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The geometry of bicharacteristics and the global existence of holomorphic solutions of systems of linear differential equations

By

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0. Introduction

In the previous paper [4] of Kawai and the author we studied the relationship between the geometry of bicharacteristics and the (semi-)global existence of holomorphic solutions of single linear differential equations. The main result of [4] is that, in order to discuss the (semi-)global existence of holomorphic solutions, we have to take into account not only the convexity of the domain in question with respect to bicharacteristic curves, but also the pseudo-convexity of some manifold given through the foliation structure determined by bicharacteristic curves. In this article we generalize this result to the case of over-determined systems of linear differential equations with one unknown function; of course, we have to replace bicharacteristic strips by bicharacteristic manifolds.

We proceed in a similar way as in [4]. When we study the existence of holomorphic solutions, we should consider the Cauchy-Riemann equations together with the linear differential equations under consideration. Then, due to the Cauchy-Riemann equations, we can apply the theory of boundary value problems for elliptic systems developed by Kashiwara-Kawai [1]. In fact, making use of this theory with a result of Sato-Kawai-Kashiwara [7], Kawai has presented in [2] and [3] some theorems on finite-dimensionality of cohomology groups attached to elliptic systems. In the situation we are considering, his results give sufficient conditions which guarantee the (semi-)global existence of holomorphic solutions. (See Theorem 2.1 below). We will investigate the geometric meaning of his conditions, supposing the second order tangency of the bicharacteristics and the boundary of the domain in question (Theorem 2.4). As a result we can obtain our main theorems (Theorem 1.5 and Theorem 1.8) which describe the relationship between the geometry of bicharacteristics and the (semi-)global existence of holomorphic solutions.

Here we should mention that the geometric conditions discussed here have its origin in the work of Suzuki [8]. He has given a complete description of the

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conditions which guarantee the global existence of holomorphic solutions of single linear differential equations of first order. Besides the work of Suzuki, for single linear differential equations there are several works closely related to our problem of the global existence of holomorphic solutions: For example, Pallu de la Barrière [6], Trépreau [9], [10], and so on. Compared with the case of single equations, almost no global existence theorems are known for general systems, as far as the present author knows.

Now let us describe briefly the plan of this paper. In §1, we prepare some notions and notations, and state our main results. In §2 we give the outline of the proof of our main results. The proof consists of two theorems: One is Kawai's theorem, which is explained in this section, and the other is Theorem 2.4, which will be proved in the subsequent three sections. First we study in §3 the geometric situations of bicharacteristics under a non-degeneracy condition. Then we prove the decomposition theorem of some Hermitian form in §4, assuming one proposition (Proposition 4.4). The main part of this paper is in a sense this decomposition theorem, from which Theorem 2.4 easily follows. And finally in §5, we give the proof of Proposition 4.4.

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1. Notations and main results

Let us first prepare some notations. For an open subset U of $\mathbf{C}^n$, T^*U denotes the cotangent bundle of U and $\mathcal{O}$ denotes the sheaf of holomorphic functions on U. We denote by z the standard coordinate of $\mathbf{C}^n$ and by ζ the corresponding cotangential coordinate of $T^*\mathbf{C}^n$. We also denote by x and y the real part and the imaginary part of z respectively. Let $P_\gamma(z, \partial_z)$ $(1 \leq \gamma \leq d)$ be linear differential operators with holomorphic coefficients defined on U. Let us denote by $p_\gamma(z, \zeta)$ the principal symbol of the operator $P_\gamma(z, \partial_z)$. Throughout this article we suppose that $1 \leq d \leq n-1$ and that $P_\gamma(z, \partial_z)$ $(1 \leq \gamma \leq d)$ satisfy the following conditions:

(1.1) $[P_\gamma, P_\delta]$, the commutator of P_γ and P_δ, identically vanishes for $\gamma, \delta = 1, \ldots, d$.

(1.2) $\operatorname{grad}_\zeta p_1(z, \zeta), \ldots, \operatorname{grad}_\zeta p_d(z, \zeta)$ are linearly independent over $\mathbf{C}$ on $\{(z, \zeta) \in T^*U; \zeta \neq 0, p_1(z, \zeta) = \cdots = p_d(z, \zeta) = 0\}$.

We denote by $\mathfrak{M}$ the coherent left $\mathcal{D}$-module determined by $P_\gamma(z, \partial_z)$ $(1 \leq \gamma \leq d)$, i.e.

$$\mathfrak{M} = \mathscr{D}/(\mathscr{D}P_1 + \cdots + \mathscr{D}P_d),$$

where $\mathscr{D}$ denotes the sheaf of linear differential operators (with real analytic coefficients) on U.

Let φ be a strictly plurisubharmonic real analytic function defined on U, and Ω be a relatively compact strongly pseudo-convex domain defined by

$$\Omega = \{z \in U;\ \varphi(z) < 0\}. \tag{1.3}$$

We suppose that

(1.4) $\partial\varphi = \mathrm{grad}_z \varphi$ never vanishes on the boundary $\partial\Omega$ of Ω.

Here, and in what follows, ∂_j and $\bar{\partial}_j$ denote

$$\partial_j = \frac{\partial}{\partial z_j} = \frac{1}{2}(\partial_{x_j} - \sqrt{-1}\,\partial_{y_j}), \qquad j = 1,\dots,n,$$

$$\bar{\partial}_j = \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\partial_{x_j} + \sqrt{-1}\,\partial_{y_j}), \qquad j = 1,\dots,n,$$

respectively. Let us denote by C_0 the set of characteristic boundary points,

$$C_0 = \{z \in U;\ \varphi(z) = 0 \text{ and } p_\gamma(z, \partial\varphi(z)) = 0,\ \gamma = 1,\dots,d\},$$

and also denote by C and C_- the following sets:

$$C = \{z \in U;\ p_\gamma(z, \partial\varphi(z)) = 0,\ \gamma = 1,\dots,d\},$$

$$C_- = \{z \in U;\ \varphi(z) < 0 \text{ and } p_\gamma(z, \partial\varphi(z)) = 0,\ \gamma = 1,\dots,d\}.$$

The purpose of this article is to find a geometric condition which guarantees the solvability of an over-determined system of linear differential equations

$$P_\gamma u = f_\gamma, \quad \gamma = 1,\dots,d \tag{1.5}$$

in the space of holomorphic functions on Ω, when $f = (f_1,\dots,f_d)$ satisfies the obvious compatibility conditions:

$$P_\gamma f_\delta = P_\delta f_\gamma, \quad \gamma, \delta = 1,\dots,d. \tag{1.6}$$

In studying this problem, bicharacteristic manifolds play an important role. Here let us recall the definition of a bicharacteristic manifold.

Definition 1.1. For a point (z_0, ζ_0) in $\{(z,\zeta) \in T^*U;\ \zeta \neq 0,\ p_1(z,\zeta) = \cdots = p_d(z,\zeta) = 0\}$, the bicharacteristic manifold of $\mathfrak{M}$ through (z_0, ζ_0) is, by definition, the (complex) d-dimensional integral manifold through (z_0, ζ_0) of the system of Hamiltonian operators

$$H_{p_\gamma} = \sum_{1 \le j \le n} \left(\frac{\partial p_\gamma}{\partial \zeta_j} \frac{\partial}{\partial z_j} - \frac{\partial p_\gamma}{\partial z_j} \frac{\partial}{\partial \zeta_j} \right), \qquad \gamma = 1,\dots,d.$$

We denote by $b_{(z_0,\zeta_0)}$ its projection to the base manifold U.

Remark 1.2. It follows from (1.1) and (1.2) that $\{H_{p_\gamma}\}_{\gamma=1,\dots,d}$ satisfies the integrability condition. In fact, they commute each other. Hence the bicharacteristic manifold really exists for any (z_0, ζ_0) by Frobenius theorem. In particular, for a given point (z_0, ζ_0) there exist a neighborhood of (z_0, ζ_0) and d complex parameters $t=(t_1,\dots,t_d)\in\mathbf{C}^d$ such that, for any point $(\tilde{z}, \tilde{\zeta})$ in that neighborhood of (z_0, ζ_0), the bicharacteristic manifold through $(\tilde{z}, \tilde{\zeta})$ is given locally by the imbedding

$$(t_1,\dots,t_d)\longmapsto(z(t_1,\dots,t_d;\tilde{z},\tilde{\zeta}),\ \zeta(t_1,\dots,t_d;\tilde{z},\tilde{\zeta})),$$

which satisfies the following equations:

$$\text{(1.7)}\qquad \begin{cases} \dfrac{\partial z_j(t;\tilde{z},\tilde{\zeta})}{\partial t_\gamma}=\dfrac{\partial p_\gamma}{\partial \zeta_j}(z(t;\tilde{z},\tilde{\zeta}),\ \zeta(t;\tilde{z},\tilde{\zeta})), & \text{for } j=1,\dots,n,\ \gamma=1,\dots,d,\\ \dfrac{\partial \zeta_j(t;\tilde{z},\tilde{\zeta})}{\partial t_\gamma}=-\dfrac{\partial p_\gamma}{\partial z_j}(z(t;\tilde{z},\tilde{\zeta}),\ \zeta(t;\tilde{z},\tilde{\zeta})), & \text{for } j=1,\dots,n,\ \gamma=1,\dots,d,\\ z(0,\dots,0;\tilde{z},\tilde{\zeta})=\tilde{z} & \\ \zeta(0,\dots,0;\tilde{z},\tilde{\zeta})=\tilde{\zeta}. & \end{cases}$$

Here we should notice that this map $(z(t;\tilde{z},\tilde{\zeta}),\ \zeta(t;\tilde{z},\tilde{\zeta}))$ is holomorphic with respect to both t and $(\tilde{z}, \tilde{\zeta})$. Let us also remark that the assumption (1.2) entails that $b_{(z_0,\zeta_0)}$ is a complex d-dimensional submanifold of U given by

$$(t_1,\dots,t_d)\longmapsto(z(t_1,\dots,t_d;z_0,\zeta_0)).$$

On the other hand, according to Kawai's theorem (whose precise statement will be found in the next section), the "boundary behavior" of $\mathfrak{M}$, especially its behavior on C_0, should be essential in order that the solvability of the system (1.5) may hold in the above sense. Hence it can be considered the most important to study how the bicharacteristics of $\mathfrak{M}$ are situated against the boundary of Ω.

First we should notice that, for a point z_0 in C_0, Euler's identity for homogeneous functions implies

$$\text{(1.8)}\qquad \sum_{1\le j\le n}\frac{\partial\varphi}{\partial z_j}(z_0)\frac{\partial p_\gamma}{\partial\zeta_j}(z_0,\partial\varphi(z_0))=0 \qquad \text{for } \gamma=1,\dots,d,$$

which show that the bicharacteristic $b_{(z_0,\partial\varphi(z_0))}$ is tangent to $\partial\Omega$ at z_0. Now let us introduce the following convexity condition:

Definition 1.3. The domain Ω is said to be bicharacteristically convex with respect to $\mathfrak{M}$ at z_0 in C_0 if

$$\text{(1.9)}\qquad \left.\frac{\partial^2\varphi(z(r\phi_1,\dots,r\phi_d;z_0,\partial\varphi(z_0)))}{\partial r^2}\right|_{r=0}>0$$

holds for any $\phi = (\phi_1, \dots, \phi_d) \in \mathbf{C}^d$ with $\|\phi\| = 1$.

Here $z(t_1, \dots, t_d; z_0, \partial\varphi(z_0))$ is the local expression of $b_{(z_0, \partial\varphi(z_0))}$ explained in Remark 1.2. It is obvious that the bicharacteristical convexity guarantees $b_{(z_0, \partial\varphi(z_0))}$ does not intersect with the closure of Ω except z_0 in a sufficiently small neighborhood of z_0.

