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On the birational structure of certain Calabi-Yau threefolds

By

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§0. Introduction

Let X be the fibre product over $\boldsymbol{P}^1$ of two rational elliptic surfaces with sections $\pi: S \to \boldsymbol{P}^1$ and $\lambda: S^+ \to \boldsymbol{P}^1$. Then X is a projective threefold with the trivial first Chern class. We shall assume the following general conditions.

(1.1) Generic fibres of π and λ are not isogenous.

(1.2) All singular fibres of π and λ are of I_1-type.

(1.3) For every point $p \in \boldsymbol{P}^1$, $\pi^{-1}(p)$ or $\lambda^{-1}(p)$ is smooth.

Under these assumptions, it is easily shown that X is nonsingular. In general there are infinitely many (-1)-curves on S and S^+. Hence if we consider the curve $\ell \times_{\boldsymbol{P}^1} m$ on X with ℓ and m (-1)-curves on S and S^+, respectively, then we can find infinitely many $(-1, -1)$-curves (i.e. rational curves of which normal bundles are isomorphic to $\mathcal{O}_{\boldsymbol{P}^1}(-1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(-1)$) on X. Let E denote the divisor $\ell \times_{\boldsymbol{P}^1} S^+$ on X. Then we have infinitely many extremal rays with respect to $K_X + \varepsilon E$ for a rational number ε, $0 < \varepsilon < 1$. In fact, $\ell \times_{\boldsymbol{P}^1} m$'s with m (-1)-curves on S^+ form extremal rays. Moreover $(-1, -1)$-curves have much to do with the birational structure of X. For example, blowing up X along some of mutually disjoint these curves, we can produce a number of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ as exceptional divisors. Blowing down them to other directions, we have a new Moishezon threefold bimeromorphic to X. If we choose suitable $(-1, -1)$-curves, then we can proceed above operations in the category of projective varieties. Hence we have many Calabi-Yau threefolds (i.e. compact Kähler threefolds with $K = 0$ and with the finite fundamental group) birational to X. Conversely, any such threefold can be obtained from X by a similar process called a flop, and these thrreefolds give the decomposition of $\overline{\mathrm{Mov}}(X) \cap \mathrm{Big}(X)$ into many chambers ([1]), where $\mathrm{Mov}(X)$ and $\mathrm{Big}(X)$ denote the movable cone of X and the cone generated by big divisors on X, respectively. In our case, the number of chambers is *infinite*. But there is a possibility that two manifolds obtained by different flops are isomorphic. As a consequence, we have a finite number of Calabi-Yau threefolds birational to X, up to isomorphisms. *The number* is

$$561203476479837773489.$$

Let us explain the contents of this paper. In §1 we shall study movable and big

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line bundles on X. Indeed, a flop in the category of projective varieties is determined by a movable and big line bundle by [1, Theorem 5.3, Theorem 5.7].

By Propositions (1.1), (1.2) we know on what curves we should operate a flop so as to obtain a *projective* threefold. In §2 first we shall determine $\mathrm{Bir}(X)$ and $\mathrm{Aut}(X)$. Next, by applying the theory of Del Pezzo surfaces (i.e. surfaces of which anti-canonical bundles are ample) we shall classify the Calabi-Yau threefolds birational up to X to isomorphisms.

Notation and Conventions

In this paper we shall work over the complex number field $\boldsymbol{C}$.

For a coherent sheaf $\mathfrak{F}$ on a complete variety X, $h^i(X, \mathfrak{F})$ denotes the dimension of the vector space $H^i(X, \mathfrak{F})$. For a nonsingular variety X, we denote by K_X the canonical line bundle of X. Let L be a line bundle on a complete variety X. Then we say that L is movable if the natural map $H^0(X, L) \otimes_{\boldsymbol{C}} \mathcal{O}_X \to L$ is surjective in codimension one, and L is called big if the Iitaka dimension $\kappa(X, L)$ is equal to $\dim X$. If $(L, C) \geq 0$ for every curve C on X, then L is called nef.

§1.

Let S and S^+ be rational elliptic surfaces with sections, and we denote by π and λ, their fibrations over $\boldsymbol{P}^1$, respectively. We shall consider the case where the following are satisfied:

(1.1) Generic fibres of π and λ are not isogenous to each other.

(1.2) All singular fibres of π and λ are of I_1-type.

(1.3) For every point $p \in \boldsymbol{P}^1$, $\pi^{-1}(p)$ or $\lambda^{-1}(p)$ is smooth.

Taking the fibre product of S and S^+ over $\boldsymbol{P}^1$, we obtain the following diagram (1.4):

$$\begin{tikzcd} & S \times_{\boldsymbol{P}^1} S^+ \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "f"] & \\ S \arrow[dr, "\pi"'] & & S^+ \arrow[dl, "\lambda"] \\ & \boldsymbol{P}^1 & \end{tikzcd}$$

By the assumption (1.3), $S \times_{\boldsymbol{P}^1} S^+$ is a nonsingular projective threefold. We denote this threefold by X in the remaining of this section.

Proposition (1.1) *A) Let $\mathscr{L}$ be a movable and big line bundle on X and T a set of irreducible reduced curves having negative interssections with $\mathscr{L}$. Then $T = \{\ell_i \times_{\boldsymbol{P}^1} m_j;\ (i, j) \in \mathfrak{S}\}$ where ℓ_j's (or m_j's) are mutually disjoint (-1)-curves on S (or S^+, resp.), and $\mathfrak{S}$ is the subset of $[1, r] \times [1, k]$ with the following properties:*

1) *There exists a descending chain of sebsets of $[1, k]$ of length p $(p \leq r)$:*

$\{j_1, \dots, j_{k_1}\} \supset \{j_1, \dots, j_{k_2}\} \supset \cdots \supset \{j_1, \dots, j_{k_p}\}$, *where* k_i*'s are positive integers with* $k_1 \geq k_2 \geq \cdots \geq k_p$.

