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Local trees in the theory of affine plane curves

By

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If S is a complete nonsingular algebraic surface and D is a divisor of S with normal crossings then the pair (S, D) determines a weighted graph which carries some information about the surface $S\setminus D$. If the divisor D with which one has to cope doesn't have normal crossings then one has to desingularize it by blowing-up the surface at the "bad" points of D. This paper develops a graph theory which relates that desingularization process to the weighted graph obtained at the end. This is done by attaching graph-theoretic devices called "local trees" to the singular points of D, in such a way that each blowing-up gives rise to a transformation of local trees (also called a blowing-up).

The first and third sections study sequences of blowings-up of local trees in a purely graph-theoretic manner, and the case where certain members of the sequence are contractible to linear trees is given particular attention. The two other sections apply these methods to geometry. The second section gives a characterization of the coordinate lines in the affine plane, in terms of the multiplicity sequence at infinity; the fourth section classifies the birational morphisms of the affine plane with one or two fundamental points.

These graph-theoretic methods have been developed, as a part of our doctoral thesis research, in order to investigate certain problems related to the geometry of the affine plane. We would like to thank our professor, K.P. Russell, for the help he provided during the time this work was done.

For the theory of weighted graphs, we use the notations and results contained in the fift section of [3]. For all geometric considerations, our ground field is an arbitrary algebraically closed field k, all curves and surfaces are irreducible and reduced, all surfaces are nonsingular and the word "point" means "closed point". If X is a (nonsingular) surface, $\mathrm{Div}(X)$ is its group of divisors; if P is a point of X and $D \in \mathrm{Div}(X)$ then $\mu(P, D)$ is the multiplicity of P on D; if $D' \in \mathrm{Div}(X)$ has no component in common with D then $(D.D')_P$ is the local intersection multiplicity at P; if X is complete then $D.D'$ is the intersection number and $D^2 = D.D$ (self-intersection number). If $D \in \mathrm{Div}(X)$ and $\tilde{X} \to X$ is a monoidal transformation (resp. $X \subsetneq X'$ is an open immersion) then *the strict*

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transform of D in $\tilde{X}$ (resp. the closure of D in X') is denoted by D whenever no confusion seems likely to arise. **N**, **Z** and **Q** denote respectively the sets of positive integers, integers and rational numbers. If a, b are two integers, their g.c.d. is sometimes denoted by (a, b).

1. Local Trees

This section introduces local trees and begins the study of their blowings-up and contractions. We refer to the last section of [3] for generalities about graphs and weighted graphs.

Definition 1.1. A *local tree* is a 4-tuple $\mathscr{T} = (T, x_0, R, \Omega)$ where:

1. T is a finite set and $x_0 \in T$;
2. R is a collection of subsets of T such that every $a \in R$ contains exactly two elements, and (T, R) is a tree;
3. Ω is a set map $T \setminus \{x_0\} \to \mathbf{Z}$.

The elements of T are called the *vertices*, and those of R the *links*; x_0 is called the *root* of $\mathscr{T}$. Given $x \in T \setminus \{x_0\}$, $\Omega(x)$ is the *weight* of x. Write $R^0 = \{a \in R \mid x_0 \in a\}$ and call the elements of R^0 the *principal links* of $\mathscr{T}$. The neighbours of the root are called the *principal vertices*. The set of neighbours of $x \in T$ is denoted $\mathscr{N}_{\mathscr{T}}(x)$.

An *isomorphism* of local trees is a bijective map between the sets of vertices, preserving the root, the links and the weights.

Definition 1.2. If $\mathscr{T} = (T, x_0, R, \Omega)$ is a local tree, a *multiplicity map for* $\mathscr{T}$ is a set map

$$\mu : R^0 \cup \{x_0\} \longrightarrow \mathrm{N}$$

(where N is the set of positive integers) such that $\mu(a) \geq \mu(x_0)$ for every $a \in R^0$.

An *m-tree* is a pair $(\mathscr{T}, \mu)$ where $\mathscr{T}$ is a local tree and μ is a multiplicity map for $\mathscr{T}$. Given an m-tree $(\mathscr{T}, \mu)$, if x is either the root or a principal link the number $\mu(x)$ is called its *multiplicity*; denote by $\mathscr{N}(\mathscr{T}, \mu)$ the set $\{x \in \mathscr{N}_{\mathscr{T}}(x_0) \mid \mu(\{x, x_0\}) = \mu(x_0)\}$.

