J. Math. Kyoto Univ. (JMKYAZ)
31-2 (1991) 583-591

A note on Γ_n^C-structures

In Memory of Professor Ryoji Shizuma

By

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Intoroduction

Let Γ_n^C denote the topological groupoid of germs of local analytic automorphisms of C^n and $B\Gamma_n^C$ denote a classifying space for Γ_n^C-structures. The differential induces a continuous homomorphism $\Gamma_n^C \to GL(n, C)$, hence also a continuous map

$$\nu: B\Gamma_n^C \longrightarrow BGL(n, C).$$

We convert this map to a fibration and write $F\Gamma_n^C$ for the homotopy fibre.

As is shown in Haefliger [5], [6], this space $F\Gamma_n^C$ is closely related to the integrability of almost complex structures on open manifolds.

Our main result concerns with the homotopy groups of this space.

Theorem 1. *Let $n<i<2n$ and $n=p+1$. Let $M=S^i \times R^{m-i}$, $m-i \geqq 1$, $m=2p+1$. Assume $i \not\equiv 2 \bmod 4$. Then there exists a one-to-one correspondence between the homotopy group $\pi_i(F\Gamma_n^C)$ and the set of all integrable homotopy classes of PC-structures of type $(p, 1)$ on M.*

Notes. (1) Haefliger-Sithananthan [7] has shown that $F\Gamma_1^C$ is 2-connected.

(2) Landweber proved that $F\Gamma_n^C$ is $(n-1)$-connected (Landweber [8]).

(3) It has been shown that $\pi_n(F\Gamma_n^C)=0$ (Adachi [1]).

(4) At present there is no information about the homotopy groups of $F\Gamma_n^C$ in the range between n and $2n$. On the other hand Bott [3] has exihibited homomorphisms of π_{2n+1} $(B\Gamma_n^C)$ onto C, hence $F\Gamma_n^C$ is at most $2n$-connected.

(5) Another reflection of the lack of information about $F\Gamma_n^C$ in dimension below $2n$ is the absence of an example of an almost complex structure on an open manifold which is not homotopic to an integrable almost complex structure.

Theorem 1 is deduced as a consequence of the following theorem.

Theorem 2. *Let $n<i<2n$ and $n=p+q$, $q \geqq 1$. Let $M=S^i \times R^{m-i}$, $m-i \geqq 1$, $m=2p+q$. Then there exists a one-to-one correspondence between the homotopy group $\pi_i(F\Gamma_n^C)$ and the set of all integrable homotopy classes of PC-structures of type (p, q) on M with the tangent bundle $T(\tilde{M}_0)$ of $\tilde{M}_0$ trivial as complex vector bundles, where $\tilde{M}_0$ denotes the partial complexification of $(M$, PC-structure).*

Received September 11, 1989

For the definition of the partial complexification, see § 1.

Theorem 2 is deduced from Transversality Theorem for PC-foliations on PC-manifolds, which we explane in § 1 and § 2. We show Transversality Theorem and Theorem 1 and 2 in § 3.

The author is grateful to his friends and Professor *H.* Imanishi for their kind encouragement and valuable criticism.

1. *PC*-manifolds

Here we recall on PC-structures on manifolds.

Let M be a real analytic manifold of dimension n. Let $\tilde{M}$ be a complex manifold. Let $f: M \to \tilde{M}$ be an immersion. Then for a point x of M, we denote by $D(f, x)$ the maximum complex subspace of $T_{f(x)}(M)$ contained in $(df)_x(T_x(M))$, i.e.,

$$D(f, x)=(df)_x(T_x(M)) \cap \sqrt{-1}(df)_x(T_x(M)).$$

Assume that $\dim_C D(f, x)$ is constant.

We define a subspace $D(x)$ of $T_x(M)$ by $(df)_x(D(x))=D(f, x)$, and a complex structure I_x on the vector space $D(x)$ by $(df)_x \circ I_x(X)=\sqrt{-1}(df)_x(X)$, for $X \in D(x)$. Then we see that $\{D(x);\ x \in M\}$ gives a subbundle D of $T(M)$, i.e., a differential system on M and $\{I_x;\ x \in M\}$ gives a cross-section I of the vector bundle $\mathrm{Hom}(D, D)$. For any local cross-sections X, Y of D, we have:

(1.1) 1) $[IX, IY]-[X, Y]$ is a local cross-section of D,

2) $[IX, IY]-[X, Y]=I([IX, Y]+[X, IY])$.