2) *There exists a subset of* $[1, r]$: $\{i_1, \dots, i_p\} \subset [1, r]$.

$$\mathfrak{S} = \{(i_1, j_1), \dots, (i_1, j_{k_1}), (i_2, j_1), \dots, (i_2, j_{k_2}), \cdots (i_p, j_1), \dots, (i_p, j_{k_p})\}.$$

B) Conversely, if we have mutually disjoint curves ℓ_i*'s,* m_j*'s and the set* $\mathfrak{S}$ *with the properties above, there is a movable and big line bundle* $\mathscr{L}$ *described in A).*

Proof of A). Consider the following exact commutative diagram:

$$\begin{array}{ccccccc}
 & & \operatorname{Pic} S \times \operatorname{Pic} S^+ & \longrightarrow & \operatorname{Pic} \pi^{-1}(\eta) \times \operatorname{Pic} \lambda^{-1}(\eta) & \longrightarrow & 0 \\
 & & \big\downarrow {\scriptstyle p^* \otimes q^*} & & \big\downarrow {\scriptstyle p^*(\eta) \otimes q^*(\eta)} & & \\
\boldsymbol{Z} f^*\mathcal{O}(1) & \longrightarrow & \operatorname{Pic} S \times_{\boldsymbol{P}^1} S^+ & \longrightarrow & \operatorname{Pic} f^{-1}(\eta) & \longrightarrow & 0,
\end{array}$$

where η is a generic point of $\boldsymbol{P}^1$. $p^*(\eta) \otimes q^*(\eta)$ is an isomorphism by (1.1). Since $f^*\mathcal{O}(1) \simeq p^*\pi^*\mathcal{O}(1)$, $p^* \otimes q^*$ is surjective. Hence for a line bundle $\mathscr{F}$ on X, we have $\mathscr{F} = p^*L \otimes q^*M$, where L (or M) is a line bundle on S (or S^+, resp.). By Künneth formula, $f_*(\mathscr{F}) \simeq \pi_*L \otimes \lambda_*M$, and if $\mathscr{F}$ is effective, we may assume that L and M are effective. In fact, if $\mathscr{F}$ is effective, then $f_*\mathscr{F}$ has a non-zero section. Since π_*L and λ_*M are vector bundles on $\boldsymbol{P}^1$, they are direct sums of line bundles. Hence we conclude that for a suitable line bundle K on $\boldsymbol{P}^1$, $\pi_*L \otimes K$ and $\lambda_*M \otimes K^{-1}$ have non-zero sections. We can write $\mathscr{F} \simeq p^*(L \otimes \pi^*K) \otimes q^*(M \otimes \lambda^*K^{-1})$. Since $L \otimes \pi^*K$ and $M \otimes \lambda^*K^{-1}$ are effective, we may assume that L and M are effective.

Let us start the proof of A). By the above argument, we can write $\mathscr{L} \simeq p^*L \otimes q^*M$, for effective divisors L and M. Let C be an irreducible reduced curve having a negative intersection with $\mathscr{L}$. Set $p(C) = \ell$, and $q(C) = m$. Then we have $(p^*L \otimes q^*M, C) = [m : \boldsymbol{P}^1](L, \ell) + [\ell : \boldsymbol{P}^1](M, m) < 0$. We may assume that (L, ℓ) is negative. The self intersection number of ℓ is negative because L is effective. The arithemetic genus $P_a(\ell)$ of ℓ can be computed by the adjunction formula; $2P_a(\ell) - 2 = \ell^2 + (\ell, K_S)$. Since S is a rational elliptic surface with sections, $(\ell, K_S) \leq 0$. This we have $P_a(\ell) = 0$ and ℓ is a (-1)-curve or a (-2)-curve. But S contains no (-2)-curves by the assumption (1.2). So ℓ is a (-1)-curve. $|\mathscr{L}|$ has an irreducible element because $\mathscr{L}$ is movable and big. This divisor defines a non-zero effective divisor on $\ell \times_{\boldsymbol{P}^1} S^+$. This shows that m is also a (-1)-curve on S^+. Note that $C = \ell \times_{\boldsymbol{P}^1} m$ in this case.

Suppose that $(\mathscr{L}, \ell^+ \times_{\boldsymbol{P}^1} m^+) < 0$ for other (-1)-curves ℓ^+ and m^+. At first we shall consider the case where $m = m^+$ and $\ell \cap \ell^+ \neq \phi$. An irreducible element of $|\mathscr{L}|$ defines a non-zero effective divisor on $S \times_{\boldsymbol{P}^1} m$; $\mathscr{L}|_{S \times_{\boldsymbol{P}^1} m} = a\ell + b\ell^+ +$ (other effective divisors), where $a, b \geq 0$. If $a \geq b$, then $(\mathscr{L}, \ell^+ \times_{\boldsymbol{P}^1} m) \geq (a\ell + b\ell^+, \ell^+) \geq 0$. This contradicts our assumption. In the case where $a < b$, we have a contradiction in the same manner. Next consider the case where $\ell \neq \ell^+$, $m \neq m^+$ and $\ell \cap \ell^+ \neq \phi$. Then we obtain the following inequalities by the same argument as above:

$$(1.5)\qquad \begin{aligned} &(L, \ell) + (M, m) < 0 \\ &(L, \ell^+) + (M, m) \geq 0 \\ &(L, \ell) + (M, m^+) \geq 0 \\ &(L, \ell^+) + (M, m^+) < 0 \end{aligned}$$

But there are no 4-uples $\{(L, \ell), (L, \ell^+), (M, m), (M, m^+)\}$ satisfying these inequalities. This is a contradiction.

Hence we conclude that $\ell \cap \ell^+ = \phi$ and $m \cap m^+ = \phi$. As a consequence it has been shown that $T = \{\ell_i \times_{\mathbf{P}^1} m_j; (i, j) \in \mathfrak{S}\}$ for mutually disjoint (-1)-curves ℓ_i's on S, and mutually disjoint (-1)-curves m_j's on S^+, where $\mathfrak{S}$ is a certain subset of $[1, r] \times [1, k]$.