Definition 1.3. Let $\mathscr{T} = (T, x_0, R, \Omega)$ be a local tree. A *blowing-up of* $\mathscr{T}$ is a local tree $\mathscr{T}' = (T', x_0', R', \Omega')$ together with a root-preserving injective set map $\beta : T \to T'$, such that if we identify T with its image in T' then the following conditions hold:

1. $T' = T \cup \{e\}$, for some $e \notin T$;
2. $R' = \{\{e, x_0\}\} \cup (R \setminus \{\{x, x_0\} \mid x \in A\}) \cup \{\{x, e\} \mid x \in A\}$ for some set $A \subseteq \mathscr{N}_{\mathscr{T}}(x_0)$ such that $|\mathscr{N}_{\mathscr{T}}(x_0) \setminus A| \leq 1$. Note that $A = \mathscr{N}_{\mathscr{T}}(x_0) \setminus \mathscr{N}_{\mathscr{T}'}(x_0)$;
3. $$\Omega'(x) = \begin{cases} -1, & \text{if } x = e \\ \Omega(x), & \text{if } x \notin \{e, x_0\} \cup \mathscr{N}_{\mathscr{T}}(x_0) \\ \Omega(x) - 1, & \text{if } x \in \mathscr{N}_{\mathscr{T}}(x_0). \end{cases}$$

A map $\beta: T \to T'$ such that the above conditions hold is called an *identification map*. A blowing-up of $\mathcal{T}$ is denoted by $\mathcal{T} \leftarrow \mathcal{T}'$ or $\mathcal{T}' \to \mathcal{T}$ and the set T is usually identified with its image in T'.

Definition 1.4. Let $(\mathcal{T}, \mu)$ be an m-tree, $\mathcal{T} = (T, x_0, R, \Omega)$. We define three notions of blowing-up of $(\mathcal{T}, \mu)$.

1. A *blowing-up of the first kind* of $(\mathcal{T}, \mu)$ is an m-tree $(\mathcal{T}', \mu')$, where $\mathcal{T}' = (T', x_0', R', \Omega')$, together with an identification map $T \to T'$ (i.e., we have $\mathcal{T} \leftarrow \mathcal{T}'$), such that (in the notation of (1.3))

   (a) $\mathcal{N}(\mathcal{T}, \mu) \subseteq \mathcal{N}_{\mathcal{T}}(x_0) \backslash \mathcal{N}_{\mathcal{T}'}(x_0)$;
   (b) $\mu'(\{x, x_0\}) = \mu(\{x, x_0\}) - \mu(x_0)$, if $x \in \mathcal{N}_{\mathcal{T}}(x_0) \cap \mathcal{N}_{\mathcal{T}'}(x_0)$;
   (c) $\mu'(\{x_0, e\}) \leq \mu(x_0)$.

   A blowing-up of the first kind of $(\mathcal{T}, \mu)$ will be denoted by $(\mathcal{T}', \mu') \rightharpoondown (\mathcal{T}, \mu)$ or by $(\mathcal{T}, \mu) \leftharpoondown (\mathcal{T}', \mu')$.
2. A *blowing-up of the second kind of* $(\mathcal{T}, \mu)$ is a blowing-up of the first kind $(\mathcal{T}, \mu) \leftharpoondown (\mathcal{T}', \mu')$ such that equality holds in (1a). That situation will be indicated either by $(\mathcal{T}', \mu') \to (\mathcal{T}, \mu)$ or by $(\mathcal{T}, \mu) \leftarrow (\mathcal{T}', \mu')$.
3. A *blowing-up of the third kind of* $(\mathcal{T}, \mu)$ is a blowing-up of the second kind $(\mathcal{T}, \mu) \leftarrow (\mathcal{T}', \mu')$ such that equality holds in (1c). That situation will be indicated either by $(\mathcal{T}', \mu') \Rightarrow (\mathcal{T}, \mu)$ or by $(\mathcal{T}, \mu) \Leftarrow (\mathcal{T}', \mu')$.

Remarks. 1. If $\mathcal{T} \leftarrow \mathcal{T}'$ then $\mathcal{T}'$ has either one or two principal link(s).
2. Any local tree $\mathcal{T}$ can be blown up; in particular, there is an essentially unique blowing-up $\mathcal{T} \leftarrow \mathcal{T}'$ such that $\mathcal{T}'$ has exactly one principal link.
3. If $(\mathcal{T}, \mu) \leftharpoondown (\mathcal{T}', \mu')$ then, in the notation of (1.3) and (1.4), $\mu(x_0) \geq \mu'(\{e, x_0\}) \geq \mu'(x_0)$.
4. A blowing-up of the second (or third) kind can be performed on an m-tree $(\mathcal{T}, \mu)$ iff $|\mathcal{N}_{\mathcal{T}}(x_0) \backslash \mathcal{N}(\mathcal{T}, \mu)| \leq 1$. If this is the case, then there is an essentially unique blowing-up $\mathcal{T} \leftarrow \mathcal{T}'$ such that $(\mathcal{T}, \mu) \Leftarrow (\mathcal{T}', \mu')$ for some μ' (where μ' is not necessarely unique).

