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The Hopf rings for connective Morava K-theory and connective complex K-theory

By

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§ 0. Introduction

Let $E_*(\ )$ be a multiplicative homology theory and $\underline{F}_* = \{\underline{F}_k\}_{k\in\mathbf{Z}}$ be a multiplicative Ω-spectrum. If the Künneth formulas hold: $E_*(\underline{F}_k \times \underline{F}_k) \simeq E_*(\underline{F}_k) \otimes E_*(\underline{F}_k)$ for all k, $E_*\underline{F}_* = \{E_*\underline{F}_*\}_{k\in\mathbf{Z}}$ is a "Hopf ring" (see [R-W$_1$]). That is, each $E_*\underline{F}_k$ is a Hopf algebra with product $*$ and there is another product $\circ$ called pairing;

$$\circ\colon E_*\underline{F}_m \otimes E_*\underline{F}_n \longrightarrow E_*\underline{F}_{m+n},$$

which is induced by the pairing of Ω-spectrum

$$\mu\colon \underline{F}_m \wedge \underline{F}_n \longrightarrow \underline{F}_{m+n}.$$

In recent years, Hopf rings have been computed for various $(E_*, \underline{F}_*)$.

$(BP_*(\ ), \underline{BP}_*)$, in [R-W$_1$],

$(K(n)_*(\ ), K(\mathbf{Z}/p^j, *))$, in [R-W$_2$],

$(H_*(\ ;\mathbf{Z}/p), K(\mathbf{Z}/p, *))$, in [W$_1$],

$(E_*(\ ), \underline{K(n)}_*)$, for a wide class of $E_*(\ )$, in [W_2].

In these cases, $E_*\underline{F}_*$ is generatd by the elements of $E_*\underline{F}_k$ with $k \leq 2$. But for $\underline{F}_* = \underline{k(n)}_*$ and $\underline{bu}_*$, these aren't generated by $E_*\underline{F}_k$ with finitely many k's where $k(n)$ and bu are the connective Morava K-theory and the connective complex K-theory respectively. In this paper we compute the Hopf rings for $E_*(\ ) = H_*(\ ;\mathbf{Z}/p)$, $\underline{F}_* = \underline{k(n)}_*$ and $\underline{bu}$, for an odd prime p. Hence H_* stands for $H_*(\ ;\mathbf{Z}/p)$ and p is an odd prime throughout this paper.

In first three sections we compute $H_*\underline{k(n)}_*$. The periodic case $H_*\underline{K(n)}_*$ is completely determined by Wilson [W$_2$]. He shows $H_*\underline{K(n)}_*$ is generated by certain elements e_1, $a_{(i)} \in H_*\underline{K(n)}_1$, $b_{(j)} \in H_*\underline{K(n)}_2$, and finds many non-trivial differentials in the bar spectral sequence $E^r_{**}(H_*\underline{K(n)}_*) \Rightarrow H_*\underline{K(n)}_{*+1}$. In Section 1 we define elements in $H_*\underline{k(n)}_*$ by the same procedure as for $H_*\underline{K(n)}_*$ in

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[W_2]. In Section 2 we construct elements $\eta_*\alpha^I \in H_*\underline{k(n)}_*$, which vanish in $H_*\underline{K(n)}_*$ and prepare lemmas to show no new differentials appear in $E^r_{**}(H_*\underline{k(n)}_*)$. In Section 3 we state the first main result (Theorem 3.1). "$H_*\underline{k(n)}_*$ has more elements $\eta_*\alpha^I \circ b^J$, $(I, J) =$ admissible." It is proved by induction coupled with Theorem 3.2, where we make use of the bar spectral sequence which is compatible with pairing:

$$\begin{array}{ccc} E^r_{**}(H_*\underline{k(n)}_s) \otimes H_*\underline{k(n)}_t & \Longrightarrow & H_*\underline{k(n)}_{s+1} \otimes H_*\underline{k(n)}_t \\ \circ\big\downarrow & & \circ\big\downarrow \\ E^r_{**}(H_*\underline{k(n)}_{s+t}) & \Longrightarrow & H_*\underline{k(n)}_{s+t+1} \end{array}$$

These tools are all prepared by [R-W_2], [T-W]. In [W_2] the induction goes on degree, but in our connective case, on m for $H_*\underline{k(n)}_m$.

In Section 4 we compute $H_*\underline{bu}_*$. $H^*(BU(2n, \dots, \infty); \mathbf{Z}/p)$ and $H^*(U(2n+1, \dots, \infty); \mathbf{Z}/p)$ were computed by Stong [St] for $p = 2$ and by [Si] for $p =$ an odd prime. And for mod p homologies, Kochman determined $\operatorname{Im}[H_*(BU(2n, \dots, \infty); \mathbf{Z}/p) \to H_*(BU; \mathbf{Z}/p)]$, essentially in [K]. He used the pairings induced by tensor products of vector bundles. By Bott periodicity, $\underline{bu}_{2n} \simeq BU(2n, \dots, \infty)$ and $\underline{bu}_{2n+1} \simeq U(2n+1, \dots, \infty)$, consequently we determine the mod p homologies of them for all n. The main results are Theorem 4.21 and Theorem 4.23. We prove them by the similar method for $H_*\underline{k(n)}_*$. First we prove the result for the periodic case $H_*\underline{K}_*$ (Theorem 4.10) by induction on degree. We make new elements $\partial\alpha^I$ in $H_*\underline{bu}_*$ and study them in Lemma 4.20 using the previous Proposition 2.4. Lastly we compute the connective case $H_*\underline{bu}_*$ by induction on m for $H_*\underline{bu}_m$. The proof of collapsing of the bar spectral sequences is easy.

Many have been interested in the problem and Wilson also had results for $H_*\underline{k(n)}_*$, $H_*\underline{bu}_*$ and $H_*(\underline{bo}_*; \mathbf{Z}/2)$ independently [W_3], but he was unable to find a nice description of the answer. (See the remark (b) of Theorem 3.1.) There was difficulty to give the legitimate name to the new generators. Our advantage is that, as we use the maps η_* and ∂ induced by geometrical maps to define them, functoriality of the homology groups and the bar spectral sequences make us simplify the proofs.

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We list the sections again:

§1. Definitions of elements in $H_*\underline{k(n)}_*$

We first review the results of Wilson on the Hopf ring structure of

$H_*\underline{K(n)}_*$. He constructs elements e_1, $a^I b^J$, $[v_n]$ in $H_*\underline{K(n)}_*$ and proves the following theorem.

Theorem 1.1. (*Theorem* 1 *of* $[\mathrm{W}_2]$ *for* $E_* = H_*(-;\mathbf{Z}/p)$).

$$H_*\underline{K(n)}_* \simeq \bigotimes_{I,J=ad.,j_0<p^n-1,k\in\mathbf{Z}} E(e_1\circ a^I b^J \circ [v_n^k])$$

$$\bigotimes_{I,J=ad.,I\neq I(1),(i_0=0 \text{ or } j_0<p^n-1),k\in\mathbf{Z}} TP_{\rho(I)}(a^I b^J\circ[v_n^k])$$

$$\bigotimes_{I=I(1),J=ad.,j_0<p^n-1,k\in\mathbf{Z},} P(a^I b^J\circ[v_n^k])$$

where $E(x)$, $TP_h(x)$ *and* $P(x)$ *are the exterior algebra, the truncated polynomial algebra with* $x^{p^h}=0$, *and the polynomial algebra, respectively, and other notation is defined below.* □

We will construct elements in $H_*\underline{k(n)}_*$ which are sent onto e_1, $a^I b^J$ and $[v_n^k]$ by $\underline{\rho}_*\colon H_*\underline{k(n)}_* \to H_*\underline{K(n)}_*$. Let $\pi\colon k(n)\to H$ be the canonical multiplicative map of spectrum and $\underline{\pi}_*\colon \underline{k(n)}_k \to \underline{H}_k$ be the induced map between the k-th spaces of the Ω-spectra. In general, for a spectrum X, $\underline{X}_k$ denotes the k-th space of the associated Ω-spectrum of X, that is, $\underline{X}_k \simeq \Omega\underline{X}_{k+1}$, and for a map of spectrum $f\colon X \to Y$, $\underline{f}\colon \underline{X}_k \to \underline{Y}_k$ denotes the induced map of f.

For $k=1$, the map $\underline{\pi}_*\colon \pi_{i-1}k(n) = \pi_i\underline{k(n)}_1 \to \pi_i\underline{H}_1 = \pi_i K(\mathbf{Z}/p,1)$ is an isomorphism for $0\le i\le 2p^n-2$ and an epimorphism for $i=2p^n-1$ because the coefficient ring is $\pi_*k(n) = \mathbf{Z}/p[v_n]$, $\deg v_n = 2p^n-2$.

Thus we have an isomorphism:

$$\pi_*\colon H_i\underline{k(n)}_1 \simeq H_i K(\boldsymbol{Z}/p,1) \quad \text{for } 0\le i\le 2p^n-2.$$

Recall $H_*K(\boldsymbol{Z}/p,1) = E(e)\otimes\Gamma(\alpha_{(0)})$, where $\Gamma(\ )$ is the divided power Hopf algebra and the elements e and $\alpha_{(i)}$ are defined as follows. Let $S^1 \xrightarrow{u} K(\boldsymbol{Z}/p,1) \xrightarrow{\delta} \boldsymbol{CP}^\infty$ be the fibration, and consider the induced maps on their mod p homologies.

$$H_*S^1 \xrightarrow{u_*} H_*K(\boldsymbol{Z}/p,1) \xrightarrow{\delta_*} H_*\boldsymbol{CP}^\infty,$$

and $H_*S^1 \simeq E(e')$, $H_*\boldsymbol{CP}^\infty \simeq \boldsymbol{Z}/p\langle\beta_0,\beta_1,\cdots\rangle$, then we can define: $e = u_*e'$, $\alpha_i = \delta_*^{-1}(\beta_i)$, $\alpha_{(i)} = \alpha_{p^i}$, $\beta_{(i)} = \beta_{p^i}$. Now we use the isomorphism $\underline{\pi}_*$ to define e_1, a_i and $a_{(i)}$ in $H_*\underline{k(n)}_*$: $e_1 = \underline{\pi}_*^{-1}e$, $a_i = \underline{\pi}_*^{-1}\alpha_i$ for $i<p^n$, and $a_{(i)} = a_{p^i}$ for $i<n$.

Let $x\colon \boldsymbol{CP}^\infty \to \underline{k(n)}_2$ represent the complex orientation of $k(n)$, b_i be the image of β_i by $x_*\colon H_*\boldsymbol{CP}^\infty \to H_*\underline{k(n)}_2$, and $b_{(i)}$ be b_{p^i}.

For $I=(i_0,i_1,\ldots)$, $J=(j_0,j_1,\ldots)$, non negative finite sequences with $i_k=0$ or 1, we define an element in $H_*\underline{k(n)}_*$:

$$a^I b^J = a_{(0)}^{\circ i_0}\circ a_{(1)}^{\circ i_1}\cdots\circ b_{(0)}^{\circ j_0}\circ b_{(1)}^{\circ j_1}\cdots$$

with convention that $a^I b^I = [1]-[0]$ if I and J are all zeros, and the $\circ$ is the pairing of the Hopf ring. Here we remark that $a^I b^J$ is definable only when I is admissible (see below). Let $[v_n^k]$ be the image of the generator by the map $H_0(*) \to H_0\underline{k(n)}_{-2(p^n-1)k}$, which is induced by the element of coefficient ring $v_n^k: * \to \underline{k(n)}_{2(p^n-1)k}$. We use the notion of "admissible" or "ad." as follows:

(1.2) I is ad. if and only if $i_s = 0$ for all $s \geq n$.
J is ad. if and only if $0 \leq j_s < p^n$ for all $s \geq 0$

Let $\Delta_i = (\underbrace{0, 0, \dots, 0}_{i}, 1)$, $I(1) = \Delta_0 + \Delta_1 + \cdots + \Delta_{n-1}$, and for $I \neq I(1)$, $\rho(I) = \min\{k \mid i_{n-k} = 0\}$.

