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Embeddings of discrete series into induced representations of semisimple Lie groups, II

—Generalized Whittaker models for $SU(2, 2)$—

By

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Introduction

This is the second part of our work on embeddings of discrete series into various, important induced modules for semisimple Lie groups. Applying the general method established in the first part [10] (referred as [I] later on), we describe in this paper (generalized) Whittaker models for the simple Lie group $SU(2, 2)$ in an explicit way.

To be precise, we consider representations smoothly induced from characters of the unipotent radical of a cuspidal parabolic subgroup. The infinitesimal embeddings of discrete series are determined almost completely for such induced modules. Among other things, through this series of works we find all the embeddings into Gelfand-Graev representations, and also the zero-th $\mathfrak{n}$-cohomologies for the discrete series of $SU(2, 2)$. Note that our group is of real rank two, and that it is locally isomorphic to the (restricted) conformal group on the Minkowski space.

Now, let G be a connected semisimple Lie group with finite center, and K a maximal compact subgroup of G. We always assume the rank condition: $\mathrm{rank}(G)=\mathrm{rank}(K)$, which is necessary and sufficient for G to have a non-empty discrete series [2]. Each discrete series π_Λ of G has a unique lowest K-type τ_λ with highest weight λ(see 1.1). Further, the representation π_Λ can be realized on the L^2-kernel of gradient-type, G-invariant differential operator D_λ(see [7], cf. [I, Th. 1.5]). This D_λ is defined on the G-vector bundle over $K\backslash G$ attached to the K-module τ_λ.

From this realization of π_Λ, we can deduce that the L^2-kernel of D_λ characterizes the embeddings of π_Λ^*, the contragredient of π_Λ, into the left regular representation of G on $L^2(G)$. In fact, the exterior tensor product $\pi_\Lambda^*\hat{\otimes}\pi_\Lambda$ occurs in the bi-regular representation of $G\times G$ just once, and the functions in $L^2\text{-Ker}(D_\lambda)$ give rise to lowest K-type vectors in $L^2(G)$ of type $\tau_\lambda^*\subset\pi_\Lambda^*|K$ with respect to the left K-action.

Suggested by this fact, we formulated in the first half of [I] a general method for describing infinitesimal embeddings of discrete series into C^∞-induced G-modules. This is done by letting the operator D_λ act on the τ_λ^*-component of the induced module $\pi(\eta)=C^\infty\text{-Ind}_N^G(\eta)$ mentioned above, in a natural way (see 1.3 for the the precise definition). We have shown that, as in the regular representation case, solutions φ of the

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resulting differential equation $D_{\lambda,\eta}\varphi=0$ characterize the embeddings of π_Λ^* into $\pi(\eta)$ as $(\mathfrak{g}_C, K)$-modules:

$$\mathrm{Hom}_{\mathfrak{g}_C\text{-}K}(\pi_\Lambda^*, \pi(\eta))\cong\mathrm{Ker}(D_{\lambda,\eta}),$$

under certain assumptions on λ and η (see Theorem 1.3). Here $\mathfrak{g}_C$ denotes the complexified Lie algebra of G.

Although $D_{\lambda,\eta}\varphi=0$ is a single equation for a vector valued function φ on $K\backslash G/N$, it can be rewritten into a system of differential difference equations for the coefficients of φ. By solving such a system of differential equations, we determined in [I] all the embeddings of discrete series into (generalized) principal series for $SU(2, 2)$.

In the present article, we continue to study the case $G=SU(2, 2)$ in more detail. Up to conjugacy, our group G has two proper cuspidal parabolic subgroups P_m and P', where P_m is minimal and $P'\supset P_m$ maximal. Let N_m, N' denote the corresponding unipotent radicals respectively. Here, in Part II of our works, we deal with the G-modules $\Gamma_{\xi,N}=C^\infty\text{-}\mathrm{Ind}_N^G(\xi)$ induced from any character ξ of the unipotent subgroup $N=N_m$ or N', and we explicitly determine the embeddings of discrete series π_Λ^* into $\Gamma_{\xi,N}$ by the method explained above.

Our main results are given in Theorems 6.1 and 6.5, which describe the multiplicities of embeddings. One can construct the embeddings concretely through the corresponding lowest K-type vectors for $\Gamma_{\xi,N}$ which we gain by solving the equation $D_{\lambda,\xi}\varphi=0$.

Our results cover, as extreme cases, embeddings into the following two types of important representations. On one hand, the representation $\Gamma_{1_N,N}$ with the trivial character $\xi=1_N$ gives rise to the (generalized) principal series, studied in [I]. On the other hand, one gets (generalized) Gelfand-Graev representations (cf. [4], [6], [8], [9]) when ξ is generic.

To catch the main flow of our study, we now state three consequences of our results which allow us to classify the whole discrete series into three subclasses through generalized Whittaker models. Fix a regular integral infinitesimal character χ. Then G has exactly six mutually inequivalent discrete series representations with the same infinitesimal character χ. Two of them are holomorphic and anti-holomorphic discrete series, and the others are non-holomorphic ones. Assume that χ is sufficiently regular Then we obtain the following.

(1) Holomorphic and anti-holomorphic discrete series are characterized by the property that they never occur in $\Gamma_{\xi,N}$ with generic ξ and $N=N_m$ or N'.

(2) There exist precisely two discrete series that appear in all $\Gamma_{\xi,N}$'s, and so in particular they have ordinary Whittaker models in the sense of [1], [5].

(3) The remaining two discrete series can be embedded, with finite multiplicity, into generalized Gelfand-Graev representations $\Gamma_{\xi,N'}$ with certain generic ξ's. This property marks off these two discrete series from the other four.

In this way, the discrete series is classified into three subcategories. This idea of classifying representations goes way back to a pioneering work of Gelfand and Graev for SL_2 early in the 1960's.

This paper is organized as follows. In § 1, we review after [I, Part A] our general theory that tells how to describe the embeddings of discrete series into induced G-

modules $\pi(\eta)$ through the gradient-type differential operators $D_{\lambda,\eta}$.

On and after §2, we concentrate on the case $G=SU(2, 2)$. Let $G=KA_pN_m$ be an Iwasawa decomposition of G, and η a continuous Fréchet space representation of the maximal unipotent subgroup N_m. Since $K\backslash G/N_m\cong A_p$, any solution φ of $D_{\lambda,\eta}\varphi=0$ is uniquely determined by its restriction to the vector subgroup $A_p\cong \boldsymbol{R}^2$. We describe in §2 the radial A_p-part of $D_{\lambda,\eta}$, and give a system $C[\lambda, \eta]$ of differential difference equations on $\boldsymbol{R}^2$, which characterizes the embeddings $\pi_\Lambda^*\hookrightarrow\pi(\eta)=\Gamma_{\eta,N_m}=C^\infty\text{-Ind}_{N_m}^G(\eta)$.

The succeeding three sections, §§3-5, are devoted to solving the system $C[\lambda, \eta]$ for each λ and for the following two types of N_m-representations η. First in §§3-4 we study the case where $\eta=\xi$ is a character of N_m, and then in §5 the case of infinite-dimensional representation $\eta_\xi=C^\infty\text{-Ind}_{N'}^{N_m}(\xi)$ induced from a character ξ of the unipotent radical N' of P'. Our results are perfect for almost all pairs (λ, ξ), and they enable us to describe in §6 (generalized) Whittaker models of discrete series π_Λ^* for the induced G-modules $\Gamma_{\xi,N}$ with $N=N_m$ or N'. In Tables 6.2 and 6.6, we give a list of multiplicities of embeddings of π_Λ^* into $\Gamma_{\xi,N}$, which seem to be very important invariants attached to the discrete series.

In a certain case with infinite-dimensional η_ξ, we construct a family of infinitely many, mutually linearly independent solutions of $C[\lambda, \eta_\xi]$ through formal power series (see 5.4 for details). This technique of construction is similar to the ones employed in [1], [3] and [6], although our object of study, i.e., the discrete series, is different from theirs.

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§1. Gradient-type differential operators and embeddings of discrete series

Let G be a connected semisimple Lie group with finite center, and K a maximal compact subgroup of G. As in Introduction, we assume that G and K are of equal rank. In this section we review a general theory for describing embeddings of discrete series into various induced G-modules, given in the first part [I] of this series of works.

To be more precise, each discrete series representation is characterized by its lowest K-type. Therefore the embeddings of discrete series may be described by determining the corresponding lowest K-type vectors in the induced modules in question. In order to specify such K-type vectors, we utilize the gradient-type differential operator D_λ on $K\backslash G$ introduced in [7] for a geometric realization of discrete series, and give (a system of) differential equations characterizing the embeddings of discrete series.

1.1. The discrete series for G. At the beginning, let us fix notation and recall briefly some fundamental facts for discrete series representations. For more detailed accounts, see [I, §1] and the papers cited there.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively, and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$. By the assumption rank(G)=rank(K), $\mathfrak{g}$ has a compact Cartan subalgebra $\mathfrak{t}$ contained in $\mathfrak{k}$. Denote by Δ the root system of the complexification $\mathfrak{g}_C=\boldsymbol{C}\otimes_R\mathfrak{g}$ of $\mathfrak{g}$ with respect to $\mathfrak{t}_C=\boldsymbol{C}\otimes_R\mathfrak{t}$. The totality $\Delta_c\subset\Delta$ of compact roots forms a root subsystem of Δ. We denote by W(resp. W_c) the Weyl group of Δ(resp. Δ_c).

Once and for all we fix a positive system Δ_c^+ of Δ_c. Let Ξ_c^+ be the set of linear forms Λ on $\mathfrak{t}_C$ satisfying the following three conditions:

(1.1) $(\Lambda, \alpha)\neq 0$ for any $\alpha\in\Delta$, i.e., Λ is Δ-regular,

(1.2) $(\Lambda, \beta)\geqq 0$ for any $\beta\in\Delta_c^+$, i.e., Λ is Δ_c^+-dominant,

(1.3) the map $\exp H\mapsto\exp\langle\Lambda+\rho, H\rangle$ $(H\in\mathfrak{t})$ gives a unitary character of $T=\exp\mathfrak{t}\subset K$, i.e., $\Lambda+\rho$ is K-integral.

Here (,) denotes the W-invariant, non-degenerate bilinear form on $\mathfrak{t}_C^*$, the dual space of $\mathfrak{t}_C$, induced canonically from the Killing form of $\mathfrak{g}_C$, and ρ is half the sum of positive roots in Δ with respect to any fixed positive system.

By Harish-Chandra, the set Ξ_c^+ parametrizes the discrete series of G as follows.

Proposition 1.1 (cf. [I, Prop. 1.1]). (1) *For each $\Lambda\in\Xi_c^+$, there exists a unique (up to equivalence) discrete series representation π_Λ of G whose character $\Theta_\Lambda=\mathrm{tr}(\pi_\Lambda)$ is expressed as*

$$\Theta_\Lambda(\exp H)=(-1)^{(\dim\mathfrak{p})/2}\frac{1}{D(H)}\{\textstyle\sum_{w\in W_c}\det(w)e^{\langle w\Lambda, H\rangle}\} \tag{1.4}$$

for $H\in\mathfrak{t}$ for which $D(H)=\prod_{\alpha\in\Delta^+}(e^{\langle\alpha, H\rangle/2}-e^{-\langle\alpha, H\rangle/2})$ does not vanish, where $\Delta^+=\{\alpha\in\Delta\mid(\Lambda, \alpha)>0\}$.

(2) *The map $\Lambda\mapsto\pi_\Lambda$ gives a bijective correspondence from Ξ_c^+ onto the set of equivalence classes of discrete series representations of G.*

We call $\Lambda\in\Xi_c^+$ the Harish-Chandra parameter of discrete series π_Λ. Note that $\Delta^+\supset\Delta_c^+$ by (1.2).

Now set for $\Lambda\in\Xi_c^+$,

$$\lambda=\Lambda-\rho_c+\rho_n=(\Lambda-2\rho_c)+\rho=(\Lambda+2\rho_n)-\rho, \tag{1.5}$$

where

$$\rho=\frac{1}{2}\textstyle\sum_{\alpha\in\Delta^+}\alpha, \qquad \rho_c=\frac{1}{2}\sum_{\alpha\in\Delta_c^+}\alpha, \qquad \rho_n=\rho-\rho_c \tag{1.6}$$

with the positive system $\Delta^+\subset\Delta$ in Proposition 1.1. Then λ is Δ_c^+-dominant and K-integral. Let $(\tau_\lambda, V_\lambda)$ be an irreducible finite-dimensional representation of K with Δ_c^+-highest weight λ. Then the discrete series π_Λ has lowest K-type τ_λ:

Proposition 1.2. (cf. [I, Prop. 1.3]). *The representation π_Λ, looked upon as a K-module, contains τ_λ with multiplicity one. Furthermore, the highest weight of any K-type in π_Λ is of the form $\lambda+\sum_{\alpha\in\Delta^+}n_\alpha\alpha$ with non-negative integers n_α.*

We call λ the Blattner parameter or the lowest highest weight of π_Λ, $\Lambda=\lambda+\rho_c-\rho_n$.

1.2. Gradient-type differential operators $D_{\lambda,\eta}$ acting on induced modules. Let N be a closed subgroup of G, and η a continuous representation of N on a Fréchet space $\mathcal{F}$. Consider the representation $\pi(\eta)=(L, C^{\infty}(G; \eta))$ of G induced from η in C^{∞}-context:

(1.7) $$C^{\infty}(G; \eta)=\{\varphi: G \overset{C^{\infty}}{\longmapsto} \mathcal{F} \mid \varphi(xn)=\tilde{\eta}(n)^{-1}\varphi(x), (n, x)\in N\times G\},$$

(1.8) $$L_g\varphi(x)=\varphi(g^{-1}x) \quad \text{for} \quad g\in G, \varphi\in C^{\infty}(G; \eta),$$

where we set $\tilde{\eta}=\delta_N^{-1/2}\eta$ with the modular function δ_N on N relative to a left Haar measure. Through differentiation, $C^{\infty}(G; \eta)$ has a compatible $(\mathfrak{g}_C, K)$-module structure. Later on we often employ the notation $C^{\infty}\text{-Ind}_N^G(\eta)$ for this induced module $\pi(\eta)$.