Definition 1.1. (1) Let D be a differential system on a real analytic manifold M and let I be a cross-section of $\mathrm{Hom}(D, D)$. Then the pair (D, I) is called an *almost PC-structure* on M if I_x is a complex structure on $D(x)$ and I satisfies (1.1), 1). Moreover, an almost PC-structure (D, I) is called *integrable* or a *PC-structure*, if it also satisfies (1.1), 2).

(2) Let (D, I), (D', I') be almost PC-structures on manifolds M, M', respectively. Let $\varphi: (M, D) \to (M', D')$ be an isomorphism, namely φ is a diffeomorphism and $d\varphi: T(M) \to T(M')$ is an isomorphism of vector bundles which maps D onto D' isomorphically. Then φ is called an *isomorphism* of $(M; D, I)$ onto $(M'; D', I')$, if $(d\varphi)_x \circ I_x(X) = I' \circ (d\varphi)_x(X)$, for $X \in D(x)$.

By the above argument, we know that with every immersion f of M into a complex manifold $\tilde{M}$ with $\dim_C D(f, x)$ constant there is associated a PC-structure (D, I) on M in a natural manner.

We refer to Tanaka [10] for the following informations.

Proposition 1.1. *Let f be a real analytic imbedding of a real analytic manifold M into a complex manifold $\tilde{M}$ such that $\dim_C D(f, x)$ is constant, and let (D, I) be the corresponding PC-structure on M. Then there is a complex submanifold $\tilde{M}_0$ of $\tilde{M}$ such that $f(M) \subset \tilde{M}_0$ and*

$$\dim M = \dim_C \tilde{M}_0 + \dim_C D(f, x), \qquad x \in M.$$

Moreover, $\tilde{M}_0$ is uniquely determined as a germ.

Proposition 1.2. *Let (D, I) be a PC-structure on a real analytic manifold M. Then there are a complex manifold $\tilde{M}$ and a real analytic imbedding f of M into $\tilde{M}$ such that the imbedding f satisfies*

$$\dim M = \dim_C \tilde{M} + \dim_C D(f, x), \qquad x \in M \tag{1.2}$$

and the given (D, I) is the PC-structure on M corresponding to the imbedding f.

Proposition 1.3. *Let f and f' be real analytic imbeddings of real analytic manifolds M and M' to complex manifolds $\tilde{M}$ and $\tilde{M}'$ satisfying* (1.2), *respectively. Let (D, I) and (D', I') be the corresponding PC-structures on M and M', respectively. Let $\varphi: M \to M'$ be a homeomorphism. Then $\varphi: (M; D, I) \to (M'; D', I')$ is an isomorphism if and only if there is a biholomorphic homeomorphism $\tilde{\varphi}$ of a neighborhood of $f(M)$ onto a neighborhood of $f'(M')$ such that $\tilde{\varphi} \circ f = f' \circ \varphi$. Moreover, $\tilde{\varphi}$ is uniquely determined by φ as a germ.*

By Proposition 1.1, Proposition 1.2 and Proposition 1.3, corresponding to a *PC*-structure (D, I) on M, the germ $\tilde{M}_0$ is uniquely determined. This complex manifold $\tilde{M}_0$ we call the *partial complex extension* or *partial complexification* of $(M; D, I)$.

Let M be a real analytic manifold of dimension $n = 2p + q$, $q \geqq 1$. By Proposition 1.1, 1.2 and 1.3 we have the following diagram:

$$\begin{tikzcd}
\left\{\begin{array}{l} (D, I);\ PC\text{-structure} \\ \qquad\quad \text{of type } (p, q) \\ \qquad\quad \text{on } M \end{array}\right\} \arrow[d, shift left, "\alpha"] & \\
 & \left\{\begin{array}{l} (f, \tilde{M});\ \tilde{M}: \text{complex manifold} \\ \qquad\qquad\quad \text{of dimension } p+q, \\ \qquad\quad f: M \to \tilde{M},\ C^\omega\text{-immersion} \\ \qquad\qquad\quad \dim_C D(f, x) = p \end{array}\right\} \arrow[ul, "\pi"'] \\
\left\{\begin{array}{l} (f, \tilde{M});\ \tilde{M}: \text{complex manifold} \\ \qquad\qquad\quad \text{of dimension } p+q \\ \qquad\quad f: M \to \tilde{M},\ C^\omega\text{-imbedding} \\ \qquad\qquad\quad \dim_C D(f, x) = p \end{array}\right\} \arrow[uu, shift left, "\beta"] \arrow[ur, "\iota"] &
\end{tikzcd},$$

$$\beta \circ \alpha = 1,$$

$$\alpha \circ \beta = 1 \quad \text{as germ},$$

$$\pi \circ \iota = \beta.$$

Here, the *PC*-structure of type (p, q) means the *PC*-structure with $\dim D(x) = 2p$ for any $x \in M$.

2. *PC*-submersions and *PC*-foliations

Let M be a real analytic manifold of dimension $n=2p+q$, $q \geqq 1$. Let $\tilde{M}$ be a complex manifold. Then we call a smooth map $f: M \to \tilde{M}$ a *PC-submersion* of type (p, q), if the following conditions are satisfied:

i) $(df)_x: T_x(M) \to T_{f(x)}(\tilde{M})$ is a monomorphism for each $x \in M$,

ii) $\dim_C D(f, x) = p$, for each $x \in M$.

When $\tilde{M}$ is a complex manifold of dimension $p+q$, the condition ii) is equivalent to the condition:

ii′) $(df)_x(T_x(M))$ is transversal to $\sqrt{-1}(df)_x(T_x(M))$ in $T_{f(x)}(\tilde{M})$.

Note. It might rather be said that f is a "partial real" immersion, when the above conditions are satisfied.

Definition 2.1. Let M be a real analytic manifold of dimension $n=2p+q$, $q \geqq 1$. Let $0 \leqq r \leqq p+q$. *A PC-foliation* $\mathcal{G}$ of codimension r of type (p, q) on M is represented by $\mathcal{G} = \{(V_\alpha, g_\alpha);\ \alpha \in A\}$ with the following conditions:

i) $M = \bigcup_{\alpha \in A} V_\alpha$ is an open covering,

ii) $g_\alpha: V_\alpha \to \boldsymbol{C}^r$ is a PC-submersion of type (p, q),

iii) for $V_\alpha \cap V_\beta \neq \emptyset$, there exists a continuous map $g_{\alpha\beta}: V_\alpha \cap V_\beta \to \Gamma_r^{\boldsymbol{C}}$ satisfying $g_\alpha = g_{\alpha\beta} \circ g_\beta$ on $V_\alpha \cap V_\beta$.

Remark that PC-foliations $\mathcal{G}$ of codimension $p+q$ on M of type (p, q) are nothing but PC-structures on M.

Let (D, I), (D', I') be PC-structures on M. We call (D, I) and (D', I') are *integrably homotopic*, if there is a PC-structure $(\hat{D}, \hat{I})$ of type $(p, q+1)$ on $M \times [0, 1]$, i.e., $\hat{D}$ is a $2p$-dimensional subbundle of $T(M \times [0, 1])$ and I is a cross-section of $\mathrm{Hom}(\hat{D}, \hat{D})$ with $\hat{I} \circ \hat{I} = -1$, satisfying condition (1.1) on each point $(x, t) \in M \times [0, 1]$, such that

i) if we denote by D_t the portion of $\hat{D}$ over $M \times \{t\}$, and by $I_t = \hat{I} \mid D_t$, $(D_0, I_0) = (D, I)$, and $(D_1, I_1) = (D', I')$,

ii) for each $t \in [0, 1]$, (D_t, I_t) is a PC-structure of type (p, q) on $M \times \{t\}$.

We denote this relation by $(D, I) \underset{i}{\simeq} (D', I')$. Then this is an equivalence relation.

Let (D, I), (D', I') be PC-structures on M. To these correspond PC-foliations $\mathcal{G}$ and $\mathcal{G}'$, respectively. If (D, I) is integrably homotopic to (D', I'), then $\mathcal{G}$ is integrably homotopic to $\mathcal{G}'$.