As stated in the introduction, we have to take account of another condition besides the bicharacteristical convexity. To formulate the condition, let us prepare more notations. Let z_0 be a point in C_0 and suppose that Ω is bicharacteristically convex at z_0. As we will prove in §3, under the assumption of the bicharacteristical convexity C and C_0 are analytic submanifolds of real codimension $2d$ and $(2d+1)$ respectively in a sufficiently small neighborhood of z_0. Furthermore, C intersects transversally with $b_{(z_0, \partial\varphi(z_0))}$ at z_0 (Proposition 3.5 and Proposition 3.8). Let us consider all bicharacteristics of the form

$$b_{(z, \partial\varphi(z))} \quad \text{with} \quad z \in C.$$

As a matter of fact, these bicharacteristics define an analytic foliation of real dimension $2d$ near z_0 (Proposition 3.9). We denote this foliation by b. Now let us define $\tilde{C}_0$ and $\tilde{C}_-$, the bicharacteristic hull of C_0 and C_- respectively, as follows:

$$\tilde{C}_0 = \bigcup_{z \in C_0} b_{(z, \partial\varphi(z))} = \bigcup_{\substack{z \in C \\ \varphi(z) = 0}} b_{(z, \partial\varphi(z))}, \tag{1.10}$$

$$\tilde{C}_- = \bigcup_{z \in C_-} b_{(z, \partial\varphi(z))} = \bigcup_{\substack{z \in C \\ \varphi(z) < 0}} b_{(z, \partial\varphi(z))}. \tag{1.11}$$

Because C intersects transversally at z_0 with a leaf $b_{(z_0, \partial\varphi(z_0))}$ of the foliation b, $\tilde{C}_0$ is a non-singular real hypersurface and $\tilde{C}_-$ is an open subset with $\tilde{C}_0$ as its boundary in a neighborhood of z_0 (Proposition 3.10).

Then our main theorem is

Theorem 1.4. *Let* $\mathfrak{M} = \mathscr{D}/(\mathscr{D}P_1 + \cdots + \mathscr{D}P_d)$ *be a coherent left* $\mathscr{D}$*-module satisfying* (1.1) *and* (1.2), *and let* Ω *be a relatively compact strongly pseudo-convex domain defined by* (1.3) *and satisfying* (1.4). *Suppose that* Ω *satisfies the following condition, that is, suppose that* (i) *and* (ii) *below hold at any point* z_0 *in* C_0.

(i) Ω *is bicharacteristically convex at* z_0 *with respect to* $\mathfrak{M}$.

(ii) *For a (complex) d-codimensional complex submanifold S pessing through* z_0 *and being transversal to* $b_{(z_0, \partial\varphi(z_0))}$, $\tilde{C}_- \cap S$ *is strongly pseudo-convex at* z_0 *in S.*

Then $\mathrm{Ext}^j_{\mathscr{D}}(\Omega; \mathfrak{M}, \mathscr{O})$ *is of finite dimension for every* $j \geq 1$. *In particular,*

$$\{(P_1 u, \dots, P_d u);\ u \in \mathscr{O}(\Omega)\}$$

is of finite codimension in

$$\{(f_1, \dots, f_d) \in \mathscr{O}(\Omega)^d;\ P_\gamma f_\delta = P_\delta f_\gamma,\ \gamma, \delta = 1, \dots, d\}.$$

Remark 1.5. In the condition (ii), "the strong pseudo-convexity of $\tilde{C}_- \cap S$ at z_0 in S" means that there exist an open neighborhood ω of z_0 and a real valued real analytic function ψ defined on $S \cap \omega$ such that

$$\tilde{C}_- \cap S \cap \omega = \{w \in S \cap \omega;\ \psi(w) < 0\}$$

holds and the Levi form $L_{z_0}(\sigma)$ ($\sigma \in \mathbf{C}^{n-d}$) of ψ at z_0, restricted to $\{\sigma \in \mathbf{C}^{n-d};\ \langle \operatorname{grad}\psi(z_0), \sigma\rangle = 0\}$, is strictly positive-definite. Note that $\tilde{C}_0 \cap S \cap \omega$, the boundary of $\tilde{C}_- \cap S \cap \omega$, is a non-singular real hypersurface in $S \cap \omega$ for a sufficiently small neighborhood ω, because S intersects transversally with $b_{(z_0, \partial\varphi(z_0))}$ and $\tilde{C}_0$ is a non-singular real hypersurface under the condition (i).

Remark 1.6. As a d-codimensional complex submanifold S through z_0, we can take an arbitrary submanifold as far as it is transversal to $b_{(z_0, \partial\varphi(z_0))}$. In fact, if the condition (ii) holds for some S, it holds for any S provided that the condition (i) is satisfied. (See Theorem 2.4 below.) However, it is very important that S must be a "smooth" complex submanifold in the complex-analytic sense. Roughly speaking, the condition (ii) is concerned with the complex-analytic structure of the domain Ω in the transversal direction with respect to the foliation b.

Further, if Ω can be contracted to one point with the condition in Theorem 1.4 being satisfied in the course of contraction, then we can obtain the following vanishing theorem:

Theorem 1.7. *Let $\mathfrak{M}$ and Ω be the same as those in Theorem* 1.4. *Suppose that there exists a point z_1 in Ω such that φ satisfies*

$$\varphi(z) \geq \varphi(z_1) \text{ holds for any } z \text{ in } U, \tag{1.12}$$

$$\bigcap_{t > \varphi(z_1)} \{z \in U;\ \varphi(z) < t\} = \{z_1\}, \tag{1.13}$$

$$\operatorname{grad}_z \varphi(z) \neq 0 \quad on \quad \{z \in U;\ z \neq z_1\}. \tag{1.14}$$

Suppose further that $\Omega_\varepsilon = \{z \in U;\ \varphi(z) < \varepsilon\}$ satisfies the condition in Theorem 1.4 *for any ε with $0 \geq \varepsilon > \varphi(z_1)$. Then $\operatorname{Ext}^j_{\mathscr{D}}(\Omega;\ \mathfrak{M}, \mathscr{O})$ vanishes for every $j \geq 1$. In particular, the system* (1.5) *of linear differential equations has a holomorphic solution u for any holomorphic functions $(f_1,\ldots,f_d)$ satisfying* (1.6).

In case $d = n - 1$ (i.e. $\mathfrak{M}$ is subholonomic), we should take a 1-dimensional complex submanifold, i.e. a holomorphic complex curve, as S in the condition (ii) in Theorem 1.4. Hence the condition (ii) becomes trivial and actually it can be shown that the condition (ii) always holds when $d = n - 1$ (cf. Remark 1.5. See also Theorem 2.4 and Theorem 4.2.). This observation leads to

Corollary 1.8. *Let $\mathfrak{M}$ and Ω be the same as those in Theorem* 1.4. *Suppose that $d = n - 1$, and that Ω satisfies the following condition:*

At any point z_0 in C_0, Ω is bicharacteristically convex with respect to $\mathfrak{M}$.

Then the same conclusions as those in Theorem 1.4 *hold.*

Corollary 1.9. *Let* $\mathfrak{M}$, Ω *and* φ *be the same as those in Theorem* 1.7. *Suppose that* $d = n - 1$, *and that* $\Omega_\varepsilon = \{z \in U; \varphi(z) < \varepsilon\}$ *satisfies the condition in the preceding corollary for any* ε *with* $0 \geq \varepsilon > \varphi(z_1)$. *Then the same conclusions as those in Theorem* 1.7 *hold.*

2. Kawai's theorem and the condition (Pos)

In order to explain how our main theorems are proved, we first recall Kawai's theorem in this section. Though his theorem deals with more general situations, we present it in a form suitable for our purpose.

Let us define a $\mathscr{D}$-module $\mathfrak{M}'$ by

$$\mathfrak{M}' = \mathscr{D}/(\sum_{1 \leq \gamma \leq d} \mathscr{D}P_\gamma + \sum_{1 \leq k \leq n} \mathscr{D}\bar{\partial}_k). \tag{2.1}$$

By the assumption P_γ and $\bar{\partial}_k$ commute for every γ and k. Since Ω is strongly pseudo-convex, we have

$$\mathrm{Ext}^j_{\mathscr{D}}(\Omega; \mathfrak{M}, \mathscr{O}) \simeq \mathrm{Ext}^j_{\mathscr{D}}(\Omega; \mathfrak{M}', \mathscr{B}) \tag{2.2}$$

for every $j \geq 0$, where $\mathscr{B}$ denotes the sheaf of hyperfunctions on $U \subseteq \mathbf{C}^n \simeq \mathbf{R}^{2n}$. Note that, since we suppose the conditions (1.1) and (1.2), we can construct a Koszul complex using $P_1, \dots, P_d$ (resp. $P_1, \dots, P_d$ and $\bar{\partial}_1, \dots, \bar{\partial}_n$) and it is a free resolution of $\mathfrak{M}$ (resp. $\mathfrak{M}'$) with length d (resp. $d + n$).

Kawai's result asserts the finite-dimensionality (or vanishing) of the right-hand side of (2.2). In fact, he has proved

Theorem 2.1 (Kawai [2], [3]). *Let* $\mathfrak{M}$ *and* Ω *be the same as those in Theorem* 1.4 *and let* $\mathfrak{M}'$ *be defined by* (2.1). *Suppose that* $\mathfrak{M}'$ *and* Ω *satisfy the following condition*:

(2.3) *The generalized Levi form of the positive tangential system* $\mathfrak{N}_+$ *on* $\partial\Omega$ *induced from* $\mathfrak{M}'$ *is positive-definite at each characteristic point of* $\mathfrak{N}_+$.

Then $\dim \mathrm{Ext}^j_{\mathscr{D}}(\Omega; \mathfrak{M}', \mathscr{B})$ *is finite for every* $j \geq 1$. *Furthermore, if* φ *satisfies the conditions* (1.12)–(1.14) *for some point* z_1 *in* Ω, *and if* $\mathfrak{M}'$ *and* $\Omega_\varepsilon = \{z \in U; \varphi(z) < \varepsilon\}$ *satisfy the above condition* (2.3) *for any* ε *with* $0 \geq \varepsilon > \varphi(z_1)$, *then* $\mathrm{Ext}^j_{\mathscr{D}}(\Omega; \mathfrak{M}', \mathscr{B})$ *vanishes for every* $j \geq 1$.

The definition of the generalized Levi form is given in [7], Chapter III, Definition 2.3.1. Concerning this theorem see Kawai [2] (Theorem 1) and [3] (Corollary of Theorem 2) for details. See also [1], [5] and [6].

Now we want to write down explicitly the generalized Levi form of $\mathfrak{N}_+$ at its characteristic point. Its explicit form is given in [6] in the case of single equations, and in [5] in the case of systems. First, by straightforward calculations, we find the characteristic variety of $\mathfrak{N}_+$ is

$$\{(z, -\sqrt{-1}\,\partial\varphi(z));\ z\in C_0\}.$$

Namely the cotangential component of each characteristic point of $\mathfrak{N}_+$ is determined by its base point z and the projection of the characteristic variety of $\mathfrak{N}_+$ to the base space coincides with C_0. For z_0 in C_0 we denote by Q_{z_0} the generalized Levi form of $\mathfrak{N}_+$ at the characteristic point $(z_0, -\sqrt{-1}\,\partial\varphi(z_0))$. To give the explicit form of Q_{z_0}, let us introduce the following symbols, which will be used repeatedly in the subsequent part of this article.

$$\alpha_{\gamma\delta}(z) = \sum_{1\le j,k\le n} p_\gamma^{(j)}(z, \partial\varphi(z))\,\overline{p_\delta^{(k)}(z, \partial\varphi(z))}\,\partial_j\bar{\partial}_k\varphi(z) \qquad \text{for } \gamma, \delta = 1,\dots,d. \tag{2.4}$$

$$\begin{aligned}\beta_{\gamma\delta}(z) = &\sum_{1\le j\le n} p_{\gamma(j)}(z, \partial\varphi(z))\,p_\delta^{(j)}(z, \partial\varphi(z)) \\ &+ \sum_{1\le j,k\le n} p_\gamma^{(j)}(z, \partial\varphi(z))\,p_\delta^{(k)}(z, \partial\varphi(z))\,\partial_j\partial_k\varphi(z) \qquad \text{for } \gamma, \delta = 1,\dots,d.\end{aligned} \tag{2.5}$$

$$\kappa_{j\gamma}(z) = \sum_{1\le k\le n} \overline{p_\gamma^{(k)}(z, \partial\varphi(z))}\,\partial_j\bar{\partial}_k\varphi(z) \qquad \text{for } j = 1,\dots,n,\ \gamma = 1,\dots,d. \tag{2.6}$$

$$\begin{aligned}\lambda_{j\gamma}(z) = p_{\gamma(j)}(z, \partial\varphi(z)) + &\sum_{1\le k\le n} p_\gamma^{(k)}(z, \partial\varphi(z))\,\partial_j\partial_k\varphi(z) \\ &\text{for } j = 1,\dots,n,\ \gamma = 1,\dots,d.\end{aligned} \tag{2.7}$$

Here, and in what follows, $p^{(j)}(z, \zeta)$ and $p_{(j)}(z, \zeta)$ denote $(\partial p/\partial\zeta_j)(z, \zeta)$ and $(\partial p/\partial z_j)(z, \zeta)$ respectively.