For any finite-dimensional K-module (τ, V), let $C_{\tau}^{\infty}(G; \eta)$ denote the space of $(V\otimes\mathcal{F})$-valued C^{∞}-functions F on G satisfying

(1.9) $$F(kxn)=(\tau(k)\otimes\tilde{\eta}(n)^{-1})F(x), \quad (k, x, n)\in K\times G\times N.$$

When τ is irreducible, the assignment

(1.10) $$V^*\otimes C_{\tau}^{\infty}(G; \eta)\ni v^*\otimes F \longmapsto \langle v^*, F(\cdot)\rangle\in C^{\infty}(G; \eta)_{\tau}$$

gives rise to a K-isomorphism from the tensor product $V^*\otimes C_{\tau}^{\infty}(G; \eta)$ onto the τ-isotypic component $C^{\infty}(G; \eta)_{\tau}$ of $C^{\infty}(G; \eta)$. Here (τ^*, V^*) is the contragredient of (τ, V), $\langle\ ,\ \rangle$ the canonical dual pairing on $V^*\times(V\otimes\mathcal{F})$ with values in $\mathcal{F}$, and we equip $C_{\tau}^{\infty}(G; \eta)$ with the trivial K-module structure.

Now let $(\tau_{\lambda}, V_{\lambda})$ be the lowest K-type of discrete series π_{Λ}, $\Lambda=\lambda+\rho_c-\rho_n\in\Xi_c^+$, and $\mathrm{Ad}=\mathrm{Ad}_{\mathfrak{p}_C}$ the adjoint representation of K on $\mathfrak{p}_C$. We are going to define a gradient-type differential operator $D_{\lambda,\eta}$: $C_{\tau_{\lambda}}^{\infty}(G; \eta)\mapsto C_{\tau_{\bar{\lambda}}}^{\infty}(G; \eta)$ through which we describe the embeddings of discrete series $\pi_{\Lambda}^*=(\pi_{\Lambda})^*$ into the induced module $\pi(\eta)$. Take an orthonormal basis $(X_i)_{1\leq i\leq 2n}$, $2n=\dim\mathfrak{p}$, of $\mathfrak{p}_C$ with respect to the hermitian inner product on $\mathfrak{p}_C$ induced from the Killing form B of $\mathfrak{g}_C$: $B(X_i, \bar{X}_j)=\delta_j^i$ (Kronecker's δ), where the bar means the conjugation of $\mathfrak{p}_C$ with respect to $\mathfrak{p}$. Then we have a canonical covariant differential operator $\nabla_{\lambda,\eta}$ from $C_{\tau_{\lambda}}^{\infty}(G; \eta)$ to $C_{\tau_{\lambda}\otimes\mathrm{Ad}}^{\infty}(G; \eta)$ by

(1.11) $$\nabla_{\lambda,\eta}F(x)=\textstyle\sum_{1\leq i\leq 2n}L_{X_i}F(x)\otimes\bar{X}_i, \quad F\in C_{\tau_{\lambda}}^{\infty}(G; \eta),$$

where

$$L_{X_i}F(x)=(d/dt)F(\exp(-tX_i^{(1)})\cdot x)|_{t=0}+\sqrt{-1}(d/dt)F(\exp(-tX_i^{(2)})\cdot x)|_{t=0}$$

with $X_i=X_i^{(1)}+\sqrt{-1}X_i^{(2)}$; $X_i^{(1)}, X_i^{(2)}\in\mathfrak{p}$. Note that $\nabla_{\lambda,\eta}$ is independent of the choice of a basis (X_i).

Let $\Delta_n=\Delta\setminus\Delta_c$ be the set of non-compact roots in Δ. Since $\mathfrak{p}_C$ decomposes into a direct sum of the non-compact root subspaces, the highest weight of any irreducible component of $V_{\lambda}\otimes\mathfrak{p}_C$ is of the form $\lambda+\beta$ with $\beta\in\Delta_n$. Let $(\tau_{\bar{\lambda}}, V_{\bar{\lambda}})$ be the sum of all irreducicle constituents of $V_{\lambda}\otimes\mathfrak{p}_C$ with highest weights $\lambda-\beta$, $\beta\in\Delta_n^+=\Delta^+\cap\Delta_n$, and P_{λ}: $V_{\lambda}\otimes\mathfrak{p}_C\to V_{\bar{\lambda}}$ be any surjective K-homomorphism. Composing $\nabla_{\lambda,\eta}$ with P_{λ}, we define a gradient-type differential operator $D_{\lambda,\eta}$ from $C_{\tau_{\lambda}}^{\infty}(G; \eta)$ to $C_{\tau_{\bar{\lambda}}}^{\infty}(G; \eta)$ by

(1.12) $$D_{\lambda,\eta}F=P_{\lambda}(\nabla_{\lambda,\eta}F(\cdot)).$$

Notice that the kernel of $D_{\lambda,\eta}$, one of the main objects of this paper, is independent of the choice of P_{λ}.

In the special case where η is the trivial character of the unit subgroup $\{1\}$, $D_{\lambda,\eta}$ reduces to Schmid's D_λ in [7], and the discrete series π_Λ can be realized on the L^2-kernel of this differential operator D_λ (cf. [I, Th. 1.5]).

1.3. The kernel of $D_{\lambda,\eta}$ and the embeddings of discrete series. For a $\Lambda\in\Xi_c^+$, let (π_Λ, H_Λ) be the discrete series representation of G with Harish-Chandra parameter Λ, and $(\pi_\Lambda^*, H_\Lambda^*)$ its contragredient. One sees easily from Proposition 1.1 that the discrete series π_Λ^* corresponds to the parameter $-w_0\Lambda\in\Xi_c^+$: $\pi_\Lambda^*\cong\pi_{-w_0\Lambda}$, where w_0 is the longest element of the compact Weyl group W_c. With this fact in mind, we study the embeddings of π_Λ^* instead of those of π_Λ.

One of our main results in Part A of [I], explained below, says that the kernel of the differential operator $D_{\lambda,\eta}$ characterizes the infinitesimal embeddings of π_Λ^* into $\pi(\eta)$ under very weak assumptions on λ and η.

Now let $(H_\Lambda^*)^0$ denote the $(\mathfrak{g}_C, K)$-module of all K-finite vectors in H_Λ^*. Since π_Λ^* contains its lowest K-type $(\tau_\lambda^*, V_\lambda^*)$ with multiplicity one, we identify V_λ^* with the τ_λ^*-isotypic component of π_Λ^*. By the isomorphism (1.10), there corresponds, to each embedding ι: $(H_\Lambda^*)^0\hookrightarrow C^\infty(G\,;\,\eta)$ as $(\mathfrak{g}_C, K)$-modules, a unique element $\Upsilon^{[\iota]}$ in $C_{\tau_\lambda}^\infty(G\,;\,\eta)$ satisfying

$$\iota(v^*)=\langle v^*, \Upsilon^{[\iota]}(\cdot)\rangle\in C^\infty(G\,;\,\eta)_{\tau_\lambda^*}$$

for all $v^*\in V_\lambda^*\subset(H_\Lambda^*)^0$. Clearly, this assignement

$$\Upsilon:\ I_{\Lambda,\eta}\equiv\mathrm{Hom}_{\mathfrak{g}_C-K}(\pi_\Lambda^*, \pi(\eta))\ni\iota\longmapsto\Upsilon^{[\iota]}\in C_{\tau_\lambda}^\infty(G\,;\,\eta) \tag{1.13}$$

is injective.

Then we have the following

Theorem 1.3 [I, Prop. 2.1 and Th. 2.4]. (1) *The function $\Upsilon^{[\iota]}$ lies in the kernel of $D_{\lambda,\eta}$: $D_{\lambda,\eta}\Upsilon^{[\iota]}=0$, for each $\iota\in I_{\Lambda,\eta}$. Therefore Υ gives an injection from $I_{\Lambda,\eta}$ to* $\mathrm{Ker}(D_{\lambda,\eta})$.

(2) *Furthermore this mapping is surjective: $I_{\Lambda,\eta}\cong\mathrm{Ker}(D_{\lambda,\eta})$, if the lowest highest weight $\lambda=\Lambda+\rho_c-\rho_n$ of π_Λ and the representation $(\eta, \mathcal{F})$ of N satisfy respectively the following conditions* (FFW) *and* (WC):

(FFW) *$\lambda-\sum_{\beta\in Q}\beta$ is Δ_c^+-dominant for any subset Q of Δ_n^+, i.e., λ is far from the walls,*

(WC) *there exists a continuous linear functional T on $\mathcal{F}$ such that, for a $v\in\mathcal{F}$, $\langle T, \eta(n)v\rangle=0\,(n\in N)$ implies $v=0$, i.e., the representation η is weakly cyclic.*

Based on this theorem, we shall solve in later sections, §§3-5, the systems of differential equations induced from $D_{\lambda,\eta}F=0$, explicitly for various types of representations of $SU(2, 2)$ induced from its unipotent subgroups. Then we can describe in §6 the corresponding embeddings of discrete series.

§2. Radial A_p-parts of differential operators $D_{\lambda,\eta}$ for the unitary group $SU(2, 2)$

Let $G=KA_pN_m$ be an Iwasawa decomposition of G, and η a continuous representation of the maximal unipotent subgroup N_m on a Fréchet space $\mathcal{F}$. Then the gradient-type differential operator $D_{\lambda,\eta}$ defined by (1.12) is uniquely determined by its restriction to the vector subgroup A_p, namely by its radial A_p-part $\mathrm{R}(D_{\lambda,\eta})$.

In this section, we describe, after [I, §§4-5], this differential operator $\mathrm{R}(D_{\lambda,\eta})$ on A_p explicitly for the special unitary group $SU(2, 2)$ of real rank two, and write down a system of differential difference equations on $A_p\cong \boldsymbol{R}^2$ whose solutions characterize the embeddings of discrete series into the induced module $\pi(\eta)=C^\infty\text{-}\mathrm{Ind}_{N_m}^G(\eta)$.

2.1. The group $SU(2, 2)$ **and its discrete series.** From now on, let G be the special unitary group $SU(2, 2)$ realized as

$$(2.1)\qquad G=\{g\in SL(4, \boldsymbol{C})\,|\,g^*I_{2,2}g=I_{2,2}\},\qquad I_{2,2}=\mathrm{diag}(1, 1, -1, -1),$$

where $g^*={}^t\bar{g}$ denotes the adjoint of a matrix g. We now fix our notation for this group and its discrete series, used throughout this paper.

Take a maximal compact subgroup $K=G\cap U(4)=S(U(2)\times U(2))$ ($U(k)$=the unitary group of degree k). We set

$$(2.2)\qquad \mathfrak{a}_p=\boldsymbol{R}H_1+\boldsymbol{R}H_2\quad\text{with}\quad H_1=X_{23}+X_{32},\quad H_2=X_{14}+X_{41},$$

where $X_{ij}=(\delta_p^i\delta_q^j)_{p,q}$ with Kronecker's δ_p^j. Then $\mathfrak{a}_p$ is a maximally split abelian subalgebra of $\mathfrak{g}$. Let Ψ denote the root system of $(\mathfrak{g}, \mathfrak{a}_p)$. Then Ψ is of type C_2, and is expressed as

$$(2.3)\qquad \Psi=\{\pm(\psi_2\pm\psi_1)/2,\ \pm\psi_1,\ \pm\psi_2\},\quad \psi_i(H_j)=2\delta_j^i\ (i,\ j=1,\ 2).$$

Choose a positive system $\Psi^+=\{(\psi_2\pm\psi_1)/2,\ \psi_1,\ \psi_2\}$ having ψ_1 and $(\psi_2-\psi_1)/2$ as its simple roots, and let $\mathfrak{n}_m=\sum_{\psi\in\Psi^+}\mathfrak{g}(\psi)$ be the corresponding maximal nilpotent Lie subalgebra of $\mathfrak{g}$. Here $\mathfrak{g}(\psi)$ is the root subspace of $\mathfrak{g}$ corresponding to $\psi\in\Psi$. Then one obtains Iwasawa decompositions of $\mathfrak{g}$ and G:

$$\mathfrak{g}=\mathfrak{k}+\mathfrak{a}_p+\mathfrak{n}_m,\quad G=KA_pN_m\quad\text{with}\quad A_p=\exp\mathfrak{a}_p,\quad N_m=\exp\mathfrak{n}_m.$$

Now we set

$$(2.4)\qquad E_1=\sqrt{-1}(H'_{23}-X_{23}+X_{32})/2,\qquad E_2=\sqrt{-1}(H'_{14}-X_{14}+X_{41})/2,$$

$$(2.5)\qquad E_3^{\pm}=(X_{13}+X_{43}\mp X_{12}\mp X_{42})/2,\qquad E_4^{\mp}=(X_{24}-X_{21}\pm X_{34}\mp X_{31})/2,$$

where $H'_{kl}=X_{kk}-X_{ll}$ for $1\leqq k,\ l\leqq 4$. Then it is easily seen that

$$(2.6)\qquad E_i\in\mathfrak{g}(\psi_i),\qquad E_j^{\pm}\in\boldsymbol{C}\otimes_{\boldsymbol{R}}\mathfrak{g}((\psi_2\pm\psi_1)/2)$$

for $i=1,\ 2,\ j=3,\ 4$, and that these six elements form a basis of the complexification $(\mathfrak{n}_m)_C$ of $\mathfrak{n}_m$.