To complete the proof, it suffices to show that $\mathfrak{S}$ has the properties in the statement. Suppose to the contrary. Then we observe that for some $i, i^+ \in [1, r]$ and some $j, j^+ \in [1, k]$, (i, j) and (i^+, j^+) belong to $\mathfrak{S}$ but neither (i^+, j) nor (i, j^+) belongs to $\mathfrak{S}$. Say $\mathscr{L} = p^*L \otimes q^*M$. Put $(L, \ell_i) = a$, $(L, \ell_{i^+}) = b$, $(M, m_j) = c$ and $(M, m_{j^+}) = d$. They must satisfy the inequalities similar to (1.5):

$$\begin{aligned} a + c &< 0 \\ a + d &\geq 0 \\ b + c &\geq 0 \\ b + d &< 0 \end{aligned}$$

This is a contradiction. The proof of A) is now completed.

Proof of B). We choose the integers $\sigma_\alpha (1 \leq \alpha \leq k_1)$ and $\tau_\beta (1 \leq \beta \leq p)$ as follows:

$$(1.6)\qquad \sigma_\alpha + \tau_\beta \quad \begin{matrix} < 0 & \text{if } \alpha \leq k_\beta \\ \geq 0 & \text{if } \alpha > k_\beta \end{matrix}$$

For a very ample line bundle H (or H^+) on S (or S^+, resp.), put

$$\begin{aligned} L &= nH + a_1 \ell_{i_1} + \cdots + a_p \ell_{i_p} \\ M &= n^+ H^+ + b_1 m_{j_1} + \cdots + b_{k_1} m_{j_{k_1}}, \end{aligned}$$

where we adjust the coefficients n, n^+, a, $b \in \mathbf{Z}_+$ in such a way $(L, \ell_{i_1}) = \tau_1, \ldots,$ $(L, \ell_{i_p}) = \tau_p$, $(M, m_{j_1}) = \sigma_1, \ldots, (M, m_{j_{k_1}}) = \sigma_{k_1}$. We shall show that $\mathscr{L} = p^*L \otimes q^*M$ is a desired one for sufficiently large n and n^+. In fact it is clear that $\mathscr{L}$ is big. Suppose that $\mathscr{L}$ is not movable. Then fixed components of $|\mathscr{L}|$ are unions of $S \times_{\mathbf{P}^1} m_{j_\alpha}$'s and $\ell_{i_\beta} \times_{\mathbf{P}^1} S^+$'s. For simplicity we may assume that $\ell_i \times_{\mathbf{P}^1} S^+$ is contained in fixed components. Note that i refers to one member of $\{i_1, \ldots, i_p\}$. Then the following diagram is obtained:

$$
(1.7)\qquad \begin{array}{ccc}
H^0(X, p^*(L-\ell_i)\otimes q^*M) & \simeq & H^0(X, p^*L\otimes q^*M) \\
\Big\| & & \Big\| \\
H^0(\boldsymbol{P}^1, \pi_*(L-\ell_i)\otimes \lambda_*M) & \simeq & H^0(\boldsymbol{P}^1, \pi_*L\otimes \lambda_*M)
\end{array}
$$

Here L depends on the positive integer n, but we have

Lemma (1.8). *$h^1(\mathcal{O}(L-\ell_i))$ is bounded above by some constant $A(\tau)$, where $A(\tau)$ depends only on $\tau=(\tau_1,\dots,\tau_p)$.*

Proof of (1.8). By abuse of notation we denote (L, ℓ_i) by τ_i.
(The case where $\tau_i+1\geq 0$)

If n is sufficiently large, $L-\ell_i$ is linearly equivalent to $F+\sum_{i_\beta\neq i,\tau_{i_\beta}<0}(-\tau_{i_\beta})\ell_{i_\beta}$, where F is a nonsingular curve on S and $F\cap\ell_{i_\beta}=\phi$ for every i_β with $i_\beta\neq i$, $\tau_{i_\beta}<0$. Set $E=F+\sum(-\tau_{i_\beta})\ell_{i_\beta}$. Then we obtain

$$H^1(\mathcal{O}_S)\longrightarrow H^1(\mathcal{O}_S(L-\ell_i))\longrightarrow H^1(\mathcal{O}_E(E)).$$

By Serre duality we have $h^1(\mathcal{O}_E(E))=h^0(\mathcal{O}_E(K_S))$. $-K_S$ is linearly equivalent to a fibre of π. Hence it follows that $h^0(F, \mathcal{O}_F(K_S))=0$ and $h^0(\mathcal{O}_E(K_S))=\sum h^0((-\tau_{i_\beta})\ell_{i_\beta}, \mathcal{O}(K_S))$. Therefore, $h^1(\mathcal{O}(L-\ell_i))\leq A(\tau)$ for some constant $A(\tau)$.
(The case where $\tau_i+1<0$)

If n is sufficiently large, $L-\ell_i$ is linearly equivalent to $F+(-1-\tau_i)\ell_i+\sum(-\tau_{i_\beta})\ell_{i_\beta}$, where F is a nonsingular curve with $F\cap\ell_i=\phi$, $F\cap\ell_{i_\beta}=\phi$. The same argument as above shows the assertion in this case, too. Q.E.D.

By the spectral sequence of Leray, we have $h^1(\pi_*\mathcal{O}(L-\ell_i))\leq h^1(\mathcal{O}(L-\ell_i))$. Decomposing $\pi_*(L-\ell_i)$ into a direct sum of line bundles $\mathcal{O}_{\boldsymbol{P}^1}(d_1)\oplus\cdots\oplus\mathcal{O}_{\boldsymbol{P}^1}(d_{k(n)})$, it can be shown by (1.8) that $\min\{d_1,\dots,d_{k(n)}\}\geq B(\tau)$, where $B(\tau)$ is a constant depending only on τ. Since L is π-very ample, $R^1\pi_*\mathcal{O}(L-\ell_i)=0$. It is obvious that $\pi_*\mathcal{O}_{\ell_i}(L)=\mathcal{O}_{\boldsymbol{P}^1}(\tau_i)$. Thus we obtain

