in \operatorname{Ker} E^r$ *for some* $r > 0$, *then* $\gamma_k \circ \theta = 0$.

Proof. Note that $B_{G*}(\gamma_k \circ \theta) = B_{G*}(\gamma_k) \circ \theta = \partial^8(\gamma_{k+8}) \circ \theta = \partial^8(\gamma_{k+8} \circ E^8\theta)$, by Kervaire [K, Lemma 1]. Iterating this, we obtain $(\Omega^{8(n-1)}B_G)_* \circ \cdots \circ B_{G*}(\gamma_k \circ \theta) = \partial^{8n}(\gamma_{k+8n} \circ E^{8n}\theta)$. If $8n \geq r$, then $E^{8n}\theta = 0$. Since the maps $(\Omega^{8k}B_G)_*$ and ∂^{8n} are isomorphisms, we see that $\gamma_k \circ \theta = 0$. ■

It is worth observing that we have an operation of $\theta \in \pi_n(S^k)$ on $\pi_m(G)$ for $m < k$, even if θ does not desuspend. For later use we state the following.

Corollary 1.3. *If* $j\colon S^3 = Sp(1) \to Sp$ *is the inclusion, then* $\operatorname{Ker} E^r \subset \operatorname{Ker} j_*$.

Proof. For $\theta \in \pi_k(S^3)$, we have $\theta = \iota_3 \circ \theta$. Thus if $E^r\theta = 0$, then $j_*(\theta) = j_*(\iota_3) \circ \theta = \beta_3 \circ \theta = 0$. ■

The following limits the degrees of primary homotopy operations one needs to consider in $\pi_*(G)$.

Theorem 1.4. Any non-trivial primary homotopy operation of positive degree in $\pi_*(G)$ is of degree $4t + 1$ or $4t + 2$. In more detail, if m is even when $G = O$

and odd when $G = Sp$, then a non-trivial primary operation must be of the form

(1) $\pi_{4m}(G) \to \pi_{4m+8t+1}(G)$;

(2) $\pi_{4k-1}(G) \to \pi_{4m-1+s}(G)$ where $s = 1, 2$;

(3) any composite operation $\theta = \theta_1 \circ \theta_2$ is of degree $8t + 2$ with θ_i of degree $8t_i + 1$ and acts in dimension $4m - 1$.

Proof. If θ is a primary operation of the form $\pi_k(G) \to \pi_{4p-1}(G)$, then θ is right composition by an element $\theta \in \pi_{4p-1}(S^k)$. By applying the Bott isomorphism $\tilde{B}_G$ n times, we calculate $\gamma_k \circ \theta$ by calculating $\gamma_{k+8n} \circ E^{8n}\theta$. But if θ is of positive degree, then $E^{8n}\theta$ is an element of finite order, so $\gamma_{k+8n} \circ E^{8n}\theta \in \pi_{4p+8n-1}(G) \cong \mathbf{Z}$ is of finite order and therefore 0.

Now suppose that θ is a primary operation $\pi_{4m-1+r}(G) \to \pi_{4m'-1+s}(G)$, where $m \equiv m' \pmod 2$. Since $\tilde{B}_G$ is an isomorphism, $\gamma_{4m-1+r} \circ \theta$ is trivial if and only if $\tilde{B}_G^p(\gamma_{4m-1+r} \circ \theta) = \gamma_{4m+8p-1+r} \circ E^{8p}\theta$ is trivial, and we may assume that $\theta \in \pi^S_{4(m'-m)+s-r}$. Then $\gamma_{4m-1+r} \circ \theta = \gamma_{4m-1} \circ \eta^r \circ \theta = \gamma_{4m-1} \circ \theta \circ \eta^r$. Since θ is of finite order, $\gamma_{4m-1} \circ \theta = 0$ when $r = s$, and since $\pi_{4m'-2}(G) = 0$, we have $\gamma_{4m-1} \circ \theta = 0$ when $r = 2$ and $s = 1$. Thus $\theta \in \pi^S_{4(m'-m)+1}$ where $m - m'$ is even and θ acts in dimension $4m$. This establishes (1), and the only remaining possibility for a non-trivial operation is of the type listed in (2).

For the statement about composite operations, an element $\gamma_k \circ \theta_1 \circ \theta_2$, we must have θ_1 of degree $4t_1 + s_1$ with $s_1 = 1, 2$ for $\gamma_k \circ \theta_1$ to be non-trivial, and then θ_2 must have degree $4t_2 + s_2$ with $s_2 = 1, 2$ for the final composite to be non-trivial. But then $\theta_1 \circ \theta_2$ has degree $4(t_1 + t_2) + s_1 + s_2$, and the only possibility is $s_1 = s_2 = 1$. Consider the commutative diagram

$$\begin{array}{ccc} \pi_{4k-1}(G) & \xrightarrow{\theta_1} & \pi_{4k+4p}(G) \\ {\scriptstyle\theta_2}\downarrow & & {\scriptstyle\theta_2}\downarrow \\ \pi_{4k+8q}(G) & \xrightarrow{\theta_1} & \pi_{4k+4p+8q+1}(G). \end{array}$$