Note that the set of multiplicity maps for a given local tree is an additive (nonempty) semigroup. We make the following trivial observation:

1.5. Let $\mathcal{T} \leftarrow \mathcal{T}'$ be a blowing-up of local trees with identification map $\beta: T \to T'$. Then, if μ' is any multiplicity map for $\mathcal{T}'$, there is a unique μ such that $(\mathcal{T}, \mu) \Leftarrow (\mathcal{T}', \mu')$ (with the same β). The map $\mu' \mapsto \mu$ so defined is a homomorphism of semigroups; denote it by β^*. In general, β^* is neither injective nor surjective. In particular, $\beta^*(\mu_1) = \beta^*(\mu_2) \Leftrightarrow \mu_1(a) = \mu_2(a)$, for all principal links a of $\mathcal{T}'$.

This can be generalized as follows:

Lemma 1.6. *If* $\mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k (k \geq 1)$, *each multiplicity map* μ_k *for* $\mathcal{T}_k$ *determines uniquely* $(\mu_0, \ldots, \mu_{k-1})$ *such that* $(\mathcal{T}_0, \mu_0) \Leftarrow \cdots \Leftarrow (\mathcal{T}_k, \mu_k)$. *Moreover, if*

$(\mu'_0, \dots, \mu'_k)$ *is such that* $(\mathcal{T}_0, \mu'_0) \Leftarrow \cdots \Leftarrow (\mathcal{T}_k, \mu'_k)$ *and for some* $q \in \mathbf{Q}$ *we have* $q\mu_k(a) = \mu'_k(a)$, *all* $a \in R_k^0$, *then* $q(\mu_0, \dots, \mu_{k-1}) = (\mu'_0, \dots, \mu'_{k-1})$.

Definition 1.7. Consider a sequence of local trees $S: \mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k$ where k is any nonnegative integer.

1. Define $\mathrm{Mul}(S)$ to be the set of $k+1$-tuples $\mu = (\mu_0, \dots, \mu_k)$ of multiplicity maps such that $(\mathcal{T}_0, \mu_0) \Leftarrow \cdots \Leftarrow (\mathcal{T}_k, \mu_k)$. Then $\mathrm{Mul}(S)$ is a semigroup and (1.6) says that the projection map $\mathrm{Mul}(S) \to \mathrm{Mul}(\mathcal{T}_k)$ is an isomorphism.
2. Suppose that $k \geq 1$ and that $\mathcal{T}_0$ has one principal link a. An element $\mu = (\mu_0, \dots, \mu_k)$ of $\mathrm{Mul}(S)$ is said to *satisfy the condition of* (1.7.2) if the following holds:
   Let the euclidean algorithm of $(\mu_0(a), \mu_0(x_0))$ *be written as*

$$\mu_0(a) = \alpha_0 \rho_0 + \rho_1 \qquad (\textit{where } \rho_0 = \mu_0(x_0))$$
$$\rho_0 = \alpha_1 \rho_1 + \rho_2$$
$$\vdots$$
$$\rho_{s-1} = \alpha_s \rho_s.$$

   Then $(\mu_0(x_0), \dots, \mu_{k-1}(x_0)) = (\rho_0, \dots, \rho_0, \rho_1, \dots, \rho_s)$ *where each* ρ_i *occurs exactly* α_i *times.*

Lemma 1.8. *Let* $S: \mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k$ *be a sequence of local trees such that* $k \geq 1$ *and such that* $\mathcal{T}_0$ *has one principal link. Then the following conditions are equivalent*:

1. $\{v \mid \mathcal{T}_v$ *has one principal link*$\} = \{0, k\}$,
2. *all* $\mu \in \mathrm{Mul}(S)$ *satisfy the condition of* (1.7.2),
3. *some* $\mu \in \mathrm{Mul}(S)$ *satisfies the condition of* (1.7.2).

