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Stochastic differential equations of jump type on manifolds and Lévy flows

By

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§ 1. Introduction

In the previous paper Fujiwara-Kunita [3], we have clarified the structure of $C(\mathbf{R}^d, \mathbf{R}^d)$-Lévy flows, i.e., stochastic processes with values in the semigroup $C(\mathbf{R}^d, \mathbf{R}^d)$ of continuous mappings on $\mathbf{R}^d$ with stationary independent increments. More concretely, we constructed those stochastic flows by some stochastic differential equations of jump type, and conversely when the stochastic flow was given, we represented it as the system of solutions of the same type of stochastic differential equation. In this way, we established a one-to-one correspondence between a general class of $C(\mathbf{R}^d, \mathbf{R}^d)$-Lévy flows and a class of stochastic differential equations which govern the flows.

The main purpose of this paper is to study a similar problem of constructing and characterizing $C(M, M)$-Lévy flows when M is a manifold. In particular, we would like to discuss geometrical aspects of the problem treated in [3]: we are interested in the problem of, first, giving the characteristic quantities which determine a flow, or equivalently giving a stochastic differential equation which governs the flow and, secondly, the problem of solving this class of stochastic differential equations. In section 2, as the characteristic quantities we introduce the notion of 'characteristic systems' for $C(M, M)$-Lévy flows which is an analogue of the one for finite dimensional Lévy processes. It is also considered as a generalization of the one introduced in Le Jan-Watanabe [7] to characterize Diffeo(M)-Brownian flows. We will get more delicate properties of characteristic systems which were unnecessary when M is an Euclidean space.

In section 3, we will discuss the construction problem, that is, given a characteristic system satisfying some regularity conditions, we construct a $C(M, M)$-Lévy flow by solving a stochastic differential equation corresponding to the characteristic system. It is our new idea to introduce the equation. See (3.1) in section 3. The main claim in the section is Theorem 3.1. Let us note that this construction problem is closely related to the problem how we can construct stochastic processes with jumps on a manifold by stochastic differential equation. To study the problem is one of major purposes of this paper. In

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Marcus [8] and [9], he studied it about some special cases. We will show in section 5 that our results in section 3 essentially contain his results. We would like to emphasize that the results of this paper give a unified approach to the problem. In section 4, we will discuss the converse problem of showing that all $C(M, M)$-Lévy flows satisfying some regularity conditions can be actually obtained by the method of section 3. That is, for a given $C(M, M)$-Lévy flow of a general class, there corresponds a characteristic system and the flow can be represented as the sysem of solutions of the stochastic differential equation corresponding to the characteristic system. The main claim in the section is Theorem 4.1.

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§ 2. Preliminaries

In this section, we explain some notions and terminology which we will use in the following sections. In particular, we will introduce the notion of 'characteristic system' for $C(M, M)$-Lévy flow. Although it is a reformulation of characteristics for $C(\mathbf{R}^d, \mathbf{R}^d)$-Lévy flow introduced in Fujiwara-Kunita [3], we give some relation between the components of the characteristic system which induces important geometrical properties. Using it, we will be able to clarify the structure of $C(M, M)$-Lévy flows.

Let M be a d-dimensional compact smooth manifold without boundary. We denote by $C(M, M)$ the space of all continuous mappings from M to itself. It can be considered as a Polish space with respect to the uniform convergence topology. We also denote by $C^r(M)$ $(r = 0, \dots, \infty)$ be the space of C^r-functions on M.

Our main objective is the following $C(M, M)$-valued stochastic process $\{\xi_{s,t};\, s \leq t\}$ defined on a probability space $(\Omega, \mathscr{F}, P)$.

Definition of $C(M, M)$-Lévy flows. (i) $\xi_{s,t} \in C(M, M)$ for each s and t. It is continuous in probability, right continuous and has lefthand limits in t with respect to the topology of $C(M, M)$.

(ii) $\xi_{s,u} = \xi_{t,u} \circ \xi_{s,t}$ P-a.s. for all $s \leq t \leq u$.
($\circ$ denotes the composition of mappings.)

(iii) For all $n \in \mathbf{N} =: \{1, 2, \dots\}$ and $t_1 \leq t_2 \leq \cdots \leq t_n$, $\{\xi_{t_i, t_{i+1}};\, i = 1, \dots, n\}$ are independent random variables.

(iv) $\{\xi_{s,t}\}$ is time homogeneous. That is, for all $u \geq 0$ the law of $\xi_{s+u, t+u}$ is equal to that of $\xi_{s,t}$.

Further, if $\{\xi_{s,t}\}$ is continuous in probability, right continuous and has lefthand limits also in s, then we call it a $C(M, M)$-Lévy flow in strong sense.

In the case of $M = \mathbf{R}^d$, we discussed in detail about $C(M, M)$-and $C^r(M, M)$-Lévy flows in [3].

The characteristic system consists of three quantities, $\langle \ , \ \rangle$, μ, and $\mathscr{L}$. In the following, we give the definition of them and a relation between them.

Definition of characteristic system ($\langle\ ,\ \rangle, \mu, \mathscr{L}$). 1) $\langle\ ,\ \rangle$ is a bilinear map on $C^\infty(M) \times C^\infty(M)$ with values in $C^0(M \times M)$, and it has the following properties.

a) $\langle f, g\rangle(p, q) = \langle g, f\rangle(q, p)$ for all $f, g \in C^\infty(M)$ and $p, q \in M$

b) $\langle f_1 f_2, g\rangle(p, q) = f_1(p)\langle f_2, g\rangle(p, q) + f_2(p)\langle f_1, g\rangle(p, q)$

for all $f_1, f_2, g \in C^\infty(M)$ and $p, q \in M$.

c) $\sum_{i,j=1}^{m} \langle f_i, f_j\rangle(p_i, p_j) \geq 0,$

for all $m \in \mathbf{N}$, $f_i \in C^\infty(M)$, and $p_i \in M$.

2) μ is a Borel measure on $C(M, M)$ satisfying the following properties.

a) $\mu(\{e\}) = 0$, where $e =$ the identity map on M.

b) there exists a sequence of Borel measurable sets $\{U^n; n \in \mathbf{N}\}$ satisfying $U^1 \subset U^2 \subset \cdots \uparrow (C(M, M)\backslash\{e\})$ and $\mu(U^n) < \infty$ for each $n \in \mathbf{N}$.

c) $\int_{C(M,M)} |f(v(p)) - f(p)|^2 \mu(dv) < \infty$ for each $f \in C^\infty(M)$ and $p \in M$.

3) $\mathscr{L}: C^\infty(M) \to C^0(M)$ is a linear map.

4) ($\langle\ ,\ \rangle, \mu, \mathscr{L}$) satisfies the relation: for all $f, g \in C^\infty(M)$ and $p \in M$

$$\mathscr{L}(fg)(p) - f(p)\mathscr{L}g(p) - g(p)\mathscr{L}f(p)$$

$$= \langle f, g\rangle(p, p) + \int_{C(M,M)} \{f(v(p)) - f(p)\}\{g(v(p)) - g(p)\}\mu(dv).$$

In particular, if $\mu = 0$, then the characteristic system is nothing but the local characteristic system (L. C. -system) introduced in Le Jan-Watanabe [7].

To clarify the geometrical properties of characteristic system, we assume the following regularity condition.

(A, I): $\langle f, g\rangle \in C^2(M \times M)$ for each $f, g \in C^\infty(M)$.

Let U be a Borel set such that $\mu(U^c) < \infty$. U^c denotes the complement of U.

(A, II_r)(U) ($r \geq 2$): for some embedding map $\iota: M \to \mathbf{R}^N$, there exists a constant $K > 0$ such that

(i) $$\sup_{p \in M} \int_U |\iota(v(p)) - \iota(p)|^2 \mu(dv) \leq K < \infty,$$

(ii) $$\int_U |\iota(v(p)) - \iota(p) - \{\iota(v(q)) - \iota(q)\}|^{r'} \mu(dv) \leq K|\iota(p) - \iota(q)|^{r'},$$

for all $r' \in [2, r]$ and $p, q \in M$, (A, III)(U): $\mathscr{L}_U f \in C^1(M)$ for $f \in C^\infty(M)$,

$$\text{where } \mathscr{L}_U f(p) = \mathscr{L} f(p) - \int_{U^c} \{f(v(p)) - f(p)\} \mu(dv).$$

In the case of $U = C(M, M)$, we omit (U).

Remark 2.1. The condition (A, II_r)(U) does not depend on the choice of embeddings because M is compact.

Furthermore, we introduce a system of smooth functions Ψ^k and smooth vector fields Z_k $\{\Psi^k, Z_k; k = 1, \dots, m\}$ for some m, which satisfies the relation:

$$\sum_{k=1}^{m} X(\Psi^k) Z_k = X \tag{2.1}$$

for all $X \in \mathfrak{X}(M) =:$ the space of all smooth vector fields on M.

