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On integral manifolds of an integrable nonlinear connection from the standpoint of the theory of Finsler connections

By

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There are recently several papers concerning Finsler spaces having the vanishing $(v)h$-torsion tensor R^1 ([1], [2]). It has been shown in a previous paper [4] that the tensor R^1 is regarded as the curvature tensor of the nonlinear connection N and the vanishing of R^1 is an integrability condition of N which is a distribution in the tangent bundle.

The purpose of the present paper is to consider an integral manifold of the integrable nonlinear connection N from the standpoint of the theory of Finsler connections. We are mainly concerned with the Cartan connection $C\Gamma = (F, N, C)$ of a Finsler space (M, L). The tangent bundle $T(M)$ becomes a Riemannian space with the metric which is obtained by lifting the Finslerian fundamental tensor. Thus we shall develope a Riemannian geometrical subspace theory of integral manifolds.

The terminology and notation are referred to the author's book [3]; the quotation from it is indicated only by putting asterisk.

§ 1. Integral manifolds of a nonlinear connection

Let $F\Gamma = (F, N, C)$ be a Finsler connection (*§9) of an n-dimensional differentiable manifold M. The N is a nonlinear connection (*Definition 8.2), that is, an n-dimensional distribution $y \in T(M) \mapsto N_y \subset T(M)_y$ in the tangent bundle $T(M)$ such that $T(M)_y = N_y \oplus T^v_y$. By lifting to the point y a tangent vector $V = (V^i)_x$ of M gives rise to a tangent vector (*(9.1))

$$(1.1) \qquad (V)^h = V^i(\partial/\partial x^i - N^j_i(x, y)\,\partial/\partial y^j)_y,$$

contained in the subspace N_y.

This distribution N is integrable (*Remark 10.2, [4]), when the differential equations

$$(1.2) \qquad \partial y^i/\partial x^j = -N^i_j(x, y)$$

are completely integrable and we get an integral manifold N^n: $y^i = \phi^i(x)$ which is

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tangent to N_y at every point y of N^n.

In general, for a function $Q(x, y)$ on a domain of $T(M)$, we have along N^n

$$\partial Q(x, \phi(x))/\partial x^i = \partial Q(x, y)/\partial x^i - \{\partial Q(x, y)/\partial y^j\} N^j_i, \tag{1.3}$$

which is usually indicated by the symbol $\delta Q/\delta x^i$ (*(9.19)). Then, from (1.2) we get $\partial(\partial y^i/\partial x^i)/\partial x^k = -\delta N^i_j/\delta x^k$, so that the integrability condition of the equation (1.2) is written as

$$R^i_{jk} = \delta N^i_j/\delta x^k - \delta N^i_k/\delta x^j = 0. \tag{1.4}$$

This tensor field R^i_{jk} is called the (v)-*torsion tensor* of the Finsler connection $F\Gamma$ (*§10, [4]).

In the following we shall denote by $V = (V^i, V^{(i)})$ (we put $(i) = n + i$) a tangent vector $V^i\partial/\partial x^i + V^{(i)}\partial/\partial y^i \in T(M)_y$. Since (x^i) may be regarded as a coordinate system of an integral manifold N^n, the n vectors

$$B_j = (\partial x^i/\partial x^j, \partial y^i/\partial x^j) = (\delta^i_j, -N^i_j), \quad j = 1, \cdots, n, \tag{1.5}$$

are independent vector fields tangent to the n-dimensional N^n. By (1.1) it is seen that these are horizontal lifts $(\partial/\partial x^i)^h$ of tangent vectors $\partial/\partial x^i$ of the base manifold M.