Remarks.
- If the conditions of (1.8) are met, $\mu \in \mathrm{Mul}(S)$ and if the principal links of $\mathcal{T}_0$ and $\mathcal{T}_k$ are a and a' respectively, then $\mu_k(a')$ is the g.c.d. of $\mu_0(a)$ and $\mu_0(x_0)$.
- If $k > 1$ and the conditions of (1.8) are met then the principal vertex of $\mathcal{T}_k$ is a branch point (for $\mathcal{T}_{k-1}$ has two principal links, while $\mathcal{T}_k$ has only one). So a branch point is created each time an euclidean algorithm terminates.

Definition 1.9. 1. Given $S: \mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k$ such that $k \geq 1$ and both $\mathcal{T}_0$ and $\mathcal{T}_k$ have one principal link, define
$\mathcal{J}(S) = \{j \mid 0 \leq j < k, \mathcal{T}_j$ has one principal link and $\mathcal{T}_{j+1}$ has two$\}$.
$\mathcal{H}(S) = \{j \mid 0 < j \leq k, \mathcal{T}_{j-1}$ has two principal links and $\mathcal{T}_j$ has one$\}$ and
$l = (\#$ of branch points of $\mathcal{T}_k) - (\#$ of branch points of $\mathcal{T}_0)$.
We see that $|\mathcal{J}(S)| = |\mathcal{H}(S)| = l$. Write

$$\mathcal{J}(S) = \{j_0, \dots, j_{l-1}\}, \qquad 0 \leq j_0 < \cdots < j_{l-1},$$
$$\mathcal{H}(S) = \{h_1, \dots, h_l\}, \qquad 0 < h_1 < \cdots < h_l \leq k;$$

then $0 \le j_0 < h_1 \le j_1 < \cdots \le j_{l-1} < h_l \le k$.

2. If $\mu = (\mu_0, \ldots, \mu_k) \in \mathrm{Mul}(S)$ then the pair (S, μ) determines the following numbers (where x_0 is the root of any $\mathscr{T}_i$ and a_i is the principal link of $\mathscr{T}_i$, whenever i is such that $\mathscr{T}_i$ has exactly one principal link):

$$\begin{cases} i_0 = \mu_{j_0}(a_{j_0}) \\ i_\nu = \mu_{h_\nu}(a_{h_\nu}) = \mu_{j_\nu}(a_{j_\nu}), & 0 < \nu < l \\ i_l = \mu_{h_l}(a_{h_l}) \end{cases}$$

$$\begin{cases} m_\nu = \mu_{j_\nu}(x_0), & 0 \le \nu < l \\ m = m(S, \mu) = m_0 + \cdots + m_{l-1}. \end{cases}$$

Then $i_0 > m_0 \ge i_1 > m_1 \ge \cdots \ge i_{l-1} > m_{l-1} \ge i_l$ and, by (1.8), we have $\gcd(i_{\nu-1}, m_{\nu-1}) = i_\nu$ for $\nu = 1, \ldots, l$.

Definition 1.10. Given an infinite sequence $S: (\mathscr{T}_0, \mu_0) \leftarrow (\mathscr{T}_1, \mu_1) \leftarrow \cdots$, there exists an $i \ge 0$ such that

- $\mathscr{T}_i$ has at most one principal link, and if it has one then its multiplicity is $\mu_i(x_0)$;
- $\forall j > i$, $\mathscr{T}_j$ has exactly one principal link, say a_j, and $\mu_j(a_j) = \mu_i(x_0)$.

The least such i will be denoted $k(S)$. Observe that if $k = k(S)$ then $(\mathscr{T}_k, \mu_k) \Leftarrow (\mathscr{T}_{k+1}, \mu_{k+1}) \Leftarrow \cdots$.

Relation to Geometry. See the last section of [3] for the definitions of *strong normal crossings* (*s.n.c.*) and of the dual graph $\mathscr{G}(S, D)$ associated to a pair (S, D).

Definition 1.11. We consider a triple (P, D, S) where

1. S is a nonsingular projective surface,
2. $D \in \mathrm{Div}(S)$ has *s.n.c.* and $\mathscr{G}(S, D)$ is a (possibly empty) tree,
3. $P \in \mathrm{supp}(D)$ if $D \ne 0$.

The *local tree of* (P, D, S) is $\mathscr{T} = (T, x_0, R, \Omega)$ where:

(a) $x_0 = P$, $T = \{P\} \cup \{D_1, \ldots, D_n\}$, where $D_1, \ldots, D_n$ are the distinct irreducible components of D,

(b) $R = \{\{D_i, D_j\} \mid i \ne j \text{ and } P \notin D_i \cap D_j \ne \emptyset\} \cup \{\{P, D_i\} \mid P \in D_i\}$.

(c) $\Omega(D_i) = D_i^2$ (self-intersection number in S).