Remark 2.2. Among these symbols, $\alpha_{\gamma\delta}$ and $\beta_{\gamma\delta}$ are independent of the choice of a holomorphic local coordinate system for any γ and δ. That is, if $\tilde{z} = (\tilde{z}_1,\dots,\tilde{z}_n)$ is another holomorphic local coordinate system, and if the above symbols calculated in this new coordinate system $\tilde{z}$ are denoted by $\tilde{\alpha}_{\gamma\delta}(\tilde{z})$, $\tilde{\beta}_{\gamma\delta}(\tilde{z})$ and so on, then we have

$$\tilde{\alpha}_{\gamma\delta}(\tilde{z}) = \alpha_{\gamma\delta}(z),\ \tilde{\beta}_{\gamma\delta}(\tilde{z}) = \beta_{\gamma\delta}(z) \qquad \text{for } \gamma, \delta = 1,\dots,d. \tag{2.8}$$

On the other hand, $\kappa_{j\gamma}$ or $\lambda_{j\gamma}$ is not so. They satisfy the following relations:

$$\tilde{\kappa}_{j\gamma}(\tilde{z}) = \sum_{1\le k\le n} \frac{\partial z_k}{\partial\tilde{z}_j}\kappa_{k\gamma}(z) \qquad \text{for } j = 1,\dots,n,\ \gamma = 1,\dots,d, \tag{2.9}$$

$$\tilde{\lambda}_{j\gamma}(\tilde{z}) = \sum_{1\le k\le n} \frac{\partial z_k}{\partial\tilde{z}_j}\lambda_{k\gamma}(z) \qquad \text{for } j = 1,\dots,n,\ \gamma = 1,\dots,d. \tag{2.10}$$

In terms of these symbols, the generalized Levi form $Q_{z_0}(z_0\in C_0)$ is given as follows:

$$Q_{z_0}(\tau) = \sum_{1 \le j,k \le n+d} q_{j,k}(z_0)\tau_j\bar{\tau}_k \tag{2.11}$$

considered with the constraint

$$\sum_{1 \le j \le n} \partial_j\varphi(z_0)\tau_j = 0, \tag{2.12}$$

where

$$\begin{cases} q_{j,k}(z_0) = \partial_j\bar{\partial}_k\varphi(z_0) & (1 \le j,\, k \le n), \\ q_{j,n+\gamma}(z_0) = \lambda_{j\gamma}(z_0) & (1 \le j \le n,\ 1 \le \gamma \le d). \\ q_{n+\gamma,j}(z_0) = \overline{\lambda_{j\gamma}(z_0)} & (1 \le j \le n,\ 1 \le \gamma \le d), \\ q_{n+\gamma,n+\delta}(z_0) = \overline{\alpha_{\gamma\delta}(z_0)} & (1 \le \gamma,\, \delta \le d). \end{cases} \tag{2.13}$$

Remark 2.3. Taking account of the transformation relations (2.8) and (2.10), we find that this generalized Levi form Q_{z_0} is independent of the choice of holomorphic local coordinates, if we view Q_{z_0} as an Hermitian form on the space of $H_{z_0} \oplus \mathbf{C}^d$, where H_{z_0} is a complex 1-codimensional subspace of $T_{z_0}\mathbf{C}^n$ given by

$$H_{z_0} = \{(\tau_1,\ldots,\tau_n) \in T_{z_0}\mathbf{C}^n;\ \sum_{1 \le j \le n} \partial_j\varphi(z_0)\tau_j = 0\}.$$

Moreover, we should consider $\mathbf{C}^d$, the other direct summand of $H_{z_0} \oplus \mathbf{C}^d$, as the complex conjugate of the tangent space at the origin of the parameter space $(t_1,\ldots,t_d)$ explained in Remark 1.2. See Remark 3.2 and Lemma 3.4 below.

As in [4], let us denote by (Pos), or more precisely by $(\mathrm{Pos})_{z_0}$, the condition that $Q_{z_0}(\tau)$ is positive-definite.

(Pos) The generalized Levi form $Q_{z_0}(\tau)$ is positive-definite, i.e., $Q_{z_0}(\tau)$ is strictly positive-definite on $\{\tau = (\tau_1,\ldots,\tau_{n+d}) \in \mathbf{C}^{n+d};\ \sum_{1 \le j \le n} \partial_j\varphi(z_0)\tau_j = 0\}$.

The following Theorem 2.4 is a generalization of Theorem 2.8.1 in [4] to the case of over-determined systems.

Theorem 2.4. *Let $\mathfrak{M}$ and Ω be the same as those in Theorem* 1.4. *For a point z_0 in C_0 the condition* $(\mathrm{Pos})_{z_0}$ *holds if and only if the following two conditions are satisfied*:

(i) *Ω is bicharacteristically convex at z_0 with respect to $\mathfrak{M}$.*

(ii) *For a (complex) d-codimensional complex submanifold S passing through z_0 and being transversal to $b_{(z_0,\partial\varphi(z_0))}$, $\tilde{C}_- \cap S$ is strongly pseudo-convex at z_0 in S.*

Corollary 2.5. *Let $\mathfrak{M}$, Ω and z_0 be the same as those in Theorem*

2.4. *Suppose that* $d = n - 1$. *Then the condition* $(\mathrm{Pos})_{z_0}$ *is equivalent to the bicharacteristical convexity of* Ω *at* z_0 *with respect to* $\mathfrak{M}$.

It is obvious that our main theorems (Theorem 1.4 and Theorem 1.7) follow from Theorem 2.1, Theorem 2.4 and the isomorphism (2.2). We will prove this Theorem 2.4 in the subsequent three sections.

3. The geometry of bicharacteristics

In this section we give several propositions which describe some geometric properties of C, C_0 and b introduced in § 1.

First let us introduce the non-degeneracy condition of the domain Ω with respect to the bicharacteristics of $\mathfrak{M}$. Let z_0 be a point in C_0 and $z(t_1,\dots,t_d;\, z_0,\, \partial\varphi(z_0))$ be the local expression of $b_{(z_0,\partial\varphi(z_0))}$ explained in Remark 1.2. We denote by $\tilde{\varphi}$ the restriction of φ to $b_{(z_0,\partial\varphi(z_0))}$:

$$\tilde{\varphi}(t_1,\dots,t_d) = \varphi(z(t_1,\dots,t_d;\, z_0,\, \partial\varphi(z_0))). \tag{3.1}$$

Now let $B_{z_0} = (b_{\gamma,\delta}(z_0))_{1\le\gamma,\delta\le 2d}$ be the Hermitian matrix defined by

$$\tag{3.2}\begin{cases} b_{\gamma,\delta}(z_0) = \dfrac{\partial^2\tilde{\varphi}}{\partial t_\gamma \partial \bar{t}_\delta}(0) & (1\le\gamma,\ \delta\le d),\\[2ex] b_{\gamma,d+\delta}(z_0) = \dfrac{\partial^2\tilde{\varphi}}{\partial t_\gamma \partial t_\delta}(0) & (1\le\gamma,\ \delta\le d),\\[2ex] b_{d+\gamma,\delta}(z_0) = \dfrac{\partial^2\tilde{\varphi}}{\partial \bar{t}_\gamma \partial \bar{t}_\delta}(0) & (1\le\gamma,\ \delta\le d),\\[2ex] b_{d+\gamma,d+\delta}(z_0) = \dfrac{\partial^2\tilde{\varphi}}{\partial \bar{t}_\gamma \partial t_\delta}(0) & (1\le\gamma,\ \delta\le d). \end{cases}$$

On the other hand, we denote by $B^{\mathbf{R}}_{z_0}$ the real Hessian of $\tilde{\varphi}$, that is, $B^{\mathbf{R}}_{z_0} = (b'_{\gamma,\delta}(z_0))_{1\le\gamma,\delta\le 2d}$ is given by

$$\tag{3.3}\begin{cases} b'_{\gamma,\delta}(z_0) = \dfrac{\partial^2\tilde{\varphi}}{\partial u_\gamma \partial u_\delta}(0) & (1\le\gamma,\ \delta\le d),\\[2ex] b'_{\gamma,d+\delta}(z_0) = \dfrac{\partial^2\tilde{\varphi}}{\partial u_\gamma \partial v_\delta}(0) & (1\le\gamma,\ \delta\le d),\\[2ex] b'_{d+\gamma,\delta}(z_0) = \dfrac{\partial^2\tilde{\varphi}}{\partial v_\gamma \partial u_\delta}(0) & (1\le\gamma,\ \delta\le d),\\[2ex] b'_{d+\gamma,d+\delta}(z_0) = \dfrac{\partial^2\tilde{\varphi}}{\partial v_\gamma \partial v_\delta}(0) & (1\le\gamma,\ \delta\le d), \end{cases}$$

where u_γ (resp. v_γ) is the real (resp. imaginary) part of t_γ. The matrix B_{z_0} is tied up with $B^{\mathbf{R}}_{z_0}$ through the following formula:

$$B^{\mathbf{R}}_{z_0} = {}^t W B_{z_0} \bar{W}, \tag{3.4}$$

where W is the $(2d) \times (2d)$ matrix given by

$$W = \begin{pmatrix} I_d & \sqrt{-1}\, I_d \\ I_d & -\sqrt{-1}\, I_d \end{pmatrix}, \tag{3.5}$$

(I_d is the $d \times d$ identity matrix).

Definition 3.1. For a point z_0 in C_0 we call the Hermitian form whose matrix is given by B_{z_0} the bicharacteristic form of $\mathfrak{M}$ at z_0. When the bicharacteristic form of $\mathfrak{M}$ at z_0 is non-degenerate, the domain Ω is said to be non-degenerate with respect to $\mathfrak{M}$ at z_0.

Remark 3.2. It is obvious that the matrix B_{z_0} does not depend on the choice of holomorphic local coordinates. (See also (2.8) and the expression (3.7) of B_{z_0} below.) In fact, according to (3.2)–(3.5), B_{z_0} should be considered as an Hermitian form on the complexification of the real tangent space $T_0^{\mathbf{R}}\mathbf{C}^d$ of the parameter space $(t_1, \ldots, t_d)$ at the origin, more precisely, as an Hermitian form on the complexification of the following real $2d$-dimensional vector space:

$$\{(\tau, \bar{\tau});\, \tau \in \mathbf{C}^d = T_0\mathbf{C}^d\}.$$

The non-degeneracy of Ω at z_0 means the second order tangency of the boundary $\partial\Omega$ of Ω and the bicharacteristic $b_{(z_0, \partial\varphi(z_0))}$.

Since the left-hand side of (3.4) is a real symmetric matrix, it follows from (3.4) and (3.5) that the positive-definiteness of B_{z_0} is equivalent to the following condition:

$$\text{For any } \phi = (\phi_1, \ldots, \phi_d) \in \mathbf{C}^d \text{ with } \phi \neq 0, \quad (\phi, \bar{\phi})\, B_{z_0}\, {}^t(\bar{\phi}, \phi) > 0. \tag{3.6}$$

As is easily seen, the condition (3.6) is nothing but the bicharacteristical convexity (1.9) of Ω at z_0. Hence we have

Lemma 3.3. *The domain Ω is bicharacteristically convex with respect to $\mathfrak{M}$ at z_0 in C_0 if and only if the bicharacteristic form B_{z_0} of $\mathfrak{M}$ at z_0 is positive-definite.*

Here let us write down B_{z_0} explicitly in terms of φ and the principal symbols $p_1, \ldots, p_d$.

Lemma 3.4. *For $B_{z_0} = (b_{\gamma,\delta}(z_0))_{1 \le \gamma, \delta \le 2d}$ defined by (3.2), we have the following formula:*

$$\begin{cases} b_{\gamma,\delta}(z_0) = \alpha_{\gamma\delta}(z_0) & \text{for } \gamma, \delta = 1, \ldots, d, \\ b_{\gamma,d+\delta}(z_0) = \beta_{\gamma\delta}(z_0) & \text{for } \gamma, \delta = 1, \ldots, d, \\ b_{d+\gamma,\delta}(z_0) = \overline{\beta_{\gamma\delta}(z_0)} & \text{for } \gamma, \delta = 1, \ldots, d, \\ b_{d+\gamma,d+\delta}(z_0) = \overline{\alpha_{\gamma\delta}(z_0)} & \text{for } \gamma, \delta = 1, \ldots, d, \end{cases} \tag{3.7}$$

where $\alpha_{\gamma\delta}$ *and* $\beta_{\gamma\delta}$ *are the symbols given by* (2.4) *and* (2.5).