Let us now parametrize the discrete series of $SU(2, 2)$. Take a compact Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ consisting of all diagonal matrices in $\mathfrak{k}$. Then the root system Δ of $(\mathfrak{g}_C,\ \mathfrak{t}_C)$, of type A_3, is expressed as $\Delta=\{\beta_{ij}\,|\,1\leqq i,\ j\leqq 4,\ i\neq j\}$, where

$$\beta_{ij}(\mathrm{diag}(h_1,\ h_2,\ h_3,\ h_4))=h_i-h_j$$

for $\mathrm{diag}(h_1, h_2, h_3, h_4)\in\mathfrak{t}_C$. Further one gets $\Delta_c=\{\pm\beta_{12},\ \pm\beta_{34}\}$. We identify the Weyl group W of Δ with the symmetric group $\mathfrak{S}_4$ of degree 4 acting on $\mathfrak{t}_C$ by permutation of diagonal entries. Then the compact Weyl group W_c is identified with the subgroup $\mathfrak{S}_2\times\mathfrak{S}_2$ in the canonical way.

As in §1, we fix a positive system $\Delta_c^+=\{\beta_{12}, \beta_{34}\}$ of Δ_c. Then Δ admits precisely six positive systems $\Delta_{\mathrm{I}}^+, \Delta_{\mathrm{II}}^+, \cdots, \Delta_{\mathrm{VI}}^+$, containing Δ_c^+:

$$\Delta_J^+=w_J\Delta_{\mathrm{I}}^+ \quad \text{with} \quad \Delta_{\mathrm{I}}^+=\{\beta_{ij}\,|\,i<j\}, \tag{2.7}$$

where the elements $w_J\in W$ are given as

$$\begin{aligned} &w_{\mathrm{I}}=1, \qquad w_{\mathrm{II}}=s_2, \qquad w_{\mathrm{III}}=s_2s_3, \\ &w_{\mathrm{IV}}=s_2s_1, \qquad w_{\mathrm{V}}=s_2s_3s_1=s_2s_1s_3, \qquad w_{\mathrm{VI}}=s_2s_1s_3s_2 \end{aligned} \tag{2.8}$$

in terms of the transpositions s_i of i and $i+1$ ($i=1, 2, 3$). Correspondingly, the space $\Xi_c^+\subset\mathfrak{t}_c^*$ of Harish-Chandra parameters are divided into six parts:

$$\begin{aligned} &\Xi_c^+=\coprod_{\mathrm{I}\leq J\leq \mathrm{VI}}\Xi_J^+, \\ &\Xi_J^+=\{\Lambda\in\Xi_c^+\,|\,\Lambda \text{ is } \Delta_J^+\text{-dominant}\}. \end{aligned} \tag{2.9}$$

We note that Ξ_{I}^+ (resp. Ξ_{VI}^+) corresponds to the holomorphic (resp. anti-holomorphic) discrete series.

2.2. Radial A_p-part $\mathrm{R}(D_{\lambda,\eta})$ of $D_{\lambda,\eta}$. As in the beginning of this section, let $(\eta, \mathcal{F})$ be a continuous Fréchet space representation of N_m, and denote by $\mathcal{F}^\infty$ the space of C^∞-vectors for η endowed with the usual Fréchet space topology for which the representation η on $\mathcal{F}^\infty$ is smooth. Consider the gradient-type differential operator $D_{\lambda,\eta}$: $C_{\tau_\lambda}^\infty(G;\eta)\to C_{\tau_{\bar\lambda}}^\infty(G;\eta)$. Noting that $G=KA_pN_m$ is diffeomorphic to the direct product $K\times A_p\times N_m$ as a C^∞-manifold, one obtains linear isomorphisms:

$$\begin{aligned} r&: C_{\tau_\lambda}^\infty(G;\eta)\xrightarrow{\sim} C^\infty(A_p, V_\lambda\otimes\mathcal{F}^\infty), \\ r'&: C_{\tau_{\bar\lambda}}^\infty(G;\eta)\xrightarrow{\sim} C^\infty(A_p, V_{\bar\lambda}\otimes\mathcal{F}^\infty), \end{aligned}$$

through restriction of functions on G to the subgroup A_p. Here $C^\infty(A_p, E)$ denotes the space of C^∞-functions on A_p with values in a Fréchet space E. We set

$$\mathrm{R}(D_{\lambda,\eta})=r'\circ D_{\lambda,\eta}\circ r^{-1}: C^\infty(A_p, V_\lambda\otimes\mathcal{F}^\infty)\longrightarrow C^\infty(A_p, V_{\bar\lambda}\otimes\mathcal{F}^\infty), \tag{2.10}$$

and call this differential operator $\mathrm{R}(D_{\lambda,\eta})$ on A_p, equivalent to $D_{\lambda,\eta}$, the radial A_p-part of $D_{\lambda,\eta}$.

In order to write down $\mathrm{R}(D_{\lambda,\eta})$ explicitly, we give a concrete realization of $(\tau_\lambda, V_\lambda)$. For a non-negative integer d, denote by (τ_d, V_d) the unique (up to equivalence) irreducible representation of $\mathfrak{sl}(2, \boldsymbol{C})$ of dimension $d+1$. Taking a basis $(f_n^{(d)})_{0\leq n\leq d}$ of V_d consisting of weight vectors, one can describe the action of $\mathfrak{sl}(2, \boldsymbol{C})=\boldsymbol{C}X+\boldsymbol{C}H'+\boldsymbol{C}\bar{X}$ on V_d as

$$\begin{aligned} &\tau_d(X)f_n=f_{n+1}, \qquad \tau_d(H')f_n=(2n-d)f_n, \\ &\tau_d(\bar{X})f_n=n(d-n+1)f_{n-1}, \end{aligned} \tag{2.11}$$

where $X=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $H'=\begin{pmatrix}0&0\\0&-1\end{pmatrix}$, $\bar{X}=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, and the vectors f_{d+1}, f_{-1} should be understood as zero.

For the lowest highest weight $\lambda\in\mathfrak{t}_c^*$ of a discrete series, we put

$$r=\lambda(H_{12}'),\qquad s=\lambda(H_{34}'),\qquad u=\lambda(I_{2,2}). \tag{2.12}$$

Then r, s, u and in addition $(r+s+u)/2$ are integers by the K-integrability of λ. Further one has $r, s\geqq 0$ because λ is Δ_c^+-dominant. Note that the complexified Lie algebra $\mathfrak{k}_C$ of K is isomorphic to $\mathfrak{sl}(2, \boldsymbol{C})\oplus\mathfrak{sl}(2, \boldsymbol{C})\oplus\boldsymbol{C}$ through

$$\mathfrak{sl}(2, \boldsymbol{C})\oplus\mathfrak{sl}(2, \boldsymbol{C})\oplus\boldsymbol{C}\ni(Y_1, Y_2, z)\longmapsto \mathrm{diag}(Y_1, Y_2)+zI_{2,2}\in\mathfrak{k}_C. \tag{2.13}$$

Then we can (and do) realize the irreducible $\mathfrak{k}_C$-module $(\tau_\lambda, V_\lambda)$ by means of the exterior tensor product $\tau_r\otimes\tau_s$ as

$$\begin{aligned} &V_\lambda=V_r\otimes V_s,\\ &\tau_\lambda(\mathrm{diag}(Y_1, Y_2))=\tau_r(Y_1)\otimes I_{V_s}+I_{V_r}\otimes\tau_s(Y_2),\\ &\tau_\lambda(zI_{2,2})=zuI_{V_\lambda}. \end{aligned} \tag{2.14}$$

Here I_V denotes the identity operator on a vector space V.

2.3. System of differential equations for the coefficients (c_{kl}). Expand a function $\varphi\in C^\infty(A_p, V_\lambda\otimes\mathcal{F}^\infty)$ in terms of the basis $f_{kl}^{(rs)}=f_k^{(r)}\otimes f_l^{(s)}$ $(0\leqq k\leqq r,\ 0\leqq l\leqq s)$ of V_λ as

$$\varphi(a)=\textstyle\sum_{k,l}f_{kl}^{(rs)}\otimes c_{kl}(a)\qquad(a\in A_p) \tag{2.15}$$

with $c_{kl}\in C^\infty(A_p, \mathcal{F}^\infty)$. As carried out in [I, §5], we can rewrite the differential equation $\mathrm{R}(D_{\lambda,\eta})\varphi=0$ for φ to a system of difference equations for the coefficients (c_{kl}), which we are going to describe.

Define (differential) operators $L_i^\pm$ $(i=1, 2)$, $S_j^\pm$ $(j=3, 4)$ acting on $C^\infty(A_p, \mathcal{F}^\infty)$ by

$$L_i^\pm h=(\partial_i\pm 2\sqrt{-1}\,e^{-\psi_i}\eta_i)h,\quad S_j^\pm h=(e^{-(\psi_2+\psi_1)/2}\eta_j^+\pm e^{-(\psi_2-\psi_1)/2}\eta_j^-)h \tag{2.16}$$

for $h\in C^\infty(A_p, \mathcal{F}^\infty)$, where $\partial_i h(a)=(d/dt)h(\exp(-tH_i)\cdot a)|_{t=0}$; $\eta_i=\eta(E_i)$, $\eta_j^\pm=\eta(E_j^\pm)$ with the basis E_i, $E_j^\pm$ of $(\mathfrak{n}_m)_C$ in (2.4), (2.5). Further we set

$$b_0=(r+s+u)/2,\quad b_1=(-r+s+u)/2,\quad b_2=(r-s+u)/2,\quad b_3=(r+s-u)/2. \tag{2.17}$$

Using these operators and constants, let us introduce 8 systems C_j^ε $(1\leqq j\leqq 4,\ \varepsilon=\pm)$ of differential difference equations for (c_{kl}) as follows.

SYSTEM C_1^-

$$(C_1^-)\quad (k+1)(l+1)(L_2^++k+l-b_0-r-s-2)c_{k+1,l+1}-2(k+1)S_3^+c_{k+1,l}$$
$$+2(l+1)S_4^+c_{k,l+1}-(L_1^++k+l-b_0)c_{kl}=0\qquad(0\leqq k\leqq r-1,\ 0\leqq l\leqq s-1),$$

SYSTEM C_2^-

$$(C_2^-: 1)\quad 2(k+1)(l+1)(s-l)c_{k+1,l+1}+2(k+1)S_3^+c_{k+1,l}$$
$$+(L_1^++k+l-b_0)c_{kl}=0\qquad(0\leqq k\leqq r-1,\ 0\leqq l\leqq s),$$

$$(C_2^-: 2)\quad (k+1)(L_2^++k-l-r-b_2-1)c_{k+1,l}+2S_4^+c_{kl}=0\qquad(0\leqq k\leqq r-1,\ 0\leqq l\leqq s),$$

SYSTEM C_3^-

$(C_3^-: 1)$ $2(k+1)(l+1)(r-k)c_{k+1,l+1}-2(l+1)S_4^+c_{k,l+1}$

$$+(L_1^+ +k+l-b_0)c_{kl}=0 \quad (0\leqq k\leqq r,\ 0\leqq l\leqq s-1),$$

$(C_3^-: 2)$ $(l+1)(L_2^+ -k+l-s-b_1-1)c_{k,l+1}-2S_3^+c_{kl}=0 \quad (0\leqq k\leqq r,\ 0\leqq l\leqq s-1),$

SYSTEM C_4^-

$(C_4^-: 1)$ $(L_1^+ +k+l-b_0)c_{kl}=0 \quad (0\leqq k\leqq r,\ 0\leqq l\leqq s),$

$(C_4^-: 2)$ $(k+1)(r-k)c_{k+1,l}-S_4^+c_{kl}=0 \quad (0\leqq k\leqq r,\ 0\leqq l\leqq s),$

$(C_4^-: 3)$ $(l+1)(s-l)c_{k,l+1}+S_3^+c_{kl}=0 \quad (0\leqq k\leqq r,\ 0\leqq l\leqq s),$

$(C_4^-: 4)$ $(L_2^+ -k-l+b_3)c_{kl}=0 \quad (0\leqq k\leqq r,\ 0\leqq l\leqq s),$

SYSTEM C_1^+

(C_1^+) $(k+1)(l+1)(L_1^- -k-l+b_0-2)c_{k+1,l+1}-2(k+1)S_3^-c_{k+1,l}$

$$+2(l+1)S_4^-c_{k,l+1}-(L_2^- -k-l-b_3-4)c_{kl}=0 \quad (0\leqq k\leqq r-1,\ 0\leqq l\leqq s-1),$$

SYSTEM C_2^+

$(C_2^+: 1)$ $(k+1)(L_1^- -k-l+b_0-1)c_{k+1,l}+2c_{k,l-1}+2S_4^-c_{kl}=0 \quad (0\leqq k\leqq r-1,\ 0\leqq l\leqq s),$

$(C_2^+: 2)$ $(L_2^- -k+l-b_3-2)c_{kl}+2(k+1)S_3^-c_{k+1,l}=0 \quad (0\leqq k\leqq r-1,\ 0\leqq l\leqq s),$

SYSTEM C_3^+

$(C_3^+: 1)$ $(l+1)(L_1^- -k-l+b_0-1)c_{k,l+1}+2c_{k-1,l}-2S_3^-c_{kl}=0 \quad (0\leqq k\leqq r,\ 0\leqq l\leqq s-1),$

$(C_3^+: 2)$ $(L_2^- +k-l-b_3-2)c_{kl}-2(l+1)S_4^-c_{k,l+1}=0 \quad (0\leqq k\leqq r,\ 0\leqq l\leqq s-1),$

SYSTEM C_4^+

$(C_4^+: 1)$ $(L_1^- -k-l+b_0)c_{kl}=0 \quad (0\leqq k\leqq r,\ 0\leqq l\leqq s),$

$(C_4^+: 2)$ $c_{k-1,l}-S_3^-c_{kl}=0 \quad (0\leqq k\leqq r,\ 0\leqq l\leqq s),$

$(C_4^+: 3)$ $c_{k,l-1}+S_4^-c_{kl}=0 \quad (0\leqq k\leqq r,\ 0\leqq l\leqq s),$

$(C_4^+: 4)$ $(L_2^- +k+l-b_3)c_{kl}=0 \quad (0\leqq k\leqq r,\ 0\leqq l\leqq s).$

Here, undefined terms, for instance $c_{k+1,s+1}$ in $(C_2^-: 1)$, should be understood as zero. We note that each system C_j^ε for (c_{kl}) is obtained by rewriting a differential equation $P_j^\varepsilon(\nabla_{\lambda,\eta}\varphi)=0$ for φ, where P_j^ε is a K-homomorphism on $V_\lambda\otimes\mathfrak{p}_C$ such that

$$\mathrm{Im}P_4^\varepsilon\supset\mathrm{Im}P_2^\varepsilon+\mathrm{Im}P_3^\varepsilon\supset\mathrm{Im}P_1^\varepsilon \quad (\mathrm{Im}P_j^\varepsilon \text{ the image of } P_j^\varepsilon).$$

See [I, 5.2] for the precise definition of P_j^ε.