The local tree of (P, D, S) is denoted $\mathscr{T}(P, D, S)$. If C is a nonzero effective divisor of S *such that*

4. $P \in \mathrm{supp}(C)$,
5. C and D have no irreducible component in common,

we define *the m-tree of* (P, C, D, S) to be $(\mathscr{T}, \mu)$, where $\mathscr{T} = \mathscr{T}(P, D, S)$ and $\mu: R^0 \cup \{x_0\} \to \mathbf{N}$ is as follows:

(d) $\mu(x_0) = \mu(P, C)$ (multiplicity of P on C),

(e) $\mu(\{x_0, D_i\}) = (C.D_i)_P$ (local intersection multiplicity at P), if $\{x_0, D_i\} \in R^0$, i.e., if $P \in D_i$.

Let $\mathscr{C}(P, D, S)$ denote the set of nonzero effective divisors C of S satisfying (4) and (5). We note that it is a semigroup and that the map $C \mapsto \mu$, determined by conditions (d-e), is a homomorphism of semigroups $\mathscr{C}(P, D, S) \to \mathrm{Mul}(\mathscr{T})$.

1.12 (Blowing-Up). Suppose (P, D, S) satisfies conditions (1–3) of (1.11) and let $\mathscr{T} = \mathscr{T}(P, D, S)$. Let $\pi: \tilde{S} \to S$ be the blowing-up of S at P, $E = \pi^{-1}(P) \in \mathrm{Div}(\tilde{S})$, let $\sim$ mean "strict transform of..." and define $D' = \tilde{D} + E \in \mathrm{Div}(\tilde{S})$. If P' is a point of E then we may consider $\mathscr{T}' = \mathscr{T}(P', D', \tilde{S})$ and we clearly have $\mathscr{T} \leftarrow \mathscr{T}'$, where the identification map is the obvious one.

If $C \in \mathscr{C}(P, D, S)$ is such that $P' \in \mathrm{supp}(\tilde{C})$ (i.e., $\tilde{C} \in \mathscr{C}(P', D', \tilde{S})$), we may consider the m-trees $(\mathscr{T}, \mu)$ of (P, C, D, S) and $(\mathscr{T}', \mu')$ of $(P', \tilde{C}, D', \tilde{S})$. We let the reader convince himself that

$$(\mathscr{T}, \mu) \longleftarrow (\mathscr{T}', \mu')$$

and that the following claims are true:

1. $(\mathscr{T}, \mu) \longleftarrow (\mathscr{T}', \mu')$ iff $E \cap \mathrm{supp}(\tilde{D}) \cap \mathrm{supp}(\tilde{C}) \subseteq \{P'\}$;
2. $(\mathscr{T}, \mu) \Longleftarrow (\mathscr{T}', \mu')$ iff $E \cap \mathrm{supp}(\tilde{C}) = \{P'\}$.

Definition 1.13. Let (S, D) satisfy conditions (1–2) of (1.11) and let C be a nonzero effective divisor of S satisfying condition (5). If $\tilde{P}$ is a place of C, i.e., a closed point of the nonsingular model of some irreducible component of C, then the triple $(\tilde{P}, C, S)$ determines an infinite sequence of monoidal transformations

$$S_0 \xleftarrow{\pi_1} S_1 \xleftarrow{\pi_2} S_2 \longleftarrow \cdots$$

where $S_0 = S$, P_i = image of $\tilde{P}$ in S_{i-1} and π_i is the blowing-up of S_{i-1} at P_i. If $P_1 \in \mathrm{supp}(D)$ or $D = 0$, we say that the 4-tuple $(\tilde{P}, C, D, S)$ *is as in* (1.13), *or satisfies the conditions of* (1.13). If that is the case, let $C^{(i)}$ be the strict transform of $C^{(0)} = C$ in S_i and let $E_i = \pi_i^{-1}(P_i)$; define $D^0 = D \in \mathrm{Div}(S_0)$, D^i = (strict transform of D^{i-1}) + $E_i \in \mathrm{Div}(S_i)$ $(i \geq 1)$.