We must have $k + p$ and $k + 2q$ even if $G = O$ and $k + p$ and $k + 2q$ odd if $G = Sp$. This implies that k and p are even if $G = O$ and k is odd and p is even if $G = Sp$. ■

2. Symplectic groups

Our next objective is to discuss some secondary homotopy operations in $\pi_*(Sp)$. For this purpose we now describe the periodic family of elements $\mu_{m,3} \in \pi_{8m+4}(S^3)$. First, $\mu_{0,3} = \eta_3 \in \pi_4(S^3)$. Next, $\mu_{1,3} \in \pi_{12}(S^3)$ is Toda's element [T, pp. 54–58], (which he denotes by μ_3). According to this description, $\mu_{1,3} \in \{\eta_3, E\beta, E^2\gamma\} \subset \pi_{12}(S^3)$ with indeterminacy $\eta_3 \circ E\pi_{11}(S^3)$. Finally, for $m > 1$, we define $\mu_{m,3} \in \{\mu_{m-1,3}, 2\iota, 8\sigma\} \subset \pi_{8m+4}(S^3)$ with indeterminacy $\mu_{m-1,3} \circ \pi^S_8 + \pi_{8m-3}(S^3) \circ (8\sigma)$, and $\pi_{8m+4}(S^3) \circ (8\sigma)$ is of odd order, since $4\pi_*(S^3)$ is of odd order [J, Corollary 1.22]. We recall that $\mu_{m,3}$ is of order 2, generates a direct summand of $\pi_{8m+4}(S^3)$, has e_C-invariant $\frac{1}{2}$ (mod 1), and suspends non-trivially to the stable $(8m+1)$-stem, *i.e.*, $E^{8m}\mu_{m,3} = \mu_m \in \pi^S_{8m+1}$. See Mori [M, p. 72 and Theorem 3.1 (ii)].

From the work of Walker [W], the inclusion $SU(2) \to SU(4m+2)$ is such that $0 \neq h_*(\mu_{m,3}) \in \pi_{8m+4}(SU(4m+2)) = \mathbf{Z}/(4m+2)!$ (and h_* maps the supplementary summand to 0). From the commutative diagram

$$\begin{array}{ccc} \pi_{8m+4}(Sp(1)) & \xrightarrow[\cong]{g'_*} & \pi_{8m+4}(SU(2)) \\ \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle h_*} \\ \pi_{8m+4}(Sp(2m+1)) & \xrightarrow{g_*} & \pi_{8m+4}(SU(4m+2)) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ \mathbf{Z}/2 & & \mathbf{Z}/(4m+2)! \end{array}$$

and the fact that $0 \neq h_* \circ g'_* = g_* \circ j_*$, we see that g_* is monomorphic and j_* is a projection onto a direct summand. Since $\pi_{8m+4}(Sp(2m+1))$ and $\pi_{8m+5}(Sp(2m+1))$ are stable homotopy groups (both isomorphic to $\mathbf{Z}/2$), we have proved the following.

Proposition 2.1. *For* $m \geq 0$,
(1) $j_*(\mu_{m,3}) = \beta_{8m+4}$;
(2) $j_*(\mu_{m,3} \circ \eta) = \beta_{8m+4} \circ \eta = \beta_{8m+5}$;
(3) j_* *is a projection onto a direct summand.* ■

If $j' : Sp(1) \to Sp(2m+1-k)$ is the inclusion map, then $j'_*(\mu_{m,3})$ generates a $\mathbf{Z}/2$ summand of $\pi_{8m+4}(Sp(2m+1-k))$ and $j'_*(\mu_{m,3} \circ \eta)$ generates a $\mathbf{Z}/2$ summand of $\pi_{8m+5}(Sp(2m+1-k))$. Thus we have the following, see [Mo, Proposition 2.4].

Corollary 2.2. *If* $0 \leq m$ *and* $0 < k < 2m+1$, *then*:
(1) $\pi_{8m+4}(Sp(2m+1-k)) \cong \mathbf{Z}/2 \oplus \pi_{8m+5}(Sp/Sp(2m+1-k))$ *with the first summand generated by* $j'_*(\mu_{m,3})$;
(2) $\pi_{8m+5}(Sp(2m+1-k) \cong \mathbf{Z}/2 \oplus \pi_{8m+6}(Sp/Sp(2m+1-k))$ *with the first summand generated by* $j'_*(\mu_{m,3} \circ \eta)$;
(3) $\pi_{8m+r}(Sp(2m+1-k)) \cong \pi_{8m+1+r}(Sp/Sp(2m+1-k))$ *for* $r = 0, 1, 3, 7$;
(4) *the sequence*

$$0 \longrightarrow \mathbf{Z} \longrightarrow \pi_{8m+1+r}(Sp/Sp(2m+1-k)) \longrightarrow \pi_{8m+r}(Sp(2m+1-k)) \longrightarrow 0$$

is exact for $r = 2, 6$. ■

We remark that the groups $\pi_{8m+r}(Sp(2m+1-k))$ are known for $r + k \leq 20$ by the work of several authors (see [L]).