§2. Metric and normal of N^n

We are concerned with a Finsler space $F^n = (M, L)$ with a fundamental metric function $L(x, y)$, from which the fundamental tensor field $g = (g_{ij}(x, y))$ is induced such that $g_{ij} = \dot\partial_i\dot\partial_j L^2/2$. It is well-known that the tangent bundle $T(M)$ is regarded as a $2n$-dimensional Riemannian space with a lifted Riemannian metric tensor $\bar g$ (*§21). Throughout the paper we shall restrict our consideration to the 0-lift alone, called the *Sasakian lift* as is well-known. Thus *(21.2′) gives $\bar g$ as

$$\bar g_{ij} = g_{ij} + g_{rs}N^r_i N^s_j, \quad \bar g_{i(j)} = N^r_i g_{rj}, \quad \bar g_{(i)(j)} = g_{ij}. \tag{2.1}$$

The reciprocal $\bar g^{\alpha\beta}$ of $\bar g_{\alpha\beta}$ ($\alpha, \beta = 1, \cdots, 2n$) is given by

$$\bar g^{ij} = g^{ij}, \quad \bar g^{i(j)} = -g^{ir}N^j_r, \quad \bar g^{(i)(j)} = g^{ij} + g^{rs}N^i_r N^j_s. \tag{2.2}$$

Every integral manifold N^n is an n-dimensional subspace of the Riemannian space $T(M)$ and the Riemannian metric $a = (a_{ij}(x))$ is induced in N^n. It follows from (1.5) and (2.1) that we get an interesting fact; the induced metric $a_{ij} = \bar g_{\alpha\beta}B^\alpha_i B^\beta_j$ is nothing less than

$$a_{ij}(x) = g_{ij}(x, \phi(x)), \tag{2.3}$$

that is, the metric of N^n is essentially the same with that of the base space F^n.

Remark. If we put $\delta y^i = dy^i + N^i_j dx^j$ (*§21), then the lifted metric is written as $d\bar s^2 = g_{ij}dx^i dx^j + g_{ij}\delta y^i \delta y^j$. On account of (1.2) δy^i vanishes along N^n and we get (2.3) immediately.

A normal vector N of N^n is given by the equations $\bar{g}_{\alpha\beta}B^\alpha_i N^\beta = 0$, which is written as $g_{ij}N^j = 0$ from (1.5) and (2.1). Thus we have $N^j = 0$. Therefore n independent normal vectors are given as

$$B_{(j)} = (0,\ B^{(i)}_{(j)}), \quad \det\,(B^{(i)}_{(j)}) \neq 0. \tag{2.4}$$

These are written as $B^{(i)}_{(j)}\,\partial/\partial y^i$, that is, the vertical lift of n tangent vectors $B^{(i)}_{(j)}\,\partial/\partial x^i$ at the base point. For the later use we shall indicate the scalar products of normals $B_{(i)}$ and $B_{(j)}$ as

$$\bar{g}_{\alpha\beta}B^\alpha_{(i)}\,B^\beta_{(j)} = g_{hk}\,B^{(h)}_{(i)}\,B^{(k)}_{(j)} = n_{(ij)} \tag{2.5}$$

and call the *characteristic numbers* of the normal frame $B_{(j)}$.

§ 3. Christoffel symbols

The strain tensor S has been defined (*Definition 21.2) as the difference of the Riemannian connection $\Gamma(\bar{g})$ given by the lifted metric $\bar{g}$ from the linear connection of Finsler type $\bar{\Gamma}$ derived from the Cartan connection $C\Gamma$. That is, we have

$$\left\{ {\bar{\alpha} \atop \beta\gamma} \right\} = \bar{\Gamma}^\alpha_{\beta\gamma} - S^\alpha_{\beta\gamma}, \tag{3.1}$$

where $\left\{ {\bar{\alpha} \atop \beta\gamma} \right\}$ are the Christoffel symbols of $\Gamma(\bar{g})$, $\bar{\Gamma}^\alpha_{\beta\gamma}$ the connection coefficients of $\bar{\Gamma}$ and $S^\alpha_{\beta\gamma}$ the components of the strain tensor S. Coefficients $\bar{\Gamma}^\alpha_{\beta\gamma}$ are given by *(20.5).