Then, for $i \geq 0$, $(P_{i+1}, C^{(i)}, D^i, S_i)$ satisfies conditions (1–5) of (1.11) and we can consider its m-tree $(\mathscr{T}_i, \mu_i)$. By (1.12), we have

$$(\mathscr{T}_0, \mu_0) \longleftarrow (\mathscr{T}_1, \mu_1) \longleftarrow \cdots$$

which we call *the infinite sequence of m-trees of* $(\tilde{P}, C, D, S)$. We denote by $k = k(\tilde{P}, C, D, S)$ the integer determined by this sequence, as defined in (1.10). Observe that $(\mathscr{T}_k, \mu_k) \Leftarrow (\mathscr{T}_{k+1}, \mu_{k+1}) \Leftarrow \cdots$ and that, as far as the place $\tilde{P}$ is concerned, the desingularization process ends with $S_{k-1} \leftarrow S_k$. What we mean, here, is that k is the least integer $i \geq 0$ which satisfies:

> P_{i+1} *belongs to exactly one irreducible component* Γ *of* $C^{(i)}$, Γ *is nonsingular at* P_{i+1} *and* $(\Gamma . D^i)_{P_{i+1}} \leq 1$.

For these reasons, the finite sequence

$$(\mathscr{T}_0, \mu_0) \longleftarrow \cdots \longleftarrow (\mathscr{T}_k, \mu_k)$$

is given special consideration: we call it *the sequence of m-trees of* $(\tilde{P}, C, D, S)$, and denote it by $\mu(\tilde{P}, C, D, S)$. The sequence

$$\mathscr{T}_0 \longleftarrow \cdots \longleftarrow \mathscr{T}_k$$

is called *the sequence of local trees of* $(\tilde{P}, C, D, S)$.

Lemma 1.14. *Let* $(\tilde{P}, C, D, S)$ *be as in* (1.13) *and consider*

$$\mu(\tilde{P}, C, D, S): \qquad (\mathscr{T}_0, \mu_0) \longleftarrow \cdots \longleftarrow (\mathscr{T}_k, \mu_k).$$

Let the notation be as in (1.13) *and assume* $k > 0$.

1. *If* $\operatorname{supp}(C^{(k)} + D^k) = \operatorname{supp}(B)$ *for some* $B \in \operatorname{Div}(S_k)$ *with s.n.c., then*

$$(\mathscr{T}_0, \mu_0) \longleftarrow \cdots \longleftarrow (\mathscr{T}_k, \mu_k).$$

2. *If* $\tilde{C}$ *is the disjoint union of the nonsingular models of the irreducible components of* C, *and if* $\tau: \tilde{C} \to \operatorname{supp}(C)$ *is the canonical surjective set map, then following are equivalent:*
   - $(\mathscr{T}_0, \mu_0) \Longleftarrow \cdots \Longleftarrow (\mathscr{T}_k, \mu_k)$
   - $\tau^{-1}(P_1) = \{\tilde{P}\}$.
3. *If* $S \backslash \operatorname{supp}(C + D)$ *has no loops at infinity* (*see* [3], *just before* (5.19)), *then the following are equivalent:*
   - $(\mathscr{T}_0, \mu_0) \Longleftarrow \cdots \Longleftarrow (\mathscr{T}_k, \mu_k)$
   - *only one irreducible component of* C *contains* P_1.

Proof. Immediate from (1.12).

Contraction of Local Trees. Given $\omega \in \mathbf{Z}$, the symbol (ω) will denote any local tree which has two vertices and such that the principal vertex has weight ω. We will now study sequences

$$(\omega) = \mathscr{T}_0 \longleftarrow \cdots \longleftarrow \mathscr{T}_k$$

of local trees such that $\mathscr{T}_k$ contracts to some simple local tree, such as (ω) or a linear local tree. First, we define the necessary notions.

Definition 1.15. Let $\mathscr{T} = (T, x_0, R, \Omega)$ be a local tree. We say that $\mathscr{T}$ is a *linear local tree* if it has exactly one principal link and if the tree (T, R) is linear.

Definition 1.16. Let $\mathscr{T} = (T, x_0, R, \Omega)$ be a local tree.

1. A *superfluous vertex* of $\mathscr{T}$ is a vertex $e \in T \backslash (\{x_0\} \cup \mathscr{N}_{\mathscr{T}}(x_0))$ which is linear and which has weight -1.
2. If e is a superfluous vertex of $\mathscr{T}$ then an *elementary contraction of* $\mathscr{T}$ *at* e is a local tree $\mathscr{T}' = (T', x_0', R', \Omega')$ together with a root-preserving injective set map $\beta: T' \to T$ such that, if we identify T' with its image in T, the following conditions hold:

$$T' = T\setminus\{e\}$$

$$R' = \begin{cases} (R\setminus\{\{e, x\} \mid x\in\mathcal{N}_{\mathcal{T}}(e)\})\cup\{\mathcal{N}_{\mathcal{T}}(e)\}, & \text{if } |\mathcal{N}_{\mathcal{T}}(e)| = 2 \\ R\setminus\{\{e, x\} \mid x\in\mathcal{N}_{\mathcal{T}}(e)\}, & \text{if } |\mathcal{N}_{\mathcal{T}}(e)| = 1, \end{cases}$$

$$\Omega'(x) = \begin{cases} \Omega(x) + 1, & \text{if } x\in\mathcal{N}_{\mathcal{T}}(e) \\ \Omega(x), & \text{if } x\in T\setminus(\{x_0, e\}\cup\mathcal{N}_{\mathcal{T}}(e)). \end{cases}$$