We next prove a lemma on the stable suspensions $\mu_m \in \pi^S_{8m+1}$ of the $\mu_{m,3}$. The proof is analogous to the proof of Toda's Theorem 14.1(v) [T, p. 190], and we use Toda's material on pp. 189–190 and p. 33 without further reference.

Lemma 2.3. *For $m \geq 1$ and $k \geq 0$, we have $\mu_k \circ \mu_m + \mu_{k+m} \circ \eta = 0$.*

Proof. We have $\mu_m \in \langle \mu_{m-1}, 2\iota, 8\sigma \rangle \equiv \langle 2\iota, 8\sigma, \mu_{m-1} \rangle + \langle 8\sigma, \mu_{m-1}, 2\iota \rangle$ (mod $\mu_{m-1} \circ \pi_8^S + \pi_{8m-3}^S \circ (8\sigma)$). Since $0 = 2\langle 2\iota, \mu_{m-1}, 2\iota \rangle$, we have $0 \in \langle 2\iota, \mu_{m-1}, 2\iota \rangle \circ (4\sigma) \subset \langle 2\iota, \mu_{m-1}, 8\sigma \rangle = \langle 8\sigma, \mu_{m-1}, 2\iota \rangle$, and $\mu_m \in \langle 2\iota, 8\sigma, \mu_{m-1} \rangle$. Forming the composition, $\mu_k \circ \mu_m \in \mu_k \circ \langle 2\iota, 8\sigma, \mu_{m-1} \rangle = \langle \mu_k, 2\iota, 8\sigma \rangle \circ \mu_{m-1}$. But $\mu_{k+1} \circ \mu_{m-1} \in \langle \mu_k, 2\iota, 8\sigma \rangle \circ \mu_{m-1}$, and

$$\mu_k \circ \mu_m + \mu_{k+1} \circ \mu_{m-1} \in \mu_k \circ \pi_8^S \circ \mu_{m-1} + \pi_{8k+2}^S \circ (8\sigma) \circ \mu_{m-1} = \mu_k \circ \pi_8^S \circ \mu_{m-1},$$

since $(8\sigma) \circ \mu_{m-1} = 0$. Adding these relations,

$$\mu_k \circ \mu_m + \mu_{k+m} \circ \mu_0 = \sum_{i=0}^{m-1} \mu_{k+i} \circ \mu_{m-i} + \mu_{k+i+1} \circ \mu_{m-i-1} \in \sum_{i=0}^{m-1} \mu_{k+i} \circ \pi_8^S \circ \mu_{m-1-i}.$$

For $m = 1$, this says $\mu_k \circ \mu_1 + \mu_{k+1} \circ \eta \in \mu_k \circ \pi_8^S \circ \eta$. For $m > 1$, we have

$$\begin{aligned} \mu_k \circ \mu_m + \mu_{k+m} \circ \eta &\in \sum_{i=0}^{m-1} \mu_{k+i} \circ \mu_{m-1-i} \circ \pi_8^S \\ &\in \sum_{i=0}^{m-1} \Big(\mu_{k+m-1} \circ \eta + \sum_{j=0}^{m-2-i} \mu_{k+i+j} \circ \pi_8^S \circ \mu_{m-1-i-j}\Big) \circ \pi_8^S \\ &\in \mu_{k+m-1} \circ \eta \circ \pi_8^S, \end{aligned}$$

since $\pi_8^S \circ \pi_8^S = 0$.

Now observe that for $n = 0$, we have $\mu_n \circ \eta \circ \pi_8^S = \eta^2 \circ \pi_8^S$, and $0 = \eta^2 \circ \varepsilon = \eta^2 \circ \bar{\nu} = \eta \circ \nu^3$, while for $n > 0$, we have $\mu_n \circ \eta \circ \pi_8^S \subset \langle \mu_{n-1}, 2\iota, 8\sigma \rangle \circ \eta \circ \pi_8^S \subset \langle \mu_{n-1}, 2\iota, 8\sigma \circ \eta \rangle \circ \pi_8^S = \langle \mu_{n-1}, 2\iota, 0 \rangle \circ 8\pi_8^S = 0$, since $8\sigma \circ \eta = 8(\bar{\nu} + \varepsilon) = 0$. Thus $\mu_k \circ \mu_m = \mu_{k+m} \circ \eta$. ■

We can now give some new non-trivial primary operations in $\pi_*(Sp)$.