The strain tensor S is given in *Remark 21.3 for the case of the 0-lift and $C\Gamma$. We shall find the components $S^\alpha_{\beta\gamma}$ of S in the similar way to the case of the torsion tensor $\bar{T}$ as shown in *p.133. For instance we have

$$S^{(i)}_{j(k)} = \bar{z}^{(i)}_\alpha\, S^\alpha_{\beta\gamma}\,(\bar{z}^{-1})^\beta_j\,(\bar{z}^{-1})^\gamma_{(k)}.$$

From *(19. 10) and *(19.10′) we have

$$\begin{aligned} S^{(i)}_{j(k)} = {} & (-\, z^r_a N^i_r S^a_{b(c)} + z^i_a S^{(a)}_{b(c)})\,(z^{-1})^b_j (z^{-1})^c_k \\ & + (-\, z^r_a N^i_r S^a_{(b)(c)} + z^i_a S^{(a)}_{(b)(c)})(z^{-1})^b_s N^s_j (z^{-1})^c_k. \end{aligned}$$

Then *Remark 21.3 gives

$$\begin{aligned} S^{(i)}_{j(k)} &= (-\, z^r_a N^i_r R_{cb\cdot}{}^a/2 + z^i_a P^a_{bc})(z^{-1})^b_j (z^{-1})^c_k + z^r_a N^i_r P^a_{bc}(z^{-1})^b_s N^s_j (z^{-1})^c_k \\ &= -\, N^i_r R_{kj\cdot}{}^r/2 + P^i_{jk} + N^i_r N^s_j P^r_{sk}, \end{aligned}$$

where the first term vanishes from (1.4). Thus we get

$$\begin{aligned} & S^i_{jk} = -\,{}'C^i_{kr}N^r_j, && S^{(i)}_{jk} = C^i_{jk} + N^i_r C^r_{ks} N^s_j + {}'P^i_{jr}N^r_k, \\ & S^i_{(j)k} = -\,{}'C^i_{jk}, && S^{(i)}_{(j)k} = N^i_r{}'C^r_{jk}, \\ & S^i_{j(k)} = -\,P^i_{kr}N^r_j, && S^{(i)}_{j(k)} = {}'P^i_{jk}, \\ & S^i_{(j)(k)} = -\,P^i_{jk}, && S^{(i)}_{(j)(k)} = N^i_r P^r_{jk}, \end{aligned} \tag{3.2}$$

where we put

$$'C^{i}_{jk} = C^{i}_{jk} + P^{i}_{jr} N^{r}_{k}, \qquad 'P^{i}_{jk} = P^{i}_{jk} + N^{i}_{r} N^{s}_{j} P^{r}_{sk}.$$

Then (3.1) and *(20.5) lead us to the Christoffel symbols $\left\{ {\bar{\alpha} \atop \beta\gamma} \right\}$; for instance,

$$\left\{ {(\bar{i}) \atop j(k)} \right\} = \bar{\Gamma}^{(i)}_{j(k)} - S^{(i)}_{j(k)} = (\dot{\partial}_k N^{i}_{j} + C^{i}_{kr} N^{r}_{j} - N^{i}_{r} C^{r}_{jk}) - (P^{i}_{jk} + N^{i}_{r} N^{s}_{j} P^{r}_{sk}).$$

Here we shall pay attention to the equation $P^{i}_{jk} = \dot{\partial}_k N^{i}_{j} - F^{i}_{kj}$(*(10.14)).