In other words, an elementary contraction of $\mathcal{T}$ at e can be obtained as follows: first, forget that x_0 is the root and assign an arbitrary weight to that vertex; then $\mathcal{T}$ becomes a weighted tree and e is a superfluous vertex of that tree; blow-down $\mathcal{T}$ at e: forget the weight of x_0 and remember that x_0 is the root. The local tree so obtained (together with the set map which came with the blowing-down) is an elementary contraction of $\mathcal{T}$ at e. Note that the elementary contraction of $\mathcal{T}$ at e is essentially unique.

3. A *contraction* of $\mathcal{T}$ is a local tree $\mathcal{T}' = (T', x_0', R', \Omega')$ together with a set map $\beta: T' \to T$, such that either β is an isomorphism or the following condition holds:

   There exist local trees $\mathcal{T}_0, \ldots, \mathcal{T}_k$ and maps $\beta_1, \ldots, \beta_k$ $(k \geq 1)$ such that $\mathcal{T}_0 = \mathcal{T}$, $\mathcal{T}_k = \mathcal{T}'$, $(\mathcal{T}_i, \beta_i)$ is an elementary contraction of $\mathcal{T}_{i-1}$ at some superfluous vertex $(1 \leq i \leq k)$, and $\beta = \beta_1 \circ\cdots\circ \beta_k$.

   In particular, we see that β is a root-preserving injective map and that β restricts to a bijection of the sets of principal vertices (we say that the two trees have the same principal vertices and principal links). A contraction as above is denoted by $\mathcal{T}' \leq \mathcal{T}$ or $\mathcal{T} \geq \mathcal{T}'$, and we say that $\mathcal{T}$ contracts to $\mathcal{T}'$.

   Since the set map $T' \to T$ determined by a contraction $\mathcal{T}' \leq \mathcal{T}$ allows us to identify $\{x_0'\}\cup R'^0$ with $\{x_0\}\cup R^0$, we can compare multiplicity maps for the two trees and define:

4. For m-trees $(\mathcal{T}, \mu)$ and $(\mathcal{T}', \mu')$, we define $(\mathcal{T}, \mu) \geq (\mathcal{T}', \mu') \Leftrightarrow \mathcal{T} \geq \mathcal{T}'$ and $\mu = \mu'$.

Remark 1.17. We deliberately avoided the term "blowing-down" for local trees, to emphasize that the contraction is not the inverse operation of blowing-up (for blowings-up happen at the root, while contractions occur away from the root). Contractions do not affect data which are "local" to the root, such as multiplicity maps. Indeed, if (P, C, D, S) satisfies conditions (1–5) of (1.11) and $(\mathcal{T}, \mu)$ is the m-tree of that 4-tuple, and if E is an irreducible component of D which is a rational curve and a superfluous vertex of $\mathcal{T}$, then the elementary contraction of $\mathcal{T}$ at E corresponds to the contraction of the curve E. More precisely, by Castelnuovo's criterion for contracting a curve, there is a monoidal transformation $\rho: S \to S'$, where S' is a nonsingular projective surface and $\rho(E)$ is a point of S'. Now let $\rho_*: \mathrm{Div}(S) \to \mathrm{Div}(S')$ be the homomorphism defined by $\rho_*(E)=0$ and $\rho_*(\Gamma)=\rho(\Gamma)$ (any curve Γ other than E). Let $P'=\rho(P)$, $C'=\rho_*(C)$ and $D'=\rho_*(D)$, then (P', C', D', S') satisfies conditions (1–5) of (1.11) and determines an m-tree $(\mathscr{T}', \mu')$ such that $(\mathscr{T}, \mu)\geq(\mathscr{T}', \mu')$. Indeed, by definition of superfluous vertex, ρ is an isomorphism in a neighbourhood of P and the multiplicities are not affected by the contraction of E.

The next fact is an easy consequence of the definitions; we omit its proof.