We also denote by $\mathfrak{X}^r(M)$ the space of all C^r-vector fields on M with the topology of uniform convergence upto the r-th derivatives.

In the case of $M = S^{d-1}$(the unit sphere at the origin in $\mathbf{R}^d$), for example, we can take

$$m = d, \ \Psi^k(x) = x^k|_{S^{d-1}}, \text{ and } Z_k = \sum_{i=1}^{d} (\delta_{ik} - x^i x^k) \frac{\partial}{\partial x^i},$$

where $(x^1, \dots, x^d)$ is the standard coordinate of $\mathbf{R}^d$ and δ_{ik} denotes Kronecker's delta.

We will see in the later that there exists such system for sufficiently large m. Associated with the system $\{\Psi^k, Z_k\}$, we can define an operator L by the relation:

$$\mathscr{L} f(p) = L f(p) + \int_U \{f(v(p)) - f(p) - \sum_{k=1}^{m} \{\Psi^k(v(p)) - \Psi^k(p)\} Z_k f(p)\} \mu(dv) \tag{2.2}$$

$$+ \int_{U^c} \{f(v(p)) - f(p)\} \mu(dv).$$

In fact, the second part of right hand side is well-defined by the integrability condition for μ and the relation (2.1). See the proof of Lemma 3.2 below. By noting the derivation property of $\{Z_k\}$, it is easy to see that ($\langle \ , \ \rangle$, L) is an L. C-system in the sense of [7]. Hence, by Collorary [7] (p. 310), L can be decomposed into $L_0 + B$, where L_0 is a second order differential operator on M defined by the Collorary in [7] and B is a continuous vector field on M satisfying

$$|\iota_*(B)(x) - \iota_*(B)(y)| \le K|x - y| \qquad \text{for all} \quad x, y \in \iota(M).$$

($*$ denotes the differential of maps.) This Lipschitz continuity of B follows from the assumptions for the characteristic system. If we assume stronger regularity condition, we can get the smoothness of B.

By the above discussion, we can see that $\mathscr{L}$ is an integro-differential opertor on M which is represented by (2.2), where $Lf(p) = L_0 f(p) + Bf(p)$. This concrete representation will be applied in the next section to construct $C(M, M)$-Lévy flows.

Remark 2.2. We should note that the representation of $\mathscr{L}$ is not uniquely determined. In fact, for any other system satisfying (2.1), it is seen by the same reason as above that $\mathscr{L}$ is represented as the same form of integro-differential operator as (2.2).

Remark 2.3. Conversely speaking, the relation (2.2) indicates the existence of characteristic systems. Indeed, for any L. C.-system $(\langle\ ,\ \rangle, L)$ and μ satisfying (i) of $(\mathrm{A}, \mathrm{II}_2)(U)$, define $\mathscr{L}$ by (2.2). Then, $(\langle\ ,\ \rangle, \mu, \mathscr{L})$ becomes a characteristic system.

In order to express stochastic differential equations in the following sections, we prepare Brownian motions with values in the space of vector fields on M and stochastic integrals based on them. See [7] for more general and rigorous discussion.

Let $M(\ , t)$ be an $\mathfrak{X}^0(M)$-valued Brownian motion with mean 0 defined on $(\Omega, \mathscr{F}, P)$. That is, it is a stationary continuous stochastic process with values in $\mathfrak{X}^0(M)$, having independent increments, and satisfying $E[Mf(p, t)] = 0$ for each $f \in C^\infty(M)$, $p \in M$, where $Mf(p, t) = M(p, t)f$. We denotes by $\langle\ ,\ \rangle$ the covariance. That is, $E[Mf(p, t)Mg(q, s)] = t \wedge s\langle f, g\rangle(p, q)$ for all $f, g \in C^\infty(M)$ and $s, t \in [0, \infty)$. Then it is easy to see that $\langle\ ,\ \rangle$ satisfies the conditions a) $\sim$ c) in 1). Here, we give the definition of stochastic integrals based on $M(\ , t)$, which we will use in the following sections.

Assume that $\langle\ ,\ \rangle$ satisfies (A, I). Let $\phi_{s,t}$ and $\psi_{s,t}$ be M-valued $\{\mathscr{F}_{s,t}\}$-adapted simple processes, where $\{\mathscr{F}_{s,t}; s \le t\}$ is an additive class of σ-fields such that $\mathscr{F}_{s,t}$ contains $\sigma[M(\ , v) - M(\ , u); s \le u \le v \le t]$. That is, for some partition $\Delta: s = t_0 < t_1 < \cdots \to \infty$, $\phi_{s,t} = \phi_{s,t_i}$ if $t \in [t_i, t_{i+1})$. For this process $\phi_{s,t}$, define

$$\int_s^t Mf(\phi_{s,u}, du) = \sum_{i=1}^{\infty} \{Mf(\phi_{s,t_i}, t_{i+1} \wedge t) - Mf(\phi_{s,t_i}, t_i \wedge t)\}.$$

Using the relation:

$$\left\langle \int_s^t Mf(\phi_{s,u}, du), \int_s^t Mf(\psi_{s,u}, du) \right\rangle = \int_s^t \langle f, f\rangle(\phi_{s,u}, \psi_{s,u})\, du, \tag{2.3}$$

we can extend (2.3) to processes $\{\phi_{s,t}\}$ which are left continuous in t.

N. B. In the following sections, we will often use K as an arbitrary positive constant, whose value could change from line to line.

§3. Construction of $C(M, M)$-Lévy flows

In this section, we will show main results for the construction problem of

$C(M, M)$-Lévy flows. Although we will construct them by some stochastic integral equations of jump type as we did in Fujiwara-Kunita [3], we have to devise the relation between the quantities which determine the stochastic integral equation so that the solutions can never leave the manifold M. For the purpose, we introduce the stochastic integral equation (cf. (3.1) below) associated with the characteristic system defined in the previous section. Such a device was not necessary in the case of $\mathbf{R}^d$. Also, we should note that the characteristic system is a set of intrinsic quantities of $C(M, M)$-Lévy flows which will be constructed. See Theorem 3.1 and Theorem 4.1 in the next section. We will use the same notations as in §2.

Our main theorem in this section is as follows. We follow the notations of Ikeda-Watanabe [5] Chapter II with respect to stochastic integrals based on Poisson point processes.

Theorem 3.1. *Let* $M(\ \ ,t)$ *be an* $\mathfrak{X}^0(M)$*-valued Brownian motion defined on* $(\Omega, \mathscr{F}, P)$ *with mean* 0 *and the covariance* $\langle\ \ ,\ \ \rangle$ *satisfying* (A, I). *Let* $\{q(t)\}$ *be a stationary Poisson point process on* $C(M, M)$ *defined on* $(\Omega, \mathscr{F}, P)$ *with the intensity measure* μ *satisfying* (A, II$_r$) *for* $r > 3(2d+1)+4$. Let $\mathscr{L}: C^\infty(M) \to C^0(M)$ *be a linear map satisfying* (A, III). *Suppose that the triple* $(\langle\ \ ,\ \ \rangle, \mu, \mathscr{L})$ *is a characteristic system. Then, the system of solutions* $\{\xi_{s,t}(p);\ s \le t,\ p \in M\}$ *of the following equation* (3.1) *constructs a* $C(M, M)$*-Lévy flow in strong sense.*

$$(3.1)\qquad f(\xi_{s,t}(p)) = f(p) + \int_s^t Mf(\xi_{s,u-}(p), du) + \int_s^t \mathscr{L}f(\xi_{s,u-}(p))du + \int_s^{t+}\int_{C(M,M)} \{f(v(\xi_{s,u-}(p))) - f(\xi_{s,u-}(p))\}\tilde{N}_q(dudv),$$

for all $f \in C^\infty(M)$ *and* $p \in M$.

Moreover, $\{\xi_{s,t}\}$ *satisfies the following conditions*:

(ξ, I): *for all* $f, g \in C^\infty(M)$ *and* $p, q \in M$, *there exists the limit*

$$\lim_{t \downarrow s} \frac{1}{(t-s)} E[\{f(\xi_{s,t}(p)) - f(p)\}\{g(\xi_{s,t}(q)) - g(q)\}].$$

In fact, it is equal to

$$\langle f, g\rangle(p, q) + \int_{C(M,M)} \{f(v(p)) - f(p)\}\{g(v(q)) - f(q)\}\mu(dv).$$

(ξ, II): *for all* $f \in C^\infty(M)$ *and* $p \in M$, *there exists the limit*

$$\lim_{t \downarrow s} \frac{1}{(t-s)} E[f(\xi_{s,t}(p)) - f(p)].$$

In fact, it is equal to $\mathscr{L}f(p)$.

