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Spectra and monads of stable bundles

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§ 1. Introduction

Let $\mathscr{E}$ be a stable vector bundle of rank two on $\mathbf{P}^3$ with $c_1 = 0$ and given c_2. Associated to $\mathscr{E}$ are its spectrum (§ 2) and its minimal monad (§ 3, Proposition 3.2). In this article, we investigate the possible spectra and minimal monads for low values of c_2.

The spectrum χ satisfies three necessary conditions ((S1)–(S3) of § 2). We show that for $c_2 \leq 19$, these conditions are also sufficient for the existence of a stable bundle with that spectrum. The question of existence for larger values of c_2 is left unresolved but it seems reasonable to conjecture, based on the evidence so far, that (S1)–(S3) form necessary and sufficient conditions on a sequence of integers for it to be the spectrum of a rank two stable bundle on $\mathbf{P}^3$ with $c_1 = 0$. When $c_2 \leq 19$, we in fact do more. For each possible spectrum, we produce a bundle $\mathscr{E}$ for which $H^0(\mathscr{E}(1)) \neq 0$. This is also the reason that we stop at $c_2 = 19$. When $c_2 \leq 18$, the bundle $\mathscr{E}$ is constructed from a double structure on a reduced curve. For $c_2 = 19$, there is one spectrum for which $\mathscr{E}$ is constructed via a double structure on a non-reduced curve (in fact, a quadruple structure on a reduced curve). We take this as an indication that the construction gets harder beyond $c_2 = 19$.

We also determine all possible minimal monads when $c_2 \leq 8$, thus completing work begun by Barth [B-1]. We find that some monads listed in his tables do not occur, while others, which had been excluded by his simplifying assumption [B-1], top of p.211, do in fact exist (see § 6). The results are tabulated in § 5.

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§ 2.

Let $\mathscr{E}$ be a stable rank two bundle with $c_1 = 0$ and given c_2 on $\mathbf{P}^3$ (over an algebraically closed field of arbitrary characteristic). Barth and Elencwajg [B-E] have defined (in characteristic zero) the *spectrum* of $\mathscr{E}$, which is a certain sequence of c_2 integers. In [H-1], § 7, a characteristic-free definition of the spectrum is given and the following properties are proved in [H-1] and [H-2]. Let $\chi = \{k_1, k_2, \ldots, k_{c_2}\}$, $k_i \in \mathbf{Z}$ be the spectrum of $\mathscr{E}$. Then χ and $\mathscr{E}$ satisfy:

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(S1) Symmetry: $\{-k_i\} = \{k_i\}$. [H-1], 7.2

(S2) Connectedness: for any two integers in χ, every integer lying between them is also in χ. [H-1], 7.6

(S3) If an integer ℓ_0 with $1 \leq \ell_0 \leq \max\{k_i\}$ appears only once in χ, then every integer ℓ such that $\ell_0 \leq \ell \leq \max\{k_i\}$ appears only once in χ. [H-2], 5.1[2]

(S4) $h^1(\mathscr{E}(-i)) - h^1(\mathscr{E}(-i-1)) = \#\{k_j \in \chi / k_j \geq i-1\}$, for $i \geq 1$.

The main result of this section is the following

2.1. Theorem. *For* $0 < c_2 \leq 19$, *any sequence* $\chi = \{k_1, k_2, \ldots, k_{c_2}\}$ *of* c_2 *integers, satisfying the conditions* (S1)-(S3), *occurs as the spectrum of a stable rank* 2 *vector bundle with* $c_1 = 0$ *on* $\mathbf{P}^3$.

Definition. If χ_1 and χ_2 are arranged in increasing order, with repetitions indicated as powers, as $\chi_i = \{n^{s_i(n)}\}_{-\infty<n<\infty}$, $i = 1, 2$, then let $\chi_1 \cup \chi_2 = \{n^{s_1(n)+s_2(n)}\}_{-\infty<n<\infty}$.

2.2. We recall the Ferrand construction. Let X be a locally complete intersection curve in $\mathbf{P}^3$ of degree d such that each connected component X' satisfies $H^0(X', \mathcal{O}_{X'}) = k$. We will denote by $\mathcal{N}_X$ the rank two bundle on X given by

$$\mathcal{N}_X = \mathscr{H}om_{\mathcal{O}_{\mathbf{P}}}(\mathscr{I}_X, \mathcal{O}_X).$$

Suppose that there is a nowhere vanishing section of $\mathcal{N}_X \otimes \omega_X(2)$ and let Y be the multiplicity 2 structure on X defined by the associated sequence

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathscr{I}_X \longrightarrow \omega_X(2) \longrightarrow 0. \tag{2.3}$$

Then by Ferrand's theorem [F] or [H-3], 1.5, $\omega_Y \simeq \mathcal{O}_Y(-2)$. So by [H-3], §1, Y is the zero scheme of a section of $\mathscr{E}(1)$ where $\mathscr{E}$ is a rank two bundle with $c_1 = 0$ and $c_2 = 2d - 1$. We get

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow \mathscr{E} \longrightarrow \mathscr{I}_Y(1) \longrightarrow 0. \tag{2.4}$$

If $X \neq \emptyset$, then $\mathscr{E}$ is stable, for otherwise Y is a plane curve of even degree with $\omega_Y \simeq \mathcal{O}_Y(-2)$, which is not possible.

Notation. If X is a curve as above and $\mathcal{N}_X \otimes \omega_X(2)$ has a nowhere vanishing section, and $\mathscr{E}$ is any bundle constructed as above, we will loosely say that "$\mathscr{E}$ is obtained from X".

2.5. Lemma. *Suppose that* $\mathscr{E}$ *is obtained from* X *as in* (2.2). *Then the*

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[2] Proposition 5.1 in [H-2] is stated and proved in characteristic 0. However the proof can be modified to show that the statement of (S3) is valid in any characteristic. First of all, if we assume that the restriction of $\mathscr{E}$ to a general hyperplane is not stable, then by [E-1] we know that $\mathscr{E}$ is either the nullcorrelation bundle or its pullback by a power of Frobenius and we know the spectrum explicitly (see the proof of Proposition 3.1 below); hence (S3) is true in this case. The second use of the characteristic zero hypothesis is in asserting the existence of an unstable plane. But the proof can be reorganised by doing this after verifying the condition (S3).

spectrum of $\mathscr{E}$ is determined by the number n of connected components of X and by $h^0(\omega_X(m))$, $m \le 2$, via

$$h^1(\mathscr{E}(-i)) = \begin{cases} h^0(\omega_X(-i+3)), & i > 1 \\ h^0(\omega_X(2)) + n - 1, & i = 1. \end{cases}$$

Proof. Use (2.3), (2.4) and the properties (S1) and (S4). □

2.6 Corollary. *Let X be as in Lemma 2.5 with connected components $X_1, X_2, \ldots, X_n$. If we can obtain (as in (2.2)) an $\mathscr{E}_i$ from each X_i with spectrum χ_i, and if $\mathscr{E}$ is obtained from X, then its spectrum is given by*

$$\chi = \chi_1 \cup \chi_2 \cup \cdots \cup \chi_n \cup \{0^{n-1}\}.$$

Proof. $\omega_X \simeq \omega_{X_1} \oplus \cdots \oplus \omega_{X_n}$ and $h^1(\mathscr{I}_X) = n - 1$, while $h^1(\mathscr{I}_{X_i}) = 0$. □