Note that, as a consequence of the assumption (1.1), we have

$$\sum_{1\le j\le n} p_\gamma^{(j)}(z,\zeta)p_{\delta(j)}(z,\zeta)=\sum_{1\le j\le n} p_{\gamma(j)}(z,\zeta)p_\delta^{(j)}(z,\zeta)\qquad \text{for } \gamma,\delta=1,\dots,d. \tag{3.8}$$

Lemma 3.4 follows from (1.7) and this formula (3.8). We do not present the detailed calculations here. But we should remark that (3.8) implies $\beta_{\gamma\delta}$ is symmetric, i.e., $\beta_{\gamma\delta}=\beta_{\delta\gamma}$ holds for every γ and δ.

From now on, let z_0 be a point in C_0 and suppose

$$\Omega \text{ is non-degenerate with respect to } \mathfrak{M} \text{ at } z_0. \tag{3.9}$$

Note that, if Ω is bicharacteristically convex at z_0, then this condition (3.9) is satisfied by Lemma 3.3. Under this condition (3.9) we have the following geometric property of C.

Proposition 3.5. *Under the assumption* (3.9), *C is a real analytic submanifold of real codimension 2d in a sufficiently small neighborhood of z_0. Furthermore, C and $b_{(z_0,\partial\varphi(z_0))}$ are transversal at z_0.*

In order to prove Proposition 3.5, we make use of the following two lemmas.

Lemma 3.6. *Let $f_\gamma(z)=f_\gamma(z,\bar z)$ $(1\le\gamma\le d)$ be complex-valued real analytic functions defined on an open subset U of $\mathbf{C}^n$, and let V denote the set $\{z\in U; f_\gamma(z)=0,\ \gamma=1,\dots,d\}$. Let z_0 be a point in V, and suppose that, if*

$$\sum_{1\le\gamma\le d} a_\gamma\partial_j f_\gamma(z_0)+\sum_{1\le\gamma\le d}\bar a_\gamma\partial_j\bar f_\gamma(z_0)=0,\qquad j=1,\dots,n \tag{3.10}$$

hold for $(a_1,\dots,a_d)\in\mathbf{C}^d$, then $(a_1,\dots,a_d)$ must be equal to zero. Then V is a real analytic submanifold of U with real codimension 2d in a small neighborhood of z_0.

Lemma 3.7. *Let f_γ, V and z_0 be those in the preceding lemma, and let Γ be a real 2d-dimensional real analytic submanifold through z_0. Suppose that, if a tangent vector $w=(w_1,\dots,w_n)\in\mathbf{C}^n$ of Γ at z_0 satisfies*

$$\sum_{1\le j\le n} w_j\partial_j f_\gamma(z_0)+\sum_{1\le j\le n}\bar w_j\bar\partial_j f_\gamma(z_0)=0,\qquad \gamma=1,\dots,d, \tag{3.11}$$

then w must be equal to zero. Then V and Γ are transversal at z_0.

Because these lemmas are almost self-evident, we do not present their proofs here.

Proof of Proposition 3.5. By definition, C is given by

$$\{z\in U;\ p_\gamma(z,\partial\varphi(z))=0,\ \gamma=1,\dots,d\}.$$

To prove the first assertion of Proposition 3.5, we choose $p_\gamma(z,\partial\varphi(z))$ as $f_\gamma(z)$

and use Lemma 3.6. Suppose that $(a_1,\dots,a_d)\in\mathbf{C}^d$ satisfies (3.10). Here let us remark that

$$\partial_j f_\gamma(z_0) = \lambda_{j\gamma}(z_0),\ \partial_j \bar{f}_\gamma(z_0) = \kappa_{j\gamma}(z_0) \tag{3.12}$$

hold for $j = 1,\dots,n$ and $\gamma = 1,\dots,d$ in the notation of (2.6) and (2.7), and that the symbols of (2.4)–(2.7) satisfy the following equalities:

$$\sum_{1\le j\le n} \lambda_{j\gamma}(z_0)p^{(j)}_\delta(z_0,\partial\varphi(z_0)) = \beta_{\gamma\delta}(z_0) \qquad \text{for } \gamma,\ \delta = 1,\dots,d, \tag{3.13}$$

$$\sum_{1\le j\le n} \kappa_{j\gamma}(z_0)p^{(j)}_\delta(z_0,\partial\varphi(z_0)) = \overline{\alpha_{\gamma\delta}(z_0)} \qquad \text{for } \gamma,\ \delta = 1,\dots,d. \tag{3.14}$$

Hence (3.10) implies that

$$\begin{aligned}0 &= \sum_{1\le j\le n}\{\sum_{1\le\gamma\le d} a_\gamma\lambda_{j\gamma}(z_0) + \sum_{1\le\gamma\le d}\bar{a}_\gamma\kappa_{j\gamma}(z_0)\}\, p^{(j)}_\delta(z_0,\partial\varphi(z_0))\\ &= \sum_{1\le\gamma\le d} a_\gamma\beta_{\gamma\delta}(z_0) + \sum_{1\le\gamma\le d}\bar{a}_\gamma\,\overline{\alpha_{\gamma\delta}(z_0)} \qquad \text{for } \delta = 1,\dots,d.\end{aligned}$$

By Lemma 3.4 this means that

$$(a_1,\dots,a_d,\ \bar{a}_1,\dots,\bar{a}_d)\,B_{z_0} = 0.$$

Since we are assuming (3.9), we obtain $(a_1,\dots,a_d) = 0$. Therefore, Lemma 3.6 guarantees that C is a real analytic submanifold of real codimension $2d$ in a neighborhood of z_0.

Next let us prove that C and $b_{(z_0,\partial\varphi(z_0))}$ are transversal at z_0, using Lemma 3.7. Any tangent vector $w = (w_1,\dots,w_n)$ of $b_{(z_0,\partial\varphi(z_0))}$ at z_0 has the following form:

$$w_j = \sum_{1\le\gamma\le d} c_\gamma p^{(j)}_\gamma(z_0,\partial\varphi(z_0)), \qquad j = 1,\dots,n,$$

where each c_γ is a complex number. Now suppose that w satisfies (3.11). It follows from (3.12)–(3.14) that

$$\begin{aligned}0 &= \sum_{1\le j\le n}(\sum_{1\le\gamma\le d} c_\gamma p^{(j)}_\gamma(z_0,\partial\varphi(z_0)))\,\lambda_{j\delta}(z_0) + \sum_{1\le j\le n}(\sum_{1\le\gamma\le d}\bar{c}_\gamma\,\overline{p^{(j)}_\gamma(z_0,\partial\varphi(z_0))})\,\overline{\kappa_{j\delta}(z_0)}\\ &= \sum_{1\le\gamma\le d} c_\gamma\beta_{\delta\gamma}(z_0) + \sum_{1\le\gamma\le d}\bar{c}_\gamma\alpha_{\delta\gamma}(z_0), \qquad \delta = 1,\dots,d,\end{aligned}$$

which means

$$B_{z_0}{}^t(\bar{c}_1,\dots,\bar{c}_d,\ c_1,\dots,c_d) = 0.$$

Again by the assumption (3.9) we find $(c_1,\dots,c_d) = 0$, i.e. $w = 0$. Hence Lemma 3.7 shows the transversality of C and $b_{(z_0,\partial\varphi(z_0))}$ at z_0. □

As is shown in § 1 (cf. the equality (1.8)), $b_{(z_0,\partial\varphi(z_0))}$ is tangent at z_0 to the boundary $\partial\Omega$ of Ω. Since C_0 is the intersection of C with $\partial\Omega$, we immediately obtain the following proposition from Proposition 3.5.

Proposition 3.8. *Under the assumption* (3.9), C_0 *is a real analytic submanifold of real codimension* $(2d+1)$ *in a neighborhood of* z_0.

Next let us study the "foliation" b, i.e., the family of bicharacteristics of the form

$$\{b_{(z,\partial\varphi(z))}\}_{z\in C}. \tag{3.15}$$

Making use of Proposition 3.5, we can prove that b is actually an analytic foliation at least locally. More precisely, we have the following

Proposition 3.9. *Under the assumption* (3.9), *the family of bicharacteristics of the form* (3.15) *defines a real* $2d$*-dimensional real analytic foliation in a neighborhood of* z_0.

Proof. Let us consider the following map F:

$$F: C \times \mathbf{C}^d \ni (\tilde{z}, (t_1,\dots,t_d)) \longmapsto z(t_1,\dots,t_d;\, \tilde{z},\, \partial\varphi(\tilde{z})) \in \mathbf{C}^n, \tag{3.16}$$

where $z(t_1,\dots,t_d;\, \tilde{z},\, \partial\varphi(\tilde{z}))$ is the local expression of $b_{(\tilde{z},\partial\varphi(\tilde{z}))}$ explained in Remark 1.2. This map F is defined and real analytic in a small neighborhood of $(z_0, (0,\dots,0))$ because C is a real analytic submanifold near z_0 and $z(t_1,\dots,t_d;\, \tilde{z}, \tilde{\zeta})$ is holomorphic with respect to $(t_1,\dots,t_d)$ and $(\tilde{z}, \tilde{\zeta})$. Moreover, the transversality at z_0 of C and $b_{(z_0,\partial\varphi(z_0))}$ implies that the differential of F at $(z_0, (0,\dots,0))$ is surjective. Hence F is a local diffeomorphism near $(z_0, (0,\dots,0))$. Since by this diffeomorphism F each bicharacteristic $b_{(z,\partial\varphi(z))}$ $(z\in C)$ is transformed into the subset

$$\{(\tilde{z}, (t_1,\dots,t_d));\ \tilde{z} = z,\ (t_1,\dots,t_d)\colon \text{arbitrary}\}$$

of $C \times \mathbf{C}^d$, we find that the family of bicharacteristics in question is a real analytic foliation of real dimension $2d$ in a neighborhood of z_0 and the above map F is its distinguished local chart. □

We have defined by (1.10) and (1.11) $\tilde{C}_0$ and $\tilde{C}_-$, the bicharacteristic hull of C_0 and C_-. The following proposition is an immediate consequence of the preceding propositions.

Proposition 3.10. *Under the assumption of* (3.9), $\tilde{C}_0$ *is a non-singular real hypersurface and* $\tilde{C}_-$ *is an open subset with* $\tilde{C}_0$ *as its boundary in a neighborhood of* z_0.

In fact, if we consider $\tilde{C}_0$ in using the distinguished chart F defined by (3.16), it is locally the image of $C_0 \times \mathbf{C}^d$, which is a non-singular real hypersurface of $C \times \mathbf{C}^d$. Similarly $\tilde{C}_-$ is locally the image of $C_- \times \mathbf{C}^d$, which is an open subset of $C \times \mathbf{C}^d$ with $C_0 \times \mathbf{C}^d$ as its boundary.

4. The decomposition theorem

In this section we investigate the relationship between the generalized Levi form Q_{z_0} introduced in §2 and the geometry of bicharacteristics. The main theorem is the decomposition theorem of Q_{z_0} (Theorem 4.2 below). Theorem 2.4 is a consequence of that decomposition theorem.

Let us begin with the following

Proposition 4.1. *Let z_0 be a point in C_0 and suppose that the generalized Levi form Q_{z_0}, considered with the constraint* (2.12), *is positive-definite. Then the bicharacteristic form B_{z_0} at z_0 is also positive-definite. In other words, if* (Pos) *is satisfied at z_0, Ω is bicharacteristically convex with respect to $\mathfrak{M}$ at z_0.*

Proof. We use the explicit forms of Q_{z_0} and B_{z_0} given by (2.11)–(2.13) and (3.7).

For $\sigma = (\sigma_1,\dots,\sigma_{2d})$ in $\mathbf{C}^{2d}$, let us define $q(z, \zeta)$ and ϕ by

$$q(z, \zeta) = \sum_{1 \le \gamma \le d} \sigma_\gamma p_\gamma(z, \zeta),$$
$$\phi = (q^{(1)}(z_0, \partial\varphi(z_0)),\dots,q^{(n)}(z_0, \partial\varphi(z_0)), \sigma_{d+1},\dots,\sigma_{2d}) \in \mathbf{C}^{n+d}.$$

Note that this ϕ satisfies the constraint (2.12), that is,

$$\sum_{1 \le j \le n} \partial_j\varphi(z_0) q^{(j)}(z_0, \partial\varphi(z_0)) = 0$$

holds, since each p_γ satisfies (1.8). Now let us calculate $Q_{z_0}(\phi)$ for ϕ. A straightforward calculation shows that

$$Q_{z_0}(\phi) = \sigma B_{z_0} {}^t\bar{\sigma}.$$

Hence $(\mathrm{Pos})_{z_0}$ entails the positive-definiteness of B_{z_0}. □

This Proposition 4.1 states a relationship between the generalized Levi form Q and the bicharacteristic form B. But, in order that the generalized Levi form Q is positive-definite, we have also to take account of the complex-analytic structure of Ω in the transversal direction with respect to the foliation b, which is the reason why the condition (ii) appears in the statement of Theorem 1.4 and Theorem 2.4. To describe that structure, let us prepare some notations.