(ξ, III): *for some embedding* $\iota: M \to \mathbf{R}^N$, *there exists a constant* $K > 0$ *such that for all* $p, q \in M$ *and* $r' \in [2, r]$,

(i) $E[|\iota(\xi_{s,t}(p)) - \iota(p) - \{\iota(\xi_{s,t}(q)) - \iota(q)\}|^{r'}] \leq K(t-s)|\iota(p) - \iota(q)|^{r'}$,

(ii) $E[|\iota(\xi_{s,t}(p)) - \iota(p)|^{r'}] \leq K(t-s)$,

(iii) $|E[\iota(\xi_{s,t}(p)) - \iota(p)) - \{\iota(\xi_{s,t}(q)) - \iota(q)\}]| \leq K(t-s)|\iota(p) - \iota(q)|$.

Remark 3.1. The meaning of the solution of (3.1) is that for each $s \geq 0$ and $p \in M$ there exists $\{\mathscr{F}_{s,t}\}$-adapted M-valued process which is right continuous, has lefthand limits with respect to t and satisfies (3.1) for all $f \in C^\infty(M)$, where $\mathscr{F}_{s,t} = \sigma[M(\ \ , v) - M(\ \ , u), N_q((u, v], A); s \leq u \leq v \leq t, A \in \mathscr{B}(C(M, M))]$.

We first prove the following propositions which are weaker versions of Theorem 3.1.

Proposition 3.2. *Let* $M(\ \ , t)$, $\{q(t)\}$, *and the characteristic system* $(\langle\ \ ,\ \ \rangle, \mu, \mathscr{L})$ *be the same as those in Theorem* 3.1 *except for their regularity conditions. Suppose that* $\langle\ \ ,\ \ \rangle$, μ, *and* $\mathscr{L}$ *satisfy* (A, I), (A, II_2)(U), *and* (A, III)(U), *respectively. Then, the solution* $\{\xi_{s,t}(p); s \leq t\}$ *of* (3.1) *exists uniquely for each initial data* $(s, p) \in [0, \infty) \times M$.

Proposition 3.3. *Let* $M(\ \ , t)$, $\{q(t)\}$, *and the characteristic system* $(\langle\ \ ,\ \ \rangle, \mu, \mathscr{L})$ *be the same as in Proposition* 3.2. *Moreover, suppose that* μ *satisfies* (A, II_r)(U) *for* $r > 2d + 1$. *Then, the system* $\{\xi_{s,t}(p); s \leq t, p \in M\}$ *of solutions of* (3.1) *constructs a* $C(M, M)$*-Lévy flow.*

Before proceeding to the proof, we show the outline of it and give a few preparation.

Since $\mathscr{L}$ can be decomposed into the sum of differential opertor and integral operator (2.2) as we saw in §2, the equation (3.1) is rewritten as follows:

$$
\begin{aligned}
(3.2)\quad f(\xi_{s,t}(p)) &= f(p) + \int_s^t Mf(\xi_{s,u-}(p), du) + \int_s^t Lf(\xi_{s,u-}(p))\, du \\
&+ \int_s^t \int_U \{f(v \circ \xi_{s,u-}(p)) - f(\xi_{s,u-}(p)) - \sum_{k=1}^m \{\Psi^k(v \circ \xi_{s,u-}(p)) - \Psi^k(\xi_{s,u-}(p))\} \\
&\qquad \times Z_k f(\xi_{s,u-}(p))\}\, du\mu(dv) \\
&+ \int_s^{t+} \int_U \{f(v \circ \xi_{s,u-}(p)) - f(\xi_{s,u-}(p))\} \tilde{N}_q(dudv) \\
&+ \int_s^{t+} \int_{U^c} \{f(v \circ \xi_{s,u-}(p)) - f(\xi_{s,u-}(p))\} N_q(dudv),
\end{aligned}
$$

where $\{\Psi^k, Z_k; k = 1, \ldots, m\}$ is a system satisfying (2.1).

Since the embedding map ι is a diffeomorphism from M onto $\iota(M)$, the equation (3.2) on M is transfered to an equivalent equation on $\iota(M)$. We next extend this equation defined on the submanifold $\iota(M)$ of $\mathbf{R}^N$ (by Whitney's theorem, we can take $N = 2d + 1$) to that defined on the whole space $\mathbf{R}^N$. At the time, we should note that we have to extend the Brownian motion on the space of vector fields on M and the Poisson point process on $C(M, M)$ to the ones defined on $\mathbf{R}^N$ and $C(\mathbf{R}^N, \mathbf{R}^N)$, respectively, in common and by the method which does not destroy the measurability. We can carry out this program with the help of the tubular neighborhood theorem. See e.g. Franks [2] or Hirsh [4]. Therefore we can get the equation (3.3) on $\mathbf{R}^N$, which will be denoted later on, and it has the unique solution. Further, it will be shown that the solution can never leave $\iota(M)$ if the initial position is on $\iota(M)$. See Lemma 3.4. It is a key part of the proof of the above proposition. The pullback of this restricted solution on $\iota(M)$ to M by ι gives the solution of (3.2) or equivalently (3.1), and we can see that the solutions define a $C(M, M)$-Lévy flow by the argument similar to [3] and by noting the submanifold structure.

We will use the following techniques of extension in the proof of Proposition 3.2.

Let us fix an embedding $\iota: M \to \mathbf{R}^N$. Since $\iota(M)$ is a closed submanifold of $\mathbf{R}^N$, by the tubular neighborhood theorem, there exist an open submanifold $V_0(M)$ which contains $\iota(M)$ and a smooth map $\pi_0: V_0(M) \to \iota(M)$ such that $\pi_0|_{\iota(M)} =$ the identity map on $\iota(M)$. Moreover, since $\iota(M)$ is compact, there exists smaller tubular neighborhood $V(M)$ such that $\overline{V(M)} \subset V_0(M)$. Let h' be a smooth function on $\mathbf{R}^N$ such that $\mathrm{Supp}[h'] \subset V_0(M)$ and $h' \equiv 1$ on $V(M)$, and define a smooth map $\pi: \mathbf{R}^N \to \mathbf{R}^N$ by $\pi(x) = h'(x)\pi_0(x)$ if $x \in V_0(M)$, $= 0$ if $x \in (\mathrm{Supp}[h'])^c$. Then $\pi \in C_0^\infty(\mathbf{R}^N, \mathbf{R}^N)$ and $\pi(x) = \pi_0(x)$ if $x \in V(M)$. Let h be a smooth function such that $0 \le h \le 1$, $\mathrm{Supp}[h] \subset V(M)$, and $h = 1$ near $\iota(M)$. Now for $f \in C^r(M)$ and $v \in C^r(M, M)$ $(0 \le r \le \infty)$, put $\bar{f}(x) = h(x) f \circ \iota^{-1} \circ \pi(x)$ and $\bar{v}(x) = h(x) v \circ \iota^{-1} \circ \pi(x)$, $x \in \mathbf{R}^N$. Next, for $X \in \mathfrak{X}^r(M)$, put $\bar{X}^k(x) = h(x) X(x^k \circ \iota)(\iota^{-1} \circ \pi(x))$, where $(x^1, \dots, x^N)$ is the standard coordinate of $\mathbf{R}^N$. Then, $\bar{X} = \sum_{k=1}^N \bar{X}^k \frac{\partial}{\partial x^k} \in \mathfrak{X}^r(\mathbf{R}^N)$ and we can see that $\bar{X}\bar{f}(\iota(p)) = Xf(p)$ for $p \in M$ and $f \in C^1(M)$.

Next, we extend the covariance $\langle \ , \ \rangle$ and $L = L_0 + B$. Recall the results in §2 about the decomposition of L. Put

$$\bar{A}^{ij}(x, y) = h(x)h(y)\langle x^i \circ \iota, x^j \circ \iota \rangle (\iota^{-1} \circ \pi(x), \iota^{-1} \circ \pi(y))$$

and

$$\bar{L} = (1/2) \sum_{i,j=1}^N \bar{A}^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + h(x) \sum_{i=1}^N \overline{L_0(x^i \circ \iota)}\, (\iota^{-1} \circ \pi(x)) \frac{\partial}{\partial x^i} + \sum_{i=1}^N \bar{B}^i(x) \frac{\partial}{\partial x^i} \qquad \text{for } x, y \in \mathbf{R}^N,$$

where we set $\bar{A}^{ij}(x) = \bar{A}^{ij}(x, x)$.