Thus we finally obtain

$$\begin{aligned}
&\left\{ {\bar{i} \atop jk} \right\} = F^{i}_{jk} + C^{i}_{jr} N^{r}_{k} + C^{i}_{kr} N^{r}_{j} + P^{i}_{rs} N^{r}_{j} N^{s}_{k}, \\
&\left\{ {(\bar{i}) \atop j(k)} \right\} = -(C^{i}_{jk} + N^{i}_{r} F^{r}_{jk}) + (C^{i}_{rs} - N^{i}_{t} P^{t}_{rs}) N^{r}_{j} N^{s}_{k} \\
&\qquad + S_{(jk)} \{ (\delta N^{i}_{j}/\delta x^{k})/2 + F^{i}_{jr} N^{r}_{k} - N^{i}_{r} C^{r}_{js} N^{s}_{k} \}, \\
&\left\{ {\bar{i} \atop j(k)} \right\} = C^{i}_{jk} + P^{i}_{kr} N^{r}_{j}, \\
&\left\{ {(\bar{i}) \atop j(k)} \right\} = F^{i}_{jk} + C^{i}_{kr} N^{r}_{j} - N^{i}_{r} (C^{r}_{jk} + P^{r}_{ks} N^{s}_{j}), \\
&\left\{ {\bar{i} \atop (j)(k)} \right\} = P^{i}_{jk}, \qquad \left\{ {(\bar{i}) \atop (j)(k)} \right\} = C^{i}_{jk} - N^{i}_{r} P^{r}_{jk},
\end{aligned} \tag{3.3}$$

where the symbol $S_{(jk)}\{\cdot\}$ stands for the interchange of indices j, k and summation.

On the other hand, the Christoffel symbols $\left\{ {i \atop jk} \right\}$ of the induced metric in N^n are easily found: (2.3) and (1.3) show $\partial a_{ij}/\partial x^k = \delta g_{ij}/\delta x^k$, so that *(17.3) leads us to

$$\left\{ {i \atop jk} \right\} = F^{i}_{jk}, \tag{3.4}$$

which is nothing less than the connection coefficients of $C\Gamma$.

§4. Gauss derivation formula

We now consider the Gauss derivation formula

$$\partial B^{\alpha}_{i}/\partial x^j + B^{\beta}_{i} \left\{ {\bar{\alpha} \atop \beta\gamma} \right\} B^{\gamma}_{j} - B^{\alpha}_{k} \left\{ {k \atop ij} \right\} = H^{(k)}_{ij} B^{\alpha}_{(k)}. \tag{4.1}$$

The left-hand side of (4.1) is usually written as $B^{\alpha}_{i;j}$ and called the curvature tensor of N^n for $T(M)$ or the Euler-Schouten curvature tensor of N^n. The $H^{(k)}_{ij}$ of the right-hand side is called the *second fundamental tensor* for the normal $B_{(k)}$. We are concerned with (4.1) in detail.

(1^0) In case of $\alpha = 1, \cdots, n$ in (4.1), we put $\alpha = h$; on account of (1.5), (2.4) and (3.4) it is written as

$$\left\{ {\bar{h} \atop ij} \right\} - \left\{ {\bar{h} \atop i(s)} \right\} N^{s}_{j} - N^{r}_{i} \left(\left\{ {\bar{h} \atop (r)j} \right\} - \left\{ {\bar{h} \atop (r)(s)} \right\} N^{s}_{j} \right) = F^{h}_{ij}.$$

It is easy to see from (3.3) that this equation becomes only trivial.

(2^0) In case of $\alpha = n+1, \cdots, 2n$ in (4.1), we put $\alpha = (h)$; on account of (1.5) and (2.4) it is written as

$$-\delta N^{h}_{i}/\delta x^j + \left\{ {(\bar{h}) \atop ij} \right\} - \left\{ {(\bar{h}) \atop i(r)} \right\} N^{r}_{j} - N^{r}_{i} \left(\left\{ {(\bar{h}) \atop (r)j} \right\} - \left\{ {(\bar{h}) \atop (r)(s)} \right\} N^{s}_{j} \right) + N^{h}_{k} F^{k}_{ij} = H^{(k)}_{ij} B^{(h)}_{(k)}.$$

