Some remarks on the multi-dimensional Borg-Levinson theorem

Dedicated to Professor Teruo Ikebe on his sixtieth birthday

By

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§1. Introduction and summary

1.1. The Borg-Levinson theorem concerns the uniqueness in inverse eigenvalue problems. We first recall the 1-dimensional case. Consider the eigenvalue problem

(1.1)
$$\begin{cases} -y'' + q(x)y = \lambda y, & 0 \le x \le 1, \\ y(0) = y(1) = 0, \end{cases}$$

where $q(x)$ is a real function. Let

$$\lambda_1(q) < \lambda_2(q) < \cdots$$

be the eigenvalues. As can be seen easily, even if

$$\lambda_i(q_1) = \lambda_i(q_2) \qquad \text{for all } i \ge 1,$$

for two potentials q_1 and q_2, it does not necessarily imply $q_1 = q_2$. Thus to derive the uniqueness of potentials having the same eigenvalues, one must add some auxiliary condition. Let $y = y(x, \lambda, q)$ be the solution of the initial value problem

(1.2)
$$\begin{cases} -y'' + q(x)y = \lambda y, & 0 \le x \le 1, \\ y(0, \lambda, q) = 0, & y'(0, \lambda, q) = 1. \end{cases}$$

Then we have ([1], [6])

Theorem (*Borg-Levinson*). *Suppose that*

$$\lambda_i(q_1) = \lambda_i(q_2) \qquad \text{for all } i \ge 1,$$

$$y'(1, \lambda_i, q_1) = y'(1, \lambda_i, q_2) \qquad \text{for all } i \ge 1,$$

where $\lambda_i = \lambda_i(q_1) = \lambda_i(q_2)$. *Then* $q_1 = q_2$.

This is a starting point of one-dimensional isospectral theories. The recent article of Pöschel-Trubowitz [9] gives a deep insight to this problem. It is proved

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that the map

$$q \longmapsto \{\lambda_i(q)\}_{i=1}^{\infty} \times \{\log|y'(1, \lambda_i, q)|\}_{i=1}^{\infty}$$

is a real analytic isomorphism from $L^2(0, 1)$ to a Hilbert space of infinite sequences ([9], p. 116), and that for any potential p

$$M(p) = \{q; \lambda_i(p) = \lambda_i(q) \text{ for all } i \geq 1\}$$

is a real analytic manifold (isospectral manifold) with the system of coordinates

$$\{\log|y'(1, \lambda_i, q)|\}_{i=1}^{\infty}$$

([9], p. 71).

Since $y(x, \lambda_i, q)$ is an eigenfunction of $-\dfrac{d^2}{dx^2} + q(x)$ with eigenvalue λ_i, this result shows that there is a one-to-one correspondence between the potential and the pair of all eigenvalues and the normal derivatives of eigenfunctions.

1.2. We turn to the n-dimensional case $(n \geq 2)$. Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary S. Consider the Dirichlet problem

$$(1.3) \qquad \begin{cases} (-\Delta + q)u = \lambda u & \text{in } \Omega, \\ u|_S = 0. \end{cases}$$

Although we consider the Dirichlet problem, all the results presented below holds for the Neumann or Robin boundary conditions by a suitable modification.

Let $\lambda_1 < \lambda_2 \leq \cdots$ be the eigenvalues associated with (1.3). To derive the uniqueness theorem corresponding to the 1-dimensional case, we consider the normal derivatives of eigenfunctions. However, we must be careful to choose a system of eigenfunctions, since in the multi-dimensional case eigenvalues are not simple in general.

Let m be the multiplicity of λ_i, and $u_1, \ldots, u_m$ be a real-valued orthonormal eigenfunctions associated with λ_i. We set

$$E_i = \left\{ \left(\frac{\partial u_1}{\partial v}, \ldots, \frac{\partial u_m}{\partial v} \right) \bigg|_S \right\},$$

v being the outer unit normal to S. One can then see that for two such system of eigenfunctions $\{u_1, \ldots, u_m\}$, $\{v_1, \ldots, v_m\}$, there exits an orthogonal matrix T such that

$$(1.4) \qquad \left(\frac{\partial u_1}{\partial v}, \ldots, \frac{\partial u_m}{\partial v} \right) = \left(\frac{\partial v_1}{\partial v}, \ldots, \frac{\partial v_m}{\partial v} \right) T \quad \text{on } S.$$

Now (1.4) defines an equivalent relation $\sim$ in the space of the functions on the boundary S. Further, (1.4) shows that for the set $\{E_i\}$, i being fixed, there corresponds only one equivalence class, which we denote by W_i, namely,

$$W_i = \{E_i\}/\sim.$$

The following theorem generalizes the result of Borg-Levinson to the multi-dimensional case.