Now we have the following map Φ:

$$\Phi: \{(D, I);\ PC\text{-structure of type } (p, q) \text{ on } M\}/\underset{i}{\simeq}$$

$$\longrightarrow \left\{ h \;\middle|\; \begin{tikzcd} & B\Gamma^{\boldsymbol{C}}_{p+q} \arrow[d, "\nu"] \\ M \arrow[ur, "h"] \arrow[r, "PC(\tau)"'] \arrow[ur, phantom, "(\simeq)", shift right=3] & BGL(p+q, \boldsymbol{C}) \end{tikzcd} \right\} \Big/ \simeq$$

where $PC(\tau)\colon M \to BGL(p+q, \boldsymbol{C})$ is the composite map $\tilde{\tau}\circ f$: to (D, I) corresponds a C^ω-imbedding $f\colon M \to \tilde{M}_0$ and $\tilde{\tau}\colon \tilde{M}_0 \to BGL(p+q, \boldsymbol{C})$ is a classifying map of the tangent bundle $T(\tilde{M}_0)$. Remark here the homotopy class of $PC(\tau)$ depends on the PC-structure (D, I).

Hereafter we take $q=1$. This does not loose generality for our purpose. Namely, given $2 \leqq n$ and $n < i < 2n$, there exist integers p and m such that $n = p+1$, $m = 2p+1$ and $m - i \geqq 1$.

Lemma 2.3. *Let $M = S^i \times R^{m-i}$, $m - i \geqq 1$. Assume $i \not\equiv 2 \bmod 4$. Then this is a real analytic manifold of dimension m. Let $m = 2p+1$, $i > 1$. In this case the tangent bundle $T(\tilde{M}_0)$ is trivial as complex vector bundle for any PC-structure on M.*

Proof. Let $f\colon M \to \tilde{M}_0$ be a real analytic imbedding corresponding to a PC-structure (D, I) on M. Now $T(M)$ is trivial and the normal bundle ν of $f(M)$ in $\tilde{M}_0$ is trivial. Identify M with $f(M)$. Let us denote the projection of the normal bundle ν by π. Then

$$T(\tilde{M}_0) \sim \pi^* T(M) \oplus \mathcal{E}^1,$$

where $\mathcal{E}^1$ is the 1-dimensional trivial bundle over $\tilde{M}_0$. Thus we have that $T(\tilde{M}_0)$ is trivial as real vector bundle. By Bott periodicity (cf. Bott [2]), we know that for $i \not\equiv 2 \bmod 4$, the homomorphism $\rho_*\colon \pi_i(BU(p+1)) \to \pi_i(BSO(2p+2))$ induced by the canonical inclusion $\rho\colon U(p+1) \to SO(2p+2)$ is a monomorphism for $i < 2p+1$. Thus we have obtained the lemma.

Remark. We take $q=1$. In our case, the definition of PC-foliation of codimension r has the meaning only when $r = p+1$.

By the above lemma, in the situation of the lemma, the map Φ defined above becomes the following form:

$$\Phi\colon \{(D, I);\ PC\text{-structure of type } (p, 1) \text{ on } M\}/\underset{i}{\simeq}$$

$$\longrightarrow \left\{ h \;\middle|\; \begin{tikzcd} & B\Gamma^C_{p+1} \arrow[d, "\nu"] \\ M \arrow[ur, "h"] \arrow[r, "*"'] \arrow[r, phantom, bend left, "(\simeq)"] & BGL(p+1, \boldsymbol{C}) \end{tikzcd} \right\} \Big/ \simeq \quad = \pi_i(F\Gamma^C_{p+1}),$$

where $*$ denotes the constant map.

3. Proof of Theorem 1 and 2.

Now we show Theorem 1 and 2. Our proof is similar to that of Theorem 2 in Haefliger [5].

First, we show that the map Φ is surjective for $q=1$. Suppose that $2 \leqq n < i < 2n$, $n = p+1$, $M = S^i \times R^{m-i}$, $m - i \geqq 1$, and that $i \not\equiv 2 \bmod 4$.