2.7. Lemma. *Suppose X_1 and X_2 are two reduced and connected curves intersecting quasi-transversally in a set of r collinear points $S = X_1 \cap X_2$. If $r \ge 3$, assume further that the surfaces of degrees $1, 2, \ldots, r-2$ cut out complete linear series on both X_1 and X_2. If $\mathscr{E}_i$ with spectrum χ_i is obtained from X_i and $\mathscr{E}$ with spectrum χ is obtained from X, then*

$$\chi \cup \{0\} = \chi_1 \cup \chi_2 \cup \{-r, r\}.$$

Proof. We have

$$0 \longrightarrow \omega_{X_1} \oplus \omega_{X_2} \longrightarrow \omega_X \longrightarrow \omega_S \longrightarrow 0$$

where the connecting homomorphism

$$\delta\colon H^0(\omega_S(m)) \longrightarrow H^1(\omega_{X_1}(m)) \oplus H^1(\omega_{X_2}(m))$$

is dual to the natural map

$$H^0(\mathscr{O}_{X_1}(-m)) \oplus H^0(\mathscr{O}_{X_2}(-m)) \longrightarrow H^0(\mathscr{O}_S(-m)).$$

By the condition on the completeness of the linear series, the image of the natural map equals the image of $H^0(\mathscr{O}_{\mathbf{P}^1}(-m))$ for $0 \le -m \le r-2$. Hence

$$\operatorname{rank}(\delta) = \begin{cases} 0 & \text{if } m > 0 \\ 1 - m & \text{if } 2 - r \le m \le 0 \\ r & \text{if } m \le 1 - r \end{cases}$$

Using Lemma 2.5, it follows that

$$h^1(\mathscr{E}(-i)) = \begin{cases} h^1(\mathscr{E}_1(-i)) + h^1(\mathscr{E}_2(-i)), & i \ge r+2 \\ h^1(\mathscr{E}_1(-i)) + h^1(\mathscr{E}_2(-i)) + r + 2 - i, & r+1 \ge i \ge 3 \\ h^1(\mathscr{E}_1(-i)) + h^1(\mathscr{E}_2(-i)) + r, & i = 1, 2 \end{cases}$$

Now use (S4). □

The following proposition allows us to realize the situation of (2.2).

2.8. Proposition. *Let X be the union in $\mathbf{P}^3$ of irreducible nonsingular curves meeting quasi-transversally in any number of points. Then the general section of $\mathcal{N}_X \otimes \omega_X(m)$ is nowhere vanishing for $m \geq 1$. The same result holds for $m = 0$ if X has the following property: Let A be a connected component of X. Then either*
1) *A is smooth and A is not a line or*
2) *we can find a sequence of connected curves $A_1, A_2, \dots, A_n = A$ where A_1 is smooth and nonrational, $A_i = A_{i-1} \cup B_i$ where B_i is smooth and further $A_{i-1} \cap B_i$ has at least 2 points if B_i is rational.*

Proof. The map $T_{\mathbf{P}^3/X} \to \mathcal{N}_X$ has cokernel supported on the nodes of X, with a one dimensional stalk at each node [H-H]. $T_{\mathbf{P}^3}$ is generated by its global sections, hence so is its image in $\mathcal{N}_X$. Therefore the general section of the image gives a section of $\mathcal{N}_X$ which is nowhere vanishing. (Geometrically, suppose a PGL-action moves X to a curve X' disjoint from X. At the infinitesimal level, this gives a section which is a nowhere vanishing image of a section of $T_{\mathbf{P}^3}$.)

We may work with a connected component of X and hence will assume that X is connected. If X is a line, the claim of the proposition (for $m \geq 1$) is clear. If X is a smooth rational curve of degree ≥ 2, the description of $\mathcal{N}_X$ in [E-V] shows that $\mathcal{N}_X \otimes \omega_X$ splits into two line bundles of degrees ≥ 0 and hence the claim (for $m \geq 0$) of the proposition is proved. If X is smooth and non-rational, then $\omega_X(m)$ is generated by its sections for $m \geq 0$. Hence the general section of $\mathcal{N}_X \otimes \omega_X(m)$ is nowhere vanishing for $m \geq 0$.

Suppose though, that X is not smooth, but is obtained by a chain of connected curves $A_1, A_2, \dots, A_n = X$ where A_1 is smooth and $A_i = A_{i-1} \cup B_i$ with B_i smooth. We have

$$0 \longrightarrow \omega_{A_{i-1}}(m) \longrightarrow \omega_{A_i}(m) \longrightarrow \omega_{B_i}(A_{i-1} \cap B_i)(m) \longrightarrow 0.$$

For $m \geq 0$, this sequence is also exact on the level of global sections. Hence if global sections generate the sheaves on the ends, $\omega_{A_i}(m)$ is also generated by global sections. If we show this for A_n, it follows that the general section of $\mathcal{N}_X \otimes \omega_X(m)$ is nowhere vanishing.

Now assume inductively that $\omega_{A_{i-1}}(m)$ is generated by global sections. $\omega_{B_i}(A_{i-1} \cap B_i)(m)$ is clearly generated by global sections if $m \geq 1$ and, under the hypothesis (2), in degree $m = 0$ as well. To get the induction started, $\omega_{A_1}(m)$ is generated by its sections for degrees $m \geq 0$ (respectively $m \geq 1$) under hypothesis (2) (respectively if A_1 is not a line). If every choice of A_1 in making a chain gives a line, then $A_2 = A_1 \cup B_2$ is a degenerate conic and so $\omega_{A_2}(m)$ is generated by its global sections for $m \geq 1$. □

2.9. Corollary. *If X is the union of irreducible nonsingular curves meeting quasi-transversally, we can obtain a rank two bunde $\mathcal{E}$ from X (as in 2.2).*

Proof. Choose $m = 2$ in Proposition 2.8. The conditions of (2.2) are then satisfied. □

We now turn to the construction of all possible spectra up to $c_2 = 19$ and the proof of Theorem 2.1. First of all, a lemma which focuses attention on odd values of c_2 and spectra with only one zero appearing.

2.10. Lemma. *If $\mathscr{E}$ has spectrum χ and $\mathscr{E}(1)$ has a section with zero scheme a curve Y, then the disjoint union of Y and a line L is the zero scheme of a section of $\mathscr{E}'(1)$ where $\mathscr{E}'$ is a bundle with spectrum*

$$\chi' = \chi \cup \{0\}.$$

Proof. Immediate from (2.4) and (S4) as

$$h^1(\mathscr{I}_{Y\cup L}(i)) = \begin{cases} h^1(\mathscr{I}_Y(i)), & i < 0 \\ h^1(\mathscr{I}_Y) + 1, & i = 0. \end{cases} \quad \square$$

The following two lemmas give the starting point of our construction.