Let $\rho: k(n) \to K(n)$ be the canonical map, i.e., the localization by v_n. To smplify notation we used the same symbols e_1, $a^I b^J$ and $[v_n^k] \in H_*\underline{K(n)}_*$ in Theorem 1.1, for the images of $\underline{\rho}_*$ of the elements of $H_*\underline{k(n)}_*$.

$\underline{\rho}: \underline{k(n)}_i \to \underline{K(n)}_i$ is a homotopy equivalence for $i < 2p^n - 2$, and for $i = 2p^n - 2$, $\boldsymbol{Z}/p \times \underline{k(n)}_{2p^n-2} \simeq \underline{K(n)}_{2p^n-2}$. Therefore some formulas are available in $H_*\underline{k(n)}_*$ as well as in $H_*\underline{K(n)}_*$.

Proposition 1.3. *(from Proposition* 1.1 $[W_2]$).
(*a*) $e_1 \circ (-)$ *is the homology suspension map.*
(*b*) *The coproduct is given by* $a_i \to \sum a_{i-j} \otimes a_j$, $b_i \to \sum b_{i-j} \otimes b_j$.
(*c*) $e_1 \circ e_1 = b_1$.
(*d*) $b_{(i)}^{*p} = 0$, *where* $*$ *means the product of algebra.*
(*e*) $a_{(i)}^{*p} = 0$, $i < n-1$.
(*f*) $a_{(n-1)}^{*p} = -a_{(0)} \circ b_{(0)}^{\circ p^n - 1} \circ [v_n]$.
(*g*) $b_{(0)}^{\circ p^n - 1} \circ e_1 \circ [v_n] = 0$.
(*h*) $b^{p^n \Delta_k} \circ [v_n] = 0$ $(k \geq 0)$.

Next, we define elements e, $\alpha^I \beta^J$ in $H_*\underline{H}_*$. Let e be the canonical generator of $H_1 K(\boldsymbol{Z}/p, 1)$, and

$$\alpha^I \beta^J = \alpha_{(0)}^{\circ i_0} \circ \alpha_{(1)}^{\circ i_1} \circ \cdots \circ \beta_{(0)}^{\circ j_0} \circ \beta_{(1)}^{\circ j_1} \circ \cdots$$

for non negative finite sequences $I = (i_0, i_1, \dots)$, $i_k = 0$ or 1, $J = (j_0, j_1, \dots)$, where $\alpha_{(i)} = \alpha_{p^i} \in H_*K(\boldsymbol{Z}/p, 1) = H_*\underline{H}_1$ as before, and $\beta_{(j)}$ is the image of $\beta_{p^j} \in H_*\boldsymbol{CP}^\infty$ by the canonical map: $H_*\boldsymbol{CP}^\infty \to H_*\underline{H}_2$. With these elements Wilson shows:

Theorem 1.4. (p. 52, $[W_1]$). *The bar spectral sequences* $H_{**}(H_*\underline{H}_*) \Rightarrow H_*\underline{H}_{*+1}$ *collapse and*

$$H_*\underline{H}_* = \bigotimes_{I,J} \{E(e \circ \alpha^I \circ \beta^J) \otimes TP_1(\alpha^I \circ \beta^J)\},$$

where the tensor product rums over all I and J. □

§2. Lemmas for the main theorems

In this section we prepare lemmas for the proof of the main theorem.

Let $x: \boldsymbol{CP}^\infty \to \underline{k(n)}_2$ be the complex orientation, and y be the composite $y: K(\boldsymbol{Z}/p, 1) \xrightarrow{\delta} K(\boldsymbol{Z}, 2) \simeq \boldsymbol{CP}^\infty \xrightarrow{x} \underline{k(n)}_2$. We use the same notation as its representing homotopy class, then

Lemma 2.1. $k(n)^*K(\boldsymbol{Z}/p, 1) \simeq k(n)^*[[y]]/(v_n \cdot y^{p^n})$. □

Proof. Consider the fibre bundle $S^1 \xrightarrow{u} K(\boldsymbol{Z}/p, 1) \xrightarrow{\delta} \boldsymbol{CP}^\infty$, and its associated Gisin exact sequence:

$$\cdots \longrightarrow k(n)^*\boldsymbol{CP}^\infty \xrightarrow{\Phi} k(n)^{*+2}\boldsymbol{CP}^\infty \xrightarrow{\delta^*} k(n)^*K(\boldsymbol{Z}/p, 1) \longrightarrow \cdots$$

where $\Phi(z) = z \cdot [p]_{k(n)}[x]$. We know $k(n)^*\boldsymbol{CP}^\infty = k(n)^*[[x]]$, and from the computation of the formal group law of BP(see [Theorem 5.5, R-W$_2$]), we get $[p]_{k(n)}[x] = v_n \cdot x^{p^n}$. Therefore Φ is injective and the exact sequence is split. We have the lemma by $\delta^* x = y$. □

We define new elements in $H_*\underline{k(n)}_*$. Let ϕ be the composite $\phi: \Sigma^{2(p^n-1)}k(n) \simeq S^{2(p^n-1)} \wedge k(n) \xrightarrow{v_n \wedge 1} k(n) \wedge k(n) \xrightarrow{\mu} k(n)$, where μ is the multiplication of $k(n)$. Consider the cofibre sequence of spectra:

$$(2.2) \qquad \cdots \longrightarrow \Sigma^{-1}H \xrightarrow{\eta} \Sigma^{2(p^n-1)}k(n) \xrightarrow{\phi} k(n) \xrightarrow{\pi} H \longrightarrow \cdots$$

and the induced fibre sequence of spaces:

$$(2.3) \qquad \cdots \longrightarrow \underline{H}_{m-1} \xrightarrow{\underline{\eta}} \underline{k(n)}_{m+2(p^n-1)} \xrightarrow{\underline{\phi}} \underline{k(n)}_m \xrightarrow{\underline{\pi}} \underline{H}_m \longrightarrow \cdots.$$

Proposition 2.4. *In* $H_*\underline{k(n)}_{2p^n}$,

$$\underline{\eta}_* \alpha^{\Delta_{l+n}} = c\, b^{p^n \Delta_l} + \textit{decomposables},$$

with non zero constant c *in* $\boldsymbol{Z}/p$. □

Proof. We must investigate the map $\underline{\eta}: K(\boldsymbol{Z}/p, 1) \to \underline{k(n)}_{2p^n}$. From the definition of ϕ, the induced map

$$(\phi)_*: k(n)^{2p^n}K(\boldsymbol{Z}/p, 1) \longrightarrow k(n)^2 K(\boldsymbol{Z}/p, 1)$$

is obtained by the multiplication by v_n. And we get $v_n \cdot y^{p^n} = 0$ from Lemma 2.1, thus

$$\underline{\phi} \circ y^{p^n} \simeq 0: K(\boldsymbol{Z}/p, 1) \longrightarrow \underline{k(n)}_2.$$

Therefore there is a map d which gives the followng homotopy commutative diagram:

$$\begin{tikzcd} K(Z/p, 1) \arrow[r, "\eta"] & \underline{k(n)}_{2p^n} \arrow[r, "\phi"] & \underline{k(n)}_2 \\ K(Z/p, 1) \arrow[u, dotted, "d"] \arrow[r, "\delta"] & CP^\infty \arrow[ul, "y^{p^n}"] \arrow[u, "x^{p^n}"] & \end{tikzcd}$$

By the way, x^{p^n} is obtained by the composite

$$x^{p^n}\colon CP^\infty \xrightarrow{\Delta} \times^{p^n} CP^\infty \xrightarrow{\times^{p^n} x} \times^{p^n}\underline{k(n)}_2 \xrightarrow{o} \underline{k(n)}_{2p^n},$$

where Δ is the iterated diagonal map and o is the iterated pairing map. For $\alpha_{(l+n)} \in H_{2p^{l+n}} K(Z/p, 1)$,

$$\begin{aligned} (y^{p^n})_* \alpha_{(l+n)} &= (x^{p^n})_* \delta_* \alpha_{(l+n)} \\ &= (x^{p^n})_* \beta_{(l+n)} \\ &= (o)_* (\times^{p^n} x)_* \Delta_* \beta_{(l+n)} \\ &= \sum (o)_* (x_* \beta_{i_1} \times x_* \beta_{i_2} \times \cdots \times x_* \beta_{i_{p^n}}) \\ &= \sum b_{i_1} \circ b_{i_2} \circ \cdots \circ b_{i_{p^n}}, \end{aligned}$$

where the summation runs over the condition: $i_1 + i_2 + \cdots + i_{p^n} = p^{l+n}$. By the commutativity of the pairings of b_i's they survive only for $i_1 = i_2 = \cdots = i_{p^n} = p^l$. Thus we have $\underline{\eta}_* d_* \alpha_{(l+n)} = b_{(l)}^{\circ p^n}$. Especially, for $l = 0$, $\underline{\eta}_* d_* \alpha_{(n)} = b_{(0)}^{\circ p^n} = e_1^{\circ 2p^n}$. By Proposition 1.3, this is a generator of $H_{2p^n}\underline{k(n)}_{2p^n} \simeq Z/p$. Therefore the map d is a non trivial map, and there is a homotopy inverse c to d. (We regard c, d as the non zero elements of $Z/p \simeq [K(Z/p, 1), K(Z/p, 1)]$.) It follows that

$$\begin{aligned} b_{(l)}^{\circ p^n} &= \underline{\eta}_* d_* \alpha_{(l+n)} \\ &= \underline{\eta}_* (d \cdot \alpha_{(l+n)} + \text{decomposables}) \\ &= d \cdot \underline{\eta}_* \alpha_{(l+n)} + \text{decomposables}. \end{aligned}$$ □

Let γ be the composite:

$$\gamma = \pi \circ \eta\colon H \xrightarrow{\eta} \sum^{2p^n-1} k(n) \xrightarrow{\pi} \sum^{2p^n-1} H$$

Proposition 2.5. $[\gamma] = c \cdot Q_n \in H^{2p^n-1}H$, *where Q_n is the Milnor's primitive element of H^*H.* □

This is a well known fact (cf. [Y]), but here we prove it by unstable calculations. We prepare a lemma.