Theorem A (*Nachman-Sylvester-Uhlmann*). *Let q_1, q_2 be real functions* $\in C^\infty(\bar{\Omega})$. *Suppose that*

$$\lambda_i(q_1) = \lambda_i(q_2) \quad \text{for all } i \geq 1,$$
$$W_i(q_1) = W_i(q_2) \quad \text{for all } i \geq 1.$$

Then $q_1 = q_2$.

For the proof, see [8], and also Ramm [10]. Suzuki [12] recently obtained an interesting generalization. Now, the one-dimensional results lead us to the following questions. Is the map

$$q \longmapsto \{\lambda_i\}_{i=1}^\infty \times \{W_i\}_{i=1}^\infty$$

a (local) isomorphism? Can $\{W_i\}_{i=1}^\infty$ be the coordinates of isospectral set of potentials? The answers are always negative. In fact, we have

Theorem B. *Let* $q_1, q_2 \in C^\infty(\bar{\Omega})$ *be real-valued. Suppose that there exists an* $N > 0$ *such that*

$$\lambda_i(q_1) = \lambda_i(q_2) \quad \text{for all } i \geq N,$$
$$W_i(q_1) = W_i(q_2) \quad \text{for all } i \geq N.$$

Then $q_1 = q_2$.

In other words, $q_1 = q_2$ provided $\lambda_i(q_1) = \lambda_i(q_2)$, $W_i(q_1) = W_i(q_2)$ except for a finite number of indices i. The above theorem means that the totality of λ_i and W_i is too much to determine the potential. One can further see that the potential $q(x)$ is uniquely reconstructed from the asymptotic properties of the eigenvalues and the eigenfunctions (Theorem 2.3). It is a common belief that, contrary to the 1-dimensional case, the multi-dimensional eigenvalue problem has a sort of rigidity (see e.g. [2], [3]). We can find one of its examples here.

1.3. The proof of Theorem B is given in §2. The essntial point of the proof is to introduce a function similar to the scattering matrix in scattering theory (Lemma 2.2). By using the idea of Born approximation in scattering theory, one can reconstruct the potential from the Neumann operator (Theorem 2.3), from which Theorem B easily follows. In §3, these results are extended to the operators of variable coefficients by introducing asymptotic solutions and Fourier integral operators, although rather strong restrictions are imposed on the coefficients. Needless to say, many problems are left open in this field of multi-dimensional inverse spectral theory. Among them, perhaps, the most interesting problem is to find a set of data related with the spectrum which has a one-to-one correspondence between the potential. At the present stage, however, the complete solution is beyond of our scope.

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§2. Proof of Theorem B

Let Δ_D be the Laplacian in Ω with the Dirichlet boundary condition. For a bounded real function $q(x)$, let $N(\lambda, q)$ be the Neumann operator:

$$(2.1) \qquad\qquad N(\lambda, q)f = \frac{\partial v}{\partial v}\Big|_S,$$

where v is the solution of the Dirichlet problem

$$\begin{cases} (-\Delta + q)v = \lambda v & \text{in } \Omega, \\ v|_S = f. \end{cases}$$

We always assume $\lambda \neq$ the eigenvalues of $-\Delta_D + q$.

We introduce the following notation

$$(2.2) \qquad\qquad (f, g) = \int_\Omega f(x)\,\overline{g(x)}\,dx,$$

$$(2.3) \qquad\qquad \langle f, g \rangle = \int_S f(x)\,\overline{g(x)}\,dS_x,$$

$$(2.4) \qquad \varphi_{\lambda,\omega}(x) = e^{i\sqrt{\lambda}\omega\cdot x}, \quad \lambda \in \mathbf{C} - (-\infty, 0), \quad \omega \in S^{n-1}.$$

Definition 2.1. Let the function $S(\lambda, \theta, \omega; q)$ be defined by

$$S(\lambda, \theta, \omega; q) = \langle N(\lambda, q)\varphi_{\lambda,\omega}, \overline{\varphi_{\lambda,-\theta}} \rangle.$$

The crucial fact is the following lemma.