Let $M(\ , t)$ be an $\mathfrak{X}(M)$-valued Brownian motion with mean 0 and the covariance $\langle \ , \ \rangle$ and $\bar{M}(\ , t)$ be the extension of it to $\mathbf{R}^N$, i.e. $\bar{M}^k(x, t) = h(x) M(x^k \circ \iota)(\iota^{-1} \circ \pi(x), t)$ $(k = 1, 2, \dots, N)$.

Then it holds that $\langle \bar{M}^i(x, t), \bar{M}^j(y, t) \rangle = t\bar{A}^{ij}(x, y)$. Moreover, it can be verified that $\bar{L}\bar{f}(\iota(p)) = Lf(p)$ for $f \in C^\infty(M)$ and $p \in M$.

Here, we answer the question of the existence of the system $\{\Psi^k, Z_k; k = 1, \dots, m\}$ satisfying the relation (2.1). Let ι and $\bar{f}$ for $f \in C^\infty(M)$ be as above. Take $m = N$ and define $\Psi^k(p) = x^k(\iota(p))$, $Z_k f(p) = \frac{\partial \bar{f}}{\partial x^k}(\iota(p))$ for $p \in M$. Then, $\{Z_k; k = 1, 2, \dots, N\}$ is a system of smooth vector fields on M and $\{\Psi^k, Z_k; k = 1, \dots, N\}$ satisfies (2.1).

We now proceed into the proof of Proposition 3.2 and 3.3 with the above preparation. Associated with the stochastic integral equation (3.2) on M, we consider the following one (3.3) on $\mathbf{R}^N$:

$$
\begin{aligned}
(3.3) \qquad \eta_{s,t}(x) = x &+ \int_s^t \bar{M}(\eta_{s,u-}(x), du) \\
&+ \int_s^t \{ \overline{L_0(x \circ \iota)}\,(\eta_{s,u-}(x)) + \bar{B}(\eta_{s,u-}(x))\}\, du \\
&+ \int_s^t \int_U \{\bar{v}(\eta_{s,u-}(x)) - \bar{e}(\eta_{s,u-}(x)) - \overline{\Psi(v)}\,(\eta_{s,u-}(x))\}\mu(dv)\, du \\
&+ \int_s^{t+} \int_U \{\bar{v}(\eta_{s,u-}(x)) - \bar{e}(\eta_{s,u-}(x))\}\tilde{N}_q(dudv) \\
&+ \int_s^{t+} \int_{U^c} \{\bar{v}(\eta_{s,u-}(x)) - \bar{e}(\eta_{s,u-}(x))\}N_q(dudv),
\end{aligned}
$$

where we set $\Psi(v)(p) = \sum_{k=1}^m \{\Psi^k(v(p)) - \Psi^k(p)\}Z_k(p) \in T_pM$.

Lemma 3.1. *The condition* (A, II$_r$)(U) *for* $r \geq 2$ *yields the followings.*

(i) $$\int_U |\bar{v}(x) - \bar{e}(x)|^{r'} \mu(dv) \leq K \text{ for all } x \in \mathbf{R}^N \text{ and } r' \in [2, r].$$

(ii) $$\int_U |\bar{v}(x) - \bar{e}(x) - \{\bar{v}(y) - \bar{e}(y)\}|^{r'} \mu(dv) \leq K|x - y|^{r'}$$

for all $x, y \in \mathbf{R}^N$ *and* $r' \in [2, r]$.

Proof. We only show the proof of (ii) because (i) can be easily shown in the similar way. If $x, y \in \operatorname{Supp}[h] \subset V(M)$, then it holds:

$$
\begin{aligned}
&\int_U |\bar{v}(x) - \bar{e}(x) - \{\bar{v}(y) - \bar{e}(y)\}|^{r'} \mu(dv) \\
&\leq 2^{r'} \left\{ \int_U |\tilde{v}(x) - \tilde{e}(x) - \{\tilde{v}(y) - \tilde{e}(y)\}|^{r'} |h(x)|^{r'} \mu(dv) \right. \\
&\qquad \left. + \int_U |h(x) - h(y)|^{r'} |\tilde{v}(y) - \tilde{e}(y)|^{r'} \mu(dv) \right\}
\end{aligned}
$$

(where we set $\tilde{v}(x) = \iota \circ v \circ \iota^{-1} \circ \pi(x)$)

$$\leq K\{|\pi(x) - \pi(y)|^{r'} + |x - y|^{r'}\} \leq K|x - y|^{r'}.$$

If $x \in \mathrm{Supp}[h]$ and $y \in (\mathrm{Supp}[h])^c$, then there exists a point $z \in (\mathrm{Supp}[h])^c \cap V(M)$ such that $z \in \overline{xy}$ ($=$ the straight line from x to y in $\mathbf{R}^N$). Noting that $\bar{v}(y) = \bar{e}(y) = \bar{v}(z) = \bar{e}(z)$ and the result for the first case, we see that

$$\int_U |\bar{v}(x) - \bar{e}(x) - \{\bar{v}(y) - \bar{e}(y)\}|^{r'} \mu(dv)$$

$$= \int_U |\bar{v}(x) - \bar{e}(x) - \{\bar{v}(z) - \bar{e}(z)\}|^{r'} \mu(dv) \leq K|x - z|^{r'} \leq K|x - y|^{r'}.$$

If $x, y \in (\mathrm{Supp}[h])^c$, the left hand side of (ii) is 0. □

Lemma 3.2. *We have the following representation: for* $x \in \mathbf{R}^N$

$$\bar{v}(x) - \bar{e}(x) - \overline{\Psi(v)}(x) = \sum_{i,j=1}^N \Phi_{ij}(x, v)(\tilde{v}^i(x) - \tilde{e}^i(x))(\tilde{v}^j(y) - \tilde{e}^j(y))$$

and $\Phi_{ij}(x, v)$ satisfies $\sup_{x \in V(M)} \sup_{v \in C(M,M)} |\Phi_{ij}(x, v)| < \infty$ for $i, j = 1, \dots, N$.

Proof. Noting that we can put off the first order term of $(\tilde{v}(x) - \tilde{e}(x))$ in Taylor's expansion by the condition (2.1), it is easy to see that we can take

$$\Phi_{ij}(x, v) = h(x) \int_0^1 \left\{ \frac{\partial^2 \pi}{\partial x^i \partial x^j}(\tilde{e}(x) + t(\tilde{v}(x) - \tilde{e}(x))) \right.$$

$$\left. - \sum_{k=1}^m \frac{\partial^2 \Psi^k}{\partial x^i \partial x^j}(\tilde{e}(x) + t(\tilde{v}(x) - \tilde{e}(x))) \times Z_k(x \circ \iota)(\iota^{-1} \circ \pi(x)) \right\} (1 - t)\, dt.$$

Then it is easy to see that it is uniformly bounded with respect to (x, v). □

Lemma 3.3. *Under the condition* (A, II_2)(U), *the function*

$$\int_U \{\bar{v}(x) - \bar{e}(x) - \overline{\Psi(v)}(x)\} \mu(dv)$$

is globally Lipschitz continuous on $\mathbf{R}^N$.

Proof. For $x, y \in V(M)$, we have

$$\left| \int_U \{\bar{v}(x) - \bar{e}(x) - \overline{\Psi(v)}(x)\} \mu(dv) - \int_U \{\bar{v}(y) - \bar{e}(y) - \overline{\Psi(v)}(y)\} \mu(dv) \right|$$

$$\leq 2 \sum_{i,j=1}^N \left\{ \int_U |\Phi_{ij}(x, v) - \Phi_{ij}(y, v)| |(\tilde{v}^i(x) - \tilde{e}^i(x))(\tilde{v}^j(x) - \tilde{e}^j(x))|^2 \mu(dv) \right.$$

$$+ \int_U |\Phi_{ij}(y, v))| |(\tilde{v}^i(x) - \tilde{e}^i(x))(\tilde{v}^j(x) - \tilde{e}^j(x))$$

$$- (\tilde{v}^i(y) - \tilde{e}^i(y))(\tilde{v}^j(y) - \tilde{e}^j(y))|\mu(dv)\Big\} .$$