Lemma 2.6. $[\gamma]$ *is primitive.* □

Proof. As η is a $k(n)$-module spectrum map, the next diagram commutes:

$$
\begin{tikzcd}
 & k(n) \wedge H \arrow[r, "1 \wedge \eta"] & k(n) \wedge \Sigma^{2p^n-1} k(n) \arrow[d, "\mu"] \\
H \wedge H \arrow[ur, "\pi \wedge 1"] \arrow[dr, "\mu_0"'] & & \\
 & H \arrow[r, "\eta"] & \Sigma^{2p^n-1} k(n)
\end{tikzcd}
$$

Here μ_0 is the multiplication of H. Apply $H_*(-)$ to the diagram, and we get

$$
\begin{tikzcd}
 & H^*k(n) \otimes H^*H & \Sigma^{-2p^n+1} H^*k(n) \otimes H^*k(n) \arrow[l] \\
H^*H \otimes H^*H \arrow[ur] & & \\
 & H^*H \arrow[ul] \arrow[uu] & \Sigma^{-2p^n+1} H^*k(n). \arrow[l] \arrow[uu]
\end{tikzcd}
$$

We chase the images of $[\pi]$ in $H^0k(n)$.

$$
\begin{tikzcd}
 & {[\pi] \otimes [\gamma]} & {[\pi] \otimes [\pi]} \arrow[l] \\
\mu_0^*[\gamma] = [\gamma] \otimes 1 + \sum x_j \otimes y_j + 1 \otimes [\gamma] \arrow[ur] & & \\
 & {[\gamma]} \arrow[ul] & {[\pi]} \arrow[l] \arrow[uu]
\end{tikzcd}
$$

On the other hand, the map $\pi^*: H^iH \to H^ik(n)$ is an isomorphism for $i \leq 2p^n - 2$, and degrees of x_j's are less than $2p^n - 2$. Thus $\pi^*x_j = 0$ implies $x_j = 0$, and we have $\mu_0^*[\gamma] = [\gamma] \otimes 1 + 1 \otimes [\gamma]$. $\square$

Proof of Proposition 2.5. By Proposition 2.4,

$$
\begin{aligned}
\underline{\gamma}_*(\alpha_{(k)}) &= \underline{\pi}_*\underline{\eta}_*(\alpha_{(k)}) \\
&= \underline{\pi}_*(c \cdot b_{(k-n)}^{\circ p^n} + \textit{decomp.}) \\
&= c \cdot \beta_{(k-n)}^{\circ p^n} + \textit{decomp.}.
\end{aligned}
$$

As $\eta \circ \pi \simeq 0$,

$$
\begin{aligned}
\underline{\gamma}_*(\beta_{(k)}) &= \underline{\gamma}_*(\underline{\pi}_* b_{(k)}) \\
&= \underline{\pi}_*\underline{\eta}_*\pi_*(b_{(k)}) = 0.
\end{aligned}
$$

Consider the suspension to the stable group:

$$H_*\underline{H}_* \longrightarrow H_*H \simeq E(\tau_0, \tau_1, \ldots) \otimes P(\xi_1, \xi_2, \ldots)$$

we can see

$$\alpha^I \beta^J \longrightarrow \tau^I \xi^{\underline{J}} \quad \text{(see below this proof)}$$

with $\underline{J} = (j_1, j_2, \ldots)$ for $J = (j_0, j_1, j_2, \ldots)$. Thus we have stably:

$$\gamma_*\tau_k = c \cdot \xi_{k-n}^{p^n} \text{ and } \gamma_*\xi_k = 0.$$

On the other hand, by the formula

$$\psi\tau_k = \tau_k \otimes 1 + \sum_j \xi_{k-j}^{p^j} \otimes \tau_j,$$

we have (cf. [SW. p.418, Prop. 17.11]):

$$Q_n(\tau_k) = -\tau_k\langle Q_n, 1\rangle + \sum_j \xi_{k-j}^{p^j}\langle Q_n, \tau_j\rangle = \xi_{k-n}^{p^n}.$$

Similarly, $Q_n(\xi_k) = 0$. Because of the primitivity of $[\gamma]$ (Lemma 2.6) and Q_n, they act on H_*H as derivations. Thus we have $\gamma_*(-) = c\cdot Q_n(-)\colon H_*H \to H_*H$. This implies $[\gamma] = c\cdot Q_n$. □

For finite sequences, $I = (i_0, i_1, \ldots)$, $\underline{J} = (j_1, j_2 \cdots)$, $i_t = 0$ or $1, j_s \geq 0$, we define $\tau^I\xi^{\underline{J}} = \tau^{i_0}\tau^{i_1}\cdots\xi^{j_1}\xi^{j_2}\cdots$. We say $(I, \underline{J})$ is <u>admissible</u> if and only if $l(I) = i_0 + i_1 + \cdots \geq 1$, $M(I) = \max\{k \mid i_k = 1\} \geq n$ and $0 \leq j_s < p^n$, for all $s > M(I) - n$. The next proposition will be used for computations of differentials of the bar spectral sequence.

Proposition 2.7. Im $\gamma_* =$ Im Q_n

$$= \boldsymbol{Z}/p < Q_n(\tau^I\xi^{\underline{J}}) \mid (I, \underline{J}) \textit{ is } \underline{\textit{admissible}} > \subset H_*H.$$

□

Proof. First we see $\{Q_n(\tau^I\xi^{\underline{J}}) \mid (I, \underline{J}) = \underline{ad.}\}$ spans Im Q_n. When $(I, \underline{J})$ is not $\underline{ad.}$, we rewrite $Q_n(\tau^I\xi^{\underline{J}})$. Assume $l(I) \geq 1$, s_0 is the maximum of s's with $\underline{j_s} \geq p^n$, and $\underline{J'} = \underline{J} - p^n\Delta_{s_0}$. Then we have:

$$\begin{aligned} Q_n(\tau^I\xi^{\underline{J}}) &= Q_n(\tau^I\xi^{\underline{J'}}\xi^{p^n\Delta_{s_0}}) \\ &= \pm Q_n(Q_n(\tau^I)\tau_{s_0+n}\xi^{\underline{J'}}) \\ &= \pm Q_n(\sum_{i\leq M(I)} \pm \tau^{I-\Delta_i}\xi_{i-n}^{p^n}\tau_{s_0+n}\xi^{\underline{J'}}) \\ &= \pm Q_n(\sum_{i\leq M(I)} \pm \tau^{I-\Delta_i+\Delta_{s_0+n}}\xi^{p^n\Delta_{i-n}+\underline{J'}}) \end{aligned}$$

Since $s_0 > M(I) - n$, we see the indices $(I - \Delta_i + \Delta_{s_0+n}, p^n\Delta_{i-n} + \underline{J'})$ are all $\underline{ad.}$ for $i \leq M(I)$. It has been done.

Secondly we see the elements are lineraly independent. Consider the map:

$$H_*H \xrightarrow{Q_n(-)} H_*H \longrightarrow H_*H/\langle \tau^I\xi^{\underline{J}} \mid (I, \underline{J}) = \underline{ad.}\rangle$$

When $(I, \underline{J_*})$ is $\underline{ad.}$ it corresponds to

$$\tau^I\xi^{\underline{J}} \longrightarrow \pm\tau^{I-\Delta_{M(I)}}\xi^{\underline{J}+p^n\Delta_{M(I)-n}}$$

the images are linearly independent clearly. This implies the independency of $\{Q_n(\tau^I\xi^{\underline{J}}) \mid (I, \underline{J}) = \underline{ad.}\}$. □

§3. The main theorem for $H_*\underline{k(n)}_*$ and the proof

In this section we state the first main theorem and the proof of it. Let $\rho(I) = \min\{k \mid i_{n-k} = 0\}$, as before.

Theorem 3.1. *Let p be an odd prime. $H = H(\ ;\boldsymbol{Z}/p)$ and $k(n)$ be the connective Morava K-theory for p. Then the following isomorphism as algebra holds:*

$$
\begin{aligned}
H_*\underline{k(n)}_* \simeq & \bigotimes_{k=0 \text{ or } j_0 < p^n-1} E(e_1 \circ a^I b^J \circ [v_n^k]) \\
& \bigotimes_{I \neq I(1), (k=0 \text{ or } i_0 = 0 \text{ or } j_0 < p^n - 1)} TP_{\rho(I)}(a^I b^J \circ [v_n^k]) \\
& \bigotimes_{I = I(1), (k = 0 \text{ or } j_0 < p^n - 1)} P(a^I b^J \circ [v_n^k]) \\
& \otimes E(e_1 \circ \underline{\eta}_*(\alpha^I) \circ b^J) \\
& \otimes TP_1(\underline{\eta}_*(\alpha^I) \circ b^J).
\end{aligned}
$$

In the first three parts, the tensor products run over all admissible I and J (see 1.2) and $k \geq 0$. In the last two parts, they run over all admissible (I, J). We say "(I, J) is admissible" if

$$l(I) \geq 1,\ M(I) \geq n \ \textit{and}\ j_s < p^n \ \textit{for all}\ s > M(I) - n,$$

where $l(I) = i_0 + i_1 + \cdots$, $M(I) = \max\{k \mid i_k = 1\}$. $\square$

Remark. (a) As the map $H \xrightarrow{\eta} \Sigma^{p^n-1} k(n)$ is a $k(n)$-module map, we have following: If $l(I) \geq 1$,
(i) $\underline{\eta}_*(\alpha^I) \circ b^J = \underline{\eta}_*(\alpha^I \beta^J)$,
(ii) $\underline{\eta}_*(\alpha^I \beta^J) \circ e_1 = \underline{\eta}_*(\alpha^I \beta^J \circ e)$,
and if I' is admissible,
(iii) $\underline{\eta}_*(\alpha^I) \circ a^{I'} = \underline{\eta}_*(\alpha^{I+I'})$,
especially, from Proposition 2.4,
(iv) $\underline{\eta}_*(\alpha^{I' + \Delta_{n+i}}) = c \cdot a^{I'} \circ b^{p^n \Delta_i} + decomp.$
(b) When we rename $\underline{\eta}_*(\alpha^I) \circ b^J$, as $a^{I - \Delta_{M(I)}} \circ b^{p^n \Delta_{M(I)-n} + J}$, then we can rewrite Theorem 3.1 as the following simple form:

$$
\begin{aligned}
H_*\underline{k(n)}_* \simeq & \otimes E(e_1 \circ a^I b^J \circ [v_n^k]) \\
& \otimes TP_{\rho(I,J)}(a^I b^J \circ [v_n^k]),
\end{aligned}
$$

where I and J runs over the followng conditions: In E-algebra part,
if $k > 0$, then I and J are *ad.*, $j_0 < p^n - 1$,
if $k = 0$, then I is *ad.* or $j_s \geq p^n (^\exists s > M(I) - n)$.
In TP-algebra part,
if $k > 0$, then I and J are *ad.*, ($i_0 = 0$ or $j_0 < p^n - 1$),
if $k = 0$, then I is *ad*, or $j_s \geq p^n$ ($^\exists s > M(I) - n$),

$$\text{with } \rho(I, J) = \begin{cases} \rho(I), & I \text{ and } J \text{ are } ad. \\ \infty\,, & I = I(1),\ J \text{ is } ad. \\ 1\,, & \text{otherwise}\,. \end{cases}$$

Of course, if I is not admissible, $a^I b^J$ is not written in Hopf ring language as defined before. □

Our proof of Theorem 3.1 is almost parallel to [W_2], but in our case we use induction on m for $\sum_{i\le m} H_*\underline{k(n)}_i$. To begin with, we observe for $m \le 2p^n - 1$, $\underline{k(n)}_m \simeq \underline{K(n)}_m$, and Theorem 3.1 is true because it coincides with the result of Theorem 1.1. The next theorem is the $k(n)$-version of [Theorem 2.1, W_2]. The induction goes implicitly like as; Theorem 3.1 ⇒ Theorem 3.2 and Theorem 3.2 ⇒ Theorem 3.1.

Theorem 3.2. *In the bar spectral sequence;*

$$E^r_{**}(H_*\underline{k(n)}_*) \Longrightarrow H_*\underline{k(n)}'_{*+1},$$

where $\underline{k(n)}'_m$ is the connective component of $\underline{k(n)}_m$, we have

(*a*) $E^2_{**}(H_*\underline{k(n)}_*) \simeq \mathrm{Tor}^{H_*\underline{k(n)}_*}_{**}(\boldsymbol{Z}/p, \boldsymbol{Z}/p) = H_{**}(H_*\underline{k(n)}_*)$

$$\simeq \bigotimes_{k=0 \text{ or } j_0 < p^n - 1} \Gamma(\sigma e_1 \circ a^I b^J \circ [v_n^k])$$

$$\bigotimes_{k=0 \text{ or } i_0 = 0 \text{ or } j_0 < p^n - 1} E(\sigma a^I b^J \circ [v_n^k])$$

$$\otimes \Gamma_{i_{n-1}=0}(\phi(a^I b^J \circ [v_n^k]))$$

$$\otimes \Gamma(\sigma e_1 \circ \underline{\eta}_*(\alpha^I) \circ b^J)$$

$$\otimes E(\sigma \underline{\eta}_*(\alpha^I) \circ b^J)$$

$$\otimes \Gamma(\phi(\underline{\eta}_*(\alpha^I) \circ b^J))$$

In the first three parts the tensor products run over all admissible I, J, and $k \ge 0$. In the rests they run over admissible (I, J).