Lemma 2.2.

$$S(\lambda, \theta, \omega; q) = -\frac{\lambda}{2}(\theta - \omega)^2 \int_\Omega e^{-i\sqrt{\lambda}(\theta-\omega)x}\,dx$$
$$+ \int_\Omega e^{-i\sqrt{\lambda}(\theta-\omega)x}q(x)\,dx - (R(\lambda)q\varphi_{\lambda,\omega}, \overline{q\varphi_{\lambda,-\theta}}),$$

where $R(\lambda) = (-\Delta_D + q - \lambda)^{-1}$.

Proof. Recall the Green's formula

$$(2.5) \qquad \int_\Omega (\Delta u \cdot v - u \cdot \Delta v)\,dx = \int_S \left(\frac{\partial u}{\partial v}v - u\frac{\partial v}{\partial v} \right)dS.$$

Let $\psi(x, \lambda, \omega)$ be defined by

$$\psi(x, \lambda, \omega) = \varphi_{\lambda,\omega}(x) - R(\lambda)(q\varphi_{\lambda,\omega})(x),$$

which is the solution to the Dirichlet problem

$$\begin{cases} (-\Delta + q)\psi = \lambda\psi & \text{in } \Omega, \\ \psi|_S = \varphi_{\lambda,\omega}. \end{cases}$$

We have by Definition 2.1

$$S(\lambda, \theta, \omega; q) = \int_S \varphi_{\lambda,-\theta}(x) \frac{\partial}{\partial\nu} \psi(x, \lambda, \omega)\, dS_x.$$

Let $u = \psi$ and $v = \varphi_{\lambda,-\theta}$ in (2.5). Then

$$\int_\Omega q(x)\psi(x, \lambda, \omega)\varphi_{\lambda,-\theta}(x)\, dx = \int_S \left(\frac{\partial\psi}{\partial\nu}\varphi_{\lambda,-\theta} - \psi\frac{\partial}{\partial\nu}\varphi_{\lambda,-\theta} \right) dS_x.$$

We have, therefore,

$$S(\lambda, \theta, \omega; q) = -i\sqrt{\lambda}\int_S \theta\cdot\nu e^{-i\sqrt{\lambda}(\theta-\omega)x}\, dS_x$$

$$+ \int_\Omega e^{-i\sqrt{\lambda}(\theta-\omega)x} q(x)\, dx - (R(\lambda)q\varphi_{\lambda,\omega}, \overline{q\varphi_{\lambda,-\theta}}).$$

Taking $u = e^{-i\sqrt{\lambda}(\theta-\omega)x}$ and $v = 1$ in (2.5), we have,

$$-\lambda(\theta-\omega)^2 \int_\Omega e^{-i\sqrt{\lambda}(\theta-\omega)x}\, dx = -i\sqrt{\lambda}\int_S (\theta-\omega)\cdot\nu e^{-i\sqrt{\lambda}(\theta-\omega)x}\, dS_x.$$

Letting $u = e^{-i\sqrt{\lambda}\theta x}$, $v = e^{i\sqrt{\lambda}\omega x}$, in (2.5), we also have

$$0 = -i\sqrt{\lambda}\int_S (\theta+\omega)\cdot\nu e^{-i\sqrt{\lambda}(\theta-\omega)x}\, dS_x.$$

Adding these two equalities we obtain

$$-i\sqrt{\lambda}\int_S \theta\cdot\nu e^{-i\sqrt{\lambda}(\theta-\omega)x}\, dS_x = -\frac{\lambda}{2}(\theta-\omega)^2\int_\Omega e^{-i\sqrt{\lambda}(\theta-\omega)x}\, dx,$$

which completes the proof. □

One should note that the formula in Lemma 2.2 is very similar to the S matrix in scattering theory (see e.g. [11]).