Let z_0 be a point in C_0 and suppose that the condition (3.9) holds, i.e., Ω is non-degenerate with respect to $\mathfrak{M}$ at z_0. Take a (complex) d-codimensional complex submanifold S passing through z_0 and being transversal to $b_{(z_0, \partial\varphi(z_0))}$. We want to consider $\tilde{C}_- \cap S$ in S. Since under the assumption (3.9) b is a real analytic foliation in a neighborhood of z_0 and C is also transversal to $b_{(z_0, \partial\varphi(z_0))}$ as well as S, we can define a real analytic local diffeomorphism g from S to C along b as follows:

$$g: S \ni w \longmapsto g(w) \in C, \tag{4.1}$$

where the image $g(w)$ of w is determined by the following property:

(4.2) $$g(w) \text{ and } w \text{ lie on the same leaf of } b.$$

Remark that, using the distinguished chart F of b defined by (3.16), we can also represent g in such a way that

(4.3) $$g = \pi_1 \circ F^{-1}|_S$$

where π_1 is the projection from $C \times \mathbf{C}^d$ onto C. Thus we have defined an analytic local diffeomorphism $g: S \to C$. By definition, the images of $\tilde{C}_- \cap S$ and its boundary $\tilde{C}_0 \cap S$ under this diffeomorphism g are C_- and C_0 respectively. Therefore, if we define a real analytic function ψ on S by

(4.4) $$\psi = \varphi|_C \circ g,$$

then we find

$$\tilde{C}_- \cap S \cap \omega = \{w \in S \cap \omega;\ \psi(w) < 0\}$$

for a sufficiently small neighborhood ω of z_0.

Now let us state the decomposition theorem.

Theorem 4.2. *Let $\mathfrak{M}$ and Ω be the same as those in Theorem* 1.4. *Let z_0 be a point in C_0 and suppose that Ω is non-degenerate with respect to $\mathfrak{M}$ at z_0. Let S be a (complex) d-codimensional complex submanifold passing through z_0 and being transversal to $b_{(z_0, \partial\varphi(z_0))}$ at z_0, and let ψ be a real analytic function on S defined by* (4.4). *Then the generalized Levi form Q_{z_0} at z_0, considered with the constraint* (2.12), *is equivalent to the direct sum of the bicharacteristic form B_{z_0} of $\mathfrak{M}$ and the Levi form L_{z_0} of ψ at z_0.*

Remark 4.3. The Levi form $L_{z_0}(\sigma)$ $(\sigma \in \mathbf{C}^{n-d})$ of ψ at z_0 is, by definition, the Hermitian form

$$\sum_{1 \le j,k \le n-d} \frac{\partial^2 \psi}{\partial w_j \partial \bar{w}_k}(z_0)\, \sigma_j \bar{\sigma}_k$$

considered with the constraint

$$\sum_{1 \le j \le n-d} \frac{\partial \psi}{\partial w_j}(z_0)\, \sigma_j = 0,$$

where $(w_1, \ldots, w_{n-d})$ denotes a holomorphic local coordinate system of S at z_0. This is a well-defined Hermitian form on the complex 1-codimensional subspace K_{z_0} of $T_{z_0}S$ defined by

$$K_{z_0} = \left\{\sigma = (\sigma_1, \ldots, \sigma_{n-d}) \in T_{z_0}S;\ \sum_{1 \le j \le n-d} \frac{\partial \psi}{\partial w_j}(z_0)\sigma_j = 0\right\},$$

that is, $L_{z_0}(\sigma)$ is an Hermitian form on K_{z_0} which is independent of the choice of holomorphic local coordinates of S. Notice that K_{z_0} is contained in the real tangent space of $\{\psi = 0\}$ at z_0. As is well-known, the Levi form of ψ at z_0 is an Hermitian form on the space of holomorphic tangent vectors of $\{\psi = 0\}$ at z_0.

It follows from Lemma 3.3 and Proposition 4.1 that, if Ω is bicharacteristically convex at z_0, or if $(\mathrm{Pos})_{z_0}$ holds, then the assumption in this theorem of non-degeneracy of Ω at z_0 is satisfied. Hence, Theorem 2.4 is an immediate consequence of this decomposition theorem and Lemma 3.3.

In order to show the decomposition theorem, we will make use of the following proposition which describes the explicit form of L_{z_0}.

Proposition 4.4. *Let $\mathfrak{M}$, Ω, z_0, S and ψ be the same as those in Theorem 4.2. Let $L'_{z_0}(\sigma) = \sum_{1 \le j,k \le n} r_{j,k}(z_0)\sigma_j\bar{\sigma}_k$ be an Hermitian form on H_{z_0} defined by*

$$r_{j,k}(z_0) = \partial_j\bar{\partial}_k\varphi(z_0) - \rho_j(z_0)B_{z_0}^{-1\,t}\,\overline{\rho_k(z_0)}, \qquad j, k = 1,\dots,n, \tag{4.5}$$

where $\rho_j(z_0)$ is a $2d$-vector given by

$$\rho_j(z_0) = (\kappa_{j1}(z_0),\dots,\kappa_{jd}(z_0), \lambda_{j1}(z_0),\dots,\lambda_{jd}(z_0)), \qquad j = 1,\dots,n, \tag{4.6}$$

and H_{z_0} is the following subspace of $T_{z_0}\mathbf{C}^n$:

$$H_{z_0} = \{\sigma = (\sigma_1,\dots,\sigma_n) \in T_{z_0}\mathbf{C}^n;\ \sum_{1 \le j \le n} \partial_j\varphi(z_0)\sigma_j = 0\}. \tag{4.7}$$

Then, when we regard $T_{z_0}S$ as a subspace of $T_{z_0}\mathbf{C}^n$ through the canonical inclusion $S \subsetneq \mathbf{C}^n$, the Levi form L_{z_0} of ψ at z_0 coincides with the restriction of L'_{z_0} to $H_{z_0} \cap T_{z_0}S$.

Remark 4.5. By (2.8)–(2.10) we find that $L'_{z_0}(\sigma)$ is independent of the choice of holomorphic local coordinates when viewed as an Hermitian form on $T_{z_0}\mathbf{C}^n$. Note also that the assumption of the transversality of S with $b_{(z_0,\partial\varphi(z_0))}$ at z_0 and the equality (1.8) imply that $H_{z_0} \cap T_{z_0}S$ is a complex 1-codimensional subspace of $T_{z_0}S$.

Remark 4.6. We can consider $L'_{z_0}(\sigma)$ itself as an Hermitian form on $T_{z_0}S$, because the tangent space of $b_{(z_0,\partial\varphi(z_0))}$ at z_0 is contained in H_{z_0} by (1.8) and

$$(\sigma_1,\dots,\sigma_n)\begin{pmatrix} r_{1,1}(z_0) & \cdots & r_{1,n}(z_0) \\ \vdots & & \vdots \\ r_{n,1}(z_0) & \cdots & r_{n,n}(z_0) \end{pmatrix} = 0$$

holds for any tangent vector $\sigma = (\sigma_1,\dots,\sigma_n)$ of $b_{(z_0,\partial\varphi(z_0))}$ at z_0. In fact, for any k and γ $(1 \le k \le n,\ 1 \le \gamma \le d)$, we have

$$\begin{aligned}&\sum_{1 \le j \le n} p_\gamma^{(j)}(z_0, \partial\varphi(z_0))\, r_{j,k}(z_0)\\ &\quad = \sum_{1 \le j \le n} p_\gamma^{(j)}(z_0, \partial\varphi(z_0))\{\partial_j\bar{\partial}_k\varphi(z_0) - \rho_j(z_0)B_{z_0}^{-1\,t}\,\overline{\rho_k(z_0)}\}\end{aligned}$$

$$= \overline{\kappa_{k\gamma}(z_0)} - (\alpha_{\gamma 1}(z_0), \dots, \alpha_{\gamma d}(z_0), \beta_{\gamma 1}(z_0), \dots, \beta_{\gamma d}(z_0)) B_{z_0}^{-1}\, {}^t\overline{\rho_k(z_0)}$$

$$= \overline{\kappa_{k\gamma}(z_0)} - (0, \dots, 0, \overset{\overset{\gamma}{\vee}}{1}, 0, \dots, 0)\, {}^t\overline{\rho_k(z_0)}$$

$$= \overline{\kappa_{k\gamma}(z_0)} - \overline{\kappa_{k\gamma}(z_0)} = 0.$$

The proof of Proposition 4.4 requires many straightforward calculations. We will prove it in the next section. Here, assuming Proposition 4.4, let us finish the proof of the decomposition theorem.

Proof of Theorem 4.2. First remark that, since the generalized Levi form Q_{z_0}, the bicharacteristic form B_{z_0} and the Levi form L_{z_0} of ψ are all independent of the choice of holomorphic local coordinates as explained in Remark 2.3, Remark 3.2 and Remark 4.3, it suffices to prove this theorem in one arbitrarily chosen holomorphic local coordinate system. Let us choose a system $(\tilde{z}_1, \dots, \tilde{z}_n)$ so that z_0 is the origin in this system and that $\{\tilde{z}_{n-d+1} = \cdots = \tilde{z}_n = 0\}$ coincides with S in a neighborhood of z_0. For simplicity we denote $\tilde{z}$ by z in what follows.

Let

$$Q_0(\tau) = \sum_{1 \le j,k \le n+d} q_{j,k}(0)\, \tau_j \bar{\tau}_k$$

be the generalized Levi form at $z_0 = 0$, where each $q_{j,k}(0)$ is given by (2.13) and $\tau = (\tau_1, \dots, \tau_{n+d})$ satisfies

$$\sum_{1 \le j \le n} \partial_j \varphi(0)\, \tau_j = 0. \tag{4.8}$$

Let us transform $\tau = (\tau_1, \dots, \tau_{n+d})$ into $\chi = (\chi_1, \dots, \chi_{n+d})$ as follows:

$$\begin{aligned}(\tau_1, \dots, \tau_n) &= (\chi_1, \dots, \chi_{n-d}, 0, \dots, 0) \\ &\quad + \sum_{1 \le \gamma \le d} \chi_{n-d+\gamma} (p_\gamma^{(1)}(0, \partial\varphi(0)), \dots, p_\gamma^{(n)}(0, \partial\varphi(0))),\end{aligned} \tag{4.9}$$

$$(\tau_{n+1}, \dots, \tau_{n+d}) = (\chi_{n+1}, \dots, \chi_{n+d}), \tag{4.10}$$

that is,

$$(\tau_1, \dots, \tau_{n+d}) = (\chi_1, \dots, \chi_{n+d}) \begin{pmatrix} 1 & & 0 & & & & \\ & \ddots & & 0 & & 0 & \\ 0 & & 1 & & & & \\ p_1^{(1)} & & \cdots & & p_1^{(n)} & & \\ \vdots & & & & \vdots & 0 & \\ p_d^{(1)} & & \cdots & & p_d^{(n)} & & \\ & & & & & 1 & & 0 \\ & & & 0 & & & \ddots & \\ & & & & & 0 & & 1 \end{pmatrix}. \tag{4.11}$$

Note that, since S is assumed to be transversal to $b_{(z_0, \partial\varphi(z_0))}$, the matrix

$$\begin{pmatrix} p_1^{(n-d+1)}(0, \partial\varphi(0)) & \cdots & p_1^{(n)}(0, \partial\varphi(0)) \\ \vdots & & \vdots \\ p_d^{(n-d+1)}(0, \partial\varphi(0)) & \cdots & p_d^{(n)}(0, \partial\varphi(0)) \end{pmatrix}$$

is non-singular. Hence the transformation (4.9)–(4.10) or (4.11) is invertible. Remark also that the transformation (4.9) is nothing but the decomposition of $T_0\mathbf{C}^n$, the tangent space of $\mathbf{C}^n$ at $z_0 = 0$, into the direct sum of $T_0 S$ and the complex tangent space at the origin of $(t_1, \ldots, t_d)$, the parameter space of $b_{(z_0, \partial\varphi(z_0))}$ explained in Remark 1.2.