(*b*) *For the Hopf ring pairing of the bar spectral sequence, we have the following relations modulo decomposables*:

For $J \ne 0$, $k = m(J) = \min\{s | j_s > 0\}$, consider

$$\circ\colon H_{**}(H_*\underline{k(n)}_{*-2}) \otimes H_*\underline{k(n)}_2 \longrightarrow H_{**}(H_*\underline{k(n)}_*).$$

(1) $\gamma_{p^i}(\sigma e_1 \circ a^I b^{J-\Delta_k}) \circ b_{(k+1)} = \gamma_{p^i}(\sigma e_1 \circ a^I b^J)$.

(2) $\gamma_{p^i}(\phi(a^I b^{J-\Delta_k})) \circ b_{(k+i+1)} = \gamma_{p^i}(\phi(a^I b^J))$.

For $J = 0$, $k = m(J)$, consider

$$\circ\colon H_{**}(H_*\underline{k(n)}_{*-1}) \otimes H_*\underline{k(n)}_1 \longrightarrow H_{**}(H_*\underline{k(n)}_*).$$

(3) $\gamma_{p^i}(\sigma e_1 \circ a^{I-\Delta_k}) \circ a_{(k+i)} = (-1)^{l(I)-1}\gamma_{p^i}(\sigma e_1 \circ a^I)$, $k + i < n$.

(4) $\gamma_{p^i}(\phi(a^{I-\Delta_k})\circ a_{(k+i+1)}=(-1)^{\iota(I)-1}\gamma_{p^i}(\phi a^I),\ i_{n-1}=0,\ k+i+1<n.$

(c) *Let* $q=\eta(I)=\min\{s \mid i_{n-s}=1\}$, $\eta(0)=n+1$, *and* $I'=I-\Delta_{n-q}$. *The differentials are determined by*:

(1) $d^{p^q-1}\gamma_{p^q}(\sigma e_1\circ a^I b^J)=r_I\cdot\sigma a^{s^qI'}b^{s^q(J+\Delta_0)+(p^n-1)\Delta_0}\circ[v_n],\ I\neq 0,\ j_0<p^n-1,$

$r_I\neq 0.$

(2) $d^{2p^{q-1}-1}\gamma_{p^{q-1}}(\phi(a^I b^J))$

$=t_I\sigma a^{s^qI'+\Delta_{q-1}}b^{s^qJ+(p^n-1)\Delta_0}\circ[v_n],\ I\neq 0,\ i_{n-1}=0,\ t_I\neq 0,$

$=t_0\sigma b^{s^qJ+(p^n-1)\Delta_0}\circ[v_n],\ I=0,\ t_0\neq 0.$

Here $s^i(k_0,k_1,\dots)$ denotes $(\underbrace{0,\dots,}_{i}0,k_0,k_1,\dots)$.

(*d*) In E^∞_{**} (modulo decomposables)

(1) $\gamma_{p^i}(\sigma e_1\circ a^I b^J\circ[v_n^k])$, $j_0<p^n-1$, *represents* $(-1)^{\iota(I)}a^{s^iI}b^{s^i(J+\Delta_0)}\circ[v_n^k]$, *where if* $I\neq 0$, *then* $i<\eta(I)$.

(2) $c\cdot\gamma_{p^i}(\sigma e_1\circ a^I b^J)$, $j_0=p^n-1$, *represents* $(-1)^{\iota(I)}\underline{\eta}_*(\alpha^{s^i(I+\Delta_n)})\circ b^{s^i(J-(p^n-1)\Delta_0)}$.

(3) $\sigma a^I b^J\circ[v_n^k]$ *represents* $e_1\circ a^I b^J\circ[v_n^k]$, *where* $k=0$ *or* $j_0<p^{n-1}$.

(4) $\gamma_{p^i}(\phi(a^I b^J\circ[v_n^k]))$, $i_{n-1}=0$ *represents* $a^{s^{i+1}I+\Delta_i}b^{s^{i+1}J}$, *where* $i<\eta(I)-1$.

(5) $\gamma_{p^i}(\sigma e_1\circ\underline{\eta}_*(\alpha^I)\circ b^J)$ *represents* $\underline{\eta}_*(\alpha^{s^iI})\circ b^{s^i(J+\Delta_0)}$.
$\sigma\underline{\eta}_*(\alpha^I)\circ b^J$ *represents* $\sigma e_1\circ\underline{\eta}_*(\alpha^I)\circ b^J$.
$\gamma_{p^i}(\phi(\underline{\eta}_*(\alpha^I)\circ b^J))$ *represents* $\underline{\eta}_*(\alpha^{s^{i+1}I+\Delta_i})\circ b^{s^{i+1}J}$.

(*e*) *As an algebra,*

$$\begin{aligned}E^\infty_{**}\simeq &\bigotimes_{k=0\,or\,j_0<p^n-1} E(e_1\circ a^I b^J\circ[v^k])\\ &\otimes TP_1(a^I b^J\circ[v_n^k])\\ &\otimes E(e_1\circ\underline{\eta}_*(\alpha^I)\circ b^J)\\ &\otimes TP_1(\underline{\eta}_*(\alpha^I)\circ b^J),\end{aligned}$$

where the first two tensor products run over admissible I, admissible J and non negative integer k and others run over admissible (I, J). □

Proof of Theorem 3.1 *from Theorem* 3.2.
This is done by the next proposition like as in [W_2].

Proposition 3.3 ([*Proposition* 1.2, W_2]).
(*a*) *If* $i_{n-1}=1$, *then*

$$(a^I b^J\circ[v_n^k])^{*p}=(-1)^{\iota(I)}a^{\Delta_0+s(I-\Delta_{n-1})}b^{(p^n-1)\Delta_0+sJ}\circ[v_n^{k+1}].$$

(*b*) *If* $i_{n-1}=0$, *then* $(a^I b^J\circ[v^k])^{*p}=0$. □

This solves all algebraic extension problems for Theorem 3.2 (*e*) because the Im $\underline{\eta}_*$-parts are clear by Theorem 1.3. □

Proof of Theorem 3.2 *from Theorem* 3.1.

Refer to the corresponding parts of the proof of [Theorem 2.1, W_2]. We show only the esssential parts of this connective case.
(a). Like as in [R-W_2], we use homological calculations:

$$H_{**}(E(x)) = \Gamma(\sigma x),$$

$$H_{**}(TP_k(x)) = E(\sigma x) \otimes \Gamma(\phi(x^{p^{k-1}})),$$

$$H_{**}(P(x)) = E(\sigma x).$$

We have to check that, in the second Γ-algebra part:
$\{(a^I b^J \circ [v_n^k])^{*p^{\rho(I)-1}} | I \neq I(1),\ k = 0 \text{ or } j_0 < p^n - 1\} = \{a^I b^J \circ [v_n^k] | i_{n-1} = 0\}$ (up to sign), by Proposition 3.3. The result follows.
(b). The same as [W_2].
(c). This is also the same as [W_2], but we have to prove no new differentials appear other than those listed in (c).

The bar spectal sequences for $H_*\underline{H}_*$ collapse. And by functoriality of the bar spectral sequences for the infinite loop map, we conclude the differentials on Im $\underline{\eta}_*$ are trivial. By homological degree reason, we may examine only the following elements;

$$\gamma_{p^i}(\sigma e_1 \circ a^I b^J),\ j_0 = p^n - 1.$$

We prove these are permanent cycles. Suppose they aren't let i be the minimum of such i, then the target T of the differential is an odd total degree, primitive element. Seeing E^2_{**} of (a), the candidate is a linear combination of the next elements.

$$\sigma a^I b^J \circ [v_n^k],\ k = 0 \text{ or } i_0 = 0 \text{ or } j_0 < p^n - 1$$

$$\sigma \underline{\eta}_*(\alpha^I) \circ b^J,\ (I, J) = ad.$$

We divide the condition "(A) $k = 0$ or $i_0 = 0$ or $j_0 < p^n - 1$" into four parts:

(B) $j_0 < p^n - 1$

(B′) $k = 0,\ j_0 = p^n - 1$

(C) $k \geq 1,\ i_0 = 0,\ j_0 = p^n - 1,\ m(I) \geq m(J - (p^n - 1)\Delta_0)$

(D) $k \geq 1,\ i_0 = 0,\ j_0 = p^n - 1,\ m(I) < m(J - (p^n - 1)\Delta_0)$

(the elements of (C) correspond to targets of differentials of (1) and the elements of (D) to targets of differentials of (2).)

We show none of the elements of (B) appear in T. Because these elements have the lowest filtration degree, they exactly correspond to the elements $e_1 \circ a^I b^J \circ [v_n^k]$, $e \circ \underline{\eta}_*(\alpha^I) \circ b^J$. Sending T by $\underline{\rho}_*$: $H_*\underline{k(n)}_* \to H_*\underline{K(n)}_*$, we see that only the elements of (B) survive linearly independently and others are killed. Thus we can exclude (B) from T.

We observe the elements of (B′) and $\sigma \underline{\eta}_*(\alpha^I) \circ b^J$, $(I, J) = ad.$. Stabilize these elements by the map

$$H_*\underline{k(n)}_{m+1} \longrightarrow H_*\underline{H}_{m+1} \xrightarrow{\text{suspension}} H_{*-m-1}H$$

We have:

$$e_1 \circ a^I b^J \circ [v_n^k],\ (\mathrm{B}') \longrightarrow \tau^I \xi^{\underline{J}} = \pm Q_n(\tau^{I+\Delta_n} \xi^{\underline{J}}),$$

$$e_1 \circ a^I b^J \circ [v_n^k],\ (\mathrm{C}),\ (\mathrm{D}) \longrightarrow 0,$$

$$e_1 \circ \underline{\eta}_*(\alpha^I) \circ b^J,\ (I, J) = ad. \longrightarrow c \cdot Q_n(\tau^I \xi^{\underline{J}}),\ (I, \underline{J}) = \underline{ad}.,$$

by Proposition 2.5, where $\underline{J} = (j_1, j_2, \ldots)$ for $J = (j_0, j_1, j_2, \ldots)$.

On the other hand, by Proposition 2.7, we know the set;

$$\{Q_n(\tau^{I+\Delta_n} \xi^{\underline{J}}) | (\mathrm{B}'),\ 1 + l(I) + 2 \cdot l(J) = m + 1\} \cup \{Q_n(\tau^I \xi^{\underline{J}}) | (I, J) = ad., 2p^n + l(I) + 2 \cdot l(J) = m + 1\}$$

consists of linearly independent elements. Hence the elements;

$$\sigma e_1 \circ a^I b^J \circ [v_n^k],\ (\mathrm{B}') \text{ and } \sigma\underline{\eta}_*(\alpha^I) \circ b^J,\ (I, J) = ad.$$

don't appear in T.

Thus we can conclude T is a linear combination of the elements of (C), (D). But

$$T = d^{p^i - 1} \gamma_{p^i}(\sigma e_1 \circ a^I b^{(p^n - 1, \ldots)})$$

implies

$$T \circ [v_n] = d^{p^i - 1} \gamma_{p^i}(\sigma e_1 \circ a^I b^{(p^n - 1, \ldots)} \circ [v_n])$$

$$= 0, \text{ by Proposition 1.3.}$$

Then sending T by the map

$$E^r_{**}\underline{\rho}\colon E^r_{**}(H_*\underline{k(n)}_*) \longrightarrow E^r_{**}(H_*\underline{K(n)}_*),$$

we see all coefficients of elements in T are trivial. Thus $T = 0$ in $E^r_{**}(H_*\underline{k(n)}_*)$.