Here we recall the Born approximation utilized in the reconstruction procedure in scattering theory. Let $0 \neq \xi \in \mathbf{R}^n$ be arbitrarily fixed. Choose $\eta \in S^{n-1}$ so that η is orthogonal to ξ. For a large parameter N, we define

$$(2.6) \quad \begin{cases} \theta_N = c_N\eta + \dfrac{\xi}{2N}, \qquad c_N = \left(1 - \dfrac{|\xi|^2}{4N^2}\right)^{1/2}, \\[2mm] \omega_N = c_N\eta - \dfrac{\xi}{2N}, \\[2mm] \sqrt{t_N} = N + i. \end{cases}$$

They have the following properties:

$$(2.7) \quad \begin{cases} \theta_N, \, \omega_N \in S^{n-1}, \\ \sqrt{t_N}(\theta_N - \omega_N) \longrightarrow \xi \quad \text{as} \quad N \longrightarrow \infty, \\ \text{Im} \, t_N \longrightarrow \infty \quad \text{as} \quad N \longrightarrow \infty, \\ \text{Im} \sqrt{t_N} \, \theta_N, \, \text{Im} \sqrt{t_N} \, \omega_N \text{ are bounded as } N \longrightarrow \infty, \end{cases}$$

where Im denotes the imaginary part. Using (2.7) and Lemma 2.2, one can easily show

Theorem 2.3.

$$\lim_{N \to \infty} S(t_N, \, \theta_N, \, \omega_N; \, q) = -\frac{|\xi|^2}{2} \int_\Omega e^{-ix\xi} \, dx + \int_\Omega e^{-ix\xi} q(x) \, dx.$$

Thus one can reconstruct the potential q from $S(\lambda, \, \theta, \, \omega; \, q)$.

Now we turn to the proof of Theorem B.

Lemma 2.4. *Under the assumptions of Theorem B, there exists a constant C > 0 such that*

$$\| N(\lambda, \, q_1) - N(\lambda, \, q_2) \|_{\mathbf{B}(L^2(S))} \leq \frac{C}{|\lambda|},$$

for large $|\lambda|$, where $\| \cdot \|_{\mathbf{B}(L^2(S))}$ denotes the norm of an operator on $L^2(S)$.

Theorem B then readily follows from Definition 2.1, Theorem 2.3 and Lemma 2.4.

Proof of Lemma 2.4. This is intuitively obvious, since $N(\lambda, \, q)$ has, formally, the integral kernel

$$\sum_{i=1}^\infty \frac{1}{\lambda_i - \lambda} \left(\frac{\partial u_i}{\partial v} \right)(x) \left(\frac{\partial u_i}{\partial v} \right)(y),$$

$u_i(x)$ being the eigenfunction associated with λ_i. To make this observation rigorous, we use the idea of [8], Lemma 3.1. Choose m large enough and set

$$r(x, \, y) = m! \sum_{i=1}^\infty (\lambda_i - \lambda)^{-m-1} \left(\frac{\partial u_i}{\partial v} \right)(x) \left(\frac{\partial u_i}{\partial v} \right)(y).$$

Then as has been proved in [8], Lemma 3.1, $r(x, \, y)$ is the integral kernel of $\left(\dfrac{d}{d\lambda} \right)^m N(\lambda; \, q)$. Therefore, by our assumption,

$$\left(\frac{d}{d\lambda} \right)^m (N(\lambda, \, q_1) - N(\lambda, \, q_2)) = m! \sum_{i=1}^N (\lambda_i - \lambda)^{-m-1} A_i,$$

A_i being a bounded operator. Integrating m times we have

$$N(\lambda, q_1) - N(\lambda, q_2) = \sum_{i=1}^{N} (\lambda_i - \lambda)^{-1} A_i + \sum_{k=0}^{m-1} \lambda^k B_k,$$

where B_k is a bounded operator. The formula (3.6) of [8] shows that $B_k = 0$, $0 \le k \le m - 1$, which proves the lemma. $\square$

Remark 2.5. From the very proof, one can see that if the eigenvalues and the normal derivatives of eigenfunctions are sufficiently close, the potentials coincide. In other words, the potential is uniquely determined by the asymptotic properties of eigenfunctions and eigenvalues. We should also remark that the main concern in [8] is to deduce the uniqueness of the potential from the Neumann operator for a fixed value λ. In [7], Nachman has obtained a construction procedure of the potential from the Neumann operator with fixed λ. See also Novikov-Khenkin [4].

§3. Operators of variable coefficients

We briefly mention the case of variable coefficients. Consider the operators

$$H_0 = - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right),$$

$$H = H_0 + q(x),$$

with Dirichlet boundary conditions. The most interesting problem in this case is to recover the coefficients $a_{ij}(x)$. However, it seems to be so difficult (for a related problem see [5]) that in this paper we content ourselves by fixing $a_{ij}(x)$ and trying to reconstruct the potential $q(x)$. We assume that $a_{ij}(x) = a_{ji}(x)$ and that

$$(3.1) \qquad \sup_{x \in \Omega} \sum_{|\alpha| \le m} |\partial^\alpha (a_{ij}(x) - \delta_{ij})| \equiv \delta \qquad \text{is sufficiently small,}$$

where $\partial^\alpha = \left(\dfrac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left(\dfrac{\partial}{\partial x_n} \right)^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and m is a sufficiently large fixed constant.