After this transformation we further transform $\chi = (\chi_1, \ldots, \chi_{n+d})$ into $\sigma = (\sigma_1, \ldots, \sigma_{n+d})$ in such a way that

$$(4.12) \qquad (\chi_1, \ldots, \chi_{n+d}) = (\sigma_1, \ldots, \sigma_{n+d}) \begin{pmatrix} 1 & & 0 & \phi_{1,1} & \cdots & \phi_{1,2d} \\ & \ddots & & \vdots & & \vdots \\ 0 & & 1 & \phi_{n-d,1} & \cdots & \phi_{n-d,2d} \\ & & & 1 & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & 1 \end{pmatrix}$$

where $\phi_j = (\phi_{j,1}, \ldots, \phi_{j,2d})$ $(1 \le j \le n-d)$ is a $2d$-vector given by

$$\begin{aligned} \phi_j &= -\rho_j(0) B_0^{-1} \\ &= -(\kappa_{j1}(0), \ldots, \kappa_{jd}(0), \lambda_{j1}(0), \ldots, \lambda_{jd}(0)) B_0^{-1}, \qquad j = 1, \ldots, n-d. \end{aligned}$$

Then, by a straightforward calculation, we find that $Q_0(\tau)$ is transformed into the form

$$\sum_{1 \le j,k \le n+d} q'_{j,k} \sigma_j \bar{\sigma}_k$$

which is defined in the following manner:

$$\begin{aligned} q'_{j,k} &= \partial_j \bar{\partial}_k \varphi(0) - \rho_j(0) B_0^{-1\,t}\,\overline{\rho_k(0)} \qquad (1 \le j,\ k \le n-d), \\ q'_{j,n-d+\gamma} &= q'_{n-d+\gamma,j} = 0 \qquad (1 \le j \le n-d,\ 1 \le \gamma \le 2d), \\ q'_{n-d+\gamma,n-d+\delta} &= \alpha_{\gamma\delta} \qquad (1 \le \gamma,\ \delta \le d), \\ q'_{n-d+\gamma,n+\delta} &= \overline{q'_{n+\delta,n-d+\gamma}} = \beta_{\gamma\delta} \qquad (1 \le \gamma,\ \delta \le d), \\ q'_{n+\gamma,n+\delta} &= \overline{\alpha_{\gamma\delta}} \qquad (1 \le \gamma,\ \delta \le d). \end{aligned}$$

Furthermore, using (1.8), we find the constraint (4.8) of $Q_0(\tau)$ is transformed into

$$(4.13) \qquad \sum_{1 \le j \le n-d} \partial_j \varphi(0) \sigma_j = 0$$

under these transformations (4.11) and (4.12).

Now Proposition 4.4 tells us that, in the coordinate system we are using now, the Levi form $L_0(\sigma)$ at $z_0 = 0$ is

$$L_0(\sigma) = \sum_{1 \leq j,k \leq n-d} (\partial_j \bar{\partial}_k \varphi(0) - \rho_j(0) B_0^{-1}\, {}^t\overline{\rho_k(0)})\, \sigma_j \bar{\sigma}_k,$$

where $\sigma = (\sigma_1, \ldots, \sigma_{n-d})$ is a tangent vector of S at $z_0 = 0$ satisfying

$$\sum_{1 \leq j \leq n-d} \partial_j \varphi(0)\, \sigma_j = 0.$$

In fact, in the current coordinate system, the tangent space $T_0 S$ of S at $z_0 = 0$ can be identified with the subspace

$$\{(\sigma_1, \ldots, \sigma_n) \in T_0 \mathbf{C}^n;\ \sigma_{n-d+1} = \cdots = \sigma_n = 0\}$$

of $T_0 \mathbf{C}^n$. Hence, the Hermitian form

$$(\sigma_1, \ldots, \sigma_{n-d}) \begin{pmatrix} q'_{1,1} & \cdots & q'_{1,n-d} \\ \vdots & & \vdots \\ q'_{n-d,1} & \cdots & q'_{n-d,n-d} \end{pmatrix} \begin{pmatrix} \bar{\sigma}_1 \\ \vdots \\ \overline{\sigma_{n-d}} \end{pmatrix}$$

with the constraint (4.13) is exactly the Levi form $L_0(\sigma)$. On the other hand, Lemma 3.4 implies that

$$\begin{pmatrix} q'_{n-d+1,n-d+1} & \cdots & q'_{n-d+1,n+d} \\ \vdots & & \vdots \\ q'_{n+d,n-d+1} & \cdots & q'_{n+d,n+d} \end{pmatrix} = B_0.$$

Thus we obtain

$$Q_0(\tau) = L_0(\sigma_1, \ldots, \sigma_{n-d}) + B_0(\sigma_{n-d+1}, \ldots, \sigma_{n+d}),$$

which means that Q_0 is equivalent to the direct sum of B_0 and L_0. □

5. Proof of Proposition 4.4

Finally, let us prove Proposition 4.4.

Proof of Proposition 4.4. As in the proof of Theorem 4.2, let us choose a holomorphic local coordinate system $(\tilde{z}_1, \ldots, \tilde{z}_n)$ on a small neighborhood W of z_0 so that z_0 is the origin in this system and that $S = \{\tilde{z}_{n-d+1} = \cdots = \tilde{z}_n = 0\}$ holds in W. For the sake of simplicity, we will denote $\tilde{z}$ by z in what follows.

In this coordinate system, every point w of $S \cap W$ is represented by $w = (w_1, \ldots, w_{n-d}, 0, \ldots, 0)$, and $w' = (w_1, \ldots, w_{n-d})$ gives a holomorphic local coordinate system of S around z_0. Hence, in this system, the Levi form $L_0(\sigma)$ of ψ at $z_0 = 0$ has the form

$$\sum_{1\le j,k\le n-d}\frac{\partial^2\psi}{\partial w_j\partial\bar{w}_k}(0)\,\sigma_j\bar{\sigma}_k$$

with the constraint

$$\sum_{1\le j\le n-d}\frac{\partial\psi}{\partial w_j}(0)\,\sigma_j=0.$$

On the other hand, in this coordinate system the restriction of the Hermitian form $L_0'(\sigma)$, given by (4.5)–(4.7), to the subspace $H_0\cap T_0S$ is expressed as follows:

$$L_0'(\sigma)=\sum_{1\le j,k\le n-d}(\partial_j\bar{\partial}_k\varphi(0)-\rho_j(0)\,B_0^{-1\,t}\,\overline{\rho_k(0)}\,)\,\sigma_j\bar{\sigma}_k$$

where

$$\rho_j(0)=(\kappa_{j1}(0),\dots,\kappa_{jd}(0),\ \lambda_{j1}(0),\dots,\lambda_{jd}(0)),\qquad j=1,\dots,n,$$

and

$$\sigma\in H_0\cap T_0S=\{\sigma=(\sigma_1,\dots,\sigma_{n-d},0,\dots,0);\ \sum_{1\le j\le n-d}\partial_j\varphi(0)\sigma_j=0\}.$$

Therefore, since both $L_0(\sigma)$ and $L_0'(\sigma)$ are independent of the choice of holomorphic local coordinate systems as stated in Remark 4.3 and Remark 4.5, it suffices for us to prove the following equalities:

$$\text{(5.1)}\qquad \frac{\partial^2\psi}{\partial w_j\partial\bar{w}_k}(0)=\partial_j\bar{\partial}_k\varphi(0)-\rho_j(0)\,B_0^{-1\,t}\,\overline{\rho_k(0)}\qquad\text{for } j,k=1,\dots,n-d$$

and

$$\text{(5.2)}\qquad \frac{\partial\psi}{\partial w_j}(0)=\partial_j\varphi(0)\qquad\text{for } j=1,\dots,n-d.$$

The proof of (5.1) and (5.2) will be done in the similar way as in [4]. We divide it into four steps.

STEP 1: First let us investigate how we can obtain an explicit form of the real analytic local diffeomorphism g defined by (4.1) and (4.2). Once we find an explicit form of g, we can easily obtain the one of ψ according to (4.4).

Let $w=(w',0,\dots,0)$ be a point in $S\cap W$, and let $\theta=(\theta_1,\dots,\theta_n)$ denote a point in a set $\{\theta\in\mathbf{C}^n;\,|\theta-\partial\varphi(0)|<c\}$ where c is a small positive constant. Let $(z(t;w,\theta),\zeta(t;w,\theta))$ $(t=(t_1,\dots,t_d)\in\mathbf{C}^d)$ be the local expression of the bicharacteristic manifold through (w,θ) explained in Remark 1.2. Let us define

$$\text{(5.3)}\qquad f_j(t;w',\theta)=\zeta_j(t;w,\theta)-\partial_j\varphi(z(t;w,\theta))\qquad(1\le j\le n),$$

$$\text{(5.4)}\qquad f_{n+\gamma}(t;w',\theta)=p_\gamma(z(t;w,\theta),\,\partial\varphi(z(t;w,\theta)))\qquad(1\le\gamma\le d),$$

and consider the simultaneous equations

(5.5) $$f_\mu(t; w', \theta) = 0, \qquad \mu = 1,\dots,n+d.$$

It is obvious that, if (5.5) holds for some $t = (t_1,\dots,t_d)$, then $z(t; w, \theta)$ belongs to C and (w, θ) lies on the bicharacteristic manifold of $\mathfrak{M}$ passing through $(z(t; w, \theta), \partial\varphi(z(t; w, \theta)))$. In other words, if (5.5) holds for some t, $b_{(w,\theta)}$ is a leaf of the foliation b. Since b is a real analytic foliation in a neighborhood of $z_0 = 0$, we can expect that for any point $(w', 0,\dots,0)$ in S near 0 there exists a unique (t, θ) near $(0, \partial\varphi(0))$ such that (t, w', θ) satisfies the equations (5.5). As a matter of fact, we will show in the next step that the Jacobian matrix of $(f_1,\dots,f_{n+d}, \bar{f}_1,\dots,\bar{f}_{n+d})$ with respect to θ, $\bar{\theta}$, t and $\bar{t}$ is non-singular at $(t, w', \theta) = (0, 0, \partial\varphi(0))$. Hence, by the implicit function theorem, (5.5) can be converted to the form

(5.6) $$(t, \theta) = (T(w'), \Theta(w'))$$

in a neighborhood of $(t, w', \theta) = (0, 0, \partial\varphi(0))$. Remark that $T(w')$ and $\Theta(w')$ are real analytic in w' but not necessarily holomorphic in w'. Then, by the definition (4.1) and (4.2) of g, we find that

$$\begin{aligned} g(w) &= g((w', 0,\dots,0)) \\ &= z(T(w'); (w', 0,\dots,0), \Theta(w')). \end{aligned}$$

Thus we obtain the following expression of ψ:

(5.7) $$\psi(w') = \varphi(z(T(w'); (w', 0,\dots,0), \Theta(w'))).$$

STEP 2: Let us now prove that the equations (5.5) can be converted to (5.6). To do so, it suffices to show that the following Jacobian matrix J is non-singular at $(t, w', \theta) = (0, 0, \partial\varphi(0))$. In what follows the evaluation of some function, say f, at $(t, w', \theta) = (0, 0, \partial\varphi(0))$ will be indicated by the symbol $f|_Y$.

$$J = \left.\begin{pmatrix} \dfrac{\partial f_1}{\partial\theta_1} & \dfrac{\partial f_1}{\partial\bar{\theta}_1} & \dfrac{\partial f_1}{\partial\theta_2} & \cdots & \dfrac{\partial f_1}{\partial\bar{\theta}_n} & \dfrac{\partial f_1}{\partial t_1} & \cdots & \dfrac{\partial f_1}{\partial\bar{t}_d} \\ \dfrac{\partial\bar{f}_1}{\partial\theta_1} & \dfrac{\partial\bar{f}_1}{\partial\bar{\theta}_1} & \dfrac{\partial\bar{f}_1}{\partial\theta_2} & \cdots & \dfrac{\partial\bar{f}_1}{\partial\bar{\theta}_n} & \dfrac{\partial\bar{f}_1}{\partial t_1} & \cdots & \dfrac{\partial\bar{f}_1}{\partial\bar{t}_d} \\ \dfrac{\partial f_2}{\partial\theta_1} & \dfrac{\partial f_2}{\partial\bar{\theta}_1} & \dfrac{\partial f_2}{\partial\theta_2} & & & \cdots & & \dfrac{\partial f_2}{\partial\bar{t}_d} \\ \vdots & \vdots & \vdots & & & & & \vdots \\ \dfrac{\partial\bar{f}_{n+d}}{\partial\theta_1} & \dfrac{\partial\bar{f}_{n+d}}{\partial\bar{\theta}_1} & \dfrac{\partial\bar{f}_{n+d}}{\partial\theta_2} & & & \cdots & & \dfrac{\partial\bar{f}_{n+d}}{\partial\bar{t}_d} \end{pmatrix}\right|_Y$$

Now the following relations are immediate consequences of (1.7).