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_*\mathcal{O}(L-\ell_i) & \longrightarrow & \pi_*L & \longrightarrow & \pi_*\mathcal{O}_{\ell_i}(L) & \longrightarrow & 0. \\
 & & \Big\| & & & & \Big\| & & \\
 & & \sum_{1\leq j\leq k(n)}\mathcal{O}_{\boldsymbol{P}^1}(d_j) & & & & \mathcal{O}_{\boldsymbol{P}^1}(\tau_i) & &
\end{array}
$$

We can write $\pi_*L=\sum\mathcal{O}(d_j+e_j)\oplus\mathcal{O}(d)$ for some non-negative integers e_j $(1\leq j\leq k(n))$ and for some integer d. If some e_j is positive, it follows that $h^0(\pi_*\mathcal{O}(L-\ell_i)\otimes\lambda_*M)<h^0(\pi_*L\otimes\lambda_*M)$ for $n^+>C(\tau)$, where $M=n^+H^+ +$ (others) and $C(\tau)$ is a constant depending only on τ. On the other hand, if $e_j=0$ for all j, we have $d=\tau_i$. In this case, also the inequality $h^0(\pi_*\mathcal{O}(L-\ell_i)\otimes\lambda_*M)<h^0(\pi_*L\otimes\lambda_*M)$ holds. This contradicts the assumption that $\mathcal{L}$ is not movable. Hence $\mathcal{L}$ is movable.

Finally, let C be an irreducible curve on X with $(C, \mathscr{L}) < 0$. By A) C is $\ell \times_{P^1} m$ for a (-1)-curve ℓ on S and for a (-1)-curve m on S^+. If ℓ does not coincide with any ℓ_i and m does not coincide with any m_j, we have $(C, \mathscr{L}) > 0$ by the construction of $\mathscr{L}$. Thus we may assume that $C = \ell_i \times_{P^1} m$. Then we obtain

$$(C, \mathscr{L}) = (m, (L, \ell_i)(-K_{S^+}) + M).$$

If m does not coincide with any m_j, then we have $(m, M) \geq (m, n^+H^+)$. For sufficiently large n^+, $n^+H^+ + (L, \ell_i)(-K_{S^+})$ is very ample. It is clear that in this case $(m, (L, \ell_i)(-K_{S^+}) + M) > 0$. This completes the proof of B).

By the notion of a *flop*, we can grasp the relationship between two minimal models birational to each other. For the definition of a *flop*, see [1, (4.4), (4.4′)]

Proposition (1.2). *With the notation in Proposition* (1.1) *the strict transform* $\mathscr{L}^+$ *of* $\mathscr{L}$ *becomes nef and big by the flop of all curves which belong to* T.

Proof. It is obvious that $\mathscr{L}^+$ *is big. For an irreducible reduced curve* C *on* X, *let* C^+ *be its strict transform by the flop. It suffices to show that* $(C^+, \mathscr{L}^+) \geq 0$. *We can write* $\mathscr{L} = p^*L \otimes q^*M$ for an effective divisor L on S and for an effective divisor M on S^+. We may assume that C is contained in $p^*L + q^*M$. In fact it is easy to see that $(\mathscr{L}^+, C^+) \geq 0$ for C which is not contained in $p^*L + q^*M$. In the situation of (1.4), we have

Lemma (1.9). *If* $P(C)$ *is contained in the fixed components of* $L^{\otimes n}$ *for all* $n > 0$, *then the inequality* $(L, p(C)) < 0$ *holds.*

Proof. By abuse of notation, let us denote $p(C)$ by C. Suppose we have $(L, C) \geq 0$ under the assumption. We shall derive a contradiction in each case of

a) C is a (-1)-curve;

b) C is not a (-1)-curve.

Case a) First of all, an irreducible reduced divisor $D \subset L$ can be chosen so that $(D + C, C) \geq 0$ holds. In fact, some irreducible reduced divisor $D \subset L$ must intersect with C because $(L, C) \geq 0$. Consider the following exact sequence.

$$0 \longrightarrow H^0(\mathcal{O}(D)) \longrightarrow H^0(\mathcal{O}(D + C)) \longrightarrow H^0(\mathcal{O}_C(D + C)) \longrightarrow H^1(\mathcal{O}(D)).$$

If D is not linearly equivalent to $-K_S$, then $H^1(\mathcal{O}(D))$ must vanish. To show this, we may apply the exact sequence

$$\begin{array}{ccccc} H^1(\mathcal{O}) & \longrightarrow & H^1(\mathcal{O}(D)) & \longrightarrow & H^1(\mathcal{O}_D(D)) \\ \| & & & & \\ 0 & & & & \end{array}$$

and the duality of $H^1(\mathcal{O}_D(D))$ and $H^0(\mathcal{O}_D(K_S))$. In fact since $-K_S$ is linearly equivalent to a fibre of π, $D \neq -K_S$ means $H^0(\mathcal{O}(K_S)) = 0$. Therefore, replacing $D + C$ with a new divisor $F \not\supset C$ one by one if $D \neq -K_S$, we come to one of the following two cases:

1) There is an effective divisor which is linearly equivalent to L and which does not contain C.

2) $L \sim m(-K_S) + nC +$ (other divisors which does not interect with C for $m \geq 0$ and $n > 0$. We derive a contradiction in case 1). In case 2) since $(L, C) = (m(-K_S) + nC, C) \geq 0$, it follows that $m(-K_S) + nC$ is nef and $m \geq n$. We have $(m(-K_S) + nC)^2 > 0$ because $m > n$. Hence $m(-K_S) + nC$ is big by [2, Lemma 3]. Moreover $N(m(-K_S) + nC)$ is free for a sufficiently large integer N by [1, Th 1.3]. This shows that there is an effective divisor linearly equivalent to NL which does not contain C. This is a contradiction. Hence Case a) does not occur. Case b). By (1.2), we have $(C^2) \geq 0$. Note that S is a 9 points blow-up of $\boldsymbol{P}^2$; $\alpha: S \to \boldsymbol{P}^2$. Let us denote the exceptional curves by $E_1, \dots, E_9$. Set $(C, E_i) = k_i \geq 0$ and $\alpha(C) = \tilde{C}$. Then we obtain