Proposition 2.4. *The composition elements $\beta_{8m+4} \circ E^{8m+1}\mu_{k,3}$, $\beta_{8m+3} \circ E^{8m}\mu_{k,3}$, and $\beta_{8m+3} \circ E^{8m}\mu_{k,3} \circ \eta$ are non trivial.*

Proof. By Lemma 2.3, we have $\mu_{m,3} \circ E^{8m+1}\mu_{k,3} = \mu_{m+k,3} \circ \eta + \phi$ and $\mu_{k,3} \circ \eta = \eta \circ E\mu_{k,3} + \phi'$, where ϕ and ϕ' are in the kernel of some iterated suspension. Applying Proposition 2.1 and Corollary 1.3, we see that $\beta_{8m+4} \circ E^{8m+1}\mu_{k,3} = \beta_{8(m+k)+4} \circ \eta = \beta_{8(m+k)+5}$. If we now use Corollary 1.3 again, $\beta_{8m+3} \circ E^{8m}\mu_{k,3} \circ \eta = \beta_{8m+3} \circ \eta \circ E^{8m+1}\mu_{k,3} = \beta_{8(m+k)+4} \circ \eta = \beta_{8(m+k)+5}$. Since right composition by η is an isomorphism $\pi_{8(m+k)+4}(Sp) \xrightarrow{\cong} \pi_{8(m+k)+5}(Sp)$, we have $\beta_{8m+3} \circ E^{8m}\mu_{k,3} = \beta_{8(m+k)+4}$. ■

Remark 2.5. An examination of generators and relations in the stable k-stem for $k \leq 30$ (see [T] [M-T] [M] [M-M-O] [O2]) and application of Theorem 1.4, shows that the only possibilities for non-trivial primary operations of degree ≤ 30 are:

(1) $n\iota$ of degree 0 acting in dimensions $4m - 1$ and in dimensions $8m + 4$ and $8m + 5$ if n is odd;

(2) η of degree 1 acting in dimensions $8m+3$ and $8m+4$;
(3) η^2 of degree 2 acting in dimension $8m+3$;
(4) μ_1 of degree 9 acting in dimensions $8m+3$ and $8m+4$;
(5) $\mu_1 \circ \eta$ of degree 10 acting in dimension $8m+3$;
(6) κ of degree 14 acting in dimension $8m-1$;
(7) μ_2 of degree 17 acting in dimensions $8m+3$ and $8m+4$;
(8) $\mu_2 \circ \eta$ and ν^* of degree 18 acting in dimension $8m+3$;
(9) μ_3 of degree 25 acting in dimensions $8m+3$ and $8m+4$;
(10) $\mu_3 \circ \eta$ of degree 26 acting in dimension $8m+3$;
(11) θ' of degree 30 acting in dimensions $8m-1$.

We do not know whether the action of κ, ν^*, or θ' is trivial.

Next, we compute some secondary operations in $\pi_*(Sp)$.

Lemma 2.6. *For* $m \geq 2$,

$$\mu_{m,3} \in \{\mu_{m-1,3}, 2\iota, 8\sigma\} \subset \{\mu_{m-1,3}, 4\iota, 4\sigma\} \subset \{\mu_{m-1,3}, 8\iota, 2\sigma\} \subset \{\mu_{m-1,3}, 16\iota, \sigma\},$$

with respective indeterminacies $\mu_{m-1,3} \circ \pi_8^S + H$, $\mu_{m-1,3} \circ \pi_8^S + H$, $\mu_{m-1,3} \circ \pi_8^S + \pi_{8m-3}(S^3) \circ (2\sigma)$, *and* $\mu_{m-1,3} \circ \pi_8^S + \pi_{8m-3}(S^3) \circ \sigma$, *where* H *is of odd order.*

Proof. From Toda [T, Proposition 1.2 (ii)] and the fact that $2^q\iota \circ 2^{4-q}\sigma = 2\iota \circ 2^{q-1}\iota \circ 2^{4-q}\sigma$ is null-homotopic for $q = 1, 2, 3, 4$, we get the string of inclusion. Since $4\pi_{8m-3}(S^3)$ is of odd order, for $q = 1, 2$, we have $\pi_{8m-3}(S^3) \circ (2^{4-q}\sigma) = H$ is of odd order. ■

Proposition 2.7. *For* $m \geq 1$, $r = 4, 5$, *and* $q = 1, 2, 3, 4$, *the maps*
(1) $\{-, E\beta, E^2\gamma\}_1 : \pi_4(Sp) \longrightarrow \pi_{12}(Sp)$,
(2) $\{-, 2\iota_5, \sigma'''\} : \pi_5(Sp) \longrightarrow \pi_{13}(Sp)$,
(3) $\{-, 2^q\iota, 2^{4-q}\sigma\}_1 : \pi_{8m+r}(Sp) \longrightarrow \pi_{8(m+1)+r}(Sp)$,
are isomorphisms.

Proof. The indeterminacy of $\{\beta_4, E\beta, E^2\beta\}_1$ is $\beta_4 \circ E\pi_{11}(S^3)$; that of $\{\beta_5, 2\iota_5, \sigma'''\}$ is $\beta_5 \circ \pi_{13}(S^5) + \pi_6(Sp) \circ 2\sigma'' = \beta_5 \circ \pi_{13}(S^5)$; and that of $\{\beta_{8m+r}, 2^q\iota, 2^{4-q}\sigma\}$ is $\beta_{8m+r} \circ \pi_8^S + \pi_{8m+r+1}(Sp) \circ 2^{4-q}\sigma$. By Theorem 1.4 all of these indeterminacies are zero, and the brackets are a single homotopy class.