We employ the method of geometrical optics. First we note that by extending $a_{ij}(x)$ suitably one can assume that $a_{ij}(x)$ is defined on $\mathbf{R}^n$ and satisfies (3.1) on $\mathbf{R}^n$. Next we construct functions $\varphi(x, \omega)$, $a_0(x, \omega)$, $a_1(x, \omega)$ satisfying

$$\sum_{i,j} a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} = 1,$$

$$L a_0 = 0,$$

$$L a_1 + M a_0 = 0,$$

$$L = 2 \sum_{i,j} a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i,j} \left(\frac{\partial a_{ij}}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + a_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right),$$

$$M = \sum_{i,j} \frac{\partial a_{ij}}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i,j} a_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j},$$

for $x \in \Omega$ and $\omega \in S^{n-1}$.

By the standard method, choosing δ sufficiently small, one can construct a solution $\varphi(x, \omega) \in C^m(\bar{\Omega} \times S^{n-1})$ such that

$$(3.2) \qquad \sup_{x \in \Omega, \omega \in S^{n-1}} \sum_{|\alpha| + |\beta| \le m} |\partial_x^\alpha \partial_\omega^\beta (\varphi(x, \omega) - x \cdot \omega)| \le C\delta,$$

and solutions $a_0(x, \omega) \in C^{m-2}(\bar{\Omega} \times S^{n-1})$, $a_1(x, \omega) \in C^{m-3}(\bar{\Omega} \times S^{n-1})$ such that

$$(3.3) \qquad \sup_{x \in \Omega, \omega \in S^{n-1}} \sum_{|\alpha| + |\beta| \le m-2} |\partial_x^\alpha \partial_\omega^\beta (a_0(x, \omega) - 1)| \le C\delta,$$

C being a constant independent of δ. We set

$$(3.4) \qquad \Phi_{\lambda,\omega}(x) = e^{i\sqrt{\lambda}\,\varphi(x,\omega)}(a_0(x, \omega) + (i\sqrt{\lambda})^{-1} a_1(x, \omega)),$$

$$(3.5) \qquad g_{\lambda,\omega}(x) = e^{-i\sqrt{\lambda}\varphi(x,\omega)}(H_0 - \lambda)\Phi_{\lambda,\omega}(x)$$
$$= (i\sqrt{\lambda})^{-1} M a_1(x, \omega).$$

Let the Neumann operator for $H = H_0 + q$ be defined in the same way as in (2.1), where the normal derivative $\dfrac{\partial}{\partial \nu}$ is replaced by

$$\frac{\partial}{\partial \nu_H} = \sum_{ij} a_{ij}(x)\nu_i \frac{\partial}{\partial x_j}.$$

We define

$$(3.6) \qquad S(\lambda, \theta, \omega; q) = \langle N(\lambda, q)\Phi_{\lambda,\omega}, \Phi_{\bar{\lambda},\theta}\rangle.$$

Using the Green's formula

$$\int_\Omega (H_0 u \cdot v - u \cdot H_0 v)\,dx = -\int_S \left(\frac{\partial u}{\partial \nu_H}v - u\frac{\partial v}{\partial \nu_H}\right)dS,$$

one can prove the following lemma in the same way as in Lemma 2.2. Note that the complex conjugate of $e^{i\sqrt{\bar{\lambda}}\varphi(x,\theta)}$ is equal to $e^{-i\sqrt{\lambda}\varphi(x,\theta)}$.