Then, owing to (3.3), the left-hand side reduces to $-C^h_{ij} - R^h_{ij}/2$ alone, so that from (1.4) we get

$$H^{(k)}_{ij} B^{(h)}_{(k)} = -C^h_{ij}. \tag{4.2}$$

This equation yields some geometrical results as follows: The normal curvautre vector $N(d)$ for a given tangent direction d^i is defined as $N(d)^\alpha = B^\alpha_{i;j} d^i d^j$. Then (4.1) and (4.2) give

$$N(d) = (0, \ -C^h_{ij} d^i d^j). \tag{4.3}$$

It follows from (2.1) that its length $|N(d)|$ is given by

$$|N(d)|^2 = C^r_{hi} C_{rjk} d^h d^i d^j d^k. \tag{4.4}$$

In particular we shall deal with two-dimensional space where we have the Berwald frame (1, m) (*§ 28). The horizontal lifts $(l)^h$ and $(m)^h$ are tangent to the integral manifold N^2 and the vertical lifts $(l)^v$ and $(m)^v$ are normal to N^2. From the equation $LC^h_{ij} = Im^h m_i m_j$ where I is the main scalar, we get

$$N(d) = (0, \ -I(m_i d^i)^2 m^h/L). \tag{4.3$_2$}$$

Thus the direction of the normal vector is orthogonal to the vertical lift $(l)^v$ of the vector l. Therefore *Definition 3.11 shows

Proposition 1. *For a two-dimensional Finsler space with $R^1 = 0$ the normal curvature vector of an integral manifold N^2 is orthogonal to the intrinsic vertical vector $(l)^v$.*

Next the second fundamental form $\Phi(dx)$ of the N^n is defined by $\Phi(dx) = |N(dx)|^2$. Thus we have

$$\Phi(dx) = C^r_{hi} C_{rjk} dx^h dx^i dx^j dx^k, \tag{4.5}$$

and two directions $d_1 x$ and $d_2 x$ are called conjugate to each other if

$$C^r_{hi} C_{rjk} d_1 x^h d_2 x^i d_1 x^j d_2 x^k = 0. \tag{4.6}$$

If dx is self-conjugate, then dx is called asymptotic. We are concerned with the intrinsic horizontal vector field $(l)^h$(*p. 126). Then the well-known equation $l^r C_{rij} = 0$ shows

Proposition 2. *In an integral manifold N^n the direction of the intrinsic horizontal vector $(l)^h$ is conjugate to arbitrary direction and asymptotic.*

Next the mean curvature normal M is defined by $M^\alpha = g^{ij} B^\alpha_{i;j}$ and its length is called the mean curvature. From (4.2) we get

$$M = (0, \ -C^i), \tag{4.7}$$

where $C^i = g^{jk} C^i_{jk}$ is Cartan's torsion vector.

Proposition 3. *The mean curvature vector of an integral manifold N^n is the vertical lift of the vector $-C^i$ where C^i is the torsion vector.*

A subspace is minimal if and only if the mean curvature normal vanishes. Thar is, $C^i = 0$, so that Deicke's Theorem (*Theorem 24.2) leads us to

Theorem 1. *All the integral manifolds are minimal if and only if the Finsler space is Riemannian.*

Further a subspace is totally geodesic if and only if $B^{\alpha}_{i;j} = 0$. Therefore (4.2) shows

Theorem 2. *All the integral manifolds are totally geodesic if and only if the Finsler space is Riemannian.*

§ 5. Weingarten derivation formula

We shall consider the Weingarten derivation formula

$$\partial B^{\alpha}_{(i)}/\partial x^j + B^{\beta}_{(i)}\left\{{\bar{\alpha} \atop \beta\gamma}\right\} B^{\gamma}_j = H^k_{(i)j} B^{\alpha}_k + H^{(k)}_{(i)j} B^{\alpha}_{(k)}. \tag{5.1}$$

The left-hand side is usually written as $B^{\alpha}_{(i);j}$. We are concerned with (5.1) in detail.