Now for the inclusion $j: S^3 \to Sp$, we have $\beta_{12} = j_*(\mu_{1,3}) = j_*(\{\eta_3, E\beta, E^2\gamma\}_1) \subset \{\beta_4, E\beta, E^2\gamma\}_1$, and since the indeterminacy is zero, $\beta_{12} = \{\beta_4, E\beta, E^2\gamma\}_1$. Similarly in the other cases. ■

3. The orthogonal groups

In order to study the homotopy of the orthogonal groups we take a more detailed look at the Bott maps in the spirit of [B2] or [D-L]. The method is to define maps $B_n: Sp(n) \to \Omega^4 SO(8n)$ and $B_n': O(n) \to \Omega^4 Sp(2n)$ which are natural with respect to the standard inclusions. Then the diagrams

$$\begin{array}{ccc} Sp(n) & \xrightarrow{B_n} & \Omega^4 SO(8n) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \Omega^4 i} \\ Sp & \xrightarrow{B} & \Omega^4 SO \end{array} \qquad \begin{array}{ccc} O(n) & \xrightarrow{B'_n} & \Omega^4 Sp(2n) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \Omega^4 j} \\ O & \xrightarrow{B'} & \Omega^4 Sp \end{array}$$

are commutative. We set $B_{Sp,n} = \Omega^4 B'_{8n} \circ B_n$ and $B_{O,n} = \Omega^4 B_{2n} \circ B'_n$. We start with the map $B_1 : S^3 = Sp(1) \to \Omega^4 SO(8)$ and observe that

$$\tilde{B}_1(\iota_3) = \partial^{-4} B_{1*}(\iota_3) = \alpha'_{7,8}, \text{ where } i_*(\alpha'_{7,8}) = \alpha_7;$$

$$\tilde{B}_1(\eta_3) = \partial^{-4} B_{1*}(\eta_3) = \alpha'_{8,8}, \text{ where } i_*(\alpha'_{8,8}) = \alpha_8;$$

and

$$\tilde{B}_1(\eta_3^2) = \partial^{-4} B_{1*}(\eta_3^2) = \alpha'_{9,8}, \text{ where } i_*(\alpha'_{9,8}) = \alpha_9.$$

If ∂ is the boundary operator in the homotopy sequence of a fibration, then $\partial(\gamma \circ E\delta) = (\partial\gamma) \circ \delta$, by [K, Lemma 1], and $\partial\{\gamma, E\delta, E\varepsilon\}_1 \subset \{\partial\gamma, \delta, \varepsilon\}$, by [Mi, Proposition 4.2]. In the case of the path-space fibration, ∂ is an isomorphism, and therefore a bijection of $\pi_{k+1}(X) \circ E\pi_r(S^k)$ with $\pi_k(\Omega X) \circ \pi_r(S^k)$. From this, the indeterminacy of $\{\gamma, E\delta, E\varepsilon\}$ is mapped bijectively onto the indeterminacy of $\{\partial\gamma, \delta, \varepsilon\}$, and hence $\partial\{\gamma, E\delta, E\varepsilon\}_1 = \{\partial\gamma, \delta, \varepsilon\} \subset \pi_*(\Omega X)$.

Proposition 3.1. *For $m \geq 1$ the composition elements $\alpha_{8m} \circ E^{8m-3}\mu_{k,3}$, $\alpha_{8m-1} \circ E^{8m-4}\mu_{k,3}$ and $\alpha_{8m-1} \circ E^{8m-4}\mu_{k,3} \circ \eta$ are non-trivial.*

Proof. Applying the map $\tilde{B}$ and using Proposition 2.4, we have $\tilde{B}(\beta_{8m+4} \circ E^{8m+1}\mu_{k,3}) = \partial^{-4}(B_*(\beta_{8m+4}) \circ E^{8m+1}\mu_{k,3}) = \tilde{B}(\beta_{8m+4}) \circ E^{8m+5}\mu_{k,3} = \alpha_{8m+8} \circ E^{8m+5}\mu_{k,3}$. Since $\tilde{B}$ is an isomorphism we have non-triviality of the element $\alpha_{8m+8} \circ E^{8m+5}\mu_{k,3}$. Similarly for the other cases. ■

Lemma 3.2. *If B'' is one of the Bott maps B or B', then $\tilde{B}''\{\gamma, \delta, \varepsilon\} = \{\tilde{B}''(\gamma), E^4\delta, E^4\varepsilon\}_4$, and if B''' is one of the Bott maps B_O or B_{Sp}, then $\tilde{B}'''\{\gamma, \delta, \varepsilon\} = \{\tilde{B}'''(\gamma), E^8\delta, E^8\varepsilon\}_8$.*

Proof. Since the map B'' is a homotopy equivalence, $\tilde{B}''\{\gamma, \delta, \varepsilon\} = \partial^{-4}\tilde{B}''_*\{\gamma, \delta, \varepsilon\} = \partial^{-4}\{B''_*(\gamma), \delta, \varepsilon\} = \{\partial^{-4}B''_*(\gamma), E^4\delta, E^4\varepsilon\}_4 = \{\tilde{B}''(\gamma), E^4\delta, E^4\varepsilon\}_4$. Similarly for B'''. ■

The following gives some non-trivial secondary operations.