Theorem 2.1 [I, Th. 5.5]. *Let $\lambda=\Lambda-\rho_c+\rho_n$ be the Blattner parameter of discrete series π_Λ. Then a function $\varphi=\sum_{k,l}f_{kl}^{(rs)}\otimes c_{kl}\in C^\infty(A_p, V_\lambda\otimes\mathcal{F}^\infty)$ lies in the kernel of*

$\mathrm{R}(D_{\lambda,\eta})$: $\mathrm{R}(D_{\lambda,\eta})\varphi=0$, *if and only if the coefficients* c_{kl} $(0\leqq k\leqq r,\ 0\leqq l\leqq s)$ *satisfy the system of differential equations* $C[\lambda, \eta]$ *specified below:*

$$C[\lambda,\eta]\qquad \begin{array}{l} C_4^- \ \textit{for}\ \Lambda\in\Xi_{\mathrm{I}}^+;\ C_4^+\ \textit{for}\ \Lambda\in\Xi_{\mathrm{VI}}^+, \\ C_1^+, C_2^-, C_3^-\ \textit{for}\ \Lambda\in\Xi_{\mathrm{II}}^+;\ C_1^-, C_2^+, C_3^+\ \textit{for}\ \Lambda\in\Xi_{\mathrm{V}}^+, \\ C_2^\pm\ \textit{for}\ \Lambda\in\Xi_{\mathrm{III}}^+;\ C_3^\pm\ \textit{for}\ \Lambda\in\Xi_{\mathrm{IV}}^+, \end{array}$$

where Ξ_J^+ $(\mathrm{I}\leqq J\leqq \mathrm{VI})$ *are the sets of Harish-Chandra parameters defined in* (2.9).

By Theorem 1.3 we can determine the embeddings of discrete series π_Λ^* into the induced module $\pi(\eta)=C^\infty\text{-}\mathrm{Ind}_{N_m}^G(\eta)$ by solving the above system $C[\lambda, \eta]$.

§ 3. Solutions of the system $C[\lambda, \eta]$ for a character η: Cases I and III

Let η be a one-dimensional representation (=a character) of N_m. In these two sections, §§ 3 and 4, we solve explicitly the system of differential equations $C[\lambda, \eta]$ in Theorem 2.1 for each lowest highest weight λ of discrete series. Among other things, our results for non-degenerate η's give a complete description of (ordinary, or non-generalized) Whittaker models for the discrete series.

As is readily seen from the expression of $C[\lambda, \eta]$ in 2.3, one can solve the systems $C[\lambda, \eta]$ for $\Lambda=\lambda+\rho_c-\rho_n\in\Xi_J^+$ $(\mathrm{I}\leqq J\leqq\mathrm{VI})$ quite analogously to those for $\Lambda\in\Xi_{J^*}^+$, $J^*=\mathrm{VI}-J+\mathrm{I}$. So we concentrate on three cases $\Lambda\in\Xi_J^+$ with $J=\mathrm{I}$, II, III. We study the cases $J=\mathrm{I}$, III in this section, and the most difficult but the most interesting case $J=\mathrm{II}$ in the next section.

3.1. Coordinates and parameters. In what follows, we identify the vector group A_p with $\boldsymbol{R}^2$:

$$\boldsymbol{R}^2\ni(t_1, t_2)\overset{\sim}{\longmapsto}\exp(-t_1H_1-t_2H_2)\in A_p, \tag{3.1}$$

using the basis $(H_i)_{i=1,2}$ of $\mathfrak{a}_p$ in (2.2). Then the differential operator ∂_i and the function $e^{-\phi_i}$ in (2.16) turn out to be $\partial/\partial t_i$ and e^{2t_i} respectively. Noting that any character of N_m is trivial on the commutator subgroup $[N_m, N_m]$, one finds

$$\eta_2\equiv\eta(E_2)=0,\qquad \eta_j^+\equiv\eta(E_j^+)=0\qquad (j=3, 4) \tag{3.2}$$

for E_2, $E_j^+\in[(\mathfrak{n}_m)_c, (\mathfrak{n}_m)_c]$. This implies that

$$L_2^+=L_2^-=\partial/\partial t_2,\qquad S_j^+=-S_j^-=e^{t_2-t_1}\eta_j^-, \tag{3.3}$$

which we denote respectively by L_2 and S_j from now on.

3.2. Case I: $\Lambda\in\Xi_{\mathrm{I}}^+$. Let us begin with the case where the papameter Λ is Δ_{I}^+-dominant, and solve the system $C[\lambda, \eta]=\{(C_4^-: j)\,|\,1\leqq j\leqq 4\}$ for (c_{kl}), $c_{kl}\in C^\infty(\boldsymbol{R}^2)$, with $0\leqq k\leqq r$ and $0\leqq l\leqq s$. Now suppose $\eta_3^-\neq 0$. Then the condition $(C_4^-: 3)$, applied for $l=s$, implies $c_{ks}=0$ for $0\leqq k\leqq r$. Again by $(C_4^-: 3)$, we find

$$c_{kl}=-\frac{s!(s-l)!}{l!}\left(\frac{1}{S_3}\right)^{s-l}c_{ks}=0\qquad \text{for}\quad 0\leqq l\leqq s.$$

Hence the system $C[\lambda, \eta]$ does not have non-zero solutions if $\eta_3^- \neq 0$. Analogously, one obtains the same conclusion for $\eta_4^- \neq 0$.

So we consider the remaining case $\eta_3^- = \eta_4^- = 0$. Then $(C_4^- : 2)$ and $(C_4^- : 3)$ are equivalent to

$$c_{kl}=0 \quad \text{unless} \quad (k, l)=(0, 0). \tag{3.4}$$

Further $(C_4^- : 1)$ and $(C_4^- : 4)$ for $(k, l)=(0, 0)$ imply that

$$c_{00}=\kappa \cdot \exp(-\sqrt{-1}\, e^{2t_1}\eta_1 + b_0 t_1 - b_3 t_2) \qquad \text{for some} \quad \kappa \in \boldsymbol{C}, \tag{3.5}$$

where $\eta_1 = \eta(E_1)$, and b_i $(i=1, 2, 3)$ are the integers defined in (2.17).

Summarizing the above discussion, one gets a complete result for $\Lambda \in \Xi_{\mathrm{I}}^+$ as follows.

Proposition 3.1. *The system of differential equation $C[\lambda, \eta]$ with $\Lambda \in \Xi_{\mathrm{I}}^+$ has a non-zero soution (c_{kl}) if and only if $\eta_3^- = \eta_4^- = 0$, or equivalently $\eta|_{\mathfrak{g}((\psi_2-\psi_1)/2)}=0$. In this case, the solutions are unique up to scalar multiples, and are given by* (3.4) *and* (3.5).

Note. This case of holomorphic discrete series has been studied by Hashizume for any simple Lie group of hermitian type.

3.3. Case III: $\Lambda \in \Xi_{\mathrm{III}}^+$. We now proceed to the cases of non-holomorphic discrete series. For Λ in Ξ_{III}^+, the system $C[\lambda, \eta]$ in question consists of four equations $(C_2^{\pm} : 1)$, $(C_2^{\pm} : 2)$.

Lemma 3.2. *Set $y_{kl}=k!\, e^{-(r+2)t_2} c_{kl}$ for $0 \leqq k \leqq r$ and $0 \leqq l \leqq s$. Then (c_{kl}) is a solution of the system $C[\lambda, \eta]$ if and only if (y_{kl}) satisfies the following system of differential equations:*

$$2S_3 y_{k+1,l} = (L_2 - k + l + b_2) y_{kl}, \tag{3.6}$$

$$2S_4 y_{kl} = -(L_2 + k + 1 - l - b_2) y_{k+1,l}, \tag{3.7}$$

$$2(l+1)(s-l) y_{k+1,l+1} = (L_1^+ + L_2 + 2l - s) y_{kl} \tag{3.8}$$

for $0 \leqq k \leqq r-1$, $0 \leqq l \leqq s$, *and*

$$2y_{kl} = -(L_1^- + L_2 - 2(l+1) + s) y_{k+1,l+1} \tag{3.9}$$

for $0 \leqq k \leqq r-1$, $-1 \leqq l \leqq s-1$.

Proof. It is easy to see that the equations $(C_2^+ : 2)$ and $(C_2^- : 2)$ for (c_{kl}) are rewritten respectively as (3.6) and (3.7) for (y_{kl}). With $S_j = S_j^+ = -S_j^-$ $(j=3, 4)$ in mind, add the both hand sides of $(C_2^+ : 2)$ and $(C_2^- : 1)$ (resp. $(C_2^+ : 1)$ and $(C_2^- : 2)$), and then transfer the resulting equation for (c_{kl}) into that for (y_{kl}). We thus get (3.8) (resp. (3.9)). Thus the system $C[\lambda, \eta]$ is equivalent to (3.6)-(3.9). Q. E. D.

We now note that the integers r, s and u in (2.12) fulfill the inequality

$$r-s-2 > |u| \tag{3.10}$$

by the $\Delta_{\mathrm{III}}^{+}$-dominancy of $\Lambda=\lambda+\rho_c-\rho_n$.

In order to solve (3.6)-(3.9), let us study two cases: $\eta_1\neq 0$ and $\eta_1=0$, separately.

3.3.1. Case of $\eta_1\neq 0$. Let k be an integer satisfying $1\leqq k\leqq r-s-1$. (Such a k actually exists by (3.10).) Using (3.8) repeatedly, one deduces

$$\Big\{\prod_{l=0}^{s}(L_1^{+}+L_2+s-2l)\Big\}y_{k0}=0. \tag{3.11}$$

Furthermore (3.9) with $l=-1$ implies

$$(L_1^{-}+L_2+s)y_{k0}=0. \tag{3.12}$$

Lemma 3.3. *One has an equality*

$$\Big\{\prod_{l=0}^{s}(L_1^{+}+L_2-2l)\Big\}y=(4\sqrt{-1}\,e^{2t_1}\eta_1)^{s+1}y \tag{3.13}$$

for any $y\in C^{\infty}(\boldsymbol{R}^2)$ satisfying $(L_1^{-}+L_2)y=0$.

Proof. We show (3.13) by the induction on s. If $s=0$, one gets

$$(L_1^{+}+L_2)y=(L_1^{-}+L_2+4\sqrt{-1}\,e^{2t_1}\eta_1)y=(4\sqrt{-1}\,e^{2t_1}\eta_1)y.$$

Now let $s>0$ and suppose that the formula holds for $s-1$. Then the left hand side of (3.13) is calculated as

$$\begin{aligned}\Big\{\prod_{l=0}^{s}(L_1^{+}+L_2-2l)\Big\}y&=(L_1^{+}+L_2-2s)\Big\{\prod_{l=0}^{s-1}(L_1^{+}+L_2-2l)\Big\}y\\&=(L_1^{+}+L_2-2s)(4\sqrt{-1}\,e^{2t_1}\eta_1)^{s}y \qquad \text{(by the hypothesis)}\\&=(4\sqrt{-1}\,e^{2t_1}\eta_1)^{s}(L_1^{+}+L_2)y \qquad (\text{by } [L_1^{+}, e^{2st_1}]=2se^{2st_1})\\&=(4\sqrt{-1}\,e^{2t_1}\eta_1)^{s+1}y.\end{aligned}$$

Thus we have proved the desired formula. Q. E. D.

The conditions (3.11) and (3.12) combined with the above proposition tell us the following fact that imposes a severe restriction on the solutions of (3.6)-(3.9).

Proposition 3.4. *If $\eta_1\neq 0$, then the coefficients y_{kl} with $1\leqq k-l\leqq r-s-1$ are identically zero for any (y_{kl}) satisfying the differential equations* (3.6)-(3.9).

Proof. Let $1\leqq k\leqq r-s-1$, and put $y_k'=e^{st_2}y_{k0}$. Then we have $(L_1^{-}+L_2)y_k'=0$ by (3.12). So, applying the formula (3.13) to y_k', one obtains

$$\begin{aligned}(4\sqrt{-1}\,e^{2t_1}\eta_1)^{s+1}y_k'&=\Big\{\prod_{l=0}^{s}(L_1^{+}+L_2-2l)\Big\}y_k'\\&=e^{st_2}\Big\{\prod_{l=0}^{s}(L_1^{+}+L_2+s-2l)\Big\}y_{k0}=0 \qquad \text{(by (3.11))}.\end{aligned}$$

Since $\eta_1\neq 0$, we conclude $y_k'=0$, or $y_{k0}=0$. This together with (3.8) proves the proposition. Q. E. D.