(1^0) In case of $\alpha = 1, \cdots, n$ in (5.1), we put $\alpha = h$; on account of (1.5) and (2.4) it is written in the form

$$B^{(r)}_{(i)}\left(\left\{{\bar{h} \atop (r)j}\right\} - \left\{{\bar{h} \atop (r)(s)}\right\} N^s_j\right) = H^h_{(i)j}.$$

It follows from (3.3) that the left-hand side of the above reduces only to $B^{(r)}_{(i)} C^h_{rj}$, so we get

$$B^{(r)}_{(i)} C^h_{rj} = H^h_{(i)j}. \tag{5.2}$$

(2^0) In case of $\alpha = n+1, \cdots, 2n$ in (5.1), we put $\alpha = (h)$; in the similar way to the case (1^0), (5.1) is written in the form

$$\partial B^{(h)}_{(i)}/\partial x^j + B^{(r)}_{(i)} F^h_{rj} \; (= B^{(h)}_{(i);j}) = H^{(k)}_{(i)j} B^{(h)}_{(k)}. \tag{5.3}$$

Next we have to pay attention to the equations which will be derived from differentiating the equations

$$(5.4)\qquad \begin{aligned} &(1)\quad \bar{g}_{\alpha\beta} B^{\alpha}_i B^{\beta}_j = g_{ij}(x, \phi(x)), \\ &(2)\quad \bar{g}_{\alpha\beta} B^{\alpha}_i B^{\beta}_{(j)} = 0, \\ &(3)\quad \bar{g}_{\alpha\beta} B^{\alpha}_{(i)} B^{\beta}_{(j)} = n_{(ij)}. \end{aligned}$$

First the equation from (1) is only $\delta g_{ij}/\delta x^k = g_{hj}\left\{{h \atop ik}\right\} + g_{ih}\left\{{h \atop jk}\right\}$. It is seen from (3.4) that this is nothing less than the metrical property of $C\Gamma$.

Secondly (2) gives rise to $\bar{g}_{\alpha\beta} B^{\alpha}_{i;k} B^{\beta}_{(j)} + \bar{g}_{\alpha\beta} B^{\alpha}_i B^{\beta}_{(j);k} = 0$, and owing to (4.1) and (5.1) it is written as

(5.5) $$n_{(jh)}H^{(h)}_{ik} + g_{ih}H^h_{(j)k} = 0.$$

Similarly (3) yields

(5.6) $$\delta n_{(ij)}/\delta x^k - n_{(jh)}H^{(h)}_{(i)k} - n_{(ih)}H^{(h)}_{(j)k} = 0.$$

We are now concerned with a special normal frame $B_{(j)} = (0, \delta^i_j)$, called the *associated normal frame* with the coordinate system. Owing to (2.1) its characteristic number $n_{(ij)}$ is equal to g_{ij}. For the associated normal frame the equations (4.2), (5.2) and (5.3) become

(5.7) $$H^{(h)}_{ij} = -C^h_{ij},\ H^h_{(i)j} = C^h_{ij},\ H^{(h)}_{(i)j} = F^h_{ij}$$

respectively. Therefore (5.5) reduces trivial and (5.6) to only the metrical condition.

§6. Gauss, Codazzi and Ricci equations

We consider the integrability conditions of the Gauss and Weingarten formulae.

The integrability condition of the Gauss formula is written as

(6.1) $$B^\alpha_{i;j;k} - j/k = H^{(h)}_{ij;k}B^\alpha_{(h)} + H^{(h)}_{ij}(H^l_{(h)k}B^\alpha_l + H^{(l)}_{(h)k}B^\alpha_{(l)}) - j/k,$$

where the symbol $-j/k$ stands for interchange of indices j, k and subtraction. It follows from the Ricci identity that the left-hand side of (6.1) is written as $B^\beta_i\bar{R}^\alpha_{\beta\gamma\delta}B^\gamma_jB^\delta_k - B^\alpha_l\overset{a}{R}{}^l_{ijk}$, where $\bar{R}^\alpha_{\beta\gamma\delta}$ is the Riemannian curvature tensor of the metric $\bar{g}_{\alpha\beta}$ given by (2.1) and $\overset{a}{R}{}^l_{ijk}$ is that of the induced metric $a_{ij}(x)$ given by (2.3), that is, (3.4) shows $\overset{a}{R}{}^l_{ijk} = \delta F^l_{ij}/\delta x^k + F^r_{ij}F^l_{rk} - j/k$, which is nothing less than the h-curvature tensor K^l_{ijk} of the Rund connection $R\Gamma$ and the h-curvature tensor R^l_{ijk} of the Cartan connection $C\Gamma$ owing to (1.4) and *(18.2). Therefore (6.1) becomes