Now our assumption results in contradiction.

(d). For (1), (3), (4) it is done by the same way as in [W$_2$]. For (5), refer to [W$_1$, p.55], and use functoriality of the bar spectral sequences.

For (2), even when $j_0 = p^n - 1$, (1) is true. However, it is the problem of the new named elements of the right hand side in (2). From now, we show:

(3.4) $\quad c \cdot \gamma_{p^i}(\sigma e_1 \circ a^I b^J),\ j_0 = p^n - 1$ *represents*

$$(-1)^{l(I)} \underline{\eta}_*(\alpha^{s^i(I + \Delta_n)}) \circ b^{s^i(J - (p^n - 1)\Delta_0)} + \textit{decomposables}.$$

Proof. We proved in (a) that this element is a permanent cycle. We show that we can chose elements $x_i^{I,J} \in H_*\underline{k(n)}_*$, $j_0 = p^n - 1$, so as to satisfy following conditions:

(i) $x_0 = (-1)^{l(I)} \underline{\eta}_*(\alpha^{I + \Delta_n}) \circ b^{J - (p^n - 1)\Delta_0} + \text{decomp.} = \text{primitive}.$

(ii) $c \cdot \gamma_{p^i}(\sigma e_1 \circ a^I b^J)$ represents $x_i^{I,J}$, for $i \geq 0$.

(iii) $V(x_{i+1}^{\underline{I},\underline{J}}) = x_i^{\underline{I},\underline{J}}$. ($V$ is the Verschiebung map)

First we chose $x_i^{I,J}$ only satisfying (ii). Let $i = 0$ in (1), then we have:

$$c \cdot \sigma e_1 \circ a^I b^J \text{ represents } (-1)^{\iota(I)} c \cdot a^I \circ b^{\circ p^n \Delta_0} b^{J-(p^n-1)\Delta_0},$$

by Proposition 2.4,

$$= (-1)^{\iota(I)} a^I \circ \underline{\eta}_*(\alpha_{(n)} + decomp.) \circ b^{J-(p^n-1)\Delta_0},$$

by remark for Theorem 3.1,

$$= (-1)^{\iota(I)} \underline{\eta}_*(\alpha^{I+\Delta_n}) \circ b^{J-(p^n-1)\Delta_0} + decomp.$$

Then (i) holds automatically because $E_{1*}^{\infty} \simeq F_{1*}$. Even when we replace $x_i^{I,J}$ by $V^k x_{i+k}^{I,J}$, (ii) still holds because $V^k \gamma_{p^{i+k}}(-) = \gamma_{p^i}(-)$ in E_{**}^{∞}-terms. We can change $x_i^{I,J} = x_i$ such that (iii) holds as follows:

Consider the finite sequences;

$$\{x_0 = V^i x_i \xleftarrow{V} V^{i-1} x_i \xleftarrow{V} \cdots \xleftarrow{V} V x_i \xleftarrow{V} x_i\}_{i \geq 0}.$$

As $H_* \underline{k(n)}_{m+1}$ has a finite order in each degree, we can find in the infinite tree an infinitely long branch;

$$x_0 \xleftarrow{V} x_1' \xleftarrow{V} \cdots \xleftarrow{V} x_i' \xleftarrow{V} x_{i+1}' \xleftarrow{V} \cdots$$

Let x_i' be x_i again.

Return to the proof of (3.4). Now we obtain:

$$x_i^{*p} = (V x_{i+1})^{*p} = [p](x_{i+1}) = 0.$$

Going to part (e), we can determine the extension of E_{**}^{∞} as algebra completely. That is, $H_* \underline{k(n)}_*$ has following generators:

$$x_i^{I,J},\ j_0 = p^n - 1 \text{ for (2) of (d)}$$

$$y_k \text{ for others of (d).}$$

We define the filtration degree of elements:

$$\mathrm{filt}(x) = \min\{s \mid x \in F_{s*}\},$$

where F_{s*} is the filtration derived from the bar spectral sequence. Then we see in E_{**}^{∞} that

(3.5) $\mathrm{filt}(y_k) \geq p^i$ implies $V^i y_k$ is also the generator of (d),

$\mathrm{filt}(y_k) < p^i$ implies $V^i y_k = 0$,

$\mathrm{filt}(x_i^{I,J}) = p^i$.

Write down the element by these generators.

$$(-1)^{\iota(I)} \underline{\eta}_*(\alpha^{s^i(I+\Delta_n)}) \circ b^{s^i(J-(p^n-1)\Delta_0)}$$
$$= \sum c_{I',J',j} \cdot x_j^{I',J'} + \sum c_j \cdot y_j + decomp.,$$

$c_{I',J',j}$ and c_j are elements of $\boldsymbol{Z}/p$.

Apply $V^i(-)$ on it, then we have by (3.5)

$$(-1)^{i(I)}\underline{\eta}_*(\alpha^{I+\Delta_n})\circ b^{J-(p^n-1)\Delta_0}$$

$$=\sum_{j\geq i} c_{I',J',j}\cdot x_{j-i}^{I',J'}+\sum_{\mathrm{filt}(y_j)\geq p^i} c_j\cdot V^i y_j+decomp..$$

The elements on the right side are clearly linearly independent. Thus we get by (i):

$$c_{I',J',j}=\begin{pmatrix} 1 \text{ if } (I',\ J',\ j)=(I,\ j,\ i) \\ 0 \text{ otherwise } (j\geq i), \end{pmatrix}$$

$$c_j=0 \text{ if } \mathrm{filt}(y_j)\geq p^i.$$

Thus we obtain

$$(-1)^{i(I)}\underline{\eta}_*(\alpha^{s^i(I+\Delta_n)})\circ b^{s^i(J-(p^n-1)\Delta_0)}$$

$$=x_i^{I,J}+\sum_{j<i} c_{I',J',j}\cdot x_j^{I',J'}+\sum_{filt(y)<p^i} c_j\cdot y_j+decomp..$$

Therefore

$$\gamma_{p^i}(\sigma e_1\circ a^I b^J) \text{ represents}$$

$$(-1)^{i(I)}\underline{\eta}_*(\alpha^{s^i(I+\Delta_n)})\circ b^{s^i(J-(p^n-1)\Delta_0)}\text{-}decomp..$$

(e). When the differentials are determined, we can compute the spectral sequence easily. (cf. [Lemma 6.9, 6.10, R-W$_2$]) As mentioned in the proof of (c), the elements of conditions (C), (D) are hitted by differentials, hence the rests are (B) and (B$'$). These are generators of the first E-algebra part. The second E-algebra part is clear. As to the TP_1-algebra parts, we get, in E_{**}^{∞}, from (a) by (c):

$$\bigotimes_{(E)\text{ or }(F)\text{ or }(G)} TP_1(\gamma_{p^i}(\sigma e_1\circ a^I b^J\circ[v_n^k]))$$

$$\bigotimes_{(H)} TP_1(\gamma_{p^i}(\phi(a^I b^J\circ[v_n^k])))$$

$$\bigotimes_{(I,J)=ad.} TP_1(\gamma_{p^i}(\sigma e_1\circ\underline{\eta}_*(\alpha^I)\circ b^J))$$

$$\bigotimes_{(I,J)=ad.} TP_1(\gamma_{p^i}(\phi(\underline{\eta}_*(\alpha^I)\circ b^J))),$$

where the conditions are following:

(E) $I\neq 0,\ j_0<p^n-1,\ 0\leq i<\eta(I)$,

(F) $I=0,\ j_0<p^n-1$,

(G) $j_0=p^n-1,\ k=0$,

(H) $i_{n-1} = 0,\ 0 \leq i < \eta(I) - 1$.

According to (d), we see the correspondeces between the elements of E^{∞}_{**}-term and those of $H_*\underline{k(n)}_*$:

1)
$$\begin{aligned}&\{\gamma_{p^i}(\sigma e_1 \circ a^I b^J \circ [v_n^k]) \mid (\mathrm{E})\}\\ &\quad \cup \{\gamma_{p^i}(\phi(a^I b^J \circ [v_n^k])) \mid (\mathrm{H})\}\\ &\quad \cup \{\gamma_{p^i}(\sigma e_1 \circ a^I b^J \circ [v_n^k]) \mid (\mathrm{F})\}\end{aligned}$$

represents (up to sign)

$$\begin{aligned}&\{a^I b^J \circ [v_n^k] \mid I \neq 0,\ m(I) \geq m(J)\}\\ &\quad \cup \{a^I b^J \circ [v_n^k] \mid I \neq 0,\ m(I) < m(J)\}\\ &\quad \cup \{a^I b^J \circ [v_n^k] \mid I = 0,\ J \neq 0\}\\ &\quad = \{a^I b^J \circ [v_n^k] \mid I \neq 0 \text{ or } J \neq 0\},\end{aligned}$$

where $m(I) = \min\{s \mid i_s > 0\}$ as before and $m(0) = \infty$. These are the generators of the first TP_1-algebra part of (d).

2)
$$\begin{aligned}&\{\gamma_{p^i}(\sigma e_1 \circ a^I b^J) \mid (G)\}\\ &\quad \cup \{\gamma_{p^i}(\sigma e_1 \circ \underline{\eta}_*(\alpha^I) \circ b^J) \mid (I, J) = ad.\}\\ &\quad \cup \{\gamma_{p^i}(\phi(\underline{\eta}_*(\alpha^I) \circ b^J)) \mid (I, J) = ad.\}\end{aligned}$$

respresents (up to non zero scalar)

$$\begin{aligned}&\{\underline{\eta}_*(\alpha^I) \circ b^J \mid (I, J) = ad.,\ m(I) < m(J),\ M(I) - m(I) \leq n\}\\ &\quad \cup \{\underline{\eta}_*(\alpha^I) \circ b^J \mid (I, J) = ad.,\ m(I) \geq m(J)\}\\ &\quad \cup \{\underline{\eta}_*(\alpha^I) \circ b^J \mid (I, J) = ad.,\ m(I) < m(J),\ M(I) - m(I) > n\}\\ &\quad = \{\underline{\eta}_*(\alpha^I) \circ b^J \mid (I, J) = ad.\}.\end{aligned}$$

These are the generators of the second TP_1-algebra part of (d). □

§4. Results on $H_*\underline{bu}_*$

In this section we compute $H_*\underline{bu}_*$.

Let $b_i \in H_{2i}\underline{bu}_2$ be the usual element which is defined by the complex orientation $u: CP^{\infty} \to \underline{bu}_2$. We define $b^J = b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \cdots$ for $J = (j_0, j_1, \ldots)$ with $b_{(i)} = b_{p^i}$ as before. And let e be the canonical generator of $H_1\underline{bu}_1$ defined by the map $S^1 \to \underline{bu}_1$ which induces the unit of the spectrum. $[t]$ is in $H_0\underline{bu}_{-2}$.

The formal group for bu is the same as for K, that is;

$$\mu_{bu}(x, y) = x + y + t \cdot xy,\ t \in \pi_2 bu = \boldsymbol{Z}[t].$$

Let $b(x) = \sum_{i \geq 0} b_i x^i$, then we have by [Theorem 3.8, R-W$_1$]

$$b(x+y) = b(x) * b(y) * ([t] \circ b(x) \circ b(y)) \tag{4.1}$$

And we have

$$[t] \circ b(x) \circ b(y) = b(x+y) * b(x)^{-1} * b(y)^{-1} \tag{4.2}$$

$$[t] \circ b(x) \circ b(y) \equiv b(x+y) - b(x) - b(y) \tag{4.3}$$

(modulo decomposables).