Here, note the inequality:

$$\begin{aligned}
&|\Phi_{ij}(x, v) - \Phi_{ij}(y, v)| \\
&\leq K\Big\{\int_0^1 |h(x)\pi_{ij}(\tilde{e}(x) + t(\tilde{v}(x) - \tilde{e}(x)) - h(y)\pi_{ij}(\tilde{e}(y) + t(\tilde{v}(y) - \tilde{e}(y)))|dt \\
&\quad + \int_0^1 |\bar{\Psi}_{ij}(\tilde{e}(x) + t(\tilde{v}(x) - \tilde{e}(x))) - \bar{\Psi}_{ij}(\tilde{e}(y) + t(\tilde{v}(x) - \tilde{e}(y)))|dt \\
&\qquad + |\bar{Z}(x) - \bar{Z}(y)|\Big\} \\
&\leq K\{|\tilde{v}(x) - \tilde{e}(x) - \{\tilde{v}(y) - \tilde{e}(y)\}| + |x - y|\},
\end{aligned}$$

where $\pi_{ij}(x)$ and $\bar{\Psi}_{ij}(x)$ denote the second order partial derivatives of π and $\bar{\Psi}$ with respect to x^i and x^j at x, respectively. Hence, under (A, II_2)(U), it holds that

$$\int_U |\Phi_{ij}(x, v) - \Phi_{ij}(y, v)||(\tilde{v}^i(x) - \tilde{e}^i(x))(\tilde{v}^j(x) - \tilde{e}^j(x))|\mu(dv) \leq K|x - y|.$$

On the other hand, by the uniform boundedness of Φ_{ij}, we easily obtain

$$\begin{aligned}
&\int_U |\Phi_{ij}(y, v)||(\tilde{v}^i(x) - \tilde{e}^i(x))(\tilde{v}^j(x) - \tilde{e}^j(x)) \\
&\qquad - (\tilde{v}^i(y) - \tilde{e}^i(y))(\tilde{v}^j(y) - \tilde{e}^j(y))|\mu(dv) \\
&\quad \leq K|x - y|, \text{ for } i, j = 1, \ldots, N.
\end{aligned}$$

Thus, we get the conclusion for $x, y \in V(M)$. Secondly, we consider the case where $x \in V(M)$ and $y \in V(M)^c$. However, it can be reduced to the first case using the same idea as in the proof of Lemma 3.1. If $x, y \in V(M)^c$, then the statement of this lemma is trivially valid. We have thus completed the proof of Lemma 3.3. □

From Lemma 3.1, 3.2, and 3.3, we obtain the following result by standard argument.

Under (A, I), (A, II_2)(U), and (A, III), the solution $\eta_{s,t}(x)$ of (3.3) exists uniquely for each initial data $(s, x) \in [0, \infty) \times \mathbf{R}^N$.

Next lemma plays an important role in the proof of Theorem 3.1. We will show it by an approximation.

Lemma 3.4. *If $x \in \iota(M)$, then the solution $\eta_{s,t}(x)$ of* (3.3) *belongs to $\iota(M)$ for all $t \geq s$ a.s., for each $s \geq 0$.*

Proof. For each $n \in \mathbf{N}$, take $V^n \subset U$ such that $V^n \uparrow U$ as $n \uparrow \infty$ and $\mu(V^n) < \infty$. For simplicity, we set $D(x) = \overline{L_0(x \circ \iota)}\,(x) + \bar{B}(x)$. We now consider the following stochastic integral equation without jumps on $\mathbf{R}^N$,

$$\psi_{s,t}(x) = x + \int_s^t \bar{M}(\psi_{s,u}(x), du) + \int_s^t D(\psi_{s,u}(x))\,du - \int_s^t \int_{V^n} \{\Psi(v)(\psi_{s,u}(x))\}\mu(dv)\,du\,.$$

It can be easily seen that it has a unique solution. Since it is an extension of stochastic integral equation based on the Brownian motion with the L. C.-system $(\langle\ ,\ \rangle, B - \int_{V^n} \Psi(v)\mu(dv))$, we see that $\psi_{s,t}(x) \in \imath(M)$ if $x \in \imath(M)$ if $x \in \imath(M)$. See Elworthy [1] Chapter VII for the detailed discussion.

Secondly, we consider the next equation with jumps,

$$\begin{aligned}\zeta^n_{s,t}(x) = x &+ \int_s^t \bar{M}(\zeta^n_{s,u-}(x), du) + \int_s^t D(\zeta^n_{s,u-}(x))\,du \\ &+ \int_s^t \int_{V^n} \{\bar{v}(\zeta^n_{s,u-}(x)) - \bar{e}(\zeta^n_{s,u-}(x)) - \overline{\Psi(v)}(\zeta^n_{s,u-}(x))\}\mu(dv)\,du \\ &+ \int_s^{t+} \int_{V^n} \{\bar{v}(\zeta^n_{s,u-}(x)) - \bar{e}(\zeta^n_{s,u-}(x))\}\tilde{N}_q(dudv) \\ = x &+ \int_s^t \bar{M}(\zeta^n_{s,u-}(x), du) \\ &+ \int_s^t \left\{D(\zeta^n_{s,u-}(x)) - \int_{V^n} \overline{\Psi(v)}(\zeta^n_{s,u-}(x))\mu(dv)\right\}du \\ &+ \int_s^{t+} \int_{V^n} \{\bar{v}(\zeta^n_{s,u-}(x)) - \bar{e}(\zeta^n_{s,u-}(x))\}N_q(dudv).\end{aligned}$$

Since $\mu(V^n) < \infty$, if $x \in \imath(M)$ then the solution $\zeta^n_{s,t}(x)$ is given by $\psi_{s_\ell,t} \circ q(s_\ell) \circ \cdots \circ q(s_1) \circ \psi_{s,s_1}(x)$, where $s < s_1 < \cdots < s_\ell \le t < s_{\ell+1}$ and $s_i \in D_q \cap V^n$ ($i = 1, 2, \ldots$). (D_q denotes the domain of the point process $\{q(t)\}$.) Hence, the solution $\zeta^n_{s,t}(x)$ can not leave $\imath(M)$ if the initial point belongs to $\imath(M)$.

Next, let $\zeta_{s,t}(x)$ be the unique solution of the following equation:

$$\begin{aligned}\zeta_{s,t}(x) = x &+ \int_s^t \bar{M}(\zeta_{s,u-})(x), du) + \int_s^t D(\zeta_{s,u-}(x))\,du \\ &+ \int_s^t \int_U \{\bar{v}(\zeta_{s,u-}(x)) - \bar{e}(\zeta_{s,u-}(x)) - \overline{\Psi(v)}(\zeta_{s,u-}(x))\}\mu(dv)du \\ &+ \int_s^{t+} \int_U \{\bar{v}(\zeta_{s,u-}(x)) - \bar{e}(\zeta_{s,u-}(x))\}\tilde{N}_q(dudv).\end{aligned}$$

Then it holds that

$$\lim_{n\uparrow\infty} E[\sup_{s\le u\le t} |\zeta^n_{s,u}(x) - \zeta_{s,u}(x)|^2] = 0 \text{ for each } x \in \mathbf{R}^N.$$

Since $\iota(M)$ is a closed set in $\mathbf{R}^N$ and the approximating processes $\{\zeta^n_{s,t}(x); n\in\mathbf{N}\}$ have its values in $\iota(M)$ if $x\in\iota(M)$, so does the limiting process $\zeta_{s,t}(x)$.

Finally, it is easily seen that the solution of (3.3) can not leave $\iota(M)$ if the initial point belongs to $\iota(M)$. Thus we have completed the proof of Lemma 3.4. □

Associated with the solution $\eta_{s,t}(x)$ of (3.3), put $\xi_{s,t}(p)=\iota^{-1}\circ(\eta_{s,t}|_{\iota(M)})\circ\iota(p)$ for $p\in M$. Then we have the following lemma.

Lemma 3.5. *$\xi_{s,t}(p)$ is the unique solution of* (3.1) *with the initial data* $(s, p)\in[0,\infty)\times M$.

Proof. By Ito's formula for $\mathbf{R}^N$-valued semimartingales, it holds that for all $f\in C^\infty(M)$

$$
\begin{aligned}
(3.4)\qquad \bar f(\eta_{s,t}(x)) &= \bar f(x)+\sum_{i=1}^N\int_s^t\frac{\partial\bar f}{\partial x^i}(\eta_{s,u-}(x))\bar M^i(\eta_{s,u-}(x),du)\\
&+\sum_{i=1}^N\int_s^t\frac{\partial\bar f}{\partial x^i}(\eta_{s,u-}(x))D^i(\eta_{s,u-}(x))\,du\\
&+\sum_{i=1}^N\int_s^t\int_U\{\bar v^i(\eta_{s,u-}(x))-\bar e^i(\eta_{s,u-}(x))\\
&\qquad-\overline{\Psi(v)}^i(\eta_{s,u-}(x))\}\frac{\partial\bar f}{\partial x^i}(\eta_{s,u-}(x))\mu(dv)\,du\\
&+(1/2)\sum_{i,j=1}^N\int_s^t\frac{\partial^2\bar f}{\partial x^i\partial x^j}(\eta_{s,u-}(x))\bar A^{ij}(\eta_{s,u-}(x))\,du\\
&+\int_s^t\int_U\{\bar f(\eta_{s,u-}(x))+\bar v(\eta_{s,u-}(x))-\bar e(\eta_{s,u-}(x)))-\bar f(\eta_{s,u-}(x))\\
&\qquad-\sum_{i=1}^N(\bar v^i(\eta_{s,u-}(x))-\bar e^i(\eta_{s,u-}(x)))\frac{\partial\bar f}{\partial x^i}(\eta_{s,u-}(x))\}\mu(dv)\,du\\
&+\int_s^{t+}\int_U\{\bar f(\eta_{s,u-}(x)+\bar v(\eta_{s,u-}(x))-\bar e(\eta_{s,u-}(x)))-\bar f(\eta_{s,u-}(x))\}\tilde N_q(dudv)\\
&+\int_s^{t+}\int_{U^c}\{\bar f(\eta_{s,u-}(x)+\bar v(\eta_{s,u-}(x))-\bar e(\eta_{s,u-}(x)))-\bar f(\eta_{s,u-}(x))\}N_q(dudv).
\end{aligned}
$$