(6.2) $$B^\beta_i\bar{R}^\alpha_{\beta\gamma\delta}R^\gamma_jB^\delta_k - B^\alpha_lR^l_{ijk} = (H^{(h)}_{ij}H^l_{(h)k} - j/k)B^\alpha_l + (H^{(l)}_{ij;k} + H^{(h)}_{ij}H^{(l)}_{(h)k} - j/k)B^\alpha_{(l)}.$$

We first transvect (6.2) by $\bar{g}_{\alpha\varepsilon}B^\varepsilon_h$. Then the left-hand side yields $-\bar{R}_{\alpha\beta\gamma\delta}B^\alpha_hB^\beta_iB^\gamma_jB^\delta_k + R_{hijk}$, and from (5.2) and (4.2) the right-hand side becomes $-C^r_{ij}C_{rhk} + C^r_{ik}C_{rhj}$, which is nothing less than the v-curvature tensor S_{ihjk} of $C\Gamma$. Therefore we get

(6.3) $$\bar{R}_{\alpha\beta\gamma\delta}B^\alpha_hB^\beta_iB^\gamma_jB^\delta_k = R_{hijk} + S_{hijk},$$

which is usually called the *Gauss equation*.

Secondly we transvect (6.2) by $\bar{g}_{\alpha\varepsilon}B^\varepsilon_{(h)}$ and get

(6.4) $$B^\beta_i\bar{R}_{\beta\varepsilon\gamma\delta}B^\varepsilon_{(h)}B^\gamma_jB^\delta_k = n_{(lh)}(H^{(l)}_{ij;k} + H^{(h)}_{ij}H^{(l)}_{(h)k} - j/k).$$

To rewrite the right-hand side of (6.4) we now differentiate (4.2):

$$(\partial H^{(l)}_{ij}/\partial x^k)B^{(h)}_{(l)} + H^{(l)}_{ij}(\partial B^{(k)}_{(l)}/\partial x^k) = -\delta C^h_{ij}/\delta x^k.$$

From (5.3) and (4.2) the left-hand side is rewritten in the form

$$(H^{(l)}_{ij;k} + H^{(l)}_{rj}F^r_{ik} + H^{(l)}_{ir}F^r_{jk})B^{(h)}_{(l)} + H^{(l)}_{ij}(H^{(r)}_{(l)k}B^{(h)}_{(r)} - B^{(r)}_{(l)}F^h_{rk})$$
$$= (H^{(l)}_{ij;k}B^{(h)}_{(l)} - C^h_{rj}F^r_{ik} - C^h_{ir}F^r_{jk}) + (H^{(l)}_{ij}H^{(r)}_{(l)k}B^{(h)}_{(r)} + C^r_{ij}F^l_{rk}),$$

and the right-hand side is rewritten as

$$-C^h_{ij|k} + C^r_{ij}F^h_{rk} - C^h_{rj}F^r_{ik} - C^h_{ir}F^r_{jk},$$

where $C^h_{ij|k}$ is the h-covariant derivative of C^h_{ij}. Thus we get

$$(H^{(l)}_{ij;k} + H^{(r)}_{ij}H^{(l)}_{(r)k})B^{(h)}_{(l)} = -C^h_{ij|k}. \tag{6.5}$$

Consequently (6.4) becomes

$$\bar{R}_{\alpha\beta\gamma\delta}B^\alpha_{(h)}B^\beta_i B^\gamma_j B^\delta_k = B^{(l)}_{(h)}(C_{lij|k} - C_{lik|j}), \tag{6.6}$$

which is usually called the *Codazzi equation.* It is noted that $C_{lij|k} - C_{lik|j}$ in the right-hand side is a part of the hv-curvature tensor P_{kjli} of the Cartan connection $C\Gamma$ (*(17.23)).