Proposition 3.3. *In $\pi_*(SO)$*

(1) $\{-, 2^q\iota, 2^{3-q}E\sigma'\}_4 : \pi_8(SO) \xrightarrow{\cong} \pi_{16}(SO)$ *for* $q = 1, 2, 3$;

(2) $\{-, 2^q\iota, 2^{4-q}\sigma\}_4 : \pi_9(SO) \xrightarrow{\cong} \pi_{17}(SO)$ *for* $k = 9$ *and* $q = 1, 2, 3, 4$;

(3) $\{-, 2^q\iota, 2^{4-q}\sigma\}_4 : \pi_{8m+r}(SO) \xrightarrow{\cong} \pi_{8(m+1)+r}(SO)$ *for* $m \geq 1$ *with* $r = 0, 1$, *and* $q = 1, 2, 3, 4$.

Proof. For (1), we have $\alpha_{16} = \tilde{B}(\beta_{12}) = \tilde{B}\{\beta_4, E\beta, E^2\gamma\}_1 = \{\tilde{B}(\beta_4), E^5\beta, E^6\gamma\}_5 = \{\alpha_8, E^5\beta, E^6\gamma\}_5$. Using [T. Lemma 6.5], we have $\{\alpha_8, E^5\beta, E^6\gamma\}_5 = \{\alpha_7 \circ \eta_7,$

$E^5\beta, E^6\gamma\} \supset \alpha_7 \circ \{E^4\eta_3, E^5\beta, E^6\gamma\} \supset \alpha_7 \circ E^4\{\eta_3, E\beta, E^2\gamma\}_1 = \alpha_7 \circ (\{\eta_7, 2\iota_7, 4E\sigma'\}_3 + \nu_7^3) = \alpha_7 \circ \{\eta_7, 2\iota_7, 4E\sigma'\}_3$, since $\alpha_7 \circ \nu_7 = 0$. Checking the indeterminacy, one sees that this is a single homotopy class and $\alpha_{16} = \alpha_7 \circ \{\eta, 2\iota, 4E\sigma'\}_3 = \alpha_7 \circ E^4\mu_{1,3} = \{\alpha_8, 2\iota, 4\sigma'\}$.

Now as in Lemma 2.6, we have $\alpha_{16} = \{\alpha_8, 2\iota, 4E\sigma'\} \subset \{\alpha_8, 4\iota, 2E\sigma'\} \subset \{\alpha_8, 8\iota, E\sigma'\}$, with respective indeterminacies (for $q=3, 2, 1$) $\alpha_8 \circ \pi_{16}(S^8) + \pi_9(SO) \circ 2^{3-q}E^2\sigma' = \alpha_8 \circ \pi_{16}(S^8) + 2^{4-q}\pi_9(SO) \circ \sigma_9 = \alpha_8 \circ \pi_{16}(S^8)$. But by Theorem 1.4, $\alpha_8 \circ \pi_{16}(S^8) = 0$, so the inclusions are equalities and $\alpha_{16} = \{\alpha_8, 2^q\iota, 2^{3-q}E\sigma'\}$ for $q = 1, 2, 3$.

For parts (2) and (3), just note that $\tilde{B}'(\alpha_{k+8}) = \beta_{k+12} = \{\tilde{B}'(\alpha_k), 2^q\iota, 2^{4-q}\sigma\} = \tilde{B}'\{\alpha_k, 2^q\iota, 2^{4-q}\sigma\}$ and $\tilde{B}'$ is an isomorphism. ■

From the commutative diagram

$$\begin{array}{ccccc} & & \pi_{8m-4+r}(Sp(1)) & \xrightarrow{\tilde{B}_1} & \pi_{8m+r}(SO(8)) \\ & & \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle i_*} \\ \mathbf{Z}/2 & \xrightarrow{\cong} & \pi_{8m-4+r}(Sp) & \underset{\cong}{\xrightarrow{\tilde{B}}} & \pi_{8m+r}(SO) \end{array}$$