$$\alpha^*(\tilde{C}) = C + k_1 E_1 + \cdots + k_9 E_9.$$

If $\mathcal{O}_{\boldsymbol{P}^2}(\tilde{C})$ is isomorphic to $\mathcal{O}_{\boldsymbol{P}^2}(a)$ for some a, then we have

$$a^2 = (C^2) + k_1^2 + \cdots + k_9^2.$$

Since $(C^2) \geq 0$, we obtain

$$a^2 \geq k_1^2 + \cdots + k_9^2.$$

The dimension of the linear space of global sections of $\mathcal{O}_{\boldsymbol{P}^2}(a)$ passing through each $\alpha(E_i)$ with the multiplicity not less than k_i, is at least

$$\begin{aligned} &(1/2\, a^2 + 3/2\, a + 1) - 1/2 \textstyle\sum k_i^2 - 1/2 \sum k_i \\ &= 1/2(a^2 - \textstyle\sum k_i^2) + 1/2(3a - \sum k_i) + 1 \geq 1. \end{aligned}$$

Since each term is not negative, the dimension is possibly *one* only if $a^2 = \sum k^2$ and $k_1 = k_2 = \cdots = k_q$. But this is the case C is a multiple of $-K_S$ and C moves in S. Therefore, we conclude that C moves in S in Case b). Hence we derive a contradiction. Thus Case b) does not occur.

Let us return to the proof of Proposition (1.2). We note here that Lemma (1.9) holds not only for L, but also for M. Hence replacing $\mathscr{L}$ with $\mathscr{L}^{\otimes n}$ for a sufficiently large integer n, we may assume that C is contained in $\ell_i \times_{\boldsymbol{P}^1} S^+$ for some $i \in [1, r]$ or C is contained in $S \times_{\boldsymbol{P}^1} m_j$ for some $j \in [1, k]$, where we take the smallest r and k such that $[1, r] \times [1, k]$ contains $\mathfrak{S}$. In fact, suppose to the contrary. Then $p(C)$ does not coincide with any ℓ_i. If $p(C)$ does not intersect with any ℓ_i, clearly we have $(\mathscr{L}^+, C^+) = (\mathscr{L}, C) \geq 0$. If $p(C)$ intersects with some ℓ_i, then the inequality $(L, p(C)) \geq 0$ holds. Otherwise we have $(L, \ell_i) \geq 0$ and $(M, m_j) < 0$ for some j such that $(i, j) \in \mathfrak{S}$. But in this case the curve $p(C) \times_{\boldsymbol{P}^1} m_j$ has a negative intersection with $\mathscr{L}$. This is a contradiction.

Since $(L, p(C)) \geq 0$, we may assume that L does not contain C by replacing L with $L^{\otimes n}$ and applying Lemma (1.9). The same argument can be used for M. Thus we may assume that $p^*L \otimes q^*M$ does not contain C, and then $(\mathscr{L}^+, C^+) \geq 0$ holds.

Let C be contained in $S \times_{P^1} m_j$. If C does not intersect with any $\ell_i \times_{P^1} m_j$ for $(i, j) \in \mathfrak{S}$, we obtain $(\mathscr{L}^+, C^+) = (\mathscr{L}, C) \geq 0$. So it suffices to consider the case where C intersects with some $\ell_i \times_{P^1} m_j$. The strict transform of $S \times_{P^1} m_j$ by the *flop* is a blow-down of $S \times_{P^1} m_j$, that is, a surface obtained by contractions of (-1)-curves. In this situation the similar argument in case b) of Lemma (1.9) can be applied and it is shown that C^+ moves in the strict transform of $S \times_{P^1} m_j$. Since $\mathscr{L}^+$ is movable, we conclude that $(\mathscr{L}^+, C^+) \geq 0$. The case where C is contained in $\ell_i \times_{P^1} S^+$ is treated in the same way. This completes the proof of Proposition (1.2). Q.E.D.

By proposition (1.1), (1.2) and [1, Th. 5.3, 5.7] for the set T as is defined in Proposition (1.1)A), we can construct, by the flop, a nonsingular projective threefold with trivial canonical bundle birational to X. Conversely, every projective threefold with the property is constructed by this procedure. In general, the threefolds obtained by flops are not isomrophic to the original X.

§2.

In this section we shall consider the birational structure of X (the birational automorphism group, the automorphism group, etc) and classify the isomrophism classes of Calabi-Yau threefolds birational to X.

Proposition (2.1). *With the same notation as in* §1.

A) *the birational automorphism group* Bir(X) *of* X *coincides with the automorphism group* Aut(X) *of* X, *and*

B) *every automorphism of* X *preserves the fibration* $f: X \to \boldsymbol{P}^1$.

Proof of A). Let γ be a birational automorphism of X. Suppose that γ is not an automorphism. Let N be the strict transform of $f^*\mathcal{O}_{P^1}(1)$ by γ. Since every element of $|f^*\mathcal{O}_{P^1}(1)|$ is irreducible, the same is true for $|N|$. As is seen in §1, γ is a flop of a number of curves. Thus these curves have negative intersections with N. We can write $N = p^*L \otimes q^*M$ for some effective divisor L on S and for some effective divisor M on S^+. (See the proof of Prop. (1.1)A).) If neither L nor M is trivial, then $|N|$ has a reducible element. This is a contradiction. If M is trivial, then we have $N = p^*L$ and $|L|$ has no fixed components. In fact, if $|L|$ has a fixed component, then $|N|$ also has. But this is a contradiction by the properties of a flop. On the other hand, if $|L|$ has no fixed components, then we have $(p^*L, C) \geq 0$ for every curve and this is also a contradiction. Q.E.D.

Proof of B) Suppose that $\gamma \in \mathrm{Aut}(X)$ does not preserve the fibration. Then there is a surjective morphism of a general fibre of f to S(or S^+). A general fibre of f is a product of elliptic curves not isogenous to each other. On the other hand, S is a nine points blow-up of $\boldsymbol{P}^2$. This is a contradiction. Q.E.D.