Now define matrices of functions $Y^{(0s)}=(y_{kl}^{(0s)})$ and $Y^{(r0)}=(y_{kl}^{(r0)})$ with $y_{kl}^{(0s)}, y_{kl}^{(r0)}\in C^{\infty}(\boldsymbol{R}^2)$ $(0\leqq k\leqq r, 0\leqq l\leqq s)$ given by

$$y_{kl}^{(0s)}=\delta_k^0\delta_l^s\cdot\exp(-\sqrt{-1}\, e^{2t_1}\eta_1+b_2t_1-b_0t_2), \tag{3.14}$$

$$y_{kl}^{(r0)}=\delta_k^r\delta_l^0\cdot\exp(\sqrt{-1}\, e^{2t_1}\eta_1-b_1t_1-b_3t_2). \tag{3.15}$$

By making use of Proposition 3.4, we can solve the system (3.6)-(3.9), equivalent to $C[\lambda, \eta]$, under the assumption $\eta_1\neq 0$.

Theorem 3.5. *Let $\Phi[\lambda, \eta]$ be the space of solutions of differential equations* (3.6)-(3.9). *If $\eta_1=\eta(E_1)$ does not vanish, then $\Phi[\lambda, \eta]$ is described as*

$$\Phi[\lambda, \eta]=\begin{cases}(0) & \text{if } \eta_3^-\neq 0, \quad \eta_4^-\neq 0,\\ \boldsymbol{C}Y^{(r0)} & \text{if } \eta_3^-=0, \quad \eta_4^-\neq 0,\\ \boldsymbol{C}Y^{(0s)} & \text{if } \eta_3^-\neq 0, \quad \eta_4^-=0,\\ \boldsymbol{C}Y^{(0s)}\oplus \boldsymbol{C}Y^{(r0)} & \text{if } \eta_3^-=\eta_4^-=0.\end{cases} \tag{3.16}$$

In particular, the system (3.6)-(3.9) *admits a non-zero solution if and only if $\eta_3^-\cdot\eta_4^-=0$.*

Proof. It follows immediately from (3.6), (3.7) and Proposition 3.4 that $\Phi[\lambda, \eta]=(0)$ if $\eta_3^-\cdot\eta_4^-\neq 0$. Now assume $\eta_3^-=0$ and $\eta_4^-\neq 0$. Then one finds from (3.6) and (3.7)

$$S_4y_{kl}=-(k+1-l-b_2)y_{k+1,l} \qquad \text{for} \quad 0\leqq k\leqq r-2, \quad 0\leqq l\leqq s.$$

Note that $1\leqq b_2\leqq r-s-1$ by (3.10). In view of Proposition 3.4, one deduces $y_{kl}=0$ unless $k=r$, and more strongly $y_{kl}=0$ for $(k, l)\neq(r, 0)$ by (3.8). So the system (3.6)-(3.9) for (y_{kl}) is reduced to the following one for y_{r0}:

$$(L_1^-+L_2+s)y_{r0}=0, \qquad (L_2+b_3)y_{r0}=0.$$

Solving these two differential equations, we get $\Phi[\lambda, \eta]=\boldsymbol{C}Y^{(r0)}$.

The remaining two cases can be treated analogously, and we obtain (3.16).

Q. E. D.

3.3.2. Case of $\eta_1=0$. In this case, L_1^+ and L_1^- both reduce to the constant coefficient differential operator $L_1=\partial/\partial t_1$, and the equations (3.8) and (3.9) are equivalent to

$$(L_1+L_2+s)y_{kl}=0 \qquad (0\leqq k\leqq r, 0\leqq l\leqq s), \tag{3.17}$$

$$y_{kl}=(l+1)y_{k+1,l+1} \qquad (0\leqq k\leqq r-1, 0\leqq l\leqq s-1). \tag{3.18}$$

First assume that $\eta_3^-\cdot\eta_4^-\neq 0$. Then, by a simple computation, we can show the following

Lemma 3.6. *The solutions $(y_{kl})\in\Phi[\lambda, \eta]$ correspond bijectively to $\tilde{y}\in C^{\infty}(\boldsymbol{R}^2)$ satisfying*

$$(L_1+L_2+s)\tilde{y}=0, \qquad ((L_2)^2+4\eta_3^-\eta_4^-e^{2t_2-2t_1})\tilde{y}=0$$

through the mapping $(y_{kl})\mapsto\tilde{y}\equiv y_{b_20}$.

Exchange the variables (t_1, t_2) for (v_1, v_2) with $v_1=t_1+t_2$, $v_2=t_1-t_2$, and put $\hat{y}=e^{s(v_1+v_2)}\tilde{y}=e^{2st_1}\tilde{y}$. Then the above two equations for $\tilde{y}$ are rewritten respectively as

$$\frac{\partial \hat{y}}{\partial v_1}=0, \qquad \left\{\left(\frac{\partial}{\partial v_2}\right)^2+16\eta_3^-\eta_4^- e^{-2v_2}\right\}\hat{y}=0.$$

This means that the solution $\hat{y}$ depends only on v_2, and are characterized by an ordinary differential equation of second order. Thus we find

Proposition 3.7. *The solution space* $\Phi[\lambda, \eta]$ *for* (3.6)-(3.9) *is two-dimensional if* $\eta_1=0$ *and* $\eta_3^-\cdot\eta_4^-\neq 0$.

Secondly, consider the case $\eta_3^-=0$, $\eta_4^-\neq 0$. Since $S_3=0$ in this case, the subsystems (3.6), (3.7) turn out to be

$$(L_2-k+l+b_2)y_{kl}=0 \qquad (0\leqq k\leqq r-1,\ 0\leqq l\leqq s), \tag{3.19}$$

$$\begin{cases} S_4 y_{kl}=-(k+1-l-b_2)y_{k+1,l} & (0\leqq k\leqq r-2,\ 0\leqq l\leqq s), \\ 2S_4 y_{r-1,l}=-(L_2+b_3-l)y_{rl} & (0\leqq l\leqq s). \end{cases} \tag{3.20}$$

We now solve the system (3.17)-(3.20) which is equivalent to (3.6)-(3.9). It follows from the first equation in (3.20) that $y_{kl}=0$ for $k-l<b_2$. By (3.17) and (3.19), one gets $y_{b_2 0}=\alpha e^{-st_1}$ for some $\alpha\in \boldsymbol{C}$.

If $\alpha\neq 0$, then the functions y_{kl} are uniquely determined by $y_{b_2 0}$ through (3.18) and should be of the form $y_{kl}=\alpha y_{kl}^{(b_2 0)}$ with

$$y_{kl}^{(b_2 0)}=\begin{cases} (-S_4)^{k-l-b_2}e^{-st_1}/(k-l-b_2)!\,l! & (k-l\geqq b_2), \\ 0 & (k-l<b_2). \end{cases} \tag{3.21}$$

Conversely, $Y^{(b_2 0)}=(y_{kl}^{(b_2 0)})$ actually satisfies (3.17)-(3.20).

If $\alpha=0$, $Y^{(r0)}=(y_{kl}^{(r0)})$ defined in (3.15) gives a unique (up to scalar multiples) solution of (3.17)-(3.20) such that $y_{b_2 0}=0$.

Thus we have gained the following

Proposition 3.8. *If* $\eta_1=\eta_3^-=0$ *and* $\eta_4^-\neq 0$, *the matrices of functions* $Y^{(b_2 0)}$ *and* $Y^{(r0)}$ *form a fundamental system of solutions of differential equations* (3.6)-(3.9). *In particular one has* $\dim \Phi[\lambda, \eta]=2$.

The case $\eta_3^-\neq 0$, $\eta_4^-=0$ can be studied analogously, and so we omit it here.

At last, let $\eta=1_{N_m}$ be the trivial character of N_m, i. e., $\eta_1=\eta_3^-=\eta_4^-=0$. In the first part [I] we have solved the system (3.6)-(3.9) in this case, and determined the embeddings of discrete series into the principal series.

Proposition 3.9 (cf. [I, Prop. 7.1]). *The space* $\Phi[\lambda, J, 1_{N_m}]$ *of solutions is three dimensional, and is described as*

$$\Phi[\lambda, 1_{N_m}]=\boldsymbol{C}Y^{(0s)}\oplus \boldsymbol{C}Y^{(r0)}\oplus \boldsymbol{C}Y^{(b_2 0)},$$

where we define $Y^{(0s)}$, $Y^{(r0)}$ *and* $Y^{(b_2 0)}$ *respectively as in* (3.14), (3.15) *and* (3.21), *with* $S_4=\eta_1=0$ *in mind.*

Now the system (3.6)-(3.9), or equivalently $C[\lambda, \eta]$ with $\Lambda=\lambda+\rho_c-\rho_n\in\Xi_{\mathrm{III}}^+$, has been solved perfectly for any character η of the maximal unipotent subgroup N_m. For later reference, we summarize the results of this subsection in the following table.

Table 3.10 (Case III).

η_1	η_3^-	η_4^-	$\dim\Phi[\lambda, \eta]$
*	*	*	0
*	* (resp. 0)	0 (resp. *)	1
*	0	0	2
0	*	*	2
0	* (resp. 0)	0 (resp. *)	2
0	0	0	3

Here * stands for a non-zero complex number, and, for instance, the first line should be understood as "$\dim\Phi[\lambda, \eta]=0$ if the numbers η_1, η_3^- and η_4^- are all non-zero".

§ 4. Solutions of the system $C[\lambda, \eta]$ for a character η: Case II

Now let us proceed to the case of Harish-Chandra parameter $\Lambda=\lambda+\rho_c-\rho_n$ in Ξ_{II}^+. Contrary to the previous two cases, we find that the system $C[\lambda, \eta]$, consisting of three subsystems C_1^+, C_2^-, C_3^-(see 2.3), has a non-zero solution for each character η of N_m. In view of Theorems 1.3 and 2.1, this shows the existence of an embedding of discrete series π_Λ^* into the induced module $\pi(\eta)=C^\infty\text{-}\mathrm{Ind}_{N_m}^G(\eta)$ for any character η (at least when Λ satisfies the condition (FFW) in Theorem 1.3).

4.1. A system for (h_{kl}). At first, we transfer the system $C[\lambda, \eta]$ for (c_{kl}), $c_{kl}\in C^\infty(\boldsymbol{R}^2)$ $(0\leqq k\leqq r,\ 0\leqq l\leqq s)$, into a more convenient form to handle. Set for each c_{kl},

$$h_{kl}=k!\,l!\exp\{\sqrt{-1}\,e^{2t_1}\eta_1+(k+l-b_0)t_1+(b_3-k-l-2)t_2\}\cdot c_{kl}, \tag{4.1}$$

where $\eta_1=\eta(E_1)$ and r, s, b_j $(0\leqq j\leqq 3)$ are the integers in (2.12), (2.17).

Proposition 4.1. *The system of functions (c_{kl}) is a solution of $C[\lambda, \eta]$ if and only if (h_{kl}) satisfies the following differential equations:*

$$e^{2(t_2-t_1)}(L_1+2L_2-4\sqrt{-1}\,e^{2t_1}\eta_1-2b_3)h_{k+1,l+1}-(L_2-2b_3-2)h_{kl}=0 \tag{4.2}$$
$$(0\leqq k\leqq r-1,\ 0\leqq l\leqq s-1),$$

$$e^{2(t_2-t_1)}(L_2+2)h_{k+1,l+1}+L_1h_{kl}=0 \quad (0\leqq k\leqq r-1,\ 0\leqq l\leqq s-1), \tag{4.3}$$

$$(L_2+2(k+1-r))h_{k+1,l}+2\eta_4^-h_{kl}=0 \quad (0\leqq k\leqq r-1,\ 0\leqq l\leqq s), \tag{4.4}$$

$$(L_2+2(l+1-s))h_{k,l+1}-2\eta_3^-h_{kl}=0 \quad (0\leqq k\leqq r,\ 0\leqq l\leqq s-1), \tag{4.5}$$

(4.6) $$2e^{2(t_2-t_1)}\eta_3^- h_{k+1,s}+L_1 h_{ks}=0 \quad (0\leqq k\leqq r-1),$$

(4.7) $$-2e^{2(t_2-t_1)}\eta_4^- h_{r,l+1}+L_1 h_{rl}=0 \quad (0\leqq l\leqq s-1),$$

where $L_i=\partial/\partial t_i$ *for* $i=1, 2$.

Proof. By an elementary computation, we see that $(C_2^-: 2)$ (resp. $(C_3^-: 2)$; $(C_2^-: 1)$ with $l=s$; and $(C_3^-: 1)$ with $k=r$) for (c_{kl}) is equivalent to (4.4) (resp. (4.5); (4.6); and (4.7)) for (h_{kl}). By using (4.4) and (4.5), (C_1^+) is rewritten as (4.2). Further, $(C_2^-: 1)$ and $(C_3^-: 1)$ with $k<r$, $l<s$, both turn out to be the same equation (4.3).

Q. E. D.

In the succeeding subsections, we study separately three cases according to the degeneracy of η, and solve the system (4.2)-(4.7) explicitly for each case.

4.2. Case of $\eta_3^-\cdot\eta_4^-\neq 0$. In this case, any solution (h_{kl}) of (4.2)-(4.7) is uniquely determined by $h=h_{rs}$ through the relations (4.4) and (4.5). By (4.5) and (4.7), h should fulfill the equation

(4.8) $$(L_1L_2-4S_3S_4)h=0.$$

Further one gets from (4.2), (4.4) and (4.5),

(4.9) $$\{(L_2-2b_3-2)L_2^2+4S_3S_4(L_1+2L_2-4\sqrt{-1}\, e^{2t_1}\eta_1-2b_3)\}h=0.$$

Conversely, it is easily checked that any $h\in C^\infty(\boldsymbol{R}^2)$ satisfying (4.8) and (4.9) can be extended uniquely to a solution (h_{kl}) of (4.2)-(4.7) through (4.4), (4.5). We thus get the following lemma.