2.11. Lemma. *Let X be a plane curve of degree $d \geq 1$. Then* (2.2) *applies and the bundle $\mathscr{E}$ obtained from X has spectrum*

$$(\cdots, 0, 1, 2, \ldots, d-1).$$

Proof. $\omega_X \simeq \mathscr{O}_X(d-3)$, hence $h^0(\omega_X(-d+3)) \neq 0$. By (2.5), $h^1(\mathscr{E}(-d)) \neq 0$ and so $d-1$ occurs in the spectrum. As $c_2 = 2d - 1$, (S1) and (S2) now determine the spectrum. $\square$

2.12. Lemma. *Let X* (*as in* 2.9) *be a curve of type* (*a, b*) *on a nonsingular quadric surface Q, with $a \geq b > 0$. The bundle $\mathscr{E}$ obtained from X has spectrum*

$$(\cdots, 0, 1^2, \ldots, (b-1)^2, b^{a-b+1}).$$

Proof. Using the bidegrees for a line bundle on Q, we have

$$0 \longrightarrow \mathscr{O}_Q(-2, -2) \longrightarrow \mathscr{O}_Q(a-2, b-2) \longrightarrow \omega_X \longrightarrow 0$$

hence

$$h^0(\omega_X(-i+3)) = \begin{cases} 0 & \text{if } i > b+1 \\ a-b+1 & \text{if } i = b+1 \\ 2(a-b+2) & \text{if } i = b \text{ and } b > 1. \end{cases}$$

Using (2.5) and (S4), we get that b occurs $a - b + 1$ times in the spectrum and if $b > 1$, $b - 1$ occurs 2 times. Since $c_2 = 2a + 2b - 1$, (S1), (S2) and (S3) uniquely determine the spectrum. $\square$

2.13. Corollary. *Any spectrum of the form*

$(\cdots, 0, 1^a, 2, \ldots, n)$, $a \geq 1$, $n \geq 1$
$(\cdots, 0, 1^a, 2^b, 3, \ldots, n)$, $a, b \geq 2$, $n \geq 2$
$(\cdots, 0, 1^a, 2^b, 3^2, \ldots, (n-1)^2, n^c)$, $a, b \geq 2$, $c \geq 1$, $n \geq 2$

is realizable through a connected curve X *as in* (2.9).

Proof. For the first, start with a plane curve of degree $n+1$, attach $a-1$ lines successively to it and use Lemma 2.7 with $X_2 = \text{line}$ (so $\chi_2 = \{0\}$ and $r=1$). For the second, add $b-1$ chords to the curve of the last sentence and use Lemma 2.7. For the third, start with a curve of type $(c+n-1, n)$ on a quadric Q, attach $a-1$ lines and $b-1$ chords in any order, and use Lemma 2.7. □

2.14. Proposition. *For* $c_2 \leq 19$, *every spectrum satisfying* (S1)–(S3) *in which* 0 *occurs just once, except for* $c_2 = 19$ *and spectrum* $(\cdots, 0, 1^2, 2^2, 3^4, 4)$, *is obtained from a connected union of nonsingular curves which meet quasi-transversally.*

Proof. By listing all possibilities, one checks easily that every such spectrum is obtained in Corollary 2.13 except for the one cited above and the following five, for which we now give constructions. Here P_d means a plane curve of degree d and "$P_d \cup P_{d'}$ meeting in r points" means two plane curves of degrees d, d' in two different planes, meeting quasi-transversally at r points (which must lie on a line). Therefore Lemma 2.7 applies. The five spectra are

$c_2 = 17.$	$(\cdots, 0, 1^2, 2^2, 3^3, 4)$	$: X = P_5 \cup P_4$ meeting in 3 points
	$(\cdots, 0, 1^1, 2^2, 3^2, 4, 5)$	$: X = P_6 \cup P_3$ meeting in 3 points
$c_2 = 19.$	$(\cdots, 0, 1^2, 2^2, 3^3, 4^2)$	$: X = P_5 \cup P_5$ meeting in 3 points
	$(\cdots, 0, 1^2, 2^2, 3^3, 4, 5)$	$: X = P_6 \cup P_4$ meeting in 3 points
	$(\cdots, 0, 1^2, 2^2, 3^2, 4, 5, 6)$	$: X = P_7 \cup P_3$ meeting in 3 points □

2.15. Proposition. *For* $c_2 = 19$, *the spectrum* $(\cdots, 0, 1^2, 2^2, 3^4, 4)$ *is obtained as in* (2.2) *from a double structure* X *on a plane quintic. The* $\mathscr{E}$ *so obtained has a section in* $\mathscr{E}(1)$ *whose zero scheme is a subcanonical quadruple structure on a plane quintic.*

Proof. Let P be a smooth plane quintic, D a divisor of 3 non-collinear points on P. As $h^0(\mathcal{O}_P(D)) = 1$,

$$\mathcal{N}_P \otimes \mathcal{O}_P(D-1) = \mathcal{O}_P(D) \oplus \mathcal{O}_P(4+D)$$

has nowhere vanishing global sections. If X is defined by one such section, we get

$$0 \longrightarrow \mathscr{I}_X \longrightarrow \mathscr{I}_P \longrightarrow \mathcal{O}_P(D-1) \longrightarrow 0. \tag{2.16}$$

We claim that $H^1(\mathcal{N}_X \otimes \omega_X(m)) = 0$ for $m \geq 0$. By Castelnuovo's theorem, this implies that $\mathcal{N}_X \otimes \omega_X(2)$ has a nowhere vanishing section, and since we notice that $H^1(\mathscr{I}_X) = 0$, Ferrand's theorem applies to give a bundle $\mathscr{E}$ as in (2.2). To see the claim, we need $H^0(\mathcal{N}_X^\vee(-m)) = 0$ for $m \geq 0$. Let $\mathscr{M} = \mathcal{N}_{X/P}^\vee$. Since

$$0 \longrightarrow \mathcal{O}_P(D-1) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_P \longrightarrow 0,$$

we get

$$0 \longrightarrow \mathcal{M} \otimes \mathcal{O}_P(D-1) \longrightarrow \mathcal{N}_X^{\vee} \longrightarrow \mathcal{M} \longrightarrow 0.$$

Since $\mathcal{O}_P(D-1)$ has negative degree and P is reduced, it is enough to show that $H^0(\mathcal{M}) = 0$. When (2.16) is restricted to P, we get the exact sequence

$$\mathcal{M} \longrightarrow \mathcal{N}_P^{\vee} \longrightarrow \mathcal{O}_P(D-1) \longrightarrow 0.$$

Since $\omega_{X/P} = \omega_P(1-D)$, $\mathcal{M}$ has determinant equal to $\omega_P^{\vee}(D-5)$. We thus get

$$0 \longrightarrow \mathcal{O}_P(2D-2) \longrightarrow \mathcal{M} \longrightarrow \omega_P^{\vee}(-D-3) \longrightarrow 0.$$

So $H^0(\mathcal{M}) = 0$.

To find the spectrum of the bundle, X has degree 10, hence $c_2 = 19$.

$$h^0(\omega_X(-2)) = h^1(\mathcal{O}_X(1)) = 1 \quad \text{as } h^0(\mathcal{O}_P(1-D)) = 0,$$
$$h_0(\omega_X(-1)) = h^1(\mathcal{O}_X(1)) = 6 \quad \text{as } h^1(\mathcal{O}_P(D)) = 3.$$

Even though X is not reduced, the formulas in Lemma 2.5 still apply as $H^1(\mathcal{I}_X(-i+1) = 0$ for $i \geq 1$. Using (S1) to (S4), we see that we have the required spectrum. □

§3.