and Proposition 2.1 (3), the map j_* is a split epimorphism and we obtain a splitting map for i_*, for $r = 0, 1$ and $m \geq 1$. From the fact that $\pi_8(V_{10,4}) = 0$ [P], we see that $\mathbf{Z}/24 \cong \pi_8(SO(6)) \xrightarrow{i_*} \pi_8(SO) \cong \mathbf{Z}/2$ is onto. Let $\alpha_{8,6}$ generate the 2-component of $\pi_8(SO(6))$, so that $\alpha_{8,6}$ is of order 8 and $i_*(\alpha_{8,6}) = \alpha_8$. Note that for the bundle projection $p: SO(6) \to S^5$, we have $p_*: \pi_8(SO(6)) \xrightarrow{\cong} \pi_8(S^5) \cong \mathbf{Z}/24$, and $\alpha_{8,6}$ can be chosen so that $p_*(\alpha_{8,6}) = \nu_5$, which generates the 2-component of $\pi_8(S^5)$. For the inclusion maps $SO(6) \xrightarrow{i'} SO(7) \xrightarrow{i''} SO(8)$, if $i'_*(\alpha_{8,6}) = \alpha_{8,7}$, we must have an element $\alpha'_{8,8}$ such that $i''_*(\alpha_{7,8}) = \alpha'_{8,8} - s_*p_*(\alpha'_{8,8})$, and the homotopy epimorphism induced by the inclusion $SO(7) \to SO$ splits under the map $\psi(\alpha_8) = i'_*(\alpha_{8,6})$.

Now $i_*(\alpha_{8,6} \circ \eta) = \alpha_8 \circ \eta = \alpha_9$, so i_* is non-trivial, and since $\pi_9(SO(6)) \cong \mathbf{Z}/2$, we see that i_* is a projection (isomorphism) onto a direct summand. Moreover, if $\alpha_{9,6} = \alpha_{8,6} \circ \eta$ then $p_*(\alpha_{9,6}) = \nu_5 \circ \eta_8$, which generates $\pi_9(S^5)$. We have proved the following.

Proposition 3.4. (1) *There is an element* $\alpha_{8,6} \in \pi_8(SO(6))$ *of order* 8 *such that* $i_*(\alpha_{8,6}) = \alpha_8$ *and* $p_*(\alpha_{8,6}) = \nu_5$;

(2) *there is an element* $\alpha_{8,7} \in \pi_8(SO(7))$ *of order* 2 *and* $i_*(\alpha_{8,7}) = \alpha_8$, *the epimorphism* $\pi_8(SO(7)) \to \pi_8(SO)$ *splits*;

(3) *there is an element* $\alpha_{9,6} \in \pi_9(SO(6))$ *of order* 2 *such that* $i_*(\alpha_{9,6}) = \alpha_9$ *and* $p_*(\alpha_{9,6}) = \nu_5 \circ \eta_8$, *the epimorphism* $\pi_9(SO(6)) \to \pi_9(SO)$ *splits.* ■

Remark 3.5. Since $\pi_8(SO(5)) = 0 = \pi_9(SO(5))$, the elements α_8 and α_9 cannot originate in the homotopy of any smaller orthogonal group.

Recall the definition of the infinite family of elements $\zeta_{m,5} \in \pi_{8(m+1)}(S^5)$ [M, p. 72] [T, p. 59]. We set $\zeta_{0,5} = \nu_5$, $\zeta_{1,5} \in \{\nu_5, 8\iota_8, E\sigma'\}_1$, and $\zeta_{m,5} \in \{\zeta_{m-1,5},$

$8\iota_{8m}, 2\sigma_{8m}\}_1$, for $m \geq 2$. The elements $\zeta_{m,5}$ are of order 8 and suspend to stable elements of order 8.

We define an infinite family of elements $\alpha_{8m,6} \in \pi_{8m}(SO(6))$ for $m \geq 1$ by choosing $\alpha_{8,6}$ as above, $\alpha_{16,6} \in \{\alpha_{8,6}, 8\iota, E\sigma'\}_1$, and $\alpha_{8m,6} \in \{\alpha_{8(m-1)}, 8\iota, 2\sigma\}_1$ for $m \geq 2$. One can choose $\alpha_{8m,6}$ so that for the projection $p: SO(6) \to S^5$, we have $p_*(\alpha_{8m,6}) = \zeta_{m-1,5}$, and by Proposition 3.3, $i_*(\alpha_{8m,6}) = \alpha_{8m}$.

Now $8\alpha_{8(m+1),6} = \alpha_{8(m+1),6} \circ 8\iota_{8(m+1)} \in \{\alpha_{8m,6}, \iota_{8m}, 2\sigma_{8m}\}_1 \circ 8\iota_{8(m+1)} = \alpha_{8m,6} \circ \{8\iota_{8m}, 2\sigma_{8m}, 8\iota_{8m+7}\}_1$. But we know $\{8\iota_{8m}, 2\sigma_{8m}, 8\iota_{8m+7}\}_1 = 8\iota_{8m} \circ E\pi_{8m+7}(S^{8m-1}) + \pi_{8m+8}(S^{8m}) \circ 8\iota_{8(m+1)}$ by [T, Corollary 3.7], and we see $\{8\iota_{8m}, 2\sigma_{8m}, 8\iota_{8m+7}\}_1 = 0$ since $2\pi_{k+8}(S^k) = 0$ for $k \geq 6$. This shows the order of $\alpha_{8m,6}$ is ≤ 8 (with a minor modification when $m = 2$). But since $p_*(\alpha_{8m,6}) = \zeta_{m-1,5}$ the order of $\alpha_{8m,6}$ is ≥ 8. Thus $\alpha_{8m,6}$ has order 8.