Iterating (4.2), we obtain

$$[t^{n-1}] \circ b(x_1) \circ b(x_2) \circ \cdots \circ b(x_n) \tag{4.4}$$

$$= \frac{\prod_{i=1}^{\binom{n}{0}} b(\sigma_{n,i}) * \prod_{i=1}^{\binom{n}{2}} b(\sigma_{n-2,i}) * \cdots\cdots}{\prod_{i=1}^{\binom{n}{1}} b(\sigma_{n-1,i}) * \prod_{i=1}^{\binom{n}{3}} b(\sigma_{n-3,i}) * \cdots\cdots},$$

where the each $\sigma_{k,i}\left(1 \leq k \leq n,\ 1 \leq i \leq \binom{n}{k}\right)$ is the sum of different k terms in $\{x_1, x_2, \ldots, x_n\}$.

Proposition 4.5. *Let p be an odd prime. In $H_*\underline{bu}_2$,*

(*a*) $[t] \circ b_i \circ b_j \equiv (i, j) b_{i+j}$ *mod decomposables,*

(b) $[t^{p-1}] \circ b_1^{\circ p} = -b_1^{*p}$,

(c) $[t^{p-1}] \circ b_{(i)}^{\circ p} = -(b_{(i)} + \textit{decomp.})^{*p^n}$,

(d) $(-1)^n [t^{n(p-1)}] \circ b^{p\Delta_0 + (p-1)(\Delta_1 + \ldots + \Delta_{n-1})} = b_1^{*p^n}$. □

Remark 4.6. We know the general facts:

(a) If $\deg x > 0$, then $[0_n] \circ x = 0$ for $[0_n] \in H_0 \underline{bu}_m$.

(b) $\Psi b_i = \sum_j b_j \otimes b_{i-j}$ for coproduct Ψ.

b_i is primitive, therefore $b_1 \circ x$ is primitive for any x, and $[t] \circ b_1 \circ b_i = (-1)^{i+1} (c_{i+1})^* \in H_* \underline{bu}_2 = H_* BU$, the dual element of the $i+1$-th chern class.

(c) $e \circ (-)\colon H_* \underline{bu}_m \to H_* \underline{bu}_{m+1}$ is equal to the suspension map, e is primitive and $e \circ e = b_1$.

(d) (primitives) $\circ$ (decomposables) $= 0$. □

Proof of Proposition 4.5.

(a). Compare the each coefficients in (4.3).

(b). This is the special case of (c).

(c). In (4.4), let $n = p$ and $x_1 = x_2 = \cdots = x_p = x$. Then we have

$$[t^{p-1}] \circ b^{\circ p}(x^p) = [t^{p-1}] \circ b(x)^{\circ p}$$

$$= \prod_{i=1}^{p-1} b((p-i)x)^{*(-1)^i \binom{p}{i}}$$

$$= \Big(\prod_{i=1}^{p-1} b((p-i)x)^{*(-1)^i\binom{p}{i}/p}\Big)^{*p}.$$

We calculate the entry of ( ) modulo decomposables.

$$\prod_{i=1}^{p-1} b((p-i)x)^{*(-1)^i\binom{p}{i}/p} \equiv \prod_{i=1}^{p-1} \sum_{k\geq 0} (-1)^i \frac{\binom{p}{i}}{p} (p-i)^k b_k x^k$$

$$\equiv \sum_{i=1}^{p-1} \sum_{k\geq 0} (-1)^i \frac{\binom{p}{i}}{p} (p-i)^k b_k x^k.$$

Let

$$\varphi(k) = \sum_{i=1}^{p-1} (-1)^i \frac{\binom{p}{i}}{p} (p-i)^k,$$

$$= \sum_{i=1}^{p-1} (-1)^i \binom{p-1}{i} (p-i)^{k-1},$$

then we have

$$\sum_{k\geq 0} [t^{p-1}] \circ b_k^{\circ p} x^{pk} = \Big(\sum_{k\geq 0} (\varphi(k) b_k + decomp.)\, x^k\Big)^{*p}$$

$$= \sum_{k\geq 0} (\varphi(k) b_k + decomp.)^{*p} x^{pk}.$$

When $k = p^s$, we see $\varphi(p^s) \equiv -1$ modulo p. Thus we get

$$[t^{p-1}] \circ b_{(s)}^{\circ p} = (-b_{(s)} + decomp.)^{*p}.$$

The result follows.

(*d*). $(b_1)^{*p^n} = (-[t^{p-1}] \circ b^{p\Delta_0})^{*p^{n-1}}$ (by (b))

$= -[t^{p-1}] \circ (b^{\Delta_0} \circ V^{n-1} b^{(p-1)\Delta_{n-1}})^{*p^{n-1}}$ (by $V^{n-1} b^{\Delta_{n-1}} = b^{\Delta_0}$)

$= -[t^{p-1}] \circ (b^{\Delta_0})^{*p^{n-1}} \circ b^{(p-1)\Delta_{n-1}}$ (by $(x \circ Vy)^{*p} = x^{*p} \circ y$)

by induction on n,

$$= -[t^{p-1}] \circ (-1)^{n-1} [t^{(n-1)(p-1)}] \circ b^{p\Delta_0 + (p-1)(\Delta_1 + \dots + \Delta_{n-2})} \circ b^{(p-1)\Delta_{n-1}}$$

$$= (-1)^n [t^{n(p-1)}] \circ b^{p\Delta_0 + (p-1)(\Delta_1 + \dots + \Delta_{n-1})} \qquad \square$$

Let g be a factor of $bu_{(p)}$, the connective ring spectrum such that: There are a map $\Phi_0 \colon bu_{(p)} \to g$ and a multplicative inclusion $\iota \colon g \to bu_{(p)}$ with $\Phi_0 \circ \iota = id_g$. And $bu_{(p)} = \coprod_{i=0}^{p-2} \Sigma^i g$.

$\pi_* g = Z_{(p)}[v] \subset \pi_* bu_{(p)} = Z_{(p)}[t]$ with $v = t^{p-1}$.

We define the complex orientation $v: CP^\infty \to \underline{g}_2$ by $v = \Phi_0 \circ u$, and let $b_i \in H_{2i}\underline{g}_2$ be the element defined by v as usual, and let e be the canonical generator of $H_1 \underline{g}_1$ defined by the map $S^1 \to \underline{g}_1$, which gives the unit of the ring spectrum: $S \to g$. Then we have

(4.7) $$\underline{\Phi}_{0*} b_i = b_i.$$

By [Theorem 1, J] we know, in $bu^*_{(p)} CP^\infty$

$$\iota_* v = \sum_{n \equiv 0 \bmod p-1} \log(1 + tu)^{n+1}/t(n+1)!$$

$$= u - \frac{tu^2}{2} + \cdots + \frac{(p-1)!+1}{p!} t^{p-1} u^p - \left(\frac{1}{p+1} + \frac{1}{2(p-1)!}\right) t^p u^{p+1} + \cdots$$

Thereore we can calculate $\underline{\iota}_* b_{(i)}$ from this.

(4.8) $\underline{\iota}_* b_{(i)} \equiv b_{(i)}$ modulo decomp. in $H_* \underline{bu}_2$. Especially $\underline{\iota}_* b_1 = b_1$.

Proposition 4.9. *Let p be an odd prime. In $H_* \underline{g}_2$*

(*a*) $[v] \circ b_1^{\circ p} = - b_1^{*p}$,

(*b*) $[v] \circ b_{(i)}^{\circ p} = - (b_{(i)} + decomp.)^{*p}$,

(c) $(-1)^n [v^n] \circ b^{p\Delta_0 + (p-1)(\Delta_1 + \cdots + \Delta_{n-1})} = b_1^{*p^n}$. □

Proof. (a), (b). By (4.8)

$$\underline{\iota}_*([v] \circ b_{(i)}^{\circ p}) = [t^{p-1}] \circ (b_{(i)} + decomp.)^{\circ p}$$
$$= [t^{p-1}] \circ b_{(i)}^{\circ p} + [t^{p-1}] \circ (decomp.)^{\circ p}.$$

On the other hand, by Proposition 4.5 we know

$$[t^{p-1}] \circ b_{(i)}^{\circ p} = - (b_{(i)} + decomp.)^{*p},$$

and we can check

$$[t^{p-1}] \circ (decomp.)^{\circ p} = (decomp.)^{*p}.$$

Thus

$$\underline{\iota}_*([v] \circ b_{(i)}^{\circ p}) = - (b_{(i)} + decomp.)^{*p} + (decomp.)^{*p}$$
$$= - (b_{(i)} + decomp.)^{*p}.$$

Apply $\underline{\Phi}_*(-)$, then

$$[v] \circ b_{(i)}^{\circ p} = - \underline{\Phi}_*(b_{(i)} + decomp.)^{*p}$$
$$= - (b_{(i)} + decomp.)^{*p}.$$

(c). It is done by the same procedure as (d) Proposition 4.5. □

Let ρ, ρ' be the localization by t, v of $bu_{(p)}$, g, then the square commutes:

$$\begin{array}{ccc} bu_{(p)} & \xrightarrow{\rho} & K_{(p)} \\ \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota'} \\ g & \xrightarrow{\rho'} & G \end{array} .$$

We define elements b_i in $H_{2i}\underline{K}_{(p)2}$ and $H_{2i}\underline{G}_2$ by $\underline{\rho}_* b_i$, $\underline{\rho}'_* b_i$, and let e be the canonical generator of $H_1\underline{K}_{(p)1}$, $H_1\underline{G}_1$, defined by $\underline{\rho}_* e$, $\underline{\rho}'_* e$. Then Proposition 4.5 is available for $H_*\underline{K}_*$ and Proposition 4.9 is also available for $H_*\underline{G}_*$.

Theorem 4.10.

$$H_*\underline{K}_{(p)*} \simeq \bigotimes_{J=ad.,k\in \boldsymbol{Z}} \{E(e\circ b^J\circ[t^k])\otimes P(b^J\circ[t^k])\}$$

$$H_*\underline{G}_* \simeq (\textit{Replace } t \textit{ by } v),$$

where, for $J=(j_0, j_1, \ldots)$, *we say* $J = ad.$ *if and only if* $0\le j_s<p$ *for all* s, *and if* $J=0$ *we regard* $P(b^J\circ[t^k])$ *as the group ring* $\boldsymbol{Z}/p[\boldsymbol{Z}_{(p)}]$. □

Theorem 4.11. *In the bar spectral sequences*

$$(a)\qquad E^2_{**} = H_{**}(H_*\underline{K}_{(p)*})$$

$$\simeq \bigotimes_{J=ad.,k\in \boldsymbol{Z}} \{\Gamma(\sigma e\circ b^J\circ[t^k])\otimes E(\sigma b^J\circ[t^k])\}$$

$$\Rightarrow H_*\underline{K}'_{(p)*+1},$$

$$E^2_{**} = H_{**}(H_*\underline{G}_*)$$

$$\simeq (\textit{Replace } t \textit{ by } v)$$

$$\Rightarrow H_*\underline{G}'_{*+1}.$$

(*b*) *The spectral sequences collapse, and*

(1) $(-1)^{n(J)}\gamma_{p^i}(\sigma e\circ b^J\circ[v^{n(J)}]) + decomp.$ *represents*

$$(b^{\Delta_i+s^i-n(J-(p-1)(\Delta_0+\cdots+\Delta_{n-1}))})^{*p^n}$$

where $n=n(J)=\max\{n|p-1=j_0=j_1=\cdots=j_{n-1}\}$, *in convention that* $\max\phi=0$.