In this section we discuss the monads associated to bundles of rank two on $\mathbf{P}^3$ and give some non-existence results. Recall the following notation of Barth [B-1].

Let $\mathcal{E}$ be a rank two bundle on $\mathbf{P}^3$ with $c_1 = 0$. Then we can find a *monad* for $\mathcal{E}$, which is a complex of direct sums of line bundles,

$$0 \longrightarrow \tilde{L}_0^{\vee} \xrightarrow{b} \tilde{L}_1 \xrightarrow{a} \tilde{L}_0 \longrightarrow 0$$

whose homology at the middle is $\mathcal{E}$ and where b and a are respectively injective and surjective as maps of vector bundles. The monad is called a *minimal* monad for $\mathcal{E}$ if a and b are both minimal; ie. as matrices of homogeneous forms, they contain no non-zero scalar entries.

Let $\mathcal{E}$ be a stable bundle of rank two on $\mathbf{P}^3$ with $c_1 = 0$ and given c_2. Let its spectrum be

$$\chi = (-k^{s(k)}, -(k-1)^{s(k-1)}, \ldots, -1^{s(1)}, 0^{s(0)}, 1^{s(1)}, \ldots, k^{s(k)}).$$

Let $M = \bigoplus_{-\infty}^{\infty} H^1(\mathcal{E}(i)) = \bigoplus_{-\infty}^{\infty} M_i$ be the first cohomology module of $\mathcal{E}$. It is a module over $S = k[X_0, X_1, X_2, X_3]$. Let

$$m_i = \dim M_i,$$
$$\rho(i) = \#\text{ of minimal generators for } M \text{ in degree } i.$$

Then (§2, S4),

$$m_{-i-1} = \sum_{j \geq i} s(j)(j - i + 1) \qquad \text{for } i \geq 0,$$

$$m_{-i} - m_{-i-1} = \sum_{j \geq i-1} s(j) \qquad \text{for } i \geq 1,$$

$$\rho(-1-k) = s(k).$$

We also have two formulas in [B-1] for which we give a proof following the characteristic-free arguments of [H-1].

3.1. Proposition (Barth).

$$\rho(-1-k) = s(k),$$

$$s(i) - 2 \sum_{j \geq i+1} s(j) \leq \rho(-1-i) \leq s(i) - 1 \qquad \text{for } 0 \leq i < k.$$

Proof. First assume that the restriction of $\mathscr{E}$ to the general plane (with equation $x = 0$) is stable. So

$$x \colon M_{-\ell-1} \longrightarrow M_{-\ell}$$

is injective for $\ell \geq 0$. Let $N_{-\ell}$ be the quotient, of dimension $n_{-\ell}$. Analyzing the proof in [H-1], Theorem 5.3, we see that $N_{-\ell-1} \otimes H^0(\mathcal{O}_{\mathbf{P}^2}(1))$ has image of dimension $> n_{-\ell-1}$ in $N_{-\ell}$, for $\ell \geq 1$ (conclusion (2) in op cit is valid). Since $\rho(-\ell)$ is also the number of generators for N in degree $-\ell$ (N is the direct sum of the N_i's), we get

$$\rho(-\ell) \leq n_{-\ell} - n_{-\ell-1} - 1 \qquad \text{for } \ell \geq 1.$$

But $n_{-\ell} - n_{-\ell-1} = s(\ell - 1)$ for $\ell \geq 1$.

For the other side of the inequality, $N_{-\ell-1} \otimes H^0(\mathcal{O}_{\mathbf{P}^2}(1))$ has image of dimension $\leq 3n_{-\ell-1}$, hence $N_{-\ell}$ must contain at least $n_{-\ell} - 3n_{-\ell-1}$ minimal generators for N, which works out to the lefthand inequality when $\ell \geq 1$.

If the restriction of $\mathscr{E}$ to a general plane is not stable, $\mathscr{E}$ is well understood. It is the null correlation bundle or a pull back by Frobenius, and $M = S/(X_0^q, X_1^q, X_2^q, X_3^q)$ for some choice of basis for linear forms, with $q =$ a power of the characteristic and with an appropriate grading shift. The spectrum is

$$(\cdots, 0^q, 1^{q-1}, 2^{q-2}, \ldots, q-1)$$

and the inequalities are certainly valid. □

Next we recall a description of minimal monads in [R-1].

3.2. Proposition. *Let $\mathscr{E}$ be a vector bundle of rank two on $\mathbf{P}^3$ with $c_1 = 0$, and let M, its cohomology module, have a minimal free resolution*

$$\cdots \longrightarrow L_2 \longrightarrow L_1 \xrightarrow{a} L_0 \longrightarrow M \longrightarrow 0.$$

Then $L_1 \simeq L_1^\vee$ and there is a summand $L_0^\vee$ of L_2 which induces

$$0 \longrightarrow \tilde{L}_0^\vee \longrightarrow \tilde{L}_1 \xrightarrow{a} \tilde{L}_0 \longrightarrow 0,$$

a minimal monad for $\mathscr{E}$.

Proof. The only statement not proved in [R-1] is that the mapping $L_0^\vee \to L_1$ is obtained from a splitting of L_2. If $G = \ker a$, we need only show that any sequence

$$0 \longrightarrow \tilde{L}_0^\vee \xrightarrow{c} \tilde{G} \longrightarrow \mathscr{E} \longrightarrow 0$$

must pick out part of a system of minimal generators for G. Suppose c picks out elements $s_1, s_2, \ldots, s_t$ of G. If these are not part of a minimal system of generators, we may assume that in $F = G/(s_1, \ldots, s_{t-1})$, $\bar{s}_t$ is not minimal generator. Now $\tilde{F}$ is a vector bundle of rank 3 and if $\bar{s}_t \in H^0(\tilde{F}(n))$, then $c_3(\tilde{F}(n)) = 0$ as $\mathscr{E}$ has rank two.

Expressing $\bar{s}_t$ as $\sum_{i=1}^{m} f_i p_i$ where the f_i's are homogeneous forms of positive degrees, we must have $m \geq 2$, as $\bar{s}_t$ is a nowhere vanishing section. Then $\sum_{i=2}^{m} f_i p_i$ is also a section of $\tilde{F}(n)$, hence it is either nowhere vanishing or it vanishes on a set S of dimension ≥ 1 (since $c_3(\tilde{F}(n)) = 0$). The latter case is impossible as $\bar{s}_t$ would vanish at $S \cap (f_1 = 0)$. Hence $\sum_{i'=2}^{m} f_i p_i$ is nowhere vanishing, and proceed inductively until we reach the contradiction that $f_m p_m$ is nowhere vanishing. □

3.3. Lemma. *Using the above notation, suppose* L_0 *has* r *summands with degrees* $\leq \ell$. *Then* L_1 *must contain at least* $r + 3$ *summands with degrees* $\geq 1 - \ell$.