Since $\eta_{s,t}(x)\in\iota(M)$ for $x=\iota(p)$, we have the following relations as seen before:

$$\bar M\bar f(\eta_{s,t}(x),t)=\bar M\bar f(\iota\circ\xi_{s,t}(p),t)=Mf(\xi_{s,t}(p),t),$$

$$
\begin{aligned}
&(1/2)\sum_{i,j=1}^N\bar A^{ij}(\eta_{s,t}(x))\frac{\partial^2\bar f}{\partial x^i\partial x^j}(\eta_{s,t}(x))+\sum_{i=1}^N D^i(\eta_{s,t}(x))\frac{\partial\bar f}{\partial x^i}(\eta_{s,t}(x))\\
&=\bar L\bar f(\eta_{s,t}(x))=\bar L\bar f(\iota\circ\xi_{s,t}(p))=Lf(\xi_{s,t}(p)),
\end{aligned}
$$

$$\bar{f}(\eta_{s,t}(x) + \bar{v}(\eta_{s,t}(x)) - \bar{e}(\eta_{s,t}(x))) = f(v(\xi_{s,t}(p))).$$

Using the above relations, it is easy to see that the equation (3.4) is nothing but (3.2) or equivalently (3.1).

On the other hand, the uniqueness follows from the one for (3.3). Thus, we have completed the proof of Lemma 3.5 and Proposition 3.2. □

We next give a proof of Proposition 3.3. But it can be shown as a simple application of the result of [3] Theorem 2.2. In fact, Lemma 3.1 says that we can apply the theorem to the system of solutions of (3.3), and we can see that the system defines a $C(\mathbf{R}^N, \mathbf{R}^N)$-Lévy flow. Moreover, since $\eta_{s,t}|_{\iota(M)}$ is an $\iota(M)$-valued process and $\iota(M)$ is a closed submanifold of $\mathbf{R}^N$, we can say that $\{\eta_{s,t}|_{\iota(M)}; s \leq t\}$ defines a $C(\iota(M), \iota(M))$-Lévy flow. Therefore, $\{\xi_{s,t}; s \leq t\}$ defines a $C(M, M)$-Lévy flow.

Finally, we give a proof of Theorem 3.1. What we have to do is to show the property of $C(M, M)$-Lévy flow in strong sense. However, it is shown by a result of Kunita [6] Theorem 4.2. Further, as regard to (ξ, I) and (ξ, II), they can be shown by the discussion similar to the one below Theorem 3.1 of [3]. (ξ, III$_r$) can be also obtained by the same calculation as in Lemma 2.1 of [3]. Thus, we have completed the proof of Theorem 3.1. □

Remark 3.2. Through the equation (3.1) under stronger regularity condition for the characteristic system, we can construct Lévy flow on the space of smooth mappings from M to itself by the similar discussion given in section 2.4 of [3] and by the methods in this paper.

Remark 3.3. We have restricted our attension to pathwise discussion. However, from the point of view of martingale problem for jump-diffusion process (= strong Markov process with jumps) on M, we can say that the characteristic system satisfying (A, I), (A, II$_2$), and (A, III) determines the distribution of $\{\xi_{0,t}(p); t \geq 0\}$.

§4. Representation of $C(M, M)$-Lévy flows

In the previous section, we have constructed the $C(M, M)$-Lévy flow through the stochastic integral equation (3.1). In this section, we consider the converse problem. That is, for a given $C(M, M)$-Lévy flow, our problem is to represent it as the system of solutions of a stochastic integral equation of the same type as (3.1). In other words, our problem is to find the stochastic infinitesimal generator of the Lévy flow. Most of ideas for finding it are contained in section 3 of [3], but we need more consideration about the geometrical properties of the infinitesimal generator.

Let $\{\xi_{s,t}; s \leq t\}$ be a $C(M, M)$-Lévy flow in strong sense defined on a probability space $(\Omega, \mathscr{F}, P)$. See §2 for the definition. We now give the statement of the main theorem in this section.

Theorem 4.1. *Assume that a given $C(M, M)$-Lévy flow $\{\xi_{s,t}; s \leq t\}$ satisfies* (ξ, I), (ξ, II), *and* (ξ, III_2). *Then, there exist an $\mathfrak{X}^0(M)$-valued Brownian motion $M(\ ,t)$ with mean 0 and the covariance $\langle\ ,\ \rangle$, a linear operator $\mathscr{L}: C^\infty(M) \to C^0(M)$, and a stationary Poisson point process $\{q(t)\}$ on $C(M, M)$ with the intensity measure μ, such that the Lévy flow is represented as the solutions of* (3.1). *Moreover, the triple $(\langle\ ,\ \rangle, \mu, \mathscr{L})$ becomes a characteristic system and μ satisfies* $(\mathrm{A}, \mathrm{II}_2)$.

We denote the limits in (ξ, I) and (ξ, II) by $\ll f, g \gg (p, q)$ and $\mathscr{L}f(p)$, respectively. (cf. Theorem 3.1) Since the proof is long, we will give it by deviding several parts.

For the given Lévy flow, define a stationary Poisson point process $\{q(t)\}$ on $C(M, M)$ by

$$q(t) = \xi_{t-,t} \text{ for } t \in \{s; \xi_{s-,s} \neq \text{the identity map } e \text{ on } M\}.$$

We denote the counting measure and the intensity measure by $N_q(dudv)$ and $\mu(dv)$, respectively.

Remark 4.1. The definition of point process $\{q(t)\}$ is somewhat different from that in [3].

First we study the properties of the intensity measure of the point process defined above.

Lemma 4.1. *Under the condition* (ξ, III_2), $(\mathrm{A}, \mathrm{II}_2)$ *holds. Further, under the condition* (ξ, III_r) $(r \geq 2)$, $(\mathrm{A}, \mathrm{II}_r)$ *holds.*

Proof. Put $\eta_{s,t}(x) = \iota \circ \xi_{s,t} \circ \iota^{-1}(x)$ for $x \in \iota(M)$. Then, it can be considered as a Lévy flow on $\iota(M)$ and therefore $\eta_{s,t}(x)$ takes the values in $\mathbf{R}^N$. Hence, we can follow the same discussion as in the proof of Lemma 3.1, 3.2, and 3.3 of [3]. At the time, we should note that $a^{ij}(x, y)$ and $b^i(x)$ in [3] correspond to $\ll x^i \circ \iota, x^j \circ \iota \gg (\iota^{-1}(x), \iota^{-1}(y))$ and $\mathscr{L}(x^i \circ \iota)(\iota^{-1}(x))$ for $x, y \in \iota(M)$, respectively. Furthermore, by Theorem 1.2 of [3], we can see that this lemma holds. □

By the first part of Lemma 4.1 the following is well-defined: for all $f, g \in C^\infty(M)$, and $p, q \in M$,

$$\langle f, g \rangle (p, q) = \ll f, g \gg (p, q) - \int_{C(M,M)} \{f(v(p)) - f(p)\}\{f(v(q)) - f(q)\}\mu(dv)$$

The next proposition was unnecessary in the case of $C(\mathbf{R}^d, \mathbf{R}^d)$-Lévy flow. But in this manifold case it is necessary for characterizing the infinitesimal generator which we want to find.