Lemma 3.1.

$$S(\lambda, \theta, \omega; q) = \int_\Omega \sum_{ij} a_{ij}\left(\frac{\partial}{\partial x_i}\Phi_{\lambda,\omega}\right)\overline{\left(\frac{\partial}{\partial x_j}\Phi_{\bar{\lambda},\theta}\right)}dx$$

$$- \lambda \int_\Omega \Phi_{\lambda,\omega}\,\overline{\Phi_{\bar{\lambda},\theta}}\,dx - \int_\Omega \Phi_{\lambda,\omega}e^{-i\lambda\varphi(x,\theta)}\,\overline{g_{\bar{\lambda},\theta}}\,dx$$

$$+ \int_\Omega \Phi_{\lambda,\omega}\,\overline{G_{\bar{\lambda},\theta}}\,dx - (R(\lambda)G_{\lambda,\omega}, G_{\bar{\lambda},\theta}),$$

where $G_{\lambda,\omega} = q\Phi_{\lambda,\omega} + e^{i\sqrt{\lambda}\,\varphi(x,\omega)}g_{\lambda,\omega}$, $R(\lambda) = (H_D - \lambda)^{-1}$, H_D being $H_0 + q$ with

Dirichlet boundary condition, and $\lambda \neq$ *the eigenvalues of* H_D.

Now, let θ_N, ω_N and t_N be as in (2.6). We compute the limit $\lim_{N \to \infty} S(t_N, \theta_N, \omega_N; q)$ in Lemma 3.1. Obviously the third and the fifth terms of the right-hand side tend to 0 as $N \to \infty$. To compute the remaining terms let $\varphi(x, \xi) = |\xi| \varphi\left(x, \dfrac{\xi}{|\xi|}\right)$ for $\xi \neq 0$. Then

$$(3.8) \qquad \lim_{N \to \infty} \sqrt{t_N}(\varphi(x, \theta_N) - \varphi(x, \omega_N)) = \xi \cdot \nabla_\xi \varphi(x, \eta).$$

In fact,

$$\sqrt{t_N}(\varphi(x, \theta_N) - \varphi(x, \omega_N)) = \sqrt{t_N}(\theta_N - \omega_N) \cdot \int_0^1 (\nabla_\xi \varphi)(x, t\theta_N + (1 - t)\omega_N)\, dt.$$

Letting N tend to infinity we obtain (3.8). We have, therefore

$$(3.9) \qquad \lim_{N \to \infty} \int_\Omega \Phi_{t_N, \omega_N} \overline{G_{t_N, \theta_N}}\, dx = \int_\Omega e^{-i\xi \cdot \nabla_\xi \varphi(x, \eta)} a_0(x, \eta)^2\, q(x)\, dx.$$

A straightforward calculation shows that

$$\frac{\partial}{\partial x_j} \Phi_{\lambda, \omega} \cdot \overline{\frac{\partial}{\partial x_k} \Phi_{\lambda, \theta}} = \lambda \frac{\partial \varphi}{\partial x_j}(x, \omega) \frac{\partial \varphi}{\partial x_k}(x, \theta) \Phi_{\lambda, \omega} \overline{\Phi_{\lambda, \theta}}$$

$$+ e^{-i\sqrt{\lambda}(\varphi(x, \theta) - \varphi(x, \omega))} B_{jk}(x, \lambda, \theta, \omega),$$

where,

$$B_{jk}(x, \lambda, \theta, \omega) = i\sqrt{\lambda} \left\{ \frac{\partial \varphi}{\partial x_j}(x, \omega) a_0(x, \omega) \frac{\partial a_0}{\partial x_k}(x, \theta) - \frac{\partial \varphi}{\partial x_k}(x, \theta) a_0(x, \theta) \frac{\partial a_0}{\partial x_j}(x, \omega) \right\}$$

$$+ \frac{\partial \varphi}{\partial x_j}(x, \omega) \left\{ a_1(x, \omega) \frac{\partial a_0}{\partial x_k}(x, \theta) - \frac{\partial a_1}{\partial x_k}(x, \theta) a_0(x, \omega) \right\}$$

$$- \frac{\partial \varphi}{\partial x_k}(x, \theta) \left\{ a_0(x, \theta) \frac{\partial a_1}{\partial x_j}(x, \omega) - \frac{\partial a_0}{\partial x_j}(x, \omega) a_1(x, \theta) \right\}$$

$$+ \frac{\partial a_0}{\partial x_j}(x, \omega) \frac{\partial a_0}{\partial x_k}(x, \theta) + O(\sqrt{\lambda}^{-1}).$$

The Taylor expansion with respect to ω and θ and a simple manipulation show the existence of the limit

$$(3.10) \qquad \lim_{N \to \infty} \sum_{j,k} a_{jk}(x) B_{jk}(x, t_N, \theta_N, \omega_N) \equiv b_2(x, \xi, \eta).$$

We introduce the inner product $\langle \, , \rangle_{A(x)}$ by

$$(3.11) \qquad \langle \xi, \eta \rangle_{A(x)} = \sum_{j,k} a_{jk}(x) \xi_j \eta_k.$$