Lemma 4.2. *The solutions* (h_{kl}) *of* (4.2)-(4.7) *correspond bijectively to* $h\in C^\infty(\boldsymbol{R}^2)$ *satisfying* (4.8) *and* (4.9), *through* $h=h_{rs}$.

Now set $q_j=L_2^jL_1h$ for $0\leqq j\leqq 3$, and consider the vector $q={}^t(q_0, q_1, q_2, q_3)$ of functions q_j on $\boldsymbol{R}^2$. Noting that $h=(1/4S_3S_4)L_2q_0$, we obtain from (4.9) a first order differential equation for q as

(4.10) $$L_2q=D^{(2)}q,$$

where $D^{(2)}=(d_{jv}^{(2)})_{0\leqq j,v\leqq 3}$ is a matrix with elements $d_{jv}^{(2)}\in C^\infty(\boldsymbol{R}^2)$ given by

(4.11) $$\begin{cases} d_{01}^{(2)}=d_{12}^{(2)}=d_{23}^{(2)}=1, \\ d_{30}^{(2)}=-16S_3^2S_4^2, \qquad d_{31}^{(2)}=8\{S_3S_4(2\sqrt{-1}\, e^{2t_1}\eta_1+b_3+2)-(b_3+2)\}, \\ d_{32}^{(2)}=-4(2b_3+5+2S_3S_4), \qquad d_{33}^{(3)}=2(b_3+4), \\ d_{jv}^{(2)}=0 \qquad \text{otherwise}. \end{cases}$$

Next we compute L_1q. Differentiating the both hand sides of (4.8): $q_1=4S_3S_4h$, first by t_1 and then by t_2 repeatedly, one deduces

(4.12) $$L_1q_1=4S_3S_4(L_1-2)h=4S_3S_4q_0-2q_1,$$

(4.13) $$L_2^j L_1 q_1 = L_1(L_2^j q_1) = 4S_3S_4 \sum_{0\leq v\leq j} \binom{j}{v} 2^{j-v} L_2^v q_0 - 2L_2^{j+1} q_0$$

for each integer j. These equalities together with $L_2 q_3 = \sum_{0\leq v\leq 3} d_{3v}^{(2)} q_v$ yield

(4.14) $$L_1 q_j = \sum_{0\leq v\leq 3} d_{jv}^{(1)} q_v \qquad (0\leqq j\leqq 3),$$

where $d_{jv}^{(1)}$ is defined by

(4.15) $$d_{jv}^{(1)} = \binom{j-1}{v} 2^{j+1-v} S_3S_4 \ (v<j), \qquad d_{jj}^{(1)} = -2, \qquad d_{jv}^{(1)} = 0 \ (v>j)$$

for $j=1, 2, 3$, and

(4.16) $$d_{00}^{(1)} = 2(2\sqrt{-1}\, e^{2t_1}\eta_1 + b_3 + 1), \qquad d_{01}^{(1)} = -2 - \frac{b_3+2}{S_3S_4},$$
$$d_{02}^{(1)} = \frac{b_3+3}{2S_3S_4}, \qquad d_{03}^{(1)} = -\frac{1}{4S_3S_4}.$$

One thus obtains

(4.17) $$L_1 q = D^{(1)} q \quad \text{with} \quad D^{(1)} = (d_{jv}^{(1)})_{0\leq j,v\leq 3}.$$

Note that the matrices $D^{(1)}$ and $D^{(2)}$ are of the form

$$D^{(1)} = \begin{pmatrix} * & * & * & * \\ * & -2 & 0 & 0 \\ * & * & -2 & 0 \\ * & * & * & -2 \end{pmatrix}, \qquad D^{(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ * & * & * & * \end{pmatrix},$$

where $*$ stands for a non-zero function on $\boldsymbol{R}^2$.

Summarizing the above discussion, we find

Proposition 4.3. *The system* (4.8), (4.9) *of differential equations for* $h\in C^\infty(\boldsymbol{R}^2)$ *is equivalent to* (4.10), (4.17) *for* $q={}^t(q_0, q_1, q_2, q_3)$ *through the relation* $q_j = L_2^j L_1 h$ $(0\leqq j\leqq 3)$.

The next lemma shows the complete integrability of the system (4.10), (4.17).

Lemma 4.4. *One gets a bracket relation* $L_2 D^{(1)} - L_1 D^{(2)} = [D^{(1)}, D^{(2)}]$.

This equality is proved by elementary but very long calculations, and so we omit the proof here.

The above complete integrability condition allows us to solve the system (4.10), (4.17), which is equivalent to $C[\lambda, \eta]$, perfectly as follows.

Theorem 4.5. *For each vector* $y\in \boldsymbol{C}^4$, *there exists a unique solution* q *of* (4.10), (4.17) *with the initial value condition* $q(0, 0)=y$ *at the origin* $(0, 0)\in \boldsymbol{R}^2$. *This* q *is given by*

(4.18) $$q(t_1, t_2) = \exp\left\{\int_0^{t_1} D^{(1)}(\nu_1, t_2)\,d\nu_1\right\} \exp\left\{\int_0^{t_2} D^{(2)}(0, \nu_2)\,d\nu_2\right\} \cdot y$$

for $(t_1, t_2)\in \boldsymbol{R}^2$. *Therefore, the space* $\Phi[\lambda, \eta]$ *of solutions of* $C[\lambda, \eta]$ *is four-dimensional.*

4.3. Case of $\eta_3^- \neq 0$, $\eta_4^- = 0$. We now put $h' = h_{0s}$. By the condition $\eta_3^- \neq 0$, any solution (h_{kl}) of (4.2)-(4.7) is uniquely determined from h' through (4.5) and (4.6). As in the beginning of 4.2, we can easily find differential equations for h' to yield the solutions (h_{kl}), as follows.

Lemma 4.6. *The systems (h_{kl}) of functions satisfying* (4.2)-(4.7) *are in bijective correspondence to $h' \in C^\infty(\boldsymbol{R}^2)$ such that*

$$(L_1 - 4\sqrt{-1}\, e^{2t_1}\eta_1 + 4r - 2 - 2b_3)L_1 h' + (L_2 - 2b_3 - 2)L_2 h' = 0\,, \tag{4.19}$$

$$(L_2 - 2r)L_1 h' = 0\,, \tag{4.20}$$

through $h' = h_{0s}$, where η_1, r and b_3 are the constants given before.

Let us solve (4.19) and (4.20). We set $h'' = (L_2 - 2r)h'$. Then h'' satisfies

$$L_1 h'' = 0, \qquad (L_2 - 2b_3 - 2)L_2 h'' = 0\,. \tag{4.21}$$

Solving these differential equations for h'', one immediately deduces that h' is of the form

$$h' = e^{2rt_2}\phi(t_1) + \mu_1 + \mu_2 e^{2(b_3+1)t_2} \tag{4.22}$$

for some $\mu_1, \mu_2 \in \boldsymbol{C}$ and $\phi \in C^\infty(\boldsymbol{R})$. Here we use the fact that the numbers $2r$, $2(b_3+1)$, 0 are distinct with each other by the condition

$$r + s + 2 > -u > |r - s| + 2$$

coming from the Δ_{II}^+-dominancy of Harish-Chandra parameter Λ.

Conversely, the function h' in (4.22) satisfies (4.19) and (4.20) if and only if ϕ fulfills

$$\{(L_1 - 4\sqrt{-1}\, e^{2t_1}\eta_1 + 4r - 2 - 2b_3)L_1 + 4r(r - b_3 - 1)\}\phi = 0\,, \tag{4.23}$$

which is a second order ordinary differential equation for ϕ and so can be easily settled.

In this way, the system (4.2)-(4.7) for (h_{kl}) has been completely settled for case $\eta_3^- \neq 0$, $\eta_4^- = 0$. One can deal with the case $\eta_3^- = 0$, $\eta_4^- \neq 0$ analogously. Thus we obtain the following

Proposition 4.7. *Assume that one and only one of the numbers η_3^- and η_4^- equals zero. Then the solution space $\Phi[\lambda, \eta]$ of the system $C[\lambda, \eta]$ is of dimension* 4 *for $\Lambda = \lambda + \rho_c - \rho_n \in \Xi_{\mathrm{II}}^+$. When $\eta_3^- \neq 0$, solutions (h_{kl}) of the system* (4.2)-(4.7), *which is equivalent to $C[\lambda, \eta]$, correspond bijectively to triples (μ_1, μ_2, ϕ) with $\mu_1, \mu_2 \in \boldsymbol{C}$ and $\phi(t_1) \in C^\infty(\boldsymbol{R})$ satisfying* (4.23), *through $h_{0s} = h'$, where h' is as in* (4.22).

4.4. Case of $\eta_3^- = \eta_4^- = 0$. In this case, the system (4.2)-(4.7) splits into $r+s+1$ number of subsystems, I_μ $(-s \leqq \mu \leqq r)$, for $(h_{kl})_{k-l=\mu}$, which have been already settled in the first part [I, 7.1.2]. To be more precise, in that place we put an additional assumption $\eta_1 = 0$, and studied not Case II but Case V. Nevertheless the same discussion goes through in the present case even if η_1 does not vanish.

Proposition 4.8 (cf. [I, Prop. 7.2]). *One has* $\dim \Phi[\lambda, \eta]=7$ *for any character* η *of* N_m *which is trivial on the root subgroup* $\exp \mathfrak{g}((\psi_2-\psi_1)/2) \subset N_m$, *or equivalently* $\eta_3^-=\eta_4^-=0$.

Now the system $C[\lambda, \eta]$ has been completely solved for each lowest highest weight λ and each character η of N_m.

§5. Solutions of the system $C[\lambda, \eta_\xi]$ for an infinite-dimensional monomial representation $\eta_\xi=C^\infty\text{-Ind}_{N'}^{N_m}(\xi)$

We now proceed to the case where η is infinite-dimensional. Let $N'=\exp \mathfrak{n}'$ be the analytic subgroup of G with Lie algebra

$$\mathfrak{n}'=\mathfrak{g}((\psi_2-\psi_1)/2)\oplus\mathfrak{g}((\psi_2+\psi_1)/2)\oplus\mathfrak{g}(\psi_2)\subset\mathfrak{n}_m. \tag{5.1}$$

Then N' is the unipotent radical of a unique (up to G-conjugacy) maximal cuspidal parabolic subgroup of G (see [I, §8]). For a character ξ of N', consider the representation $\eta_\xi=C^\infty\text{-Ind}_{N'}^{N_m}(\xi)$ of N_m induced from ξ in C^∞-context.

In this section, we solve the system $C[\lambda, \eta_\xi]$ of differential equations in Theorem 2.1, whose solutions give rise to embeddings of discrete series into the induced module

$$C^\infty\text{-Ind}_{N_m}^G(\eta_\xi)\cong C^\infty\text{-Ind}_{N'}^G(\xi).$$

Although our result here is not perfect for all the cases of (λ, ξ), we can specify and study precisely the most interesting case where the solution space for $C[\lambda, \eta_\xi]$ turns to be non-zero and finite-dimensional.

5.1. Operators $L_i^\pm$ and $S_j^\mp$ in coordinates (t_1, t_2, y). Set $N_m'=\exp \boldsymbol{R}E_1\subset N_m$. Then one gets a semidirect product decomposition $N_m=N_m'\ltimes N'$, so we can realize the monomial representation η_ξ on $\mathcal{F}\equiv C^\infty(N_m')$ as

$$\eta_\xi(g)\varphi(x)=\xi(n'(g, x))^{-1}\varphi(n_m'(g, x)) \qquad (x\in N_m') \tag{5.2}$$

for $g\in N_m$ and $\varphi\in C^\infty(N_m')$, where $g^{-1}x=n_m'(g, x)n'(g, x)$ with $n_m'(g, x)\in N_m'$, $n'(g, x)\in N'$.

Let us introduce coordinates of the direct product space $A_p\times N_m'$:

$$\boldsymbol{R}^3\ni(t_1, t_2, y)\longmapsto(\exp(-t_1H_1-t_2H_2),\quad \exp(-yE_1))\in A_p\times N_m',$$

and regard an element $c\in C^\infty(A_p, \mathcal{F})$ as a function in (t_1, t_2, y) in such a way that $\exp(-yE_1)\mapsto c(t_1, t_2, y)$ expresses the value of c (in $\mathcal{F}$) at the point $\exp(-t_1H_1-t_2H_2)\in A_p$. Using (5.2), one finds easily that the differential operators $L_i^\pm$ $(i=1, 2)$, $S_j^\mp$ $(j=3, 4)$ on $\boldsymbol{R}^3=\boldsymbol{R}^3_{t_1,t_2,y}$, defined by (2.16), are expressed as

$$L_1^\pm=\partial_1\pm2\sqrt{-1}\, e^{2t_1}\partial_y, \qquad L_2^\pm=\partial_2 \quad (=L_2(\text{put})), \tag{5.3}$$

$$S_j^\mp=e^{t_2-t_1}(\pm\xi_j^-+(e^{2t_1}\pm\sqrt{-1}\, y)\xi_i^+), \tag{5.4}$$

where $\partial_y=\partial/\partial y$ and $\xi_j^\pm$ denotes the value of differential of ξ at the element $E_j^\pm\in\mathfrak{g}((\psi_2\pm\psi_1)/2)_c\subset\mathfrak{n}_c'$.

In the succeeding subsections, we study separately three cases of parameter $\Lambda\in\Xi_J^+$ in order of $J=$I, III, II (according to the difficulty), and solve the system $C[\lambda, \eta_\xi]$ of differential equations for $c_{kl}\in C^\infty(\boldsymbol{R}^3)$. As noted before, the results for remaining three cases $J=$VI, IV and V can be derived from those for $J^*=$VI$-J+$I$=$I, III, II by certain substitution of parameters.

5.2. Case I: $\Lambda\in\Xi_{\mathrm{I}}^+$. In this case, discussing just as in 3.2, we immediately obtain the following complete result.