Let $h\colon M \to B\Gamma^C_{p+1}$ be a lifting of the constant map $*\colon M \to BGL(p+1, C)$. Correspondong to the map h we have a Γ^C_{p+1}-structure $\mathcal{Q}$ on M:

$$\mathcal{G}=\{(V_\lambda, \phi_\lambda);\ \lambda\in\Lambda\}.$$

Associated with this Γ^{C}_{p+1}-structure $\mathcal{G}$, we have a Γ^{C}_{p+1}-foliated microbundle $(E, \xi, \mathcal{E})$:

$$\xi:\ M\xrightarrow{\ i\ }E\xrightarrow{\ j\ }M,$$

$$\mathcal{E}:\ \Gamma^{C}_{p+1}\text{-foliation on } E$$

(cf. Haefliger [5]). Here E can be considered as a PC-manifold of type $(2p+1, 1)$, and $\mathcal{E}$ can be considered as a PC-foliation of codimension $p+1$ of type $(2p+1, 1)$.

Let $\mathcal{E}=\{(U_\lambda, \varphi_\lambda);\ \lambda\in\Lambda\}$ and $f:\ M\to E$ be a C^∞-map. For each $\lambda\in\Lambda$, put $W_\lambda=f^{-1}(U_\lambda)$. Then we say f is *PC-transversal* to $\mathcal{E}$, if for each $\lambda\in\Lambda$

$$\varphi_\lambda\circ(f\,|\,W_\lambda):\ W_\lambda\longrightarrow U_\lambda\longrightarrow \boldsymbol{C}^{p+1}$$

is a PC-submersion. We denote by PC-Trans $(M, \mathcal{E})$ the set of all C^∞-maps $f:\ M\to E$ which are PC-transversal to $\mathcal{E}$ with C^∞-topology. Let PC-Trans ${}^\omega(M, \mathcal{E})$ be the subspace of PC-Trans $(M, \mathcal{E})$ which consists of real analytic maps. Clearly PC-Trans ${}^\omega(M, \mathcal{E})$ is dense in PC-Trans $(M, \mathcal{E})$.

Let $\nu\mathcal{E}$ be the normal bundle of Γ^{C}_{p+1}-foliation $\mathcal{E}$ on E. This is a complex vector bundle with fibre $\boldsymbol{C}^{p+1}$. Let $\phi:\ T(M)\to\nu\mathcal{E}$ be a homomorphism of vector bundles. We say ϕ is a *PC-epimorphism*, if for each $x\in M$,

i) $\phi_x:\ T_x(M)\to\nu\mathcal{E}$ is a monomorphism into a fibre,

ii) putting

$$D(\phi, x)=\phi_x(T_x(M))\cap\sqrt{-1}\,\phi_x(T_x(M)),$$

$$\dim_C D(\phi, x)=p.$$

Let PC-Epi$(T(M), \nu\mathcal{E})$ be the space of all PC-epimorphisms $\phi:\ T(M)\to\nu\mathcal{E}$ with compact-open topology.

Then we have the following continuous map:

$$\pi\circ d:\ PC\text{-Trans }(M, \mathcal{E})\longrightarrow PC\text{-Epi}(T(M), \nu\mathcal{E}),$$

which maps f to $\pi\circ df$. Here π denotes the natural projection $\pi:\ T(E)\to\nu\mathcal{E}$.

Now we state the PC-Transversality Theorem. This corresponds to the $\boldsymbol{C}$-Transversality Theorem in Landweber [8].

Theorem 3 (***PC*-Transversality Theorem**). *In the above situation, the continuous map*

$$\pi\circ d:\ PC\text{-}Trans\ (M, \mathcal{E})\longrightarrow PC\text{-}Epi(T(M), \nu\mathcal{E})$$

induces the surjection

$$(\pi\circ d)_*:\ \pi_0(PC\text{-Trans }(M, \mathcal{E}))\longrightarrow \pi_0(PC\text{-Epi}(T(M), \nu\mathcal{E}).$$

Proof. Now M is open and "PC-submersion" is an open condition. Therefore, we can apply Gromov's Theorem [4] (cf. Haefliger [5], [6]).