Proposition (2.2). *Let* $\pi: S \to \boldsymbol{P}^1$ *be the same as in* §1. *Then we have* Aut(S)

$= \mathrm{Aut}_\pi(S)$ *for a general* S, *where* $\mathrm{Aut}_\pi(S)$ *is the* π*-automorphism group of* S.

Proof. Consider the following exact sequence of groups:

$$1 \longrightarrow \mathrm{Aut}_\pi(S) \longrightarrow \mathrm{Aut}(S) \longrightarrow \mathrm{Coker} \longrightarrow 1.$$

Let ℓ be a distinguished section of $\pi\colon S \to \boldsymbol{P}^1$. For $\gamma \in \mathrm{Aut}(S)$, $\gamma(\ell)$ is a (-1)-curve on S. There is a π-automorphism of S which sends $\gamma(\ell)$ to ℓ. Hence we can choose the representative of an element of Coker so that it sends ℓ to ℓ. Let γ be such an automorphism of S. Then we have

$$\begin{array}{ccccc}
\boldsymbol{P}(\pi_*\mathcal{O}(3\ell)) & \xrightarrow{\tilde\alpha} & \boldsymbol{P}(\bar\gamma^*\pi_*\mathcal{O}(3\ell)) & \xrightarrow{\tilde\beta} & \boldsymbol{P}(\pi_*\mathcal{O}(3\ell)) \\
\uparrow{\scriptstyle j} & & \uparrow & & \uparrow{\scriptstyle j} \\
S & \xrightarrow{\alpha} & \bar S & \xrightarrow{\beta} & S \\
\downarrow & & \downarrow{\scriptstyle \bar\gamma\times_{\boldsymbol{P}^1}\pi} & & \downarrow{\scriptstyle \pi} \\
\boldsymbol{P}^1 & \xrightarrow{id} & \boldsymbol{P}^1 & \xrightarrow{\bar\gamma} & \boldsymbol{P}^1,
\end{array}$$

where j is a closed immersion and $\gamma = \beta \circ \alpha$. Since $\pi_*\mathcal{O}(3\ell) \simeq \mathcal{O} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-3)$, we denote by Z, X and Y the following natural injections, rspectively:

$$\begin{aligned}
\mathcal{O} \quad &\longrightarrow \mathcal{O} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-3), \\
\mathcal{O}(-2) &\longrightarrow \mathcal{O} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-3), \\
\mathcal{O}(-3) &\longrightarrow \mathcal{O} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-3).
\end{aligned}$$

We can choose these sections in such a way that

$$S = \{(X, Y, Z);\ Y^2Z - (X^3 + aXZ^2 + bZ^3)\grave{}\,0\},$$

where a and b are global sections of $\mathcal{O}_{\boldsymbol{P}^1}(4)$ and $\mathcal{O}_{\boldsymbol{P}^1}(6)$ respectively. Denoting $\bar\gamma^*(X)$, $\bar\gamma^*(Y)$ and $\bar\gamma^*(Z)$ by $\bar X$, $\bar Y$ and $\bar Z$, respectively, we obtain

$$\bar S = \{(\bar X, \bar Y, \bar Z);\ \bar Y^2\bar Z - (\bar X^3 + \bar\gamma^*(a)\bar X\bar Z^2 + \bar\gamma^*(b)\bar Z^3) = 0\}.$$

We can write

$$\tilde\alpha^*(\bar X) = cX + dZ,\ \tilde\alpha^*(\bar Y) = eX + fY + gZ \text{ and } \tilde\alpha^*(\bar Z) = hZ,$$

where c, f and g are non-zero constants, $d \in H^0(\mathcal{O}(2))$, $e \in H^0(\mathcal{O}(1))$, $g \in H^0(\mathcal{O}(3))$. Since $\tilde\alpha$ sends S to $\bar S$, we have $d = e = g = 0$, $a = \bar\gamma^*(a)h^2/c^2$ and $b = \bar\gamma^*(b)h^3/c^3$. This shows that $\bar\gamma$ is the identity map if a and b are general.

Q.E.D.

Corollary (2.3). *Let* $\pi\colon S \to \boldsymbol{P}^1$ *and* $\lambda\colon S^+ \to \boldsymbol{P}^1$ *be the same as in* § 1. *Assume that* S^+ *and* S *are not isomrophic to each other, and are general in the sense of Proposition* (2.2). *Then we have* $\mathrm{Aut}(X) = \mathrm{Aut}_f(X) = \mathrm{Aut}_\pi(S) \times \mathrm{Aut}_\lambda(S^+)$, *where* $\mathrm{Aut}_f(X)$ *is the* f*-automorphism group of* X.

Proof. Let γ be an automorphism of X, and ℓ a (-1) curve on S. Then we have $\gamma(\ell \times_{\mathbf{P}^1} S^+) = \ell^+ \times_{\mathbf{P}^1} S^+$ for some (-1)-curve ℓ^+ on S or $\gamma(\ell \times_{\mathbf{P}^1} S^+) = S \times_{\mathbf{P}^1} m$ for some (-1)-curve m on S^+. In fact, since $\ell \times_{\mathbf{P}^1} S^+$ is an irreducible, isolated divisor on X, it follows from Prop. (2.1), the proof of case b) in Prop. (1.2), that $\gamma(\ell \times_{\mathbf{P}^1} S^+)$ is a pull back of a (-1)-curve on S or on S^+. But $\gamma(\ell \times_{\mathbf{P}^1} S^+) = S \times_{\mathbf{P}^1} m$ does not occur because S is not isomrophic to S^+. Thus $\gamma(\ell \times_{\mathbf{P}^1} S^+) = \ell^+ \times_{\mathbf{P}^1} S^+$. Similarly $\gamma(S \times_{\mathbf{P}^1} m) = S \times_{\mathbf{P}^1} m^+$. Prop. (2.1) B) and Prop. (2.2) implies that $\gamma = \gamma_1 \times \gamma_2$ for some $\gamma_1 \in \mathrm{Aut}_\pi(S)$ and for some $\gamma_2 \in \mathrm{Aut}_\lambda(S^+)$.