Proof. $L_0^\vee$ has r summands with degrees $\geq -\ell$. These must embed (as a vector bundle) into the summand of L_1 consisting of terms with degrees $\geq 1 - \ell$ (as the monad is minimal). The quotient of this embedding must clearly have rank 3 or more. □

Barth's formulas give us a method to calculate the summands of L_1 and L_0 with degrees > 0, in the monad. The last lemma allows us to limit the summands of L_0 in degrees ≤ 0.

3.4. Example. Consider the case $c_2 = 8$, $\chi = (-1^3, 0^2, 1^3)$. We have $m_{-2} = 3$, $m_{-1} = 8$, $\rho(-2) = 3$, $0 \leq \rho(-1) \leq 1$.

i) $\rho(-1) = 0$.

The 3 generators in M_{-2} have 4 relations in degree -1, i.e., L_1 contains $4S(1)$ as a summand. By (3.3), L_0 can contain at most 1 summand in degree ≤ 0, in fact in degree $= 0$. Hence there are just 2 possibilities for monads, with the mapping $L_1 \to L_0$ given as

$$4S(-1) \oplus 4S(1) \longrightarrow 3S(2)$$
$$4S(-1) \oplus 2S \oplus 4S(1) \longrightarrow 3S(2) \oplus S.$$

ii) $\rho(-1) = 1$.

There are now 5 minimal relations in degree -1 and so L_0 can have at most 2 summands in degree 0. The possibilities for $L_1 \to L_0$ are

$$5S(-1) \oplus 5S(1) \longrightarrow 3S(2) \oplus S(1)$$
$$5S(-1) \oplus 2S \oplus 5S(1) \longrightarrow 3S(2) \oplus S(1) \oplus S$$
$$5S(-1) \oplus 4S \oplus 5S(1) \longrightarrow 3S(2) \oplus S(1) \oplus 2S.$$

The problem facing us is whether all these five possibilities exist. The following technique is a generalization of Lemma 3.3 and it can be applied to show that some monads do not exist.

3.5. Lemma. *Let* $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{G} \to 0$ *be an exact sequence of vector bundles on* $\mathbf{P}^3$ *where* $\mathscr{A} = \bigoplus_1^r \mathcal{O}_{\mathbf{P}}(a_i)$, $\mathscr{B} = \bigoplus_1^s \mathcal{O}_{\mathbf{P}}(b_i)$, *with* $b_1 \geq b_2 \geq \cdots \geq b_s$. *A general projection gives*

$$0 \longrightarrow \mathscr{A} \longrightarrow \bigoplus_1^{r+1} \mathcal{O}_{\mathbf{P}}(b_i) \longrightarrow \mathscr{I}_C(b-a) \longrightarrow 0$$

where C *is a curve (smooth in characteristic zero),* $a = c_1(\mathscr{A})$ *and* $b = \sum_1^{r+1} b_i$. *There is an induced map* $\bigoplus_{r+2}^s \mathcal{O}_C(-b_i) \to \omega_C(4+a-b) \to 0$. *If* $\mathscr{K}$ *is the kernel of this map and if* $n < b - a$ *and* $h^0(\mathscr{I}_C(n-b_s)) = 0$, *then* $h^0(\mathscr{G}^\vee(n)) = h^0(\mathscr{K}'(n))$.

Proof. A Bertini theorem gives us the curve C. An alternative resolution for $\mathscr{I}_C(b-a)$ is

$$0 \longrightarrow \bigoplus_{r+2}^s \mathcal{O}_{\mathbf{P}}(b_i) \longrightarrow \mathscr{G} \longrightarrow \mathscr{I}_C(b-a) \longrightarrow 0.$$

We get the diagram

$$\begin{array}{ccccccccc} & & & & \bigoplus_{r+2}^s \mathscr{I}_C(-b_i) & & & & \\ & & & & \downarrow & & & & \\ 0 \longrightarrow & \mathcal{O}_{\mathbf{P}}(a-b) & \longrightarrow & \mathscr{G}^\vee \longrightarrow & \bigoplus_{r+2}^s \mathcal{O}_{\mathbf{P}}(-b_i) & \longrightarrow & \omega_C(4+a-b) & \longrightarrow & 0 \\ & & & & \downarrow & & \| & & \\ & & 0 \longrightarrow & \mathscr{K} \longrightarrow & \bigoplus_{r+2}^s \mathcal{O}_C(-b_i) & \longrightarrow & \omega_C(4+a-b) & \longrightarrow & 0 \end{array}$$

from which the result can be obtained. □

As an application, we have

3.6. Corollary. The monad with $L_1 \to L_0$ given as

$$5S(-1) \oplus 4S \oplus 5S(1) \longrightarrow 3S(2) \oplus S(1) \oplus 2S$$

does not exist.

Proof. From the mapping $L_0^\vee \to L_1$, we obtain an exact sequence of vector bundles

$$0 \longrightarrow 2\mathcal{O}_{\mathbf{P}} \longrightarrow 5\mathcal{O}_{\mathbf{P}}(1) \longrightarrow \mathcal{G} \longrightarrow 0.$$

Using Lemma 3.5, we get C a smooth twisted cubic curve (in any characteristic), and

$$0 \longrightarrow \mathcal{K} \longrightarrow 2\mathcal{O}_C(-1) \longrightarrow \omega_C(1) \longrightarrow 0.$$

So $\mathcal{K} = \omega_C^\vee(-3)$, and $H^0(\mathcal{K}(2)) = 0$. Hence $H^0(\mathcal{G}^\vee(2)) = 0$. This says that the 5S(1) in L_1 maps to zero, contradicting the minimality of the monad. □

The following lemma is also useful.

3.7. Lemma. *Let Q be a rank r subsheaf of $\mathcal{O}_{\mathbf{P}}(a_1) \oplus \mathcal{O}_{\mathbf{P}}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}}(a_s)$ where $a_1 \geq a_2 \geq \cdots \geq a_s$. Then $h^0(Q) \leq h^0(\mathcal{O}_{\mathbf{P}}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}}(a_r))$.*

Proof. This is immediate since under a generic projection $\bigoplus_{i=1}^{s} \mathcal{O}_{\mathbf{P}}(a_i) \to \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbf{P}}(a_i)$, Q injects into $\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbf{P}}(a_i)$. □

3.8. Corollary. *There is no stable bundle $\mathcal{E}$ whose monad has the part $L_1 \to L_0$ of the form*

$$5S(-1) \oplus 5S(1) \longrightarrow 3S(2) \oplus S(1).$$

Proof. Since the monad is minimal, the summand 5S(1) maps to 3S(2). Call the mapping ϕ. ϕ has rank 1, 2 or 3.

If ϕ has rank 1 (respectively 2), the subsheaf $\mathrm{Im}(\phi)$ of $3\mathcal{O}_{\mathbf{P}}(2)$ has $h^0(\mathrm{Im}(\phi)(1)) \leq 20$ (respectively 40) and hence the sheaf $\mathrm{Ker}(\phi)$ has $h^0(\mathrm{Ker}(\phi)(1)) \geq 30$ (respectively 10). But then $h^0(\mathcal{E}(1)) \geq 29$ (respectively 9) and $\mathcal{E}$ cannot be stable. In fact, if $h^0(\mathcal{E}(1)) \geq 9$, then a general section of $\mathcal{E}(1)$ gives a curve Y of degree 9 and at least 8 linearly independent quadrics containing Y. Since two of the 8 quadrics must intersect properly, this implies that Y has degree at most 4, a contradiction.