Next we define a family of elements $\alpha_{8m,7} \in \pi_{8m}(SO(7))$ by $\alpha_{8,7} = i'_*(\alpha_{8,6})$, and $\alpha_{8m,7} \in \{i'_*(\alpha_{8(m-1),6}, 8\iota, 2\sigma\}_1 = \{\alpha_{8(m-1),7}, 8\iota, 2\sigma\}_1$ for $m \geq 2$. Then we see that $2\alpha_{8m,7} \in \{2\alpha_{8(m-1),7}, 8\iota, 2\sigma\}_1$, and one inductively obtains $2\alpha_{8m,7} = 0$. The map $\psi: \pi_{8m}(SO) \to \pi_{8m}(SO(7))$ defined by $\psi(\alpha_{8m}) = \alpha_{8m,7}$ is a splitting map.

Finally, set $\alpha_{8m+1,6} = \alpha_{8m,6} \circ \eta \in \pi_{8m+1}(SO(6))$. Then it follows that $\alpha_{8m+1,6}$ is of order 2, $i_*(\alpha_{8m+1,6}) = \alpha_{8m+1}$, and $p_*(\alpha_{8m+1,6}) = \zeta_{m-1,5} \circ \eta$. The map $\psi(\alpha_{8m+1}) = \alpha_{8m+1,6}$ is a splitting map for $i_*: \pi_{8m+1}(SO(6)) \to \pi_{8m+1}(SO)$.

We collect these definitions and results in the following.

Theorem 3.6. *For $m \geq 1$*

(1) *there is an element $\alpha_{8m,6} \in \pi_{8m}(SO(6))$ of order 8 such that $i_*(\alpha_{8m,6}) = \alpha_{8m}$ and $p_*(\alpha_{8m,6}) = \zeta_{m-1,5} \in \pi_{8m}(S^5)$;*

(2) *there is a generator $\alpha_{8m,7} \in \pi_{8m}(SO(7))$ of order 2 such that $i_*(\alpha_{8m,7}) = \alpha_{8m}$;*

(3) *there is a generator $\alpha_{8m+1,6} \in \pi_{8m+1}(SO(6))$ of order 2 such that $i_*(\alpha_{8m+1,6}) = \alpha_{8m+1}$ and $p_*(\alpha_{8m+1,6}) = \zeta_{m-1,5} \circ \eta$.* ■

If we now use the inclusion maps $SO(6) \to SO(n)$ on these generators we can state the following.

Corollary 3.7. *For $m \geq 1$*

(1) $0 \longrightarrow \pi_{8m+1}(SO/SO(n)) \longrightarrow \pi_{8m}(SO(n)) \longrightarrow \pi_{8m}(SO) \longrightarrow 0$ *is exact for $n \geq 6$ and split exact for $n \geq 7$;*

(2) $0 \longrightarrow \pi_{8m+2}(SO/SO(n)) \longrightarrow \pi_{8m+1}(SO(n)) \longrightarrow \pi_{8m+1}(SO) \longrightarrow 0$ *is split exact for $n \geq 6$.* ■

Remarks 3.8. (1) If $\pi_{8m}(SO(6)) \to \pi_{8m}(SO)$ splits for some m_0, then it splits for all $m \geq m_0$.

(2) If $\pi_{8m+r}(SO(k)) \longrightarrow \pi_{8m+r}(SO)$ is onto (splits) for $k = 3, 4$ or 5, $r = 0$ or 1 and $m = m_0$, then it is onto (splits) for all $m \geq m_0$.

(3) If $\pi_{8m}(SO(k)) \longrightarrow \pi_{8m}(SO)$ is onto (splits) for $k = 3, 4$ or 5 and $m = m_0$, then composition with η shows that $\pi_{8m+1}(SO(k)) \longrightarrow \pi_{8m+1}(SO)$ is onto (splits) for $m \geq m_0$.

(4) One can see that $\pi_{17}(SO(5)) \longrightarrow \pi_{17}(SO(6))$ is trivial. Thus $\pi_{16}(SO(5)) \longrightarrow \pi_{16}(SO)$ is trivial.

(5) We do not know whether $\pi_{8m+r}(SO(5)) \longrightarrow \pi_{8m+r}(SO)$ is an epimorphism for $r=0$ or 1 and $m \geq 3$.

(6) We have a reprint [D-M] confirming that the element α_{8m} is the image of an element in $\pi_{8m}(SO(6))$. This preprint states that $\pi_{8m+r}(SO(5)) \longrightarrow \pi_{8m+r}(SO)$ is trivial for $r=0$ or 1.

University of Colorado
Department of Mathematics, Box 426
Boulder, Colorado 80309

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