Namely, let $X=M\times E$, $V=M$ and $p:\ X\to V$ be the projection to the first factor. Let Sect(X) be the space of all smooth cross-sections of (X, p, V) with C^∞-topology.

Let us denote by (X^1, p^1, V) be the 1-jet bundle of germs of local smooth cross-sections of (X, p, V). Let $\mathrm{Sect}(X^1)$ be the space of all continuous cross-sections of (X^1, p^1, V) with compact-open topology. Then by taking 1-jet, we have the following continuous map

$$J^1\colon \mathrm{Sect}(X) \longrightarrow \mathrm{Sect}(X^1).$$

Let $C^\infty(M, E)$ be the space of all C^∞-maps of M into E with C^∞-topology, $\mathrm{Hom}(T(M), T(E))$ be the space of all homomorphisms of $T(M)$ into $T(E)$ with compact-open topology and $\mathrm{Hom}(T(M), \nu\mathcal{E})$ be the space of all homomorphisms of $T(M)$ into $\nu\mathcal{E}$ with compact-open topology. Then we have the following commutative diagram:

$$\begin{tikzcd}
C^\infty(M, E) \arrow[r, "d"] \arrow[d, "\varphi"'] & \mathrm{Hom}(T(M), T(E)) \arrow[r, "\pi"] \arrow[d, "\phi"'] & \mathrm{Hom}(T(M), \nu\mathcal{E}) \\
\mathrm{Sect}(X) \arrow[r, "J^1"] & \mathrm{Sect}(X^1), & PC\text{-}\mathrm{Epi}(T(M), \nu\mathcal{E}) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

where d is the map which sends f to its differential df, and π is the map which sends ϕ to $\pi\circ\phi$. The vertial arrows φ, ϕ are natural homeomorphisms.

Let $\square$ denote $\pi^{-1}(PC\text{-}\mathrm{Epi}(T(M), \nu\mathcal{E}))$. Then we have the following commutative diagram:

$$\begin{tikzcd}
C^\infty(M, E) \arrow[r, "d"] & \mathrm{Hom}(T(M), T(E)) \arrow[r, "\pi"] & \mathrm{Hom}(T(M), \nu\mathcal{E}) \\
PC\text{-}\mathrm{Trans}(M, \mathcal{E}) \arrow[r, "d"] \arrow[u, "\cup"'] & \square \arrow[r, "\pi|\square"] \arrow[u, "\cup"'] & PC\text{-}\mathrm{Epi}(T(M), \nu\mathcal{E}), \arrow[u, "\cup"']
\end{tikzcd}$$

where $\pi|\square$ is surjective. Therefore, $\pi_*\colon \pi_0(\square)\to\pi_0(PC\text{-}\mathrm{Epi}(T(M), \nu\mathcal{E}))$ is surjective. Consequently, for the proof of the theorem, it is sufficient to prove that

$$d_*\colon \pi_0(PC\text{-}\mathrm{Trans}(M, \mathcal{E})) \longrightarrow \pi_0(\square)$$

is surjective.

However, as is stated above, the condition "PC-submersion" is an open relation. Therefore, $\phi(\square)=\mathrm{Sect}(X^1, \Omega)$ for an open subset $\Omega\subset X^1$, where

$$\mathrm{Sect}(X^1, \Omega)=\{\sigma\in\mathrm{Sect}(X^1) \mid \sigma(M)\subset\Omega\},$$

and Ω is invariant under the transformation induced by diffeomorphism of M. To say more precisely, the fibre of (X^1, p^1, M) is $J^1(m, m+2(p+1))\times E=M(m+2(p+1), m; \boldsymbol{R})\times E$. Let $\Omega_0\subset M(m+2(p+1), m; \boldsymbol{R})$ be the open subset of the following matrices:

$$\begin{array}{cl} \begin{pmatrix} A \\ B \end{pmatrix} & \begin{array}{l}]m \\ ]2(p+1) \end{array} \end{array} \quad \text{with rank } B=m=p+1,$$

and $\Omega=\Omega_0\times E$. Then by Lemma 3.1 in the following we have $\phi(\square)=\mathrm{Sect}(X^1, \Omega)$. Let us denote by $\mathrm{Sect}(X, \Omega)$ the space $(J^1)^{-1}(\mathrm{Sect}(X^1, \Omega))$ in $\mathrm{Sect}(X)$. Then we have

$$\mathrm{Sect}(X, \Omega)=\varphi(PC\text{-}\mathrm{Trans}(M, \mathcal{E})).$$

By Gromov's Theorem, we obtain that

$$d_*: \pi_0(\mathrm{Sect}(X, \Omega) \longrightarrow \pi_0(\mathrm{Sect}(X^1, \Omega))$$

is bijective. Thus we have proved the theorem.