In the third case where ϕ has rank 3, let $\mathcal{G} = \mathrm{coker}(\mathcal{O}(-1) \to 5\mathcal{O}(1))$ in the monad. The determinantal sequence

$$\mathcal{O}_{\mathbf{P}} \xrightarrow{\wedge^3 \bar{\phi}} \mathcal{G} \xrightarrow{\bar{\phi}} 3\mathcal{O}(2)$$

gives a nonzero section of $\mathscr{E}$. □

3.9. Remark. We do not know if there is a semistable bundle with the monad of (3.8).

§ 4.

In the last section we gave some techniques for eliminating the existence of some monad types. This section gives some techniques for constructing bundles with various monad types.

4.1. Notation. A monad

$$0 \longrightarrow \tilde{L}_0^\vee \longrightarrow \tilde{L}_1 \longrightarrow \tilde{L}_0 \longrightarrow 0$$

will be denoted by integer sequences ℓ_+ and ℓ_0 where ℓ_0 enumerates the degrees of the summands of L_0 and ℓ_+ enumerates the degrees of the summands of L_1 which are nonnegative.

4.2. Example. For stable bundles with $c_2 = 4$ and $\chi = (-1, 0^2, 1)$, there are two possible monad types which may exist, viz

ℓ_+	ℓ_0
0^4	2
0^4, 1	2, 1

It is easy to see that the second is obtained when we use Lemma 2.8 and add a line to the curve for the bundle with $\chi = (-1, 0, 1)$. It is also easy to see that the constructions of § 2 will never produce the first monad type, since for such a bundle $H^0(\mathscr{E}(1)) = 0$.

Hence we need some new construction techniques for bundles and monads.

4.3. Lemma. *Let X be a connected nonplanar curve of degree d satisfying the conditions* (1) *or* (2) *of Proposition* 2.8. *Then the general section of $\mathscr{N}_X \otimes \omega_X$ gives a double structure Y on X which is the zero scheme of a section of $\mathscr{E}(2)$, where $\mathscr{E}$ is a stable rank two bundle with $c_1 = 0$, $c_2 = 2d - 4$.*

Proof. We showed in Proposition 2.8 and its proof that $\mathscr{N}_X \otimes \omega_X$ is generated by its global sections at the smooth points of X and that the general section of $\mathscr{N}_X \otimes \omega_X$ is nowhere vanishing. This yields a sequence

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathscr{I}_X \longrightarrow \omega_X \longrightarrow 0 \tag{4.4}$$

and since $H^1(\mathscr{I}_X) = 0$ (as X is connected), Ferrand's theorem applies to give $\omega_Y \simeq \mathscr{O}_Y$. We thus get $\mathscr{E}$ with $c_1 = 0$, $c_2 = 2d - 4$. Since

$$0 \longrightarrow \mathscr{O}_{\mathbf{P}}(-2) \longrightarrow \mathscr{E} \longrightarrow \mathscr{I}_Y(2) \longrightarrow 0 \tag{4.5}$$

and since X and hence Y is nonplanar, $H^0(\mathscr{E}(-1)) = 0$. If $\mathscr{E}$ is not stable, we must have $h^0(\mathscr{E}) = 1$, or Y is contained in a quadric surface Q. But then X lies on the quadric surface which therefore must be reduced. If Q is a cone, then Y being $2X$ on Q is a complete intersection, or $\mathscr{E}$ is decomposable. But this cannot happen as $H^1(\mathscr{E}(-1)) = H^1(\mathscr{I}_Y(1))$ contains $H^0(\omega_X(1))$ which is nonzero. If Q is a smooth quadric, X has some type (a, b) on Q and then Y has type $(2a, 2b)$, hence $\omega_Y \simeq \mathscr{O}_Y(2a-2, 2b-2)$. But since $\omega_Y \simeq \mathscr{O}_Y$, we get $a = b = 1$, or X is planar. Finally, suppose $Q = H \cup H'$, the union of 2 planes. If X lies on more than one quadric, let us assume that H is a fixed component of these quadrics. Q induces a mapping $\mathscr{N}_Y \otimes \omega_Y \to \omega_Y(2)$. We may choose a section of $\mathscr{N}_Y \otimes \omega_Y$ which does not lie in the kernel of this map at least one smooth point of X on H. The corresponding Y will not be locally contained in H at that point. Hence this Y will not lie on any quadric. Lastly if the quadrics containing X have no fixed plane, X is the union of 2 plane curves and X has degree ≤ 4. It is easy to see that condition (2) of (2.8) is violated. □

4.6. Corollary. *If X is a nonplanar smooth curve of type (a, b) on a smooth quadric with $a \geq b > 0$, then the bundle $\mathscr{E}$ obtained in Lemma 4.3 has spectrum*

$$(\cdots, 0^2, 1^2, 2^2, \ldots, (b-2)^2, (b-1)^{a-b+1})$$

Proof. Similar to the proof of Lemma 2.12. □

4.7. Example. When $c_2 = 8$ and $\chi = (-1^3, 0^2, 1^3)$, we had 3 monad types which were not ruled out, viz.

ℓ_+	ℓ_0
1^4	2^3
$0^2, 1^4$	$2^3, 0$
$0^2, 1^5$	$2^3, 1, 0.$

The first is obtained if we apply Lemma 4.3 to a curve X of degree 6 and genus 3 which is linked to a twisted cubic curve by 2 cubic surfaces. The second is obtained if X is chosen instead as a curve of type (4, 2) on a quadric surface. The differences in the monads arise from the sequence

$$H^0(\omega_X(2)) \longrightarrow H^1(\mathscr{E}) \longrightarrow H^1(\mathscr{I}_X(2)) \longrightarrow 0$$

obtained from (4.4) and (4.5). For the second curve, $h^1(\mathscr{I}_X(2)) = 1$ and this gives a new generator for M in degree 0. The third monad is obtained when we construct the spectrum $(-1^3, 0^2, 1^3)$ as described in §2. The reason for its monad type follows from the following lemma.

4.8. Lemma. *Let $\mathscr{E}$ be a rank two bundle on $\mathbf{P}^3$ with $c_1 = 0$, $c_2 = n$ and let its cohomology module M have minimal free presentation*

$$L_1 \longrightarrow L_0 \longrightarrow M \longrightarrow 0.$$

If $Y = (s)_0$ is a curve for $s \in H^0(\mathscr{E}(r))$, $r > 0$ and X is the complete intersection of

2 surfaces of degrees a and b, where $a + b = 2r$ and $Y \cap X = \emptyset$, then $Y \cup X = (s')_0$ where $s' \in H^0(\mathscr{E}'(r))$ for some $\mathscr{E}'$ with $c_1 = 0$, $c_2 = n + \deg(X)$. The cohomology module M' of $\mathscr{E}'$ has minimal presentation

$$S(r-a) \oplus S(r-b) \oplus L_1 \longrightarrow S(r) \oplus L_0 \longrightarrow M' \longrightarrow 0.$$

$\mathscr{E}'$ is stable if $\mathscr{E}$ is semistable and X is sufficiently general.