Proposition 5.1. (1) *If the character ξ is non-trivial, the system $C[\lambda, \eta_\xi]$ does not admit any non-zero solutions.*

(2) *Assume that ξ be the trivial character of N'. Then the solutions $(c_{kl})_{k,l}$ of $C[\lambda, \eta_\xi]$ are in bijective correspondence to $\varphi\in C^\infty(\boldsymbol{R}^3)$ satisfying $(L_1^+-b_0)\varphi=(L_2+b_3)\varphi=0$ through $c_{kl}=\delta_k^0\delta_l^0\cdot\varphi$ (Kronecker's δ_k^0) In particular, the solution space $\Phi[\lambda, \eta_\xi]$ is infinite-dimensional.*

5.3. Case III: $\Lambda\in\Xi_{\mathrm{III}}^+$. We define functions Θ_j $(j=3, 4)$ on $\boldsymbol{R}^3$ by

$$\Theta_j=S_j^-/S_j^+ \quad \text{if} \quad S_j^+\not\equiv 0, \qquad \Theta_j=0 \quad \text{if} \quad S_j^+\equiv 0. \tag{5.5}$$

Note that $S_j^\mp$ is identically zero if and only if $\xi_j^+=\xi_j^-=0$. Furthermore, Θ_j is a function of two variables (t_1, y) and independent of t_2.

Put $d_{kl}(t_1, t_2, y)=k!\,e^{-(r+2)t_2}c_{kl}(t_1, t_2, y)$ for $0\leqq k\leqq r$ and $0\leqq l\leqq s$. As in Lemma 3.2, the system $C[\lambda, \eta_\xi]$ for $(c_{kl})_{k,l}$, consisting of four equations $(C_2^\pm:1)$, $(C_2^\pm:2)$ (see 2.3), is transferred into the following system (5.6)-(5.9) for $(d_{kl})_{k,l}$:

$$2S_3^-d_{k+1,l}=-(L_2-k+l+b_2)d_{kl}, \tag{5.6}$$

$$2S_4^+d_{kl}=-(L_2+k+1-l-b_2)d_{k+1,l}, \tag{5.7}$$

$$2(l+1)(s-l)d_{k+1,l+1}=-\{(L_1^++k+l-b_0)-\Theta_3^{-1}(L_2-k+l+b_2)\}d_{kl}, \tag{5.8}$$

$$2d_{k,l-1}=-\{(L_1^--k-1-l+b_0)-\Theta_4(L_2+k+1-l-b_2)\}d_{k+1,l}, \tag{5.9}$$

where $0\leqq k\leqq r-1$ and $0\leqq l\leqq s$.

To solve the above system for $(d_{kl})_{k,l}$, we go into the case-by-case study depending on the vanishing of the functions $S_j^\pm$.

CASE 1. First assume that $S_j^\pm\not\equiv 0$ or equivalently $|\xi_j^-|+|\xi_j^+|\neq 0$, for $j=3, 4$. Let $Z_j^\pm$ denote the set of zeros of functions $S_j^\pm$:

$$Z_j^\pm=\{(t_1, t_2, y)\in\boldsymbol{R}^3\,|\,(e^{2t_1}\pm\sqrt{-1}\,y)\xi_j^+=\mp\xi_j^-\}, \tag{5.10}$$

which is empty if $\pm\mathrm{Re}(\xi_j^-\overline{\xi_j^+})\geqq 0$, and otherwise it forms a line vertical to the (t_1, y)-plane.

Let Ω be any simply connected domain in $\boldsymbol{R}^3$ contained in $\Omega_\xi\equiv\boldsymbol{R}^3\setminus(Z_3^-\cup Z_4^+)$. We solve the system (5.6)-(5.9) restricted on Ω. Set $d_l=d_{b_2+l,l}$, $0\leqq l\leqq s$, with in mind the inequality $1\leqq b_2\leqq b_0=b_2+s\leqq r-1$ (by (3.10)). Then $(d_l)_l$ satisfies for $0\leqq l\leqq s$,

$$L_2^2d_l=4S_3^-S_4^+d_l \qquad \text{(by (5.6), (5.7))}, \tag{5.11}$$

(5.12) $$2(l+1)(s-l)d_{l+1}=-\{(L_1^+ +2l-s)-\Theta_3^{-1}L_2\}d_l \quad (\text{by } (5.8)),$$

(5.13) $$2d_{l-1}=-\{(L_1^- -2l+s)-\Theta_4 L_2\}d_l \quad (\text{by } (5.9)),$$

Conversely, one sees easily that any $(d_l)_l$, $d_l\in C^\infty(\Omega)$, satisfying (5.11)-(5.13) can be extended uniquely to a solution of (5.6)-(5.9) through the relations (5.6), (5.7). Thus we get

Lemma 5.2. *The system of differential equations* (5.6)-(5.9) *for* $(d_{kl})_{k,l}$ *on* Ω *is equivalent to* (5.11)-(5.13) *for* (d_l) *through* $d_l=d_{b_2+l,l}$, $0\leqq l\leqq s$.

Now put $h_l=L_2d_l$ and introduce a function p with values in $C^{2(s+1)}$ by

(5.14) $$p={}^t(d_0, d_1, \cdots, d_s, h_0, h_1, \cdots, h_s).$$

Then (5.11)-(5.13) is rewritten into the following system of first order differential equations for p:

(5.15) $$(L_1^+-D_1^+)p=(L_1^- -D_1^-)p=(L_2-D_2)p=0.$$

Here $D_1^\pm$ and D_2 are the matrices of functions defined by

(5.16) $$D_1^+=\begin{bmatrix} X & \Theta_3^{-1}\cdot I \\ 4S_3^+S_4^+I & X\end{bmatrix},\quad D_1^-=\begin{bmatrix} Y & \Theta_4\cdot I \\ 4S_3^-S_4^-I & Y\end{bmatrix},\quad D_2=\begin{bmatrix} 0 & I \\ 4S_3^-S_4^+I & 0\end{bmatrix},$$

with I (resp. 0) the identity (resp. zero) matrix of degree $s+1$, and

$$X=\begin{bmatrix} \alpha_0 & \beta_0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & & \cdot & 0 & 0 \\ \vdots & & \cdot & \cdot & \cdot & & \vdots \\ \vdots & & & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdots & & 0 & \alpha_{s-1} & \beta_{s-1} \\ 0 & 0 & \cdots & & 0 & 0 & \alpha_s \end{bmatrix},\quad Y=\begin{bmatrix} -\alpha_0 & 0 & \cdots & 0 & 0 \\ -2 & -\alpha_1 & & & 0 \\ 0 & -2 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & -2 & -\alpha_s \end{bmatrix},$$

with $\alpha_l=s-2l$, $\beta_l=2(l+1)(l-s)$. In view of (5.3), we immediately see that (5.15) is equivalent to

(5.17) $$\begin{cases} (\partial_1-B)p=(\partial_y-B')p=(\partial_2-D_2)p=0 \\ \text{with } B=(D_1^+ +D_1^-)/2, \quad B'=-\sqrt{-1}\, e^{-2t_1}(D_1^+-D_1^-)/4. \end{cases}$$

Lemma 5.3. *One has the bracket relations of differential operators*

(5.18) $$[L_1^\pm-D_1^\pm, L_2-D_2]=0,\quad [L_1^+-D_1^+, L_1^- -D_1^-]=2(L_1^- -D_1^-)-2(L_1^+-D_1^+),$$

which imply that the operators ∂_1-B, ∂_y-B' *and* ∂_2-D_2 *in* (5.17) *commute with one another.*

The relation (5.18) is proved by an elementary but little lengthy calculation, so we omit the proof.

This lemma shows that the system (5.17) is completely integrable on Ω, and thus we get the following consequences.

Proposition 5.4. *If $S_3^-\not\equiv 0$ and $S_4^+\not\equiv 0$, the system* (5.17) *has exactly* $2(s+1)$*-number of linearly independent solutions on any simply connected domain Ω contained in $\Omega_\xi = \boldsymbol{R}^3 \setminus (Z_3^- \cup Z_4^+)$.*

Theorem 5.5. *Let $\Phi[\lambda, \eta_\xi]$ be the space of solutions of the system $C[\lambda, \eta_\xi]$ on $\boldsymbol{R}^3$. Then one has*

$$\dim \Phi[\lambda, \eta_\xi] \leqq 2(s+1) \tag{5.19}$$

for any character ξ of N' such that $|\xi_j^+|+|\xi_j^-|\neq 0$ $(j=3, 4)$. Furthermore, the equality holds in (5.19) *if ξ satisfies in addition* $\mathrm{Re}(\xi_3^-\overline{\xi_3^+})\leqq 0$ *and* $\mathrm{Re}(\xi_4^-\overline{\xi_4^+})\geqq 0$ (*since $\Omega_\xi = \boldsymbol{R}^3$ in this case*).

We do not discuss here on the behavior of solutions p of (5.17) at the set $Z_3^- \cup Z_4^+$ of singular points of the system, and we leave it open.

CASE 2. Let us consider the remaining case $S_3^- \cdot S_4^+ \equiv 0$. We may assume $S_3^- \equiv 0$ without loss of generality. Then, it is readily verified that, for any $\varphi \in C^\infty(\boldsymbol{R}^3)$ such that

$$(L_1^- + b_1)\varphi = (L_2 + b_3)\varphi = 0,$$

the matrix of functions $(y_{kl})_{k,l}$ with $y_{kl} = \delta_k^r \delta_l^0 \cdot \varphi$ satisfies the system (5.6)-(5.9) in question. So one has

Proposition 5.6. *If either the function S_3^- or S_4^+ on $\boldsymbol{R}^3$ is identically zero, the solution space $\Phi[\lambda, \eta_\xi]$ for the system $C[\lambda, \eta_\xi]$ is infinite-dimensional.*

Remark 5.7. We can solve the system (5.6)-(5.9) perfectly on any simply connected domain in $\boldsymbol{R}^3$ on which both functions Θ_3^{-1} and Θ_4 have no singular points (Θ_3^{-1} should be understood as zero if $S_3^- \equiv 0$). This is done through an argument similar to that in 3.3, so we do not carry it here again.

5.4. Case II: $\Lambda \in \Xi_{\mathrm{II}}^+$. In this case we obtain the following result which allows us to say that the discrete series π_Λ^* with Λ (FFW), occurs in the induced G-module $\pi(\eta_\xi)$ with infinite multiplicity.

Theorem 5.8. *If $\Lambda = \lambda + \rho_c - \rho_n$ is Δ_{II}^+-dominant, the system of differential equations $C[\lambda, \eta_\xi]$ has infinitely many linearly independent solutions for any character ξ of N'.*

The assertion for the trivial $\xi = 1_{N'}$ follows from [I, Prop. 9.5], where we have solved the system $C[\lambda, 1_{N'}]$ completely. In general, we can construct infinitely many solutions of $C[\lambda, \eta_\xi]$ in an explicit way.

In what follows, we assume that ξ is generic: $|\xi_j^-|+|\xi_j^+|\neq 0$ $(j=3, 4)$, and we shall prove the above theorem by constructing solutions through power series. With the argument in 4.3 in mind, one can deal with the remaining case in a similar way, for which the details are omitted here.

Note that the system $C[\lambda, \eta_\xi]$ consists of five equations (C_1^+), $(C_2^-: i)$, $(C_3^-: i)$ $(i=1, 2)$ in 2.3, and that the function S_4^+ (resp. S_3^+) in $(C_2^-: 2)$ (resp. in $(C_3^-: 2)$) is not identically zero by the genericness of ξ. So any solution $(c_{kl})_{0\leq k\leq r, 0\leq l\leq s}$ of $C[\lambda, \eta_\xi]$ is uniquely determined by the single c_{rs} through $(C_2^-: 2)$ and $(C_3^-: 2)$. We set

$$q=\exp(b_3t_1-(b_0+2)t_2)\cdot c_{rs}. \tag{5.20}$$

(Compare with (4.1) for $(k, l)=(r, s)$.) Then, just as in Lemma 4.2, we can specify a system of differential equations for q, equivalent to $C[\lambda, \eta_\xi]$, as follows.

Lemma 5.9. *The function q satisfies*

$$(L_1^+L_2-4S_3^+S_4^+)q=0, \tag{5.21}$$

$$\{(L_2-2b_3-2)L_2^2+4S_3^+S_4^+(L_1^--(\Theta_3+\Theta_4)L_2-2b_3)\}q=0. \tag{5.22}$$

Conversely, any $q\in C^\infty(\boldsymbol{R}^3)$ satisfying (5.21)-(5.22) *gives rise to a unique solution of $C[\lambda, \eta_\xi]$ through* (5.20), $(C_2^-: 2)$ *and* $(C_3^-: 2)$.

5.4.1. Construction of formal solutions. Let us change the variables (t_1, t_2, y) into (z, w) as

$$z=e^{2t_1}+\sqrt{-1}\,y, \qquad w=e^{t_2}, \tag{5.23}$$

and consider the system (5.21)-(5.22) on the domain $\{(z, w)\in \boldsymbol{C}\times\boldsymbol{R} \mid \mathrm{Re}z>0,\ w>0\}$. Then one finds

$$\begin{aligned} &L_1^+=2(z+\bar{z})\cdot\partial/\partial\bar{z}, \qquad L_1^-=2(z+\bar{z})\cdot\partial/\partial z, \qquad L_2=w\cdot\partial/\partial w,\\ &S_j^+=w\cdot\left(\frac{2}{z+\bar{z}}\right)^{1/2}(\xi_j^-+z\xi_j^+), \qquad \Theta_j=(-\xi_j^-+\bar{z}\xi_j^+)/(\xi_j^-+z\xi_j^+). \end{aligned} \tag{5.24}$$

Now we look for the formal solutions q of the form

$$q=\sum_{j=0}^{\infty}\frac{1}{j!}q_j(z)w^{2j} \quad \text{with functions } q_j \text{ in } z. \tag{5.25}$$

Since $L_2w^{2j}=2jw^{2j}$, (5.21) and (5.22) are transferred into the following differential difference equations for q_j:

$$(z+\bar{z})^2\partial q_j/\partial\bar{z}-2s(z)q_{j-1}=0, \tag{5.26}$$

$$jj'q_j+2s(z)\{\partial/\partial z-(b_3+(j-1)(\Theta_3+\Theta_4))/(z+\bar{z})\}q_{j-1}=0, \tag{5.27}$$

where $s(z)=(\xi_3^-+z\xi_3^+)(\xi_4^-+z\xi_4^+)$ and $j'=j-b_3-1$.