Proposition 4.2. *The triple $(\langle\ ,\ \rangle, \mu, \mathscr{L})$ is a characteristic system.*

Proof. We first note that under the conditions (ξ, I), (ξ, II), and (ξ, III_2) we can see that $\langle f, g \rangle$ and $\mathscr{L}f$ are continuous for each $f, g \in C^\infty(M)$. Hence, we can consider $\langle\ ,\ \rangle$ and $\mathscr{L}$ as the operations $\langle\ ,\ \rangle: C^\infty(M) \times C^\infty(M) \to C^0(M \times M)$ and $\mathscr{L}: C^\infty(M) \to C^0(M)$, respectively. From the definitions, it is obvious that $\langle\ ,\ \rangle$ is bilinear and $\mathscr{L}$ is linear. Also, it is obvious that the condition 1)-a) of characteristic system is satisfied. See §2 about the condition. To see b), we consider the process

$$M_{s,t}f(p) = f(\xi_{s,t}(p)) - f(p) - \int_s^t \mathscr{L}f(\xi_{s,u}(p))\,du$$

By the same discussion as in Lemma 3.1 in [3], it can be easily seen that $M_{s,t}f(p)$ is an L^2-martingale with mean 0 for each s, p, and $f \in C^\infty(M)$. We denote the discontinuous martingale part of it by $M^d_{s,t}f(p)$. Then, it holds:

$$M^d_{s,t}f(p) = \int_s^{t+} \int_{C(M,M)} \{f(v(\xi_{s,u-}(p))) - f(\xi_{s,u-}(p))\} \tilde{N}_q(du dv).$$

Put $M^c_{s,t}f(p) = M_{s,t}f(p) - M^d_{s,t}f(p)$, which is the continuous part of $M_{s,t}f(p)$. By the orthogonality on the space of L^2-martingales, it holds:

$$\begin{aligned}
\langle M^c_{s,t}f(p), M^c_{s,t}g(q)\rangle &= \langle M_{s,t}f(p), M_{s,t}g(q)\rangle - \langle M^d_{s,t}f(p), M^d_{s,t}g(q)\rangle \\
&= \int_s^t \langle\!\langle f, g\rangle\!\rangle (\xi_{s,u}(p), \xi_{s,u}(q))\,du \\
&\quad - \int_s^t \int \{f(v(\xi_{s,u}(p))) - f(\xi_{s,u}(p))\}\{f(v(\xi_{s,u}(q))) - f(\xi_{s,u}(q))\}\,du\mu(dv) \\
&= \int_s^t \langle f, g\rangle (\xi_{s,u}(p), \xi_{s,u}(q))\,du.
\end{aligned}$$

We should see the second part of the proof of Lemma 3.1 of [3] about the equality:

$$\langle M_{s,t}f(p), M_{s,t}g(q)\rangle = \int_s^t \langle\!\langle f, g\rangle\!\rangle (\xi_{s,u}(p), \xi_{s,u}(q))\,du\,.$$

Now, for $f_i \in C^\infty(M)$ $(i = 1, 2)$, by Ito's formula, we have:

$$\begin{aligned}
&f_1(\xi_{s,t}(p))f_2(\xi_{s,t}(p)) \\
&= f_1(p)f_2(p) + \int_s^t f_1(\xi_{s,u}(p))\,dM^c_{s,u}f(p) + \int_s^t f_2(\xi_{s,u}(p))\,dM^c_{s,u}f_1(p) \\
&\quad + \text{a bounded variation process} + \text{a discontinuous martingale.}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
f_1f_2(\xi_{s,t}(p)) = f_1f_2(p) + M^c_{s,t}(f_1f_2)(p) &+ \text{a bounded variation process} \\
&+ \text{a discontinuous martingale.}
\end{aligned}$$

Hence, by the uniqueness of the decomposition, we have:

$$M^c_{s,t}(f_1 f_2)(p) = \int_s^t f_1(\xi_{s,u}(p))\, dM^c_{s,u} f_2(p) + \int_s^t f_2(\xi_{s,u}(p)) dM^c_{s,u} f_1(p).$$

Moreover, it holds:

$$\int_s^t \langle f_1 f_2, g\rangle(\xi_{s,u}(p), \xi_{s,u}(q))\, du = \langle M^c_{s,t}(f_1 f_2)(p), M^c_{s,t} g(q)\rangle$$

$$= \int_s^t f_1(\xi_{s,u}(p))\, d\langle M^c_{s,u} f_2(p), M^c_{s,u} g(q)\rangle$$

$$+ \int_s^t f_2(\xi_{s,u}(p)) d\langle M^c_{s,u} f_1(p), M^c_{s,u} g(q)\rangle$$

$$= \int_s^t \{f_1(\xi_{s,u}(p))\langle f_2, g\rangle(\xi_{s,u}(p), \xi_{s,u}(q))$$

$$+ f_2(\xi_{s,u}(p))\langle f_1, g\rangle(\xi_{s,u}(p), \xi_{s,u}(q))\}\, du.$$

Deviding the expectation of both sides by $(t-s)$ and then taking limit $t \to s$, we obtain the property b).

The positive definiteness c) follows from the relation:

$$\int_s^t \langle f_i, f_j\rangle(\xi_{s,u}(p_i), \xi_{s,u}(p_j)) du = \langle M^c_{s,t} f_i(p_i), M^c_{s,t} f_j(p_j)\rangle.$$

With regard to the property 2), we have already seen in Lemma 4.1. We next show the property 4). By using Ito's formula, it holds:

$$E[(fg)(\xi_{s,t}(p)) - (fg)(p)]$$

$$= \int_s^t E[\langle f, g\rangle(\xi_{s,u}(p), \xi_{s,u}(p)) + f(\xi_{s,u}(p))\mathscr{L} g(\xi_{s,u}(p)) + g(\xi_{s,u}(p))\mathscr{L} f(\xi_{s,u}(p))$$

$$+ \int \{f(v(\xi_{s,u}(p))) - f(\xi_{s,u}(p))\}\{g(v(\xi_{s,u}(p))) - g(\xi_{s,u}(p))\}\mu(dv)]\, du.$$

By deviding both sides by $(t-s)$ and taking limit $t \to s$ as before, we get the property 4). We have thus completed the proof of Proposition 4.2. □

What we have to show for completing the proof of Theorem 4.1 is only to construct $M(\ \ , t)$ with mean 0 and the covariance $\langle\ \ ,\ \ \rangle$ satisfying (3.1). For a partition of $[s, \infty)$ $\Delta: s = t_0 < t_1 \cdots$, put

$$Y^{\Delta}_{s,t} f(p) = \sum_{i=1}^{\infty} M^c_{t_i, t_{i+1} \wedge t} f(p).$$

Let $\{\Delta_n: n = 1, 2, \dots\}$ be a sequence of partitions such that Δ_{n+1} is a fine partition of Δ_n and $|\Delta_n| =: \max_i |t^n_{i+1} - t^n_i| \to 0$ as $n \to \infty$. Then, by the similar way in Lemma 3.2 of [3], we can see that $\{Y^{\Delta_n}_{s,t} f(p);\, n = 1, 2, \dots\}$ defines a Cauchy sequence in $L^2(P)$ for each $s \le t$, $p \in M$, and $f \in C^\infty(M)$. We denote by $Y_{s,t} f(p)$ the limit. Then, we have:

$$\langle Y_{s,t}f(p),\ Y_{s,t}g(q)\rangle = (t-s)\langle f, g\rangle(p, q)$$

and

$$Y_{s,u}f(p) = Y_{s,t}f(p) + Y_{t,u}f(p) \qquad \text{for } s \leq t \leq u.$$

See also Lemma 3.2 in [3]. Now, put $Mf(p, t) = Y_{0,t}f(p)$. By the above properties, it can be easily seen that $Mf(t, p)$ has independent increments and has a continuous modification with respect to (p, t). (Recall the first notice about the continuity of $\langle f, g\rangle$ in the proof of Proposition 4.2.) Moreover, by the derivation property of $\langle\ ,\ \rangle$, we get that of $M(\ , t)$. That, is, it holds:

$$M(fg)(p, t) = f(p)Mg(p, t) + g(p)Mf(p, t).$$

Therefore, $M(\ , t)$ is an $\mathfrak{X}^0(M)$-valued Brownian motion. It also holds:

$$M^c_{s,t}f(p) = \int_s^t Mf(\xi_{s,u}(p), du).$$

Thus we get the equation (3.1) which we want to show:

$$f(\xi_{s,t}(p)) - f(p) - \int_s^t \mathscr{L}f(\xi_{s,u}(p))du$$

$$= \int_s^t Mf(\xi_{s,u}(p), du) + \int_s^{t+}\int \{f(v(\xi_{s,u-}(p))) - f(\xi_{s,u-}(p))\}\tilde{N}_q(dudv).$$

We have thus completed the proof of Theorem 4.1. □

In this section, we have discussed from a point of the theory of stochastic differential equations and stochastic flows. In the remainder of this section, we will mention the relation to the classical theory of Markov processes.