(5.8) $$\frac{\partial z_j}{\partial t_\gamma}\Big|_Y = p_\gamma^{(j)}(0, \partial\varphi(0)), \qquad j = 1,\dots,n,\ \gamma = 1,\dots,d,$$

$$\frac{\partial \zeta_j}{\partial t_\gamma}|_Y = -p_{\gamma(j)}(0, \partial\varphi(0)), \qquad j = 1,\dots,n,\ \gamma = 1,\dots,d, \tag{5.9}$$

$$\frac{\partial z_j}{\partial w_k}|_{t=0} = \delta_{j,k}, \qquad j = 1,\dots,n,\ k = 1,\dots,n-d, \tag{5.10}$$

$$\frac{\partial \zeta_j}{\partial w_k}|_{t=0} = 0, \qquad j = 1,\dots,n,\ k = 1,\dots,n-d, \tag{5.11}$$

$$\frac{\partial z_j}{\partial \theta_l}|_{t=0} = 0, \qquad j,\ l = 1,\dots,n, \tag{5.12}$$

$$\frac{\partial \zeta_j}{\partial \theta_l}|_{t=0} = \delta_{j,l}, \qquad j,\ l = 1,\dots,n, \tag{5.13}$$

where $\delta_{j,k}$ denotes the Kronecker δ. Moreover, since $z(t; w, \theta)$ and $\zeta(t; w, \theta)$ are holomorphic functions of (t, w, θ), we have

$$\frac{\partial z_j}{\partial \bar{t}_\gamma} = \frac{\partial \zeta_j}{\partial \bar{t}_\gamma} = 0, \qquad j = 1,\dots,n,\ \gamma = 1,\dots,d, \tag{5.14}$$

$$\frac{\partial z_j}{\partial \bar{w}_k} = \frac{\partial \zeta_j}{\partial \bar{w}_k} = 0, \qquad j = 1,\dots,n,\ k = 1,\dots,n-d, \tag{5.15}$$

$$\frac{\partial z_j}{\partial \bar{\theta}_l} = \frac{\partial \zeta_j}{\partial \bar{\theta}_l} = 0, \qquad j,\ l = 1,\dots,n. \tag{5.16}$$

Using these relations, we can easily calculate each component of J. For example,

$$\frac{\partial f_j}{\partial \theta_l}|_Y = \delta_{j,l}, \qquad j,\ l = 1,\dots,n,$$

$$\frac{\partial f_\mu}{\partial \bar{\theta}_l}|_Y = 0, \qquad \mu = 1,\dots,n+d,\ l = 1,\dots,n,$$

$$\begin{aligned}
\frac{\partial f_j}{\partial \bar{t}_\gamma}|_Y &= -\sum_{1\le k\le n} \partial_j\bar{\partial}_k\varphi(0)\ \overline{\frac{\partial z_k}{\partial t_\gamma}|_Y} \\
&= -\sum_{1\le k\le n} \partial_j\bar{\partial}_k\varphi(0)\ \overline{p_\gamma^{(k)}(0, \partial\varphi(0))} \\
&= -\kappa_{j\gamma}(0), \qquad j = 1,\dots,n,\ \gamma = 1,\dots,d,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f_{n+\gamma}}{\partial t_\delta}|_Y &= \sum_{1\le j\le n} p_{\gamma(j)}(0, \partial\varphi(0))\frac{\partial z_j}{\partial t_\delta}|_Y \\
&\quad + \sum_{1\le j,k\le n} p_\gamma^{(j)}(0, \partial\varphi(0))\,\partial_j\partial_k\varphi(0)\frac{\partial z_k}{\partial t_\delta}|_Y \\
&= \sum_{1\le j\le n} p_{\gamma(j)}(0, \partial\varphi(0))\,p_\delta^{(j)}(0, \partial\varphi(0))
\end{aligned}$$

$$+\sum_{1\leq j,k\leq n} p_\gamma^{(j)}(0,\partial\varphi(0))p_\delta^{(k)}(0,\partial\varphi(0))\,\partial_j\partial_k\varphi(0)$$
$$=\beta_{\gamma\delta}(0),\qquad \gamma,\ \delta=1,\dots,d,$$
$$\frac{\partial f_{n+\gamma}}{\partial\bar t_\delta}\Big|_Y=\sum_{1\leq j,k\leq n} p_\gamma^{(j)}(0,\partial\varphi(0))\,\partial_j\bar\partial_k\varphi(0)\ \overline{\frac{\partial z_k}{\partial t_\delta}\Big|_Y}$$
$$=\sum_{1\leq j,k\leq n} p_\gamma^{(j)}(0,\partial\varphi(0))\,\overline{p_\delta^{(k)}(0,\partial\varphi(0))}\,\partial_j\bar\partial_k\varphi(0)$$
$$=\alpha_{\gamma\delta}(0),\qquad \gamma,\ \delta=1,\dots,d,$$

and so on. Thus we find

$$J=\begin{pmatrix}
1 & 0 & \cdots & 0 & -\lambda_{11}(0) & -\kappa_{11}(0) & \cdots & -\kappa_{1d}(0)\\
0 & \ddots & & \vdots & -\overline{\kappa_{11}(0)} & -\overline{\lambda_{11}(0)} & \cdots & -\overline{\lambda_{1d}(0)}\\
\vdots & & \ddots & 0 & \vdots & \vdots & & \vdots\\
0 & \cdots & 0 & 1 & -\overline{\kappa_{n1}(0)} & -\overline{\lambda_{n1}(0)} & \cdots & -\overline{\lambda_{nd}(0)}\\
 & & & & \beta_{11}(0) & \alpha_{11}(0) & \cdots & \alpha_{1d}(0)\\
 & & 0 & & \overline{\alpha_{11}(0)} & \overline{\beta_{11}(0)} & \cdots & \overline{\beta_{1d}(0)}\\
 & & & & \vdots & \vdots & & \vdots\\
 & & & & \overline{\alpha_{d1}(0)} & \overline{\beta_{d1}(0)} & \cdots & \overline{\beta_{dd}(0)}
\end{pmatrix}.$$

Since B_0 is non-degenerate by the assumption, it follows from Lemma 3.4 that the matrix

$$\begin{pmatrix}
\beta_{11}(0) & \alpha_{11}(0) & \cdots & \alpha_{1d}(0)\\
\overline{\alpha_{11}(0)} & \overline{\beta_{11}(0)} & \cdots & \overline{\beta_{1d}(0)}\\
\vdots & \vdots & & \vdots\\
\overline{\alpha_{d1}(0)} & \overline{\beta_{d1}(0)} & \cdots & \overline{\beta_{dd}(0)}
\end{pmatrix}$$

is non-singular. Hence, J is so, too. Thus we have proved that the equations (5.5) have a unique solution $(t,\theta)=(T(w'),\Theta(w'))$ for each w' in a neighborhood of $(t,w',\theta)=(0,0,\partial\varphi(0))$.

STEP 3: Before proving (5.1) and (5.2), let us calculate the first derivatives of $T(w')$ at the origin. Besides this we prove some equalities which the first derivatives of $\Theta(w')$ satisfy at the origin. They will be used in the calculation of $\partial^2\psi/\partial w_j \partial \bar{w}_k(0)$. It is a little amazing that, though we have to calculate the second derivative of ψ, we need not know the second derivative of $T(w')$ or $\Theta(w')$ as we will see in the final step.

First let us consider the derivatives of $T(w')$. By the definition of $T(w')$ and $\Theta(w')$, we have

$$f_{n+\gamma}(T(w');\, w',\, \Theta(w')) = 0 \qquad \text{for } \gamma = 1,\dots,d. \tag{5.17}$$

We differentiate these equalities by w_k and evaluate at $w' = 0$, then we find

$$\begin{aligned}&\sum_{1\le\delta\le d}\left\{\frac{\partial f_{n+\gamma}}{\partial t_\delta}\Big|_Y \frac{\partial T_\delta}{\partial w_k}(0) + \frac{\partial f_{n+\gamma}}{\partial \bar{t}_\delta}\Big|_Y \frac{\partial \bar{T}_\delta}{\partial w_k}(0)\right\} + \frac{\partial f_{n+\gamma}}{\partial w_k}\Big|_Y \\ &\quad + \sum_{1\le l\le n}\left\{\frac{\partial f_{n+\gamma}}{\partial \theta_l}\Big|_Y \frac{\partial \Theta_l}{\partial w_k}(0) + \frac{\partial f_{n+\gamma}}{\partial \bar{\theta}_l}\Big|_Y \frac{\partial \bar{\Theta}_l}{\partial w_k}(0)\right\} = 0, \\ &\quad \gamma = 1,\dots,d,\ k = 1,\dots,n-d.\end{aligned}$$

Now we have already known the derivatives of $f_{n+\gamma}$ with respect to t, $\bar{t}$, θ and $\bar{\theta}$ at $(t, w', \theta) = (0, 0, \partial\varphi(0))$. Using (5.10) and (5.15), we can also easily calculate $(\partial f_{n+\gamma}/\partial w_k)\,(0, 0, \partial\varphi(0))$. Thus we obtain

$$\begin{aligned}&\sum_{1\le\delta\le d}\beta_{\gamma\delta}(0)\frac{\partial T_\delta}{\partial w_k}(0) + \sum_{1\le\delta\le d}\alpha_{\gamma\delta}(0)\frac{\partial \bar{T}_\delta}{\partial w_k}(0) + \lambda_{k\gamma}(0) = 0, \\ &\gamma = 1,\dots,d,\ k = 1,\dots,n-d.\end{aligned} \tag{5.18}$$

Similarly, by differentiating (5.17) by $\bar{w}_k$, we obtain

$$\begin{aligned}&\sum_{1\le\delta\le d}\beta_{\gamma\delta}(0)\frac{\partial T_\delta}{\partial \bar{w}_k}(0) + \sum_{1\le\delta\le d}\alpha_{\gamma\delta}(0)\ \overline{\frac{\partial T_\delta}{\partial w_k}}\ (0) + \overline{\kappa_{k\gamma}(0)} = 0, \\ &\gamma = 1,\dots,d,\ k = 1,\dots,n-d.\end{aligned} \tag{5.19}$$

Since $\alpha_{\gamma\delta} = \overline{\alpha_{\delta\gamma}}$ and $\beta_{\gamma\delta} = \beta_{\delta\gamma}$ hold for any γ and δ (cf. (3.8)), it follows from (5.18) and (5.19) that

$$\begin{aligned}&\left(\frac{\partial T_1}{\partial w_k}(0),\dots,\frac{\partial T_d}{\partial w_k}(0),\ \frac{\partial \bar{T}_1}{\partial w_k}(0),\dots,\frac{\partial \bar{T}_d}{\partial w_k}(0)\right)B_0 \\ &\quad = -\,(\kappa_{k1}(0),\dots,\kappa_{kd}(0),\ \lambda_{k1}(0),\dots,\lambda_{kd}(0)) \\ &\quad = -\,\rho_k(0), \qquad k = 1,\dots,n-d.\end{aligned}$$

Hence we have

$$\begin{aligned}&\left(\frac{\partial T_1}{\partial w_k}(0),\dots,\frac{\partial T_d}{\partial w_k}(0),\ \frac{\partial \bar{T}_1}{\partial w_k}(0),\dots,\frac{\partial \bar{T}_d}{\partial w_k}(0)\right) \\ &\quad = -\,\rho_k(0)\,B_0^{-1} \qquad \text{for } k = 1,\dots,n-d.\end{aligned} \tag{5.20}$$

Let us next consider the derivatives of $\Theta(w')$. We begin with the following equalities:

$$(5.21)\qquad p_\gamma(z(T(w'); w, \Theta(w')), \zeta(T(w'); w, \Theta(w'))) = 0, \qquad \gamma = 1,\dots,d.$$

These are immediate consequences of (5.3)–(5.5) and the definition of $T(w')$ and $\Theta(w')$. Then we carry on our calculations in the same way as we did in the case of the derivatives of $T(w')$. That is, we differentiate (5.21) by w_k and $\bar{w}_k$, and evaluate at $w' = 0$. Thus, making use of (3.8), we obtain the following equalities:

(5.22)

$$\sum_{1\le l\le n} p_\gamma^{(l)}(0, \partial\varphi(0))\frac{\partial\Theta_l}{\partial w_k}(0) = -p_{\gamma(k)}(0, \partial\varphi(0)) \qquad \text{for } \gamma = 1,\dots,d,\ k = 1,\dots,n-d,$$

(5.23)

$$\sum_{1\le l\le n} p_\gamma^{(l)}(0, \partial\varphi(0))\frac{\partial\Theta_l}{\partial \bar{w}_k}(0) = 0 \qquad \text{for } \gamma = 1,\dots,d,\ k = 1,\dots,n-d.$$

STEP 4: Now let us calculate $\partial^2\psi/\partial w_j\partial\bar{w}_k(0)$ and $\partial\psi/\partial w_j(0)$. In what follows, we will keep the notations $\partial\psi/\partial w_j$ etc. to denote the differentiation in w-variables, and the symbol $\partial_j\varphi$ etc. always refer to the one in z-variables. We will also abbreviate $p_\gamma^{(j)}(0, \partial\varphi(0))$ etc. to $p_\gamma^{(j)}$ etc. for the sake of simplicity of notations.