We are now in a position to clarify how may different Calabi-Yau manifolds birational to $X = S \times_{\mathbf{P}^1} S^+$ exist. We have remarked in § 1 that every Calabi-Yau manifold birational to X is constructed by a flop of X. In general, they are not isomorphic to each other, but there is some possibility that two manifolds obtained by different flops are isomorphic to each other. To see in what cases such things happen, we may consider the following diagram:

$$\begin{tikzcd} X^+ \arrow[r, "\delta"'] & X^{++} \\ X \arrow[u, dotted, "\gamma_1"] \arrow[r, "\sigma"'] & X \arrow[u, dotted, "\gamma_2"] \end{tikzcd}$$

, where X^+ and X^{++} are threefolds obtained by different flops γ_1 and γ_2, respectively. If there is an isomorphism δ between X^+ and X^{++}, then we have a birational automorphism σ of X satisfying the above commutative diagram. But σ is an automorphism of X by Prop. (2.1) A). As is proved in Prop. (1.1), γ_1 and γ_2 are flops of mutually disjoint rational curves. We denote by T_1(or T_2) the set of such curves with respect to γ_1(or γ_2, resp.). Then σ must send each curve in T_1 to a curve in T_2. Since $\mathrm{Aut}(X) = \mathrm{Aut}_f(X) = \mathrm{Aut}_\pi(S) \times \mathrm{Aut}_\lambda(S^+)$ for general S and S^+ by Colorally (2.3), we can check this condition explicity. Before classifying isomorphism classes of Calabi-Yau threefolds birational to X, we must prove the following lemma.

Lemma (2.4). *Let $\ell_1, \dots, \ell_r$ be mutually disjoint (-1)-curves on S for $r \geq 2$, where S is the same as above. If $\sigma \in \mathrm{Aut}_\pi(S)$ induces a permutation of $\{\ell_1, \dots, \ell_r\}$, then one of the following cases occurs*:

1) *σ is the identity map.*

2) *$r = 2$ and $\sigma: x \to -x + \ell_2$, where ℓ_2, where* $+$ denote the addition with respect to the group structure in which ℓ_1 is the identity.

Proof. Let ℓ_1 be a ditinguished section, and we denote by $+$ and $-$ the addition and the subtraction with respect to this section. If σ is a translation, then we have $\sigma: x \to x - \ell_i$ for the ℓ_i such that $\sigma(\ell_i) = \ell_1$. But in this case we obtain $\sigma(\ell_1) = -\ell_i$. $-\ell_i$ and ℓ_i always have intersections for $i \neq 1$. Hence $\sigma(\ell_i)$ does not coincide with any ℓ_i and this contradicts the assumption of the lemma. If σ is a composition of the involution and a translation, then we have $\sigma: x \to -x + \ell_i (i \geq 2)$. We may assume that $\ell_i = \ell_2$. Since $\sigma^2 = id$, σ is $(\ell_{i_1}, \ell_{i_2})(\ell_{i_3},$

$\ell_{i_4})\cdots(\ ,\ )$ as an element of the permutation group. Note that $\sigma(\ell_1)=\ell_2$ and $\sigma(\ell_2)=\ell_1$. Suppose that r is larger than 2. If $\sigma(\ell_i)=\ell_i$ $(i\geq 3)$, then we have $-\ell_i+\ell_2=\ell_i$. Hence we obtain $\ell_2=2\ell_i$. But $2\ell_i$ intersects with ℓ_1, and this is a contradiction. If $\sigma(\ell_i)=\ell_j$ $(i\neq j$ and $i,\ j\geq 3)$, then we have $\ell_j=-\ell_i+\ell_2$. Hence we obtain $\ell_2=\ell_i+\ell_j$. On the other hand, $-\ell_i$ and ℓ_j have intersections because ℓ_i does not intersect with ℓ_j. Thus $\ell_i+\ell_j$ intersects with ℓ_1. This implies that ℓ_2 intersects with ℓ_1, and this contradicts the assumption of the lemma. Q.E.D.

Let T and $\mathfrak{S}$ be the same as those in Proposition (1.1). Let p_1(or p_2) denote the projection of $[1, r]\times[1, k]$ onto $[1, r]$ (or $[1, k]$, resp.). We shall denote by X^+, the threefold corresponding to T. Writting $n=\#(p_1(\mathfrak{S}))$ and $m=\#(p_2(\mathfrak{S}))$, we call X^+ of type(n, m). From the above considerations, it is clear that two threefolds of different types are not isomorphic. For simplicity, we may assume that $p_1(\mathfrak{S})=\{1,\dots,n\}$ and $p_2(\mathfrak{S})=\{1,\dots,m\}$. We define for $n,\ m\leq 9$
$\boldsymbol{L}:=\{\{\ell_1,\dots,\ell_n\};\ \ell_i\text{'s}$ are mutually disjoint (-1)-curves on $S\}$ modulo Aut(S),
$\boldsymbol{M}:=\{\{m_1,\dots,m_n\};\ m_i\text{'s}$ are mutually disjoint (-1)-curves on $S^+\}$ modulo Aut(S^+),
To comute $\#\boldsymbol{L}$, we may assume that ℓ_1 is equal to a ditinguished section ℓ of S. In the case where $n=1$, $\#\boldsymbol{L}=1$. Suppose that $n\geq 2$. Since the blow-down of ℓ is a Del Pezzo surface, we have $\varphi(8)\cdots\varphi(10-n)/(n-1)!$ ways of choosing the n-uples $\{\ell,\ell_2,\dots,\ell_n\}$ if $2\leq n\leq 8$, and we have $\varphi(8)\cdots\varphi(3)\cdot 2\cdot 1/8!$ ways if $n=9$ by [3, Theorem 26.2], where $\varphi(k)$ denote the number of (-1)-curves on a Del Pezzo surface V with $(K_V)^2=9-k$. Consider the automorphisms of S $\sigma_i\colon x\to x-\ell_i$ and $\tau_i\colon x\to -x+\ell_i$. Then, in the case where $n>2$, we obtain from Lemma (2.4)

$$\begin{aligned}\#\boldsymbol{L}&=\frac{1}{(n-1)!}\,\frac{\varphi(8)\cdots\varphi(10-n)}{2n}\quad \text{if } 2<n\leq 8 \text{ and}\\ &=\frac{1}{8!}\,\frac{\varphi(8)\cdots\varphi(3)\cdot 2\cdot 1}{18}\qquad \text{if } n=9.\end{aligned}$$