Next we deal with the integrability condition of the Weingarten formula. Owing to the Ricci identity we have $B^\alpha_{(i);j;k} - j/k = B^\beta_{(i)}\bar{R}^\alpha_{\beta\gamma\delta}B^\gamma_j B^\delta_k$ and so we get

$$B^\beta_{(i)}\bar{R}^\alpha_{\beta\gamma\delta}B^\gamma_j B^\delta_k = (H^r_{(i)j;k} + H^{(s)}_{(i)j}H^r_{(s)k} - j/k)B^\alpha_r$$
$$+ (H^{(r)}_{(i)j;k} + H^s_{(i)j}H^{(r)}_{sk} + H^{(s)}_{(i)j}H^{(r)}_{(s)k} - j/k)B^\alpha_{(r)}. \tag{6.7}$$

First, transvecting (6.7) by $\bar{g}_{\alpha\varepsilon}B^\varepsilon_h$, we get an equation, but it is easy to show that it is equivalent to (6.4). Secondly, transvecting (6.7) by $\bar{g}_{\alpha\varepsilon}B^\varepsilon_{(h)}$, we get

$$\bar{R}_{\beta\varepsilon\gamma\delta}B^\beta_{(i)}B^\varepsilon_{(h)}B^\gamma_j B^\delta_k = n_{(hr)}(H^{(r)}_{(i)j;k} + H^s_{(i)j}H^{(r)}_{sk} + H^{(s)}_{(i)j}H^{(r)}_{(s)k} - j/k). \tag{6.8}$$

To rewrite the right-hand side of (6.8), from (2.5), (4.2) and (5.2) we have

$$n_{(hr)}H^s_{(i)j}H^{(r)}_{sk} = g_{ml}B^{(m)}_{(h)}B^{(l)}_{(r)}H^s_{(i)j}H^{(r)}_{sk} = -B^{(r)}_{(i)}B^{(m)}_{(h)}C^s_{rj}C_{smk}.$$

Thus we get

$$n_{(hr)}(H^s_{(i)j}H^{(r)}_{sk} - j/k) = B^{(r)}_{(i)}B^{(m)}_{(h)}S_{rmjk}. \tag{6.9}$$

Next (5.3) leads us to

$$B^{(h)}_{(i);j;k} - j/k = B^{(r)}_{(i)}K^h_{rjk} = (H^{(r)}_{(i)j;k} + H^{(s)}_{(i)j}H^{(r)}_{(s)k} - j/k)B^{(h)}_{(r)}.$$

Transvecting this by $g_{hm}B^{(m)}_{(l)}$ and paying attention to $K^h_{rjk} = R^h_{rjk}$, we have

$$R_{rmjk}B^{(r)}_{(i)}B^{(m)}_{(h)} = n_{(hr)}(H^{(r)}_{(i)j;k} + H^{(s)}_{(i)j}H^{(r)}_{(s)k} - j/k). \tag{6.10}$$

Therefore (6.9) and (6.10) lead us to the following form of the right-hand side of (6.8):

(6.11) $$\bar{R}_{\alpha\beta\gamma\delta}B^{\alpha}_{(h)}B^{\beta}_{(i)}B^{\gamma}_{j}B^{\delta}_{k} = B^{(r)}_{(h)}B^{(s)}_{(i)}(R_{rsjk} + S_{rsjk}).$$

This is usually called the *Ricci equation.*

Proposition 4. *We have the Gauss, Codazzi and Ricci equations of the forms* (6.3), (6.6) *and* (6.11) *respectively.*

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