Since a $C(M, M)$-Lévy flow $\{\xi_{s,t}\}$ has independent increments, n-point process $\{(\xi_{s,t}(p_1), \dots, \xi_{s,t}(p_n))\}$ defines a Markov process on the product manifold $M^n =: M \times \cdots \times M$ (n-folds), for each $n \in \mathbf{N}$. In particular, 1-point process $\{\xi_{s,t}(p);\ p \in M\}$ is a Markov provess with the infinitesimal generator $\mathscr{L}$. Since we know that $(\langle\ ,\ \rangle, \mu, \mathscr{L})$ is the characteristic system, the representation of $\mathscr{L}$ as an integro-differential operator is not unique as we saw in §2. Thus, we see that $\mathscr{L}$ itself is intrinsic to the Markov process $\{\xi_{s,t}(p);\ p \in M\}$ but the representation of it is not so. On the other hand, in [3], we saw that, in the case of $M = \mathbf{R}^d$, $\mathscr{L}$ was uniquely decomposed into the differential operator part and the integral operator part. It seems like a contradiction, but it is not so. In the case of $\mathbf{R}^d$, we fixed automatically the system as

$$m = d,\ \Psi^k = x^k, \text{ and } Z_k = \frac{\partial}{\partial x^k}.$$

This is the reason why $\mathscr{L}$ is considered to be uniquely represented.

Now, $\mathscr{L}$ can be rewritten as the following classical representation:

$$\mathscr{L}f(p) = Lf(p) + \int_{M\setminus\{p\}} \{f(q) - f(p) - \sum_{k=1}^{m} \{\Psi^k(q) - \Psi^k(p)\} Z_k f(p)\} \mu_p(dq),$$

where $\mu_p(dq) = \mu(ev_p^{-1}(dq) \cap \{v \in C(M, M); v(p) \neq p\})$ and $ev_p(v) = v(p)$ for $p \in M$. On the other hand, we know the representation of the infinitesimal generators of more general Markov processes on M. See [10] Chapter XII, section 7. The Theorem in the section (p. 408) clarifies properties of the generator in local, but not on global. Our Theorem 4.1 says that in the case of 1-point process of $C(M, M)$-Lévy flow, we can see the global property of the quantities which determine the infinitesimal generator $\mathscr{L}$. We should note the delicate difference between the representations of the above and (6) of [10] in section 7.

§ 5. Example

In this section, we will mention the relation between the result in Marcus [8], [9] and that in § 3. It is well-known that Stratonovich's differential equation is suited to construcing a diffusion process on a manifold. Then, it is natural that the following question arises: what type of equation should we consider to construct a discontinuous process on a manifold? In [8] and [9], he answer to this question for a simple case, extending Stratonovich's differential equation for continuous case. We will show that his result including the extension which will be given below provides us an example of Theorem 3.1. In order to concentrate our attention on construcing a jump-diffusion on a manifold, we restrict his statement as follows. However, we believe that we do not miss the essence of his idea.

Let z_t be an 1-dimensional Lévy process represented by

$$z_t = B_t + \int_0^{t+} \int_{|z|\leq 1} z\tilde{N}_p(dudz) + \int_0^{t+} \int_{|z|>1} zN_p(dudz) \tag{5.1}$$

where B_t is a Brownian motion and we denote by $N_p(dudz)$ the Poisson random measure associated with the point process p possesing the intensity measure $du\nu(dz)$, where ν is a σ-finite measure on $\mathbf{R}^1\setminus\{0\}$ satisfying $\int |z|^2 \wedge 1\nu(dz) < \infty$. We also set $\tilde{N}_p(dudz) = N_p(dudz) - du\nu(dz)$. Though in [9] z_t is restricted to the case where z_t has a finite number of jumps on each bounded interval, we would like to emphasize that we can remove the restriction by the discussion below. We now consider the following stochastic differential equation associated with the Lévy process z_t.

$$\begin{aligned} \xi_{s,t}(x) = x &+ \int_s^t g(\xi_{s,u-}(x)) \circ dB_u \\ &+ \int_s^{t+} \int_{|z|\leq 1} g(\xi_{s,u-}(x)) z\tilde{N}_p(dudz) + \int_s^{t+} \int_{|z|>1} g(\xi_{s,u-}(x)) zN_p(dudz) \\ &+ \sum_{s<u\leq t} \{\phi(\xi_{s,u-}(x), \Delta z_u) - \xi_{s,u-}(x) - g(\xi_{s,u-}(x))\Delta z_u\}, \end{aligned} \tag{5.2}$$

where $g \in C_b^1(\mathbf{R}^d, \mathbf{R}^d) =$: the space of all continuously differentiable mappings from $\mathbf{R}^d$ into itself posessing bounded derivatives, and $\phi(x, z)$ is the flow generated by g:

$$(5.3)\qquad \begin{cases} \dfrac{d}{dz}\phi(x, z) = g(\phi(x, z)) \\ \\ \phi(x, 0) = x \in \mathbf{R}^d. \end{cases}$$

In the above, $\circ dB$ denotes the Stratonovich differential of B and we set $\Delta z_u = z_u - z_{u-}$.

In [9], he calls this equation (5.2) the canonical extension of the equation: $d\xi_t = g(\xi_{t-})dz_t$. (cf. (9) in [9].) Further, it is shown in [9] that the solution ξ_t is invariant under the change of coordinate system which makes g a vector field on a manifold embedded in $\mathbf{R}^d$. See Theorem 6 in [9]. However, we will see that it is an obvious consequence of our results.

In the sequel, we first give an intrinsic interpretation of (5.2). To this end, we introduce a stochastic differential equation on a compact smooth manifold M associated with z_t of (5.1): for all $f \in C^\infty(M)$ and $p \in M$

$$(5.4)\qquad f(\xi_{s,t}(p)) = f(p) + \int_s^t Gf(\xi_{s,u-}(p)) \circ dB_u$$
$$+ \int_s^{t+}\int \{f(\phi(\xi_{s,u-}(p), zI_{\{|z|\le 1\}})) - f(\xi_{s,u-}(p))\}\tilde{N}_p(dudz)$$
$$+ \int_s^{t+}\int \{f(\phi(\xi_{s,u-}(p), zI_{\{|z|>1\}})) - f(\xi_{s,u-}(p))\}N_p(dudz)$$
$$+ \int_s^{t}\int \{f(\phi(\xi_{s,u-}(p), zI_{\{|z|\le 1\}})) - f(\xi_{s,u-}(p))$$
$$- Gf(\xi_{s,u-}(p))zI_{\{|z|\le 1\}}\}\, du\nu(dz),$$

where $G \in \mathfrak{X}(M)$ and ϕ is the flow generated by G with the initial condition $\phi(p, 0) = p \in M$.

It is easy to see that (5.4) is equivalent to (5.2) in the case of $M = \mathbf{R}^d$ if we identify $G \in \mathfrak{X}(M)$ with $\sum_{i=1}^d g^i \frac{\partial}{\partial x^i}$. Since all quantities in (5.4) are coordinate free, we can say that (5.4) gives us an intrinsic form of (5.2).

Here, we should note that (5.4) is an exmple of (3.1). In fact, in order to deduce the expression (3.1) from (5.4), we define a measure $\mu(dv)$ on $C(M, M)$ by

$$(5.5)\qquad \mu(dv) = \nu \circ F^{-1}(dv),$$

where we set $F(z) = \phi(\cdot, z)$ for $z \in \mathbf{R}^1 \backslash \{0\}$. We also define a point process q on $C(M, M)$ by $q(t) = \phi(\cdot, p(t))$, $D_q = \{t; q(t) \neq e\}$. Then, we can easily see that (5.4) is equivalent to the equation:

(5.6) $$f(\xi_{s,t}(p)) = f(p) + \int_s^t Mf(\xi_{s,u-}(p), du) + \int_s^t \mathscr{L}f(\xi_{s,u-}(p))\,du + \int_s^{t+}\int \{f(v \circ \xi_{s,u-}(p)) - f(\xi_{s,u-}(p))\}\tilde{N}_q(dudv),$$

where $Mf(p, t) = Gf(p)B_t$ and

(5.7) $$\mathscr{L}f(p) = (1/2)G(Gf)(p) + \int \{f(\phi(p, z)) - f(p) - Gf(p)zI_{\{|z| \le 1\}}\}\, \nu(dz).$$

This is nothing but (3.1) corresponding to the characteristic system $\langle f, g\rangle(p, q) = Gf(p)Gg(q)$, $\mathscr{L}$ of (5.7), and μ of (5.5). Also, we can see that the characterstic system satisfies (A, I), (A, II$_r$)(U), and (A, III)(U) for $U = \{\phi(\cdot, z); |z| \le 1\}$ and for any $r \ge 2$. Therefore, according to Proposition 3.3, the solution of (5.4) defines a jump-diffusion process on M and further a $C(M, M)$-Lévy flow.

By the above consideration, we can claim that our result gives a new interpretation of the equation (9) in [9] and a generalization of [9] in the direction of constructing jump-diffusion processes on a manifold.

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