(2) σb^J *represents* $e\circ b^J$. □

Lemma 4.12. *The filtration degree induced by the bar spectral sequences is given as following*:

$$\operatorname{filt}\{(b^{J_1})^{*e_1} * (b^{J_2})^{*e_2} * \cdots * (b^{J_k})^{*e_k}\}$$

$$= p^{m(J_1)}\sigma_p(e_1) + p^{m(J_2)}\sigma_p(e_2) + \cdots + p^{m(J_k)}\sigma_p(e_k)$$

with admiassible J_i's which are all different, where $m(J) = \min\{s | j_s > 0\}$, $\sigma_p(e) =$ *the sum of entries of the p-adic expansion of the integer e.* □

Remark. Theorem 4.10 is available for $H_*(-\,;\,\boldsymbol{Z}_{(p)})$. □

As we know $H_0\underline{K}_{(p)*}$ (resp. $H_0\underline{G}_*$), the bar spectral sequences give information of $H_{*+1}\underline{K}_{(p)*+1}$ (resp. $H_{*+1}\underline{G}_{*+1}$) from those of $H_*\underline{K}_{(p)*}$ (resp. $H_*\underline{G}_*$). We prove 4.10, 4.11, 4.12 simultaneously, but these are done by implicit inductions on degrees, modulo $\circ[t]$, (resp. $\circ[v]$) similatly to [W_2].

Proof of Theorem 4.11 *from Theorem* 4.10.

(a). It is obtained directly.

(b). In each spectral sequence, the generators of E^r_{**} are concentrated in odd or even degrees on each stage, therefoere the differentials are trivial. Next we examine representatives in E^∞_{**}-term. By degree reason, σe represents b^{Δ_0}. By compatibility between filtrations and pairings, we have

$$(-1)^n\sigma(e\circ b^{(p-1)(\Delta_0+\cdots+\Delta_{n-1})}\circ[v^n]) \text{ represents}$$
$$(-1)^n\, b^{p\Delta_0+(p-1)(\Delta_1+\cdots+\Delta_{n-1})}\circ[v^n]$$
$$=(b^{\Delta_0})^{*p^n} \text{ (by Proposition 4.5, 4.9).}$$

Next we show:

(4.13) $$(-1)^n\gamma_{p^i}(\sigma e\circ b^{(p-1)(\Delta_0+\cdots+\Delta_{n-1})}\circ[v^n]) + \textit{decomp.}$$
$$\textit{represents } (b^{\Delta_i})^{*p^n}.$$

By Lemma 4.12 (cf. 4.15), filt $(b_{(i)})^{*p^n} = p^i$, thus we can use the following commutative diagram:

$$\begin{array}{ccc} F_{p^i,*} & \xrightarrow{V^i} & F_{1,*} \\ \downarrow & & \downarrow\simeq \\ E^\infty_{p^i,*} & \xrightarrow{V^i} & E^\infty_{1,*} \end{array} .$$

Since $V^i(b^{\Delta_i})^{*p^n} = (V^ib^{\Delta_i})^{*p^n} = (b^{\Delta_0})^{*p^n}$ and $V^i\gamma_{p^i}(\sigma -) = \sigma -$, the images of the both elements of (4.13) coincide in E^∞_{1*}. While, we know Ker V^i = decomposables, by the form of E^∞_{**}-term. Therefore we obtain (4.13).

Apply $(-)\circ b^{s^i+{}^nJ}$ to (4.13), then

$$(-1)^n\gamma_{p^i}(\sigma e\circ b^{(p-1)(\Delta_0+\cdots+\Delta_{n-1})+s^nJ}\circ[v^n]) + \textit{decomp.}$$
$$=(-1)^n\gamma_{p^i}(\sigma e\circ b^{(p-1)(\Delta_0+\cdots+\Delta_{n-1})}\circ[v^n] + \textit{decomp.})\circ b^{s^i+{}^nJ}$$
$$\text{represents } (b^{\Delta_i})^{*p^n}\circ b^{s^i+{}^nJ} = (b^{\Delta_i+s^iJ})^{*p^n}.$$

This we get (1). (2) is already done. □

Proof of Theorem 4.10 *from Theorem* 4.11.

We only observe the following:

$$\{\Delta_i + s^{i-n(J)}(J-(p-1)(\Delta_0+\cdots+\Delta_{n(J)-1}))\,|\,i\geq 0,\ J=ad.\}$$
$$=\{J\,|\,J=ad.,\ J\neq 0\}. \qquad \square$$

Proof of Lemma 4.12. This is done by the induction on the degree together with Theorem 4.10, 4.11. Furthermore we use the induction on t, where $t = \max\{m(J_1), m(J_2), \dots\}$ for $m(J) = \min\{s\,|\,j_s > 0\}$.

When $t = 0$, by Proposition 4.5 (b) and Proposition 4.9 (a), we know filt $(b^{\Delta_0})^{*p^n} = 1$ and the pairing preserves the filtrations, thus we get

$$\text{filt}\,(b^{\Delta_0+J})^{*p^n} = \text{filt}\,(b^{\Delta_0})^{*p^n}\circ b^{s^nJ} = 1.$$

On the other hand, we see that if $x_1, x_2, \dots$ are represented by different generators of an algebra E^{∞}_{**}, then filt $(x_1^{e_1}x_2^{e_2}\cdots) = \sum e_s\cdot$filt x_s for $0 < e_s < p$. Therefore we affirm the statement for $t = 0$.

Suppose the statement is true for all $t(<i)$. By Proposition 4.5 (c) and 4.9 (b), we know

$$[v]\circ b^{p\Delta_i} = -(b^{\Delta_i})^{*p} + x^{*p} \tag{4.14}$$

with decomposable elements x. And we can also see x is written by b_j with $j < p^i$. By assumption of the induction on degrees, we may suppose Theorem 4.10 and 4.11 are true. Thus we conclude that x^{*p} can be written uniquely as a linear combination of

$$(b^{J_1})^{*e_1} * (b^{J_2})^{*e_2} * \cdots * (b^{J_k})^{*e_k}$$

with $m(J_t) < i$, $p\,|\,e_t$. Now, from the equation (4.14), we have

$$2p^{i+1} = \deg\,(b^{J_1})^{*e_1} * (b^{J_2})^{*e_2} * \cdots * (b^{J_k})^{*e_k}$$
$$= 2|J_1|\cdot e_1 + 2|J_2|\cdot e_2 + \cdots + 2|J_k|\cdot e_k,$$

where $|J| = \sum_s j_s p^s$.

$$p\,|\,e_s \text{ implies } e_s \geq p\cdot\sigma_p(e_s). \quad \text{Likewise, } |J_s| \geq p^{m(J_s)}.$$

We have

$$\text{filt}\,\{(b^{J_1})^{*e_1} * (b^{J_2})^{*e_2} * \cdots * (b^{J_k})^{*e_k}\}$$
$$= p^{m(J_1)}\sigma_p(e_1) + p^{m(J_2)}\sigma_p(e_2) + \cdots + p^{m(J_k)}\sigma_p(e_k)$$
$$\leq |J_1|\cdot\frac{e_1}{p} + |J_2|\cdot\frac{e_2}{p} + \cdots + |J_k|\cdot\frac{e_k}{p} = p^i.$$

The equation needs for $k = 1$, $J_1 = \Delta_{i-1}$, $e_1 = p$. But this element is excluded in x^{*p} of (4.14). This leads $x^{*p}\in F_{p^i-1}$. Again by (4.14), we get

$$\text{filt}\,(b^{\Delta_i})^{*p} = \text{filt}\,b^{p\Delta_i} = \text{filt}\,b_{(i)} = p^i. \tag{4.15}$$

Thus the lemma is true in the case for $k=1$, $J=\Delta_i$, $e_1=p$.

Next apply $(-)^{*p^{n-1}}$ to (4.14), then we have

$$(4.16)\qquad (b^{\Delta_i})^{*p^n} = ([v]\circ b^{p\Delta_i})^{*p^{n-1}} + x^{*p^n}.$$

Observe the filtration degree of the elements on the right side;

$$\begin{aligned}\operatorname{filt}([v]\circ b^{p\Delta_i})^{*p^{n-1}} &= \operatorname{filt}([v]\circ b^{\Delta_i})^{*p^{n-1}}\circ b^{(p-1)\Delta_{i+n-1}}\\ &= \operatorname{filt}(b^{\Delta_i})^{*p^{n-1}}\\ &= p^i, \text{ by induction on } n.\end{aligned}$$

From the assumption of the induction on t of this lemma, we know filt x^{*p^n} = filt $x^{*p} < p^i$. Thus we obtain from (4.16):

$$(4.17)\qquad \operatorname{filt}(b^{\Delta_i})^{*p^n} = p^i.$$

This is the lemma for $k=1$, $J=\Delta_i$ and $\sigma(e_1)=1$.

(4.17) goes the induction of Theorem 4.10, 4.11 on degrees. (See the proof of (4.13)). Observing the form of E^{∞}_{**}-term, like as for $t=0$, we can affirm the entire statement of this lemma for $m\leq i$. We finish the inductions. □

We generalize Proposition 4.5 (d) and Proposition 4.9 (c).

Corollary 4.18.

$(-1)^n[v]\circ b^{p\Delta_i+(p-1)(\Delta_{i+1}+\cdots+\Delta_{i+n-1})} \equiv (b^{\Delta_i})^{*p^n}$ *modulo* $F_{p^i-1,*}$. □

Proof. We see $[v]\circ b^{p\Delta_i} \equiv -(b^{\Delta_i})^{*p}$ mod $F_{p^i-1,*}$ from (4.14), so we can do the same process of the proof for Proposition 4.5 (d), since the pairing preserves filtrations. □

Consider the cofibre sequence of spectra:

$$\cdots \longrightarrow g \xrightarrow{\times p} g \xrightarrow{r} k(1) \xrightarrow{\delta'} \textstyle\sum g \longrightarrow \cdots$$

Definition 4.19. Let the map ∂ be the composite

$$\partial = \underline{\delta}'_*\underline{\eta}_*: H_*\underline{H}_* \xrightarrow{\underline{\eta}_*} H_*\underline{k(1)}_{*+2p-1} \xrightarrow{\underline{\delta}'_*} H_*\underline{g}_{*+2p}.$$ □

Lemma 4.20. $\partial(\alpha^{\Delta_0+\Delta_{q+1}}) = c\cdot b^{\Delta_0+p\Delta_q}$ *for* $q\geq 0$. □

Proof. From Proposition 2.4,

$$\underline{\eta}_*(a^{\Delta_{q+1}}) = c\cdot b^{p\Delta_q} + decomp..$$

Thus we have

$$\begin{aligned}\partial(\alpha^{\Delta_0+\Delta_{q+1}}) &= \underline{\delta}'_*\underline{\eta}_*(\alpha^{\Delta_0}\circ\alpha^{\Delta_{q+1}})\\ &= \underline{\delta}'_*\underline{\eta}_*(\underline{\pi}_*a_{(0)}\circ\alpha^{\Delta_{q+1}}),\end{aligned}$$

since η is a $k(1)$-module map,

$$= \underline{\delta}'_*(a_{(0)} \circ \underline{\eta}_*(\alpha^{\Delta_{q+1}}))$$

$$= \underline{\delta}'_*(a_{(0)} \circ (c \cdot b^{p\Delta_q} + decomp.))$$

$$= \underline{\delta}'_*(a_{(0)} \circ (c \cdot b^{p\Delta_q}))$$

$$= \underline{\delta}'_*(a_{(0)} \circ \underline{r}_*(c \cdot b^{p\Delta_q})),$$

since δ' is a g-module map,

$$= \underline{\delta}'_*(a_{(0)}) \circ (c \cdot b^{p\Delta_q}).$$

From the definition of $a_{(0)}$, $\underline{\delta}'_* a_{(0)} = b_{(0)}$, finally we obtain

$$\partial(\alpha^{\Delta_0 + \Delta_{q+1}}) = b_{(0)} \circ (c \cdot b^{p\Delta_q}). \qquad \square$$

We use the above lemma essentially to prove the next main theorem.