In the case where $n=2$, taking the case 2) of Lemma (2.4) into consideration, we have

$$\#\boldsymbol{L}=\frac{\varphi(8)}{2}.$$

The same results also hold for $\#\boldsymbol{M}$. Next we shall consider how many different $\mathfrak{S}$ can be chosen for a fixed element $\{\ell,\ell_2,\dots,\ell_n\}\times\{m,m_2,\dots,m_m\}$ of $\boldsymbol{L}\times\boldsymbol{M}$. Note here that $\mathfrak{S}$ is a subset of $[1, n]\times[1, m]$ with the properties 1), 2), 3) of Prop. (1.1) and that $p_1(\mathfrak{S})=[1, n]$, $p_2(\mathfrak{S})=[1, m]$. An easy claculation shows that the number of such subsets is given by

$$\sum_{r=1}^{\min\{n,m\}}\ \sum_{\substack{n_1+\cdots+n_r=n\\ m_1+\cdots+m_r=m\\ n_i>0,m_j>0}}\frac{n!}{n_1!\cdots n_r!}\,\frac{m!}{m_1!\cdots m_r!}.$$

If $n > 2$ and $m > 2$, then each subset calculated above defines a different Calabi-Yau threefold. In the case where $n = 2$ or $m = 2$, by Lemma (2.4) it is shown that some different subsets define the same Calabi-Yau threefold. Hence the number P_{nm} of isomorphism classes of type (n, m) is given by the following:

$$P_{nm} = P_{mn} = \begin{cases} \displaystyle\sum_{r=1}^{\min\{n,m\}} \sum_{\substack{n_1+\cdots+n_r=n \\ m_1+\cdots+m_r=m \\ n_i>0, m_j>0}} \frac{n!}{n_1!\cdots n_r!} \frac{m!}{m_1!\cdots m_r!} \frac{\psi(n-1)\psi(m-1)}{4mn} & \text{if } n>2 \text{ and } m>2, \\ \displaystyle\sum_{r=1}^{m} \sum_{\substack{n_1+\cdots+n_r=n \\ m_1+\cdots+m_r=m \\ n_i>0, m_j>0}} \frac{n!}{n_1!\cdots n_r!} \frac{\psi(n-1)\psi(m-1)}{2mn} & \text{if } n>2 \text{ and } m \leq 2, \\ \psi(1)^2 & \text{if } n=2 \text{ and } m=2, \\ \psi(1) & \text{if } n=2 \text{ and } m=1, \\ 1 & \text{if } n=1 \text{ and } m=1, \end{cases}$$

where
$$\psi(k) := \frac{1}{k!}\varphi(8)\cdots\varphi(9-k) \quad \text{if } 1 \leq k < 8$$
$$:= \frac{1}{8}\varphi(8)\cdots\varphi(3)\cdot 2\cdot 1 \quad \text{if } k = 8,$$
$$:= 1 \quad \text{if } k = 0.$$

$\varphi(k) :=$ the number of (-1)-curves on a Del Pezzo surface V with $(K_V)^2 = 9 - k$.

k	8	7	6	5	4	3	2	1
$\varphi(k)$	240	56	27	16	10	6	3	1

The birational automorphism group of the threefold of each type is isomorphic to $\mathrm{Bir}(X)$, of which order is infinite. Finally we shall write out the automorphism group of the threefold of each type and concrete values of P_{nm}.

$$\mathrm{Aut}(X^+) = \begin{cases} \boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z} & \text{if } n \leq 2 \text{ and } m \leq 2, \\ \boldsymbol{Z}/2\boldsymbol{Z} & \text{if } n \leq 2 \text{ and } m > 2 \text{ or} \\ & \text{if } n > 2 \text{ and } m \leq 2, \\ \{id\} & \text{otherwise.} \end{cases}$$

$P_{11} = 1$, $P_{21} = 240$, $P_{21} = 57600$

$P_{31} = 1120$, $P_{32} = 940800$, $P_{41} = 7560$, $P_{42} = 13608000$

$P_{51} = 24192$, $P_{52} = 89994240$, $P_{61} = 40320$, $P_{62} = 304819200$,

$P_{71} = 34560$, $P_{72} = 526694400$, $P_{81} = 12960$, $P_{82} = 396576000$,

$P_{91} = 960$, $P_{92} = 58867200$,

$P_{33} = 91571200$,

$P_{43} = 2548627200$,

$P_{53} = 29289738240$,

$P_{63} = 163157299200$,

$P_{73} = 448732569600$,

$P_{83} = 526916275200$,

$P_{93} = 120380467200$,

$P_{44} = 118250798400$,

$P_{54} = 2118066693120$,

$P_{64} = 17603003980800$,

$P_{74} = 70120870041600$,

$P_{84} = 116788221446400$,

$P_{94} = 37275476313600$,

$P_{55} = 55833708478464$,

$P_{65} = 656713214115840$,

$P_{75} = 3600710556549120$,

$P_{85} = 8087394351697920$,

$P_{95} = 3427617028792320$,

$P_{66} = 10546654906368000$,

$P_{76} = 76937694046617600$,

$P_{86} = 225492404652748800$,

$P_{96} = 122850607548211200$,

$P_{77} = 728691787328716800$,

$P_{87} = 2721984964453785600$,

$P_{97} = 1863036528902553600$,

$P_{88} = 12730939089698534400$,

$P_{98} = 10759573724427417600$,

$P_{99} = 11078327049524121600$,

Total $= 56120347647983773489$.

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