On the other hand, by the construction of $\mathcal{G}$ and $(E, \xi, \mathcal{E})$, we have

(1) $i^*\mathcal{E}\sim\mathcal{G}$ as Γ^C_{p+1}-structure,

(2) $h^*\mathcal{U}\sim\mathcal{G}$, where $\mathcal{U}$ is the universal Γ^C_{p+1}-structure on $B\Gamma^C_{p+1}$,

(3) $\nu\mathcal{G}$ is trivial as complex vector bundle.

Consequently, if we denote $\nu=\nu\mathcal{U}$, we have

$$\nu(i^*\mathcal{E}) \sim \nu\mathcal{G} \sim \nu(h^*\mathcal{U}) \sim h^*\nu .$$

So we have

$$i^*(\nu\mathcal{E}) \sim h^*\nu$$

and the following homomorphisms of vector bundles:

$$\begin{array}{ccccc} T(M) & \xrightarrow{\phi} & h^*\nu & \xrightarrow{\tilde{i}} & \nu\mathcal{E} \\ \downarrow & & \downarrow & & \downarrow \\ M & \longrightarrow & M & \xrightarrow{i} & E, \end{array}$$

where $\tilde{i}$ is a homomorphism induced by i and ϕ is a natural monomorphism into a trivial complex vector bundle on M; i. e. on each fibre the natural inclusion $\boldsymbol{R}^{2p+1}=\boldsymbol{C}^p\times\boldsymbol{R}\to\boldsymbol{C}^p\times\boldsymbol{C}=\boldsymbol{C}^{p+1}$. Then $\tilde{i}\circ\phi$ is a PC-epimorphism which induces $i: M\to E$.

By the PC-transversality Theorem, we obtain a C^∞-map $f: M\to E$, which is PC-transversal to $\mathcal{E}$. By the approximation theorem, we have a C^ω-map $f': M\to E$, which is PC-transversal to $\mathcal{E}$.

Then $f'^*\mathcal{E}$ is an analytic PC-foliation on M of codimension $p+1$, namely a PC-structure (D, I) on M of type $(p, 1)$. By the construction, $f'^*\mathcal{E}$ corresponds to the given map $h: M\to B\Gamma^C_{p+1}$. Thus the surjectivity of Φ is proved.

Injectivity of Φ is proved in quite a parallel way as Haefliger [5].

Lemma 3.1. *Let $\phi: \boldsymbol{R}^{2p+1}\to\boldsymbol{C}^{p+1}$ be an $\boldsymbol{R}$-monomorphism. Then $\phi(\boldsymbol{R}^{2p+1})$ and $\sqrt{-1}\phi(\boldsymbol{R}^{2p+1})$ are transversal in $\boldsymbol{C}^{p+1}$.*

Proof. Let us denote $X=\phi(\boldsymbol{R}^{2p+1})$. Let S^{2p+1} be the unit sphere in $\boldsymbol{R}^{2p+2}=\boldsymbol{C}^{p+1}$. Let $P_X\in S^{2p+1}$ be the point such that $(\overrightarrow{OP_X}, X)=0$ and $\overrightarrow{OP_X}$, X is positively related to the given orientation of $\boldsymbol{R}^{2p+2}$, where (,) denotes the inner product of $\boldsymbol{R}^{2p+2}=\boldsymbol{C}^{p+1}$. Let $P_{IX}\in S^{2p+1}$ be the point corresponding to $IX=\sqrt{-1}X$ as above.

Now $X\neq IX$, so we have $P_X\neq P_{IX}$. Therefore, we have $\dim(X\cap IX)=2p$. This shows that X is transversal to IX. Thus we have obtained the lemma.

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