Notice that b_3+1 is a positive integer. With (5.27) in mind we put an additional assumption on q_j:

$$q_j=0 \qquad \text{for} \quad j<b_3+1. \tag{5.28}$$

Then, by (5.26), q_{b_3+1} is holomorphic in z, and by (5.27) each q_j is determined recursively from the first q_{b_3+1}. Conversely, we find that any holomorphic function q_{b_3+1} gives a solution (q_j) through (5.27). More exactly, one gets

Proposition 5.10. *The systems of functions* $q_j \in C^\infty(\boldsymbol{R}^3)$, $j=0, 1, 2, \cdots$, *satisfying* (5.26)-(5.28) *correspond bijectively to holomorphic functions* φ *in the right half plane* $D=\{z\in \boldsymbol{C} \mid \mathrm{Re}\, z>0\}$ *through*

$$q_j=\frac{(-2s(z))^j}{j!\, j'!\,(z+\bar{z})^{2j-b_3}}\cdot I_{j,\varphi}(z) \qquad \text{for} \quad j\geqq b_3+1, \tag{5.29}$$

where $I_{j,\varphi}(z)$ *is given by*

$$I_{j,\varphi}(z)=\Big((z+\bar{z})^2\cdot\frac{\partial}{\partial z}\Big)^{j'}(z+\bar{z})^{b_3+2}s(z)^{-b_3-1}\varphi(z). \tag{5.30}$$

Proof. It rests only to show the expression (5.29). Noting $s^{-1}(\partial s/\partial z)=(\Theta_3+\Theta_4+2)/(z+\bar{z})$, one sees easily a relation of differential operators:

$$\Big(\frac{\partial}{\partial z}-\frac{b_3+(j-1)(\Theta_3+\Theta_4)}{z+\bar{z}}\Big)\cdot s(z)^{j-1}(z+\bar{z})^{b_3-2(j-1)}$$
$$=s(z)^{j-1}(z+\bar{z})^{b_3-2(j-1)}\frac{\partial}{\partial z}$$

for each $j>0$. Define a function $\tilde{q}_j$ through $q_j=(-2s(z))^j(z+\bar{z})^{b_3-2j}\tilde{q}_j/j!\, j'!$. Then (5.27) is rewritten as

$$\tilde{q}_j=(z+\bar{z})^2(\partial\tilde{q}_{j-1}/\partial z), \tag{5.31}$$

and thus we obtain the desired expression (5.29) with $\varphi(z)=(b_3+1)!/(-2)^{b_3+1}\cdot q_{b_3+1}(z)$. Q. E. D.

5.4.2. Convergence of the formal power series. Let $\hat{\varphi}$ be a polynomial in z and put $\varphi(z)=s(z)^{b_3+1}\hat{\varphi}(z)$. We show that the formal power series (5.25) with q_j in (5.29) converges and gives a solution of (5.21)-(5.22).

In order to evaluate $|q_j(z)|$ for $z\in D$, we need the following

Lemma 5.11. *For any non-negative integer* k, *the differential operator* $((z+\bar{z})^2\cdot\partial/\partial z)^k$ *is expanded as*

$$((z+\bar{z})^2\cdot\partial/\partial z)^k=\textstyle\sum_{1\leq i\leq k}c_i^k(z+\bar{z})^{k+i}(\partial/\partial z)^i, \tag{5.32}$$

where the coefficients c_i^k *are given recursively by*

$$c_i^{k+1}=c_{i-1}^k+(k+i)c_i^k, \qquad c_1^1=1, \tag{5.33}$$

and they are estimated as

$$0<c_i^k\leqq 2^k\binom{k}{i}k\,!. \tag{5.34}$$

The proof of this lemma is straightfoward by the induction on k, so we omit it here.

By means of (5.32), $I_{j,\varphi}(z)$ is expanded as

$$\sum_{i=0}^{j'}\sum_{k=0}^{m(i)}c_i^{j'}\binom{i}{k}\cdot\frac{(b_3+2)!}{(b_3+2-k)!}\cdot(z+\bar{z})^{j+i-k+1}\Big(\frac{\partial}{\partial z}\Big)^{i-k}\hat{\varphi}(z),$$

where $m(i)=\min(b_3+2, i)$, and one finds from (5.34),

$$\sum_{i,k} c_i^{j'}\binom{i}{k} \leqq \sum_{i,k} 2^{j'} j'!\binom{j'}{i}\binom{i}{k} \leqq 2^{j'} j'^2 (j'!)^2.$$

We thus obtain the estimate

$$(5.35) \qquad |I_{j,\varphi}(z)| \leqq 2^{j'} j'^2 (j'!)^2 (z+\bar{z}+1)^{2j} \Big\{(b_3+2)! \cdot \sum_{k=0}^{\infty} \Big|\Big(\frac{\partial}{\partial z}\Big)^k \hat{\varphi}(z)\Big|\Big\}$$

for $z \in D$, where the sum in the right hand side is finite since $\hat{\varphi}$ is, by assumption, a polynomial in z. This together with (5.29) and (5.26)-(5.27) implies the following

Proposition 5.12. *The series $q=\sum_{j\geq 0}(1/j!)q_j(z)w^{2j}$, and also its term-by-term derivatives converge absolutely and uniformly on any compact subset of the domain $\{(z, w) \in \boldsymbol{C}\times\boldsymbol{R} \mid \mathrm{Re}\, z>0,\ w>0\}$, and q gives a sotution of the system of differential equations* (5.21)-(5.22).

In this way we have obtained a system of infinite linearly independent solutions of $C[\lambda, \eta_\xi]$, and our Theorem 5.8 is now completely proved.

§6. (Generalized) Whittaker models for the discrete series

Let π_Λ be the discrete series representation of G with lowest highest weight $\lambda=\Lambda-\rho_c+\rho_n$ and π_Λ^* denotes its contragredient. Gathering our results in the preceding sections, we now determine (generalized) Whittaker models for the discrete series π_Λ^* (Theorems 6.1 and 6.5). We give our results on embeddings under a slight assumption on regularity of λ: (FFW) in Theorem 1.3. Nevertheless, one would be able to show that the results remain true for any λ by using Zuckerman's translation functor [11]. See [I, §3] for the embeddings into the principal series.

Our group $G=SU(2, 2)$ has, up to G-conjugacy, two proper cuspidal parabolic subgroups. We describe the embeddings of π_Λ^* into G-modules $\Gamma_{\xi,N}=C^\infty\text{-}\mathrm{Ind}_N^G(\xi)$ smoothly induced from characters ξ of the unipotent radical N of such a parabolic subgroup. These representations $\Gamma_{\xi,N}$ include so-called Gelfand-Graev representations and some of their generalizations (see [4], [5], [6], [9]).

6.1. Embeddings of discrete series into Γ_{η,N_m}. First consider the case $N=N_m$, the maximal unipotent subgroup of G in 2.1. By Theorem 1.3, embeddings of π_Λ^* into Γ_{η,N_m} as $(\mathfrak{g}_C, K)$-modules correspond bijectively to solutions of the system of differential difference equations $C[\lambda, \eta]$, given in §§3-4. Here $\xi=\eta$ is a character of N_m. We thus establish our first main result on embeddings as follows.

Theorem 6.1. *Let η be a character of N_m, and denote by $\eta_1=\eta(E_1)$ and $\eta_j^-=\eta(E_j^-)$ $(j=3, 4)$ the values of η at the elements $E_1, E_j^- \in (\mathfrak{n}_m)_C$ in* 2.1. *Assume that the Blattner parameter λ of discrete series π_Λ satisfies the condition (FFW) in Theorem* 1.3. *Then the representation π_Λ^* with $\Lambda \in \Xi_J^+$* (I$\leqq J \leqq$VI, *see* (2.9)) *occurs in $\Gamma_{\eta,N_m}=C^\infty\text{-}\mathrm{Ind}_{N_m}^G(\eta)$ as a $(\mathfrak{g}_C, K)$-submodule with multiplicity $m(J, \eta)$ given in Table* 6.2 *and $m(J^*, \eta)=m(J, \eta)$*

for $J^*=\mathrm{VI}-J+\mathrm{I}$. *In the table,* $*$ *means any non-zero complex number, and, for example, the first row should be understood as: if* $\eta_1\neq 0$ *and* $\eta_j^-\neq 0$ $(j=3, 4)$ *then* $m(J, \eta)=0, 4$ *or* 0 *according as* $J=\mathrm{I}$, II *or* III.

Table 6.2. Multiplicity $m(J, \eta)$

η_1	η_3^-	η_4^-	I	II	III
*	*	*	0	4	0
*	* (resp. 0)	0 (resp. *)	0	4	1
*	0	0	1	7	2
0	*	*	0	4	2
0	* (resp. 0)	0 (resp. *)	0	4	2
0	0	0	1	7	3

Remark 6.3. (1) The first row in Table 6.2 describes the embeddings of π_Λ^* into Gelfand-Graev representations, and the last one shows the number of embeddings of π_Λ^* into the principal series induced from the minimal parabolic subgroup P_m containing N_m (see [I, §6]).

(2) Note that the function $\Lambda\mapsto\dim \mathrm{Hom}_{\mathfrak{g}_C-K}(\pi_\Lambda^*, \Gamma_{\eta,N_m})$ is constant as far as Λ in the above theorem lies in a fixed Weyl chamber.

Examining the columns of Table 6.2, we find the following fact.

Corollary 6.4. *The discrete series* π_Λ^* *appears in the induced representation* $\Gamma_{\eta,N_m}=C^\infty\text{-}\mathrm{Ind}_{N_m}^G(\eta)$ *for every character* η *of* N_m *if and only if* Λ *is* Δ_J^+*-dominant with* $J=\mathrm{II}$ *or* V.

Although in Theorem 6.1 we have written down the multiplicities only, we can describe the embeddings $\pi_\Lambda^*\overset{\iota}{\hookrightarrow}\Gamma_{\eta,N_m}$ explicitly using the lowest K-type vectors in $\iota(\pi_\Lambda^*)$ which have been determined in §§3-4 by solving the system of differential equations $C[\lambda, \eta]$.

6.2. Embeddings of discrete series into $\Gamma_{\xi,N'}$**.** Secondly, let N' be as in 5.1, the unipotent radical of maximal cuspidal parabolic subgroup $P'\supset P_m$, and ξ be a character of N'. Since $\Gamma_{\xi,N'}=C^\infty\text{-}\mathrm{Ind}_{N'}^G(\xi)\cong C^\infty\text{-}\mathrm{Ind}_{N_m}^G(\eta_\xi)$ with $\eta_\xi=C^\infty\text{-}\mathrm{Ind}_{N'}^{N_m}(\xi)$, the system of differential equations $C[\lambda, \eta_\xi]$, studied in §5, characterizes the embeddings of π_Λ^* into the induced module $\Gamma_{\xi,N'}$. Summarizing the results in §5, we immediately get the following

Theorem 6.5. (1) *For a character* ξ *of* N', *set* $\xi_j^\pm=\xi(E_j^\pm)$ $(j=3, 4)$ *as in* 5.1. *Under the assumption* (FFW) *on* λ, *the multiplicity* $m'(\Lambda, \xi)=\dim \mathrm{Hom}_{\mathfrak{g}_C-K}(\pi_\Lambda^*, \Gamma_{\xi,N'})$ *of* π_Λ^* *in* $\Gamma_{\xi,N'}$ *is given in Table* 6.6. *In the table,* r *and* s *are the non-negative integers in* (2.12), *and other conventions are the same as in Table* 6.2.

(2) *Furthermore there holds the equality* $m'(\Lambda, \xi)=2(s+1)$ *for* $\Lambda\in\Xi_{\mathrm{III}}^{+}$ (*resp.* $2(r+1)$ *for* $\Lambda\in\Xi_{\mathrm{IV}}^{+}$) *if* $\mathrm{Re}(\xi_3^-\overline{\xi_3^+})\leqq 0$ *and* $\mathrm{Re}(\xi_4^-\overline{\xi_4^+})\geqq 0$ (*resp.* $\mathrm{Re}(\xi_3^-\overline{\xi_3^+})\geqq 0$ *and* $\mathrm{Re}(\xi_4^-\overline{\xi_4^+})\leqq 0$), *where the bar means the complex conjugation.*

Table 6.6. Multiplicity $m'(\Lambda, \xi)$

$\lvert\xi_3^-\rvert+\lvert\xi_3^+\rvert$	$\lvert\xi_4^-\rvert+\lvert\xi_4^+\rvert$	I, VI	II, V	III (resp. IV)
*	*	0	∞	bounded by $2(s+1)$ (resp. $2(r+1)$)
* (resp. 0)	0 (resp. *)	0	∞	∞
0	0	∞	∞	∞

This is our second main result on embeddings of discrete series. From the above table we find

Corollary 6.7. *A discrete series representation of G occurs in some induced module* $\Gamma_{\xi,N'}$ *with finite* (*non-zero*) *multiplicity if and only if the corresponding Harish-Chandra parameter* Λ *is in* $\Xi_{\mathrm{III}}^{+}\cup\Xi_{\mathrm{IV}}^{+}$.

This type of embeddings, with finite multiplicity, is of particular importance for classifying irreducible representations of a semisimple group through generalized Whittaker models.

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