For $J = (j_0, j_1, \dots)$, $I = (i_0, i_1, \dots)$, $i_s = 0$ or 1. We say $J = ad.$ if and only if $0 \le j_s < p$ for all $s \ge 0$, and $(I, J) = ad.'$ if and only if the following conditions satisfies:

$$l(I) = i_0 + i_1 + \cdots \ge 2,$$

$$M_1(I) < s < M(I) \text{ implies } j_s = 0,$$

$$s \ge M(I) \text{ implies } j_s < p,$$

where $M(I)$ is defined by the largest integer s such that $i_s = 1$, and $M_1(I)$ is the second one, i.e., $M_1(I) = M(I - \Delta_{M(I)})$.

Theorem 4.21. *Let $H_* = H_*(\ ; \boldsymbol{Z}/p)$ for an odd prime p.*

$$H_*\underline{g}_* = \bigotimes_{\substack{J = ad.,k \ge 0, \\ or J = p\Delta_i + s^{i+1}K = ad.,k = 0, i \ge 0}} \{E(e \circ b^J \circ [v^k]) \otimes P(b^J \circ [v^k])\}$$

$$\bigotimes_{(I,J) = ad'.} \{E(e \circ \partial\alpha^I \circ b^J) \otimes TP_1(\partial\alpha^I \circ b^J)\}$$

where if $J = 0$ then $P(b^J \circ [v^k]) \simeq \boldsymbol{Z}/p[\boldsymbol{Z}_{(p)}]$. $\square$

Theorem 4.22. *In the bar spectral sequence $E^r_{**}(H_*\underline{g}_*) \Rightarrow H_*\underline{g}'_{*+1}$,*

(a) $E^2_{**} = H_{**}(H_*\underline{g}_*)$

$$\simeq \bigotimes_{\substack{J = ad.,k \ge 0 \\ or J = p\Delta_i + s^{i+1}K, K = ad.,k = 0, i \ge 0}} \{\Gamma(\sigma e \circ b^J \circ [v^k]) \otimes E(\sigma b^J \circ [v^k])\}$$

$$\bigotimes_{(I,J) = ad'.} \{\Gamma(\sigma e \circ \partial\alpha^I \circ b^J) \otimes E(\sigma\, \partial\alpha^I \circ b^J) \otimes TP_1(\phi(\partial\alpha^I \circ b^J))\}$$

The spectral sequence collapses, and we have follows: In E^{∞}_{**}-term (modulo decomposables),

(1) $\gamma_{p^i}(\sigma e \circ b^J \circ [v^k])$ *represents*

$$(-1)^h (b^{\Delta_i + s^{i-h}(J-(p-1)(\Delta_0 + \cdots + \Delta_{h-1}))} \circ [v^{k-h}])^{*p^h}$$

where $h = h(J, k) = \min\{n(J), k\}$.

(2) $c \cdot \gamma_{p^i}(\sigma e \circ b^{p\Delta_l + s^{l+1}K})$ *represents* $\partial\alpha^{\Delta_i + \Delta_{l+i+1}} \circ b^{s^{l+i+1}K}$.

(3) $\sigma b^J \circ [v]$ *represents* $e \circ b^J \circ [v^k]$.

(4) $\gamma_{p^i}(\sigma e \circ \partial\alpha^I \circ b^J)$ *represents* $(-1)^{i(I)} \partial\alpha^{s^i I} \circ b^{s^i(\Delta_0 + J)}$.

(5) $\sigma\partial\alpha^I \circ b^J$ *represents* $e \circ \partial\alpha^I \circ b^J$.

(6) $\gamma_{p^i}(\phi(\partial\alpha^I \circ b^J))$ *represents* $\partial\alpha^{\Delta_i + s^{i+1}I} \circ b^{s^{i+1}J}$. □

Consider the map ρ': $\underline{g}_i$ it is homotopy equivalence for $i < 2p-2$, and for $i = 2p-2$, $\underline{g}_{2p-2} \times \boldsymbol{Z}_{(p)} \simeq \underline{G}_{2p-2}$. Therefore we conclude that Theorem 4.21 and Theorem 4.22 are true for $m \leq 2p-2$, comparing Theorem 4.10 and Theorem 4.11. Our proofs for Theorem 4.21 and Theorem 4.22 are done by induction on m for $\sum_{i \leq m} H_* \underline{g}_i$ like as done for $H_* \underline{k(n)}_*$-case. The inductions go implicitly.

Proof of Theorem 4.22 *from Theorem* 4.21.

(a) Clear.

(b) Collapsing is clear because on $\partial\alpha^I$ it comes from collapsing of the bar spectral sequence for $H_* \underline{H}_*$, on $\gamma_{p^i}(\sigma e)$ it can be examined by collapsing for $H_* \underline{G}_*$, and other cases follow. Next we see what the elements of E^{∞}_{**} represent.

(1). For $J = (p-1)(\Delta_0 + \Delta_1 + \cdots + \Delta_{n-1})$, $k = n$ it is true since, sending to $H_* \underline{G}_2$ by $\underline{\rho}_*$ (this map is monic in this stage), we know the same result, Theorem 4.11, (b), (1). Genearlly we use the compatibility of the pairing between the E^{∞}_{**}-term and filtrations.

(2). For $i = 0$, Lemma 4.20 implies

$$c \cdot \sigma e \circ b^{p\Delta_l + s^{l+1}K} \text{ represents } c \cdot b^{\Delta_0 + p\Delta_l + s^{l+1}K} = \partial\alpha^{\Delta_0 + \Delta_{l+1}} b^{s^{l+1}K}.$$

Thus that is true. For $i > 0$, we can reduce them to this case, using V^i like as we did in the case for $H_* \underline{k(n)}_*$. (See the proof of Theorem 3.2 (d) (2).)

(3), (4), (5), (6). Clear. (cf. [W$_1$]) □

Proof of Theorem 4.21 *from Theorem* 4.22. We have to solve the extension problems of algebras. However, all we need now is to count the following sets.

P-algebra part.

$$\{(b^{\Delta_i + s^{i-h}(J-(p-1)(\Delta_0 + \cdots + \Delta_{h-1}))} \circ [v^{k-h}])^{*p^h}$$

$$| J = ad.,\ i \geq 0,\ k \geq 0,\ h = h(J, k)\}$$

$$= \{(b^J \circ [v^k])^{*p^n} \mid J = ad.,\ k \geq 0$$
$$\text{or } J = p\Delta_i + s^{i+1}K,\ K = ad.,\ i \geq 0,\ k = 0;\ k \geq 0\}.$$

TP_1-algebra part. Recall $m(J) = \{s \mid j_s > 0\}$, $m(0) = \infty$.

$$\{\partial\alpha^{\Delta_i + \Delta_{i+\iota+1}} \circ b^{i+\iota+1K} \mid i \geq 0,\ \iota \geq 0,\ K = ad.\}$$
$$\cup \{\partial\alpha^{s^i I} \circ b^{s^i(\Delta_0 + J)} \mid i \geq 0,\ (I, J) = ad.'\}$$
$$\cup \{\partial\alpha^{\Delta_i + s^{i+1}J} \mid i \geq 0,\ (I, J) = ad.'\}$$
$$= \{\partial\alpha^I \circ b^J \mid (I, J) = ad.',\ l(I) = 2,\ m(I) < m(J)\}$$
$$\cup \{\partial\alpha^I \circ b^J \mid (I, J) = ad.',\ m(I) \geq m(J)\}$$
$$\cup \{\partial\alpha^I \circ b^J \mid (I, J) = ad.',\ l(I) \geq 3,\ m(I) < m(J)\}$$
$$= \{\partial\alpha^I \circ b^J \mid (I, J) = ad.'\}. \qquad \square$$

We know

$$H_*\underline{bu}_{(p)*} \simeq \sum_{i=0}^{p-2} \underline{\iota}_* H_*\underline{g}_* \circ [t^i]$$

and

$$\underline{\iota}_* b^J = b^J + \textit{decomposables},$$

by (4.8). Thus we have the next theorem as a corollary.

Theorem 4.23. *Let $H = H(\ ; \boldsymbol{Z}/p)$ for an odd prime p.*

$$H_*\underline{bu}_{(p)*} \simeq \bigotimes_{\substack{J = ad.,k \geq 0, \\ or J = p\Delta_i + s^{i+1}K, K = ad., 0 \leq k < p-1, i \geq 0}} \{E(e \circ b^J \circ [t^k]) \otimes P(b^J \circ [t^k])\}$$

$$\bigotimes_{(I,J) = ad'., 0 \leq k < p-1} \{E(e \circ \partial\alpha^I \circ b^J \circ [t^k]) \otimes TP_1(\partial\alpha^I \circ b^J \circ [t^k])\}.$$

If $I = 0$, $P(b^J \circ [t^k])$ is considered to be the group ring $\boldsymbol{Z}/p[\boldsymbol{Z}_{(p)}]$, as usual.

The next theorem is an easy application.

Theorem 4.24. *Let $H_* = H_*(\ ; \boldsymbol{Z}/p)$ for an odd prime p. For the connective coverings;*

$$c\colon \underline{g}_0(2(p-1)n, \dots, \infty) \longrightarrow \underline{g}_0,$$
$$c'\colon BU(2n, \dots, \infty) \longrightarrow \boldsymbol{Z} \times BU,$$

we have

$$\operatorname{Im}[c_*\colon H_*\underline{g}_0(2(p-1)n, \dots, \infty) \longrightarrow H_*\underline{g}_0]$$
$$= \bigotimes_{\substack{J = ad., l(J) = k(p-1), k \geq n \\ or J = p\Delta_i + s^{i+1}K, K = ad., l(J) = n(p-1), i \geq 0,}} P(\bar{b}^J)$$

and

$$\mathrm{Im}\,[c'_*: H_*BU(2n, \ldots, \infty) \longrightarrow H_*Z \times BU]$$

$$= \bigotimes_{\substack{J = ad., l(J) \geq n \\ or J = p\Delta_i + s^{i+1}K, K = ad., 0 \leq l(J) - n < p-1, i \geq 0.}} P(\bar{b}^J)$$

Here for $J = (j_0, j_1, \ldots)$, $J = ad.$ *if and only if* $0 \leq j_s < p$. $l(J) = j_0 + j_1 + \cdots$, *and*

$$\bar{b}^J = b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \cdots [v^{\frac{l(J)}{p-1}}]$$

$$\text{or } = b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \circ \cdots \circ [t^{l(J)}].$$ □

Proof. We see that

$$\mathrm{Im}\ c \simeq \mathrm{Im}\ [[v^n] \circ : H_*\underline{g}_{2(p-1)n} \longrightarrow H_*\underline{g}_0]$$

and

$$[v] \circ \partial\alpha^I = [v] \circ \underline{\delta}'_*(\underline{\eta}(\alpha^I))$$
$$= \underline{\delta}'_*([v] \circ \underline{\eta}_*(\alpha^I))$$
$$= \underline{\delta}'_*(\underline{\phi}_*(\underline{\eta}_*(\alpha^I))) = 0,$$

because $\underline{\phi} \circ \underline{\eta} \simeq 0$. (see (2.3).) Thus $\mathrm{Im}\,[v] \circ (-)$ is spanned by the generators algebraically. On the other hand, we can see these generators are algebraically independent using filtration induced by the bar spectral sequence (Lemma (4.12)). The BU case is similar. □

Remark. (a) Our result for g agrees with Kochman's result. In $[K]$ he uses different notation; $bu_{p,0}$ for g, $L(k)$ for $\dfrac{l(J)}{p-1}$, $M(k)$ for $m(J)$, with $k = |J|$, and ${}_pB_*(n)$ for $\mathrm{Im}\,[c_*: H_*\underline{g}_{2(n+1)(p-1)} \to H_*\underline{g}_0]$.

(b) We can investigate easily:

$$\mathrm{Im}\,[H_*BU(2m, \ldots, \infty) \longrightarrow H_*BU(2n, \ldots, \infty)]$$

for $m > n$. The cohomological case is done by Singer [Si].

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