Since $\varphi(x, \omega)$ satisfies the eikonal equation it follows that

$$\sum_{j,k} a_{jk}(x)\frac{\partial\varphi}{\partial x_j}(x, \omega)\frac{\partial\varphi}{\partial x_k}(x, \theta) - 1$$

$$= -\frac{1}{2}\langle V_x\varphi(x, \omega) - V_x\varphi(x, \theta), V_x\varphi(x, \omega) - V_x\varphi(x, \theta)\rangle_{A(x)},$$

which implies the existence of the limit

$$(3.12) \qquad \lim_{N\to\infty} t_N\left(\sum_{j,k} a_{jk}(x)\frac{\partial\varphi}{\partial x_j}(x, \omega_N)\frac{\partial\varphi}{\partial x_k}(x, \theta_N) - 1\right) \equiv b_1(x, \xi, \eta).$$

We thus arrive at

Lemma 3.2. *Let θ_N, ω_N and t_N be as in (2.6). Then*

$$\lim_{N\to\infty} S(t_N, \theta_N, \omega_N; q) = \int_\Omega e^{-i\xi\cdot V_\xi\varphi(x,\eta)}(b_1(x, \xi, \eta) + b_2(x, \xi, \eta))\,dx$$

$$+ \int_\Omega e^{-i\xi\cdot V_\xi\varphi(x,\eta)} a_0(x, \eta)^2\, q(x)\,dx,$$

where $b_1(x, \xi, \eta)$ and $b_2(x, \xi, \eta)$ are defined by (3.12) and (3.10), respectively.

Our next aim is to reconstruct the potential from the expression

$$(3.13) \qquad \int_\Omega e^{-i\xi\cdot V_\xi\varphi(x,\eta)} a_0(x, \eta)^2\, q(x)\,dx.$$

If η can be chosen as a smooth function of ξ, (3.13) can be viewed as a Fourier integral operator applied to the potential $q(x)$. Since $\eta(\xi)$ is orthogonal to ξ and of absolute value 1, it defines a tangential vector field on the unit sphere. However, as is well-known, whether or not we can choose a smooth non-singular vector field on the unit sphere depends upon the space dimension. If $n =$ the space dimension is even, there exists a C^∞-function $\eta(\xi)\in C^\infty(\mathbf{R}^n - \{0\})$, homogeneous of degree 0, such that $\eta(\xi)$ is orthogonal to ξ. Consider the Fourier integral operator

$$Af(\xi) = \int e^{-i\psi(x,\xi)} a_0(x, \eta(\xi))^2 f(x)\,dx,$$

$$\psi(x, \xi) = \xi\cdot(V_\xi\varphi)(x, \eta(\xi)).$$

Using (3.2) and (3.3), one can easily see that

$$A^*A = I + O(\delta) \quad \text{as} \quad \delta \longrightarrow 0,$$

which shows that one can reconstruct $q(x)$ from (3.13) when δ is sufficiently small. If the space dimension is odd, we introduce a partion of unity $\{\chi_k(\xi)\}$ such that $\chi_k(\xi)\in C^\infty(\mathbf{R}^n - \{0\})$, $\sum_k\chi_k(\xi)^2 = 1$, $\chi_k(\xi) = \chi_k\left(\dfrac{\xi}{|\xi|}\right)$ and on $\operatorname{supp}\chi_k(\xi)$, η can

be chosen as a smooth function of $\xi \neq 0$ which is denoted by $\eta_k(\xi)$. Define the Fourier integral operator A_k by

$$A_k f(\xi) = \int e^{-i\psi_k(x,\xi)} a_0(x, \eta_k(\xi))^2 \chi_k(\xi) f(x) \, dx,$$

$$\psi_k(x, \xi) = \xi \cdot (\nabla_x \varphi)(x, \eta_k(\xi)).$$

Then we have

$$\sum_k A_k^* A_k = \sum_k \chi_k(D_x)^2 + O(\delta)$$

$$= I + O(\delta),$$

and hence, $\sum_k A_k^* A_k q = (I + O(\delta))q$, which shows that we can reconstruct q from (3.13).

We have thus proved

Theorem 3.3. *Theorem B also holds for the operator H_D, where $\dfrac{\partial}{\partial v}$ is replaced by $\dfrac{\partial}{\partial v_H}$*

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