First we note the following relations, which follow from (5.8)–(5.16):

$$(5.24)\qquad \frac{\partial^2 z_j}{\partial t_\gamma\partial t_\delta}\Big|_Y = \sum_{1\le\mu\le n}\{p_{\gamma(\mu)}^{(j)}p_\delta^{(\mu)} - p_\gamma^{(j,\mu)}p_{\delta(\mu)}\},$$
$$j = 1,\dots,n,\ \gamma,\ \delta = 1,\dots,d,$$

$$(5.25)\qquad \frac{\partial^2 z_j}{\partial t_\gamma\partial w_k}\Big|_Y = p_{\gamma(k)}^{(j)}, \qquad j = 1,\dots,n,\ k = 1,\dots,n-d,\ \gamma = 1,\dots,d,$$

$$(5.26)\qquad \frac{\partial^2 z_j}{\partial t_\gamma\partial \theta_l}\Big|_Y = p_\gamma^{(j,l)}, \qquad j,\ l = 1,\dots,n,\ \gamma = 1,\dots,d,$$

$$(5.27)\qquad \frac{\partial^2 z_j}{\partial w_k\partial \theta_l}\Big|_{t=0} = 0, \qquad j,\ l = 1,\dots,n,\ k = 1,\dots,n-d,$$

$$(5.28)\qquad \frac{\partial^2 z_j}{\partial \theta_l\partial \theta_{l'}}\Big|_{t=0} = 0, \qquad j,\ l,\ l' = 1,\dots,n.$$

(5.29) Every second derivative of z_j ($1\le j\le n$) containing the differentiation with respect to $\bar{t}_\gamma$, $\bar{w}_k$ or $\bar{\theta}_l$ ($1\le\gamma\le d$, $1\le k\le n-d$, $1\le l\le n$) is equal to zero.

Using (5.8)–(5.16) and (5.24)–(5.29) together, we obtain the following equality:

$$
\begin{aligned}
(5.30)\qquad \frac{\partial^2\psi}{\partial w_j\partial\bar w_k}(0) ={}& \partial_j\bar\partial_k\varphi + \sum_{\gamma,\mu} p_\gamma^{(\mu)}\partial_j\partial_\mu\varphi\frac{\partial T_\gamma}{\partial\bar w_k} + \sum_{\gamma,\mu}\overline{p_\gamma^{(\mu)}}\,\partial_j\bar\partial_\mu\varphi\,\overline{\frac{\partial T_\gamma}{\partial w_k}}\\
&+ \sum_{\gamma,\mu} p_\gamma^{(\mu)}\partial_\mu\bar\partial_k\varphi\frac{\partial T_\gamma}{\partial w_j} + \sum_{\gamma,\mu}\overline{p_\gamma^{(\mu)}}\,\bar\partial_\mu\bar\partial_k\varphi\frac{\partial\bar T_\gamma}{\partial w_j}\\
&+ \sum_{\gamma,\delta,\mu,\nu} p_\gamma^{(\mu)}p_\delta^{(\nu)}\partial_\mu\partial_\nu\varphi\frac{\partial T_\gamma}{\partial w_j}\frac{\partial T_\delta}{\partial\bar w_k}\\
&+ \sum_{\gamma,\delta,\mu,\nu} p_\gamma^{(\mu)}\,\overline{p_\delta^{(\nu)}}\,\partial_\mu\bar\partial_\nu\varphi\frac{\partial T_\gamma}{\partial w_j}\,\overline{\frac{\partial T_\delta}{\partial w_k}}\\
&+ \sum_{\gamma,\delta,\mu,\nu}\overline{p_\gamma^{(\mu)}}\,p_\delta^{(\nu)}\bar\partial_\mu\partial_\nu\varphi\frac{\partial\bar T_\gamma}{\partial w_j}\frac{\partial T_\delta}{\partial\bar w_k}\\
&+ \sum_{\gamma,\delta,\mu,\nu}\overline{p_\gamma^{(\mu)}p_\delta^{(\nu)}}\,\bar\partial_\mu\bar\partial_\nu\varphi\frac{\partial\bar T_\gamma}{\partial w_j}\,\overline{\frac{\partial T_\delta}{\partial w_k}}\\
&+ \sum_{\gamma,\delta,\mu,\nu}\{p_{\gamma(\nu)}^{(\mu)}p_\delta^{(\nu)} - p_\gamma^{(\mu,\nu)}p_{\delta(\nu)}\}\,\partial_\mu\varphi\frac{\partial T_\gamma}{\partial w_j}\frac{\partial T_\delta}{\partial\bar w_k}\\
&+ \sum_{\gamma,\delta,\mu,\nu}\{\overline{p_{\gamma(\nu)}^{(\mu)}p_\delta^{(\nu)} - p_\gamma^{(\mu,\nu)}p_{\delta(\nu)}}\}\,\bar\partial_\mu\varphi\frac{\partial\bar T_\gamma}{\partial w_j}\,\overline{\frac{\partial T_\delta}{\partial w_k}}\\
&+ \sum_{\gamma,\mu} p_{\gamma(j)}^{(\mu)}\partial_\mu\varphi\frac{\partial T_\gamma}{\partial\bar w_k} + \sum_{\gamma,\mu}\overline{p_{\gamma(k)}^{(\mu)}}\,\bar\partial_\mu\varphi\frac{\partial\bar T_\gamma}{\partial w_j}\\
&+ \sum_{\gamma,\mu,\nu} p_\gamma^{(\mu,\nu)}\partial_\mu\varphi\left\{\frac{\partial T_\gamma}{\partial w_j}\frac{\partial\Theta_\nu}{\partial\bar w_k} + \frac{\partial\Theta_\nu}{\partial w_j}\frac{\partial T_\gamma}{\partial\bar w_k}\right\}\\
&+ \sum_{\gamma,\mu,\nu}\overline{p_\gamma^{(\mu,\nu)}}\,\bar\partial_\mu\varphi\left\{\frac{\partial\bar T_\gamma}{\partial w_j}\,\overline{\frac{\partial\Theta_\nu}{\partial w_k}} + \frac{\partial\bar\Theta_\nu}{\partial w_j}\,\overline{\frac{\partial T_\gamma}{\partial w_k}}\right\}\\
&+ \sum_{\gamma,\mu} p_\gamma^{(\mu)}\partial_\mu\varphi\frac{\partial^2 T_\gamma}{\partial w_j\partial\bar w_k} + \sum_{\gamma,\mu}\overline{p_\gamma^{(\mu)}}\,\bar\partial_\mu\varphi\,\frac{\partial^2\bar T_\gamma}{\partial w_j\partial\bar w_k},\\
& j,\ k = 1,\dots,n-d.
\end{aligned}
$$

Applying Euler's identity to (5.30), and then using $p_\gamma(0, \partial\varphi(0)) = 0 (1 \le \gamma \le d)$ and (3.8), we find

$$
\begin{aligned}
(5.31)\qquad \frac{\partial^2\psi}{\partial w_j\partial\bar w_k}(0) ={}& \partial_j\bar\partial_k\varphi + \sum_\gamma \lambda_{j\gamma}\frac{\partial T_\gamma}{\partial\bar w_k} + \sum_\gamma \kappa_{j\gamma}\,\overline{\frac{\partial T_\gamma}{\partial w_k}}\\
&+ \sum_\gamma \overline{\kappa_{k\gamma}}\frac{\partial T_\gamma}{\partial w_j} + \sum_\gamma \overline{\lambda_{k\gamma}}\frac{\partial\bar T_\gamma}{\partial w_j}
\end{aligned}
$$

$$+\sum_{\gamma,\delta}\beta_{\gamma\delta}\frac{\partial T_\gamma}{\partial w_j}\frac{\partial T_\delta}{\partial \bar{w}_k}+\sum_{\gamma,\delta}\alpha_{\gamma\delta}\frac{\partial T_\gamma}{\partial w_j}\overline{\frac{\partial T_\delta}{\partial w_k}}$$

$$+\sum_{\gamma,\delta}\overline{\alpha_{\gamma\delta}}\frac{\partial \bar{T}_\gamma}{\partial w_j}\frac{\partial T_\delta}{\partial \bar{w}_k}+\sum_{\gamma,\delta}\overline{\beta_{\gamma\delta}}\frac{\partial \bar{T}_\gamma}{\partial w_j}\overline{\frac{\partial T_\delta}{\partial w_k}}$$

$$+\sum_{\gamma}(m_\gamma-1)\left\{p_{\gamma(j)}\frac{\partial T_\gamma}{\partial \bar{w}_k}+\overline{p_{\gamma(k)}}\frac{\partial \bar{T}_\gamma}{\partial w_j}\right.$$

$$+\sum_{l}p_\gamma^{(l)}\left(\frac{\partial T_\gamma}{\partial w_j}\frac{\partial \Theta_l}{\partial \bar{w}_k}+\frac{\partial \Theta_l}{\partial w_j}\frac{\partial T_\gamma}{\partial \bar{w}_k}\right)$$

$$\left.+\sum_{l}\overline{p_\gamma^{(l)}}\left(\frac{\partial \bar{T}_\gamma}{\partial w_j}\overline{\frac{\partial \Theta_l}{\partial w_k}}+\frac{\partial \bar{\Theta}_l}{\partial w_j}\overline{\frac{\partial T_\gamma}{\partial w_k}}\right)\right\},$$

$$j,\ k=1,\dots,n-d.$$

Here m_γ denotes the order of the operator P_γ.

Remark that (5.22) and (5.23) imply the last term $\sum_\gamma(m_\gamma-1)\ \{\cdots\}$ of (5.31) vanishes. Moreover, denoting the vector

$$\left(\frac{\partial T_1}{\partial w_k},\dots,\frac{\partial T_d}{\partial w_k},\frac{\partial \bar{T}_1}{\partial w_k},\dots,\frac{\partial \bar{T}_d}{\partial w_k}\right)$$

by ω_k, we have the following equality by (5.20):

$$\omega_k=-\rho_k B_0^{-1}\qquad \text{for } k=1,\dots,n-d.$$

Hence we obtain

$$\begin{aligned}\frac{\partial^2\psi}{\partial w_j\partial \bar{w}_k}(0)&=\partial_j\bar{\partial}_k\varphi+\rho_j{}^t\bar{\omega}_k+\omega_j{}^t\bar{\rho}_k+\omega_jB_0{}^t\bar{\omega}_k\\&=\partial_j\bar{\partial}_k\varphi-\rho_jB_0^{-1\,t}\bar{\rho}_k-\rho_jB_0^{-1\,t}\bar{\rho}_k+\rho_jB_0^{-1}B_0B_0^{-1\,t}\bar{\rho}_k\\&=\partial_j\bar{\partial}_k\varphi-\rho_jB_0^{-1\,t}\bar{\rho}_k\qquad \text{for } j,\ k=1,\dots,n-d.\end{aligned}$$

Thus we have proved (5.1).

Finally let us prove (5.2). It follows from (5.7) and (5.8)–(5.16) that we have

$$\frac{\partial\psi}{\partial w_j}(0)=\partial_j\varphi(0)+\sum_{l,\gamma}p_\gamma^{(l)}\partial_l\varphi\frac{\partial T_\gamma}{\partial w_j}(0)$$

$$+\sum_{l,\gamma}\overline{p_\gamma^{(l)}}\ \bar{\partial}_l\varphi\frac{\partial \bar{T}_\gamma}{\partial w_j}(0),\qquad j=1,\dots,n-d.$$

Since

$$\sum_l p_\gamma^{(l)}(0,\partial\varphi(0))\,\partial_l\varphi(0)=0$$

holds for $\gamma = 1,\dots,d$ by Euler's identity, we obtain

$$\frac{\partial\psi}{\partial w_j}(0) = \partial_j\varphi(0) \qquad \text{for } j = 1,\dots,n-d,$$

that is, we have (5.2). This completes the proof of Proposition 4.4. □

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