Proof. Let $Y' = Y \cup X$. $\mathscr{E}'$ and s' exist as $\omega_{Y'} \simeq \mathcal{O}_{Y'}(2r-4)$. If $\mathscr{E}'(e)$ with $e \leq 0$ has a section, Y' lies on a surface of degree $r + e$, hence so does Y. Hence $\mathscr{E}(e)$ has a section. If $\mathscr{E}$ is semistable, we have $e = 0$ and the surface containing Y is the unique one of that degree. Choose X not lying on this surface. We get a stable $\mathscr{E}'$.

The monad for $\mathscr{E}$ is

$$0 \longrightarrow \tilde{L}_0^\vee \longrightarrow \tilde{L}_1 \longrightarrow \tilde{L}_0 \longrightarrow 0.$$

The section $\mathcal{O}_{\mathbf{P}}(-r) \to \mathscr{E}$ which gives Y induces an exact sequence

$$0 \longrightarrow \tilde{L}_0^\vee \oplus \mathcal{O}_{\mathbf{P}}(-r) \longrightarrow \tilde{L}_1 \longrightarrow \tilde{L}_0 \oplus \mathcal{O}_{\mathbf{P}}(r) \longrightarrow \mathcal{O}_Y(r) \longrightarrow 0$$

which gives a minimal resolution for $H^0_*(\mathcal{O}_Y(r))$. [R-1]. Consider the sequence

$$0 \longrightarrow \mathscr{I}_{Y'} \longrightarrow \mathscr{I}_X \stackrel{a}{\longrightarrow} \mathcal{O}_Y \longrightarrow 0$$

where a is the natural map

$$\mathscr{I}_X \longrightarrow \mathcal{O}_{\mathbf{P}} \longrightarrow \mathcal{O}_Y.$$

It follows that the induced map $I(X) \to H^0_*(\mathcal{O}_Y)$ does not map onto any minimal generator of $H^0_*(\mathcal{O}_Y)$. Hence the minimal presentation of $H^1_*(\mathscr{I}_{Y'})$ can be obtained from the mapping cone on the Koszul resolution of $I(X)$ and the above resolution of $H^0_*(\mathcal{O}_Y)$. The result follows as $H^1_*(\mathscr{I}_{Y'}) = M'(-r)$. □

4.9. Example. The lemma allows us to build up monads for higher Chern classes. When $c_2 = 5$ and $\chi = (-1, 0^3, 1)$, the 2 possible monad types are

ℓ_+	ℓ_0
0^6	2, 1
0^6, 1	2, 1^2.

The lemma tells us that the second is obtained by the process of §2, viz construct $\chi = (-1, 0, 1)$ and add 2 lines, one after the other. To obtain the first monad, observe that we can pick out $2\mathcal{O}_{\mathbf{P}} \to \mathcal{O}_{\mathbf{P}}(2)$ in $L_1 \to L_0$, which indicates an elliptic quartic. Lemma 4.8 asks for a bundle with $c_2 = 1$, and monad $\ell_+ = 0^4$, $\ell_0 = 1$. Such a bundle certainly exists, being the null-correlation bundle, and for such a bundle $\mathscr{E}$, $\mathscr{E}(2)$ has sections s giving a curve Y. The lemma gives our monad type from this Y and a general elliptic quartic curve.

§5.

In this section, we will tabulate all the existing minimal monads for stable rank two bundles with $c_1 = 0$, $c_2 \leq 8$.

5.1. Notation. The columns of the table are interpreted as follows:

c_2 : the second Chern class of a stable bundle $\mathscr{E}$ with $c_1 = 0$

χ : the spectrum of $\mathscr{E}$

ℓ_+: the nonnegative degrees of the middle term L_1 of a minimal monad for $\mathscr{E}$. Superscripts indicate repetitions of a summand.

ℓ_0 : the degrees of the right-hand term L_0 of the minimal monad for $\mathscr{E}$

r : degrees in which a global section exists for our constructed $\mathscr{E}$, giving a curve Y as zero scheme. When we indicate ≤ 2, we may have to construct two distinct bundles $\mathscr{E}$, one with a section for $\mathscr{E}(1)$ and the other with a section for $\mathscr{E}(2)$ in order to allow later constructions. The constructions are indicated in the last column.

Construction: There are 4 categories of constructions.

Instanton–a bundle with spectrum $\{0^{c_2}\}$. One such bundle is constructed most easily by picking $c_2 + 1$ skew lines on a quadric surface. Y is subcanonical and hence equals $(s)_0$, $s \in H^0(\mathscr{E}(1))$ for a bundle $\mathscr{E}$. As Y lies on a quadric, $h^0(\mathscr{E}(1)) \geq 2$. Observe therefore that $\mathscr{E}(2)$ also has a section whose zero scheme is a curve.

a) This is the use of Corollary 2.7. We describe the curve X from which $\mathscr{E}$ is obtained (via a double structure given by $\mathscr{I}_X \to \omega_X(2) \to 0$).

b) This is the use of Lemma 4.4. We describe the curve X from which $\mathscr{E}$ is obtained (via a double structure given by $\mathscr{I}_X \to \omega_X \to 0$).

c) This is the use of Lemma 4.8. We indicate the monad of the bundle with smaller Chern class (by referring to its table entry) and also describe the complete intersection curve X. The twist r is in the previous column.

5.2. P_d denotes a smooth plane curve of degree d.

$C_{a,b}$ denotes a smooth curve of type (a, b) on a nonsingular quadric.

Note that if $X = P_d$ and we apply construction (a) the curve Y lies on the doubled plane. Hence $\mathscr{E}$ has at least 2 sections in degree 1 giving curves and therefore has a section in degree 2 giving a curve.

5.3. Table

c_2		χ	ℓ_+	ℓ_0	r	construction
1	(1)	0	0^4	1	≤ 2	instanton
2	(1)	0^2	0^6	1^2	≤ 2	instanton
3	(1)	0^3	0^8	1^3	≤ 2	instanton
	(2)	$\cdots 0, 1$	$0^2, 1$	2	≤ 2	(a): P_2

c_2		χ	ℓ_+	ℓ_0	r	construction
4	(1)	0^4	0^{10}	1^4	≤ 2	instanton
	(2)	$\cdots 0^2, 1$	i) 0^4	2	2	(b): $C_{2,2}$
			ii) $0^4, 1$	$2, 1$	≤ 2	(c): 3(2), P_1 or 1(1), P_3
5	(1)	0^5	0^{12}	1^5	≤ 2	instanton
	(2)	$\cdots 0^3, 1$	i) 0^6	$2, 1$	2	(c): 1(1), $C_{2,2}$
			ii) $0^6, 1$	$2, 1^2$	≤ 2	(c): 4(2, ii), P_1 or 2(1), P_3
	(3)	$\cdots 0, 1^2$	1^3	2^2	1	(a): $C_{2,1}$
	(4)	$\cdots 0, 1, 2$	$0^2, 2$	3	≤ 2	(a): P_3
6	(1)	0^6	0^{14}	1^6	≤ 2	instanton
	(2)	$\cdots 0^4, 1$	i) 0^8	$2, 1^2$	2	(c): 2(1), $C_{2,2}$
			ii) $0^8, 1$	$2, 1^3$	1	(c): 5(2, ii), P_1
	(3)	$\cdots 0^2, 1^2$	i) $0^2, 1^2$	2^2	2	(b): $C_{3,2}$
			ii) $0^2, 1^3$	$2^2, 1$	1	(c): 5(3), P_1
	(4)	$\cdots 0^2, 1, 2$	$0^4, 2$	$3, 1$	1	(c): 5(4), P_1
7	(1)	0^7	0^{16}	1^7	≤ 2	instanton
	(2)	$\cdots 0^5, 1$	i) 0^{10}	$2, 1^3$	2	(c): 3(1), $C_{2,2}$
			ii) $0^{10}, 1$	$2, 1^4$	1	(c): 6(2, ii), P_1
	(3)	$\cdots 0^3, 1^2$	i) $0^4, 1$	2^2	2	(c): 4(2, i), P_3
			ii) $0^4, 1^2$	$2^2, 1$	2	(c): 4(2, ii), P_3
			iii) $0^4, 1^3$	$2^2, 1^2$	1	(c): 6(3, ii), P_1
	(4)	$\cdots 0, 1^3$	1^5	$2^3, 0$	1	(a): $C_{3,1}$
	(5)	$\cdots 0^3, 1, 2$	$0^6, 2$	$3, 1^2$	1	(c): 6(4), P_1
	(6)	$\cdots 0, 1^2, 2$	i) 1^2	3	1	(a): $C_{2,2}$
			ii) $1^2, 2$	$3, 2$	1	(a): $P_3 \cup P_1$ joined at a point
	(7)	$\cdots 0, 1, 2, 3$	$0^2, 3$	4	≤ 2	(a): P_4
8	(1)	0^8	0^{18}	1^8	≤ 2	instanton
	(2)	$\cdots 0^6, 1$	i) 0^{12}	$2, 1^4$	2	(c): 4(1), $C_{2,2}$
			ii) $0^{12}, 1$	$2, 1^5$	1	(c): 7(2, ii), P_1
	(3)	$\cdots 0^4, 1^2$	i) 0^6	2^2	2	(c): 4(2, i), $C_{2,2}$
			ii) $0^6, 1$	$2^2, 1$	2	(c): 5(2, i), P_3
			iii) $0^6, 1^2$	$2^2, 1^2$	2	(c): 5(2, ii), P_3
			iv) $0^6, 1^3$	$2^2, 1^3$	1	(c): 7(3, iii); P_1
	(4)	$\cdots 0^2, 1^3$	i) 1^4	2^3	2	(b): X of $d = 6$, $g = 3$, $\not\subset$ quadric
			ii) $0^2, 1^4$	$2^3, 0$	2	(b): $C_{4,2}$
			iii) $0^2, 1^5$	$2^3, 1, 0$	1	(c): 7(4), P_1
	(5)	$\cdots 0^4, 1, 2$	$0^8, 2$	$3, 1^3$	1	(c): 7(5), P_1
	(6)	$\cdots 0^2, 1^2, 2$	i) $0^2, 1$	3	2	(b): $C_{3,3}$
			ii) $0^2, 1^2$	$3, 1$	1	(c): 7(6, i), P_1
			iii) $0^2, 1, 2$	$3, 2$	2	(b): $P_4 \cup P_2$, joined at 2 points
			iv) $0^2, 1^2, 2$	$3, 2, 1$	1	(c): 7(6, ii), P_1
	(7)	$\cdots 0^2, 1, 2, 3$	$0^4, 3$	$4, 1$	1	(c): 7(7), P_1

§6.

We would like to make some observations. It is known that the family of instanton bundles for $c_2 = 1, 2, 3, 4$ is irreducible. [H-3, E-S, B-2]. It is known that any family of generalized null-correlation bundles is an open, irreducible and smooth set in a component of the moduli space [E-2]. Examples in the table are 3(2), 4(2, i), 5(4) etc. Mei-Chu Chang has shown ([C]) that the family 4(2, ii) is in the closure of 4(2, i). In [R-2], it is shown that the family 5(3) is irreducible and lies in the closure of 5(1). A natural question is to ask if the result in [C] generalizes:

(Q1): If a spectrum is fixed, is there a "smallest" monad type and are the larger monad types specializations of the smallest? (Consider for example 8(6) in the table.)

Our table of the last section updates Barth's tables in [B-1]. He had made the assumption that the cohomology module M has minimal generators only in negative degrees (or ℓ_0 has only positive entries). We see that this is a valid assumption for $c_2 \leq 6$, but for $c_2 = 7$ and 8, we obtain generators in degree 0. (See 7(4) and 8(4).) In fact, as c_2 grows, M can obtain generators in arbitrarily large degrees, as we see from

6.1. Proposition. *Let X be a smooth curve of type $(a, a+t)$, $a \geq 1$, $t \geq 2$ on a nonsingular quadric surface Q. The bundle $\mathscr{E}$ obtained from X as in* (2.2) *has cohomology module M with $t-1$ minimal generators in degree $a-1$.*

Proof. Since

$$0 \longrightarrow \mathcal{O}_Q(-a, -a-t) \longrightarrow \mathcal{O}_Q \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

it is clear that the first value of n for which $H^1(\mathscr{I}_X(n)) \neq 0$ is $n = a$, and

$$h^1(\mathscr{I}_X(a)) = h^1(\mathcal{O}_Q(0, -t)) = t - 1.$$

Using (2.3) and (2.4), we get

$$H^0(\omega_X(2+a)) \longrightarrow H^1(\mathscr{E}(a-1)) \longrightarrow H^1(\mathscr{I}_X(a)) \longrightarrow 0,$$

hence the result. $\square$

The case when $a = 1$ and $t = 2$ of the proposition, i.e., X of type (1, 3) on Q, gives the first such example of the table, where $c_2 = 7$, $\chi = (-1^3, 0, 1^3)$. Let $\mathscr{N}$ be the family of all such bundles obtained from smooth curves X of type (1, 3) on nonsingular quadrics, via construction (a). We will quickly sketch a proof that $\mathscr{N}$ lies in the closure of the family of bundles with spectrum $(-1, 0^5, 1)$ (for proofs, [R-2] contains similar arguments).

Analyze the monad 7(4) of the table to find that if $\mathscr{E}$ has such a monad then $h^0(\mathscr{E}(1)) = 1$ and the corresponding curve Y is a double structure obtained as in (2.2) from a curve X of type (1, 3) on a quadric or a degeneration thereof. Hence $\mathcal{N}$ is open in the family of all bundles with this spectrum, and the dimension of $\mathcal{N}$ is computed to be 50. Since this is too small to constitute a component of the moduli space of bundles with $c_2 = 7$ (as $8c_2 - 3 = 53$), $\mathcal{N}$ is in the closure of a larger family. Considering the invariance of the α-invariant and the upper semi-continuity of cohomology, the only candidate for the larger family is the family of bundles with spectrum $(-1, 0^5, 1)$.

We conclude with some questions about the spectrum:

(Q2): Are the 3 conditions (S1)–(S3) sufficient for a stable bundle to exist with the spectrum χ?
(Q3): For every spectrum of an existing stable bundle, can we find a representative bundle $\mathscr{E}$ with that spectrum for which $H^0(\mathscr{E}(1)) \neq 0$?

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