Lemma 1.18. *Let $\mathscr{T}$, $\mathscr{T}'$ and $\mathscr{T}''$ be local trees with sets of vertices T, T' and T'' respectively. If $\mathscr{T}'\leq\mathscr{T}$ and $\mathscr{T}''\leq\mathscr{T}$ then the following are equivalent:*

1. *The maps $T'\to T$ and $T''\to T$ have the same image.*
2. *There exists an isomorphism $\mathscr{T}'\cong\mathscr{T}''$ that commutes with $T'\to T$ and $T''\to T$, i.e., the two contractions are essentially the same.*

Hence we can refer to a contraction process by specifying which vertices disappear and which survive. In view of that, let us adopt the following language:

Let $\mathscr{T}=(T, x_0, R, \Omega)$ be a local tree, v a vertex of $\mathscr{T}$ other than the root and B a branch of $\mathscr{T}$ at v, not containing the root. Suppose that $\mathscr{T}\geq\mathscr{T}'=(T', x_0', R', \Omega')$, where $T'=T\backslash B$ (after identification of T' with its image in T). Then we say that *B is absorbed by v* or that *v absorbs B*.

Definition 1.19. Let ω, i, i' be positive integers. A *sequence of type* (ω, i, i') is a finite sequence of positive integers, of the form

$$m_0,\dots,m_0,\quad i_1,\dots,i_1,\quad m_1,\dots,m_{l-1},\quad i_l,\dots,i_l$$

where $l\geq 1$,
m_{v-1} occurs ω times $(1\leq v\leq l)$,
i_v occurs $2n_v$ times, for some $n_v\in\mathbf{N}$ $(1\leq v\leq l-1)$,
i_l occurs n_l times, for some $n_l\in\mathbf{N}$,
and such that the following conditions hold (where we define $i_0=i$):

1. $i_l=i'$
2. $m_{v-1}=n_v i_v, \qquad 1\leq v\leq l$
3. $i_{v-1}=\omega m_{v-1}+i_v, \qquad 1\leq v\leq l.$

Remark. Consider a sequence of type (ω, i, i'), with notation as above. Then:

1. $i_0>m_0\geq i_1>m_1\geq\cdots\geq i_{l-1}>m_{l-1}\geq i_l$
2. $i_{v-1}=(\omega n_v+1)i_v, \qquad 1\leq v\leq l$
3. $\gcd(i_{v-1}, m_{v-1})=i_v, \qquad 1\leq v\leq l.$

Lemma 1.20. *Let ω be a positive integer and let*

$$\mathscr{S}:\mathscr{T}_0\longleftarrow\cdots\longleftarrow\mathscr{T}_k\qquad(k\geq 0)$$

be a sequence of local trees, such that $\mathcal{T}_0$ has one principal link a. Then the following are equivalent:

1. $\exists \mu = (\mu_0, \ldots, \mu_k) \in \mathrm{Mul}(\mathcal{S})$ *such that* $(\mu_0(x_0), \ldots, \mu_{k-1}(x_0))$ *is a sequence of type* $(\omega, \mu_0(a), i')$, *for some* i'.
2. $\forall \mu = (\mu_0, \ldots, \mu_k) \in \mathrm{Mul}(\mathcal{S})$, $(\mu_0(x_0), \ldots, \mu_{k-1}(x_0))$ *is a sequence of type* $(\omega, \mu_0(a), i')$, *for some* i'.

Moreover, if these equivalent conditions are met then $k \geq \omega + 1$, $\mathcal{T}_k$ *has one principal link* a', $i' = \mu_k(a')$ *(in the notation of (1) or (2)) and the principal vertex of* $\mathcal{T}_k$ *is a branch point.*

Proof. Follows from (1.6) and (1.8).

Definition 1.21. Let ω be a positive integer and let $\mathcal{S}: \mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k$ be a sequence of local trees. We say that $\mathcal{S}$ *is of type* ω if $\mathcal{T}_0$ has one principal link and if the equivalent conditions of (1.20) are met. When that is the case, we have in particular $k \geq \omega + 1$, $\mathcal{T}_k$ has one principal vertex and that vertex is a branch point of $\mathcal{T}_k$.

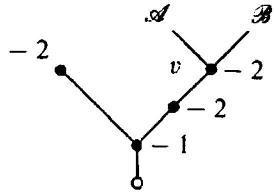
Remarks. 1. If $\mathcal{S}: \mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k$ is of type ω, $\mu \in \mathrm{Mul}(\mathcal{S})$ and if we write $(\mu_i(x_0))_{i=0,\ldots,k-1} = (m_0, \ldots, i_l)$ according to (1.9), then this sequence satisfies the conditions of (1.19), with the same m_v's and i_v's.

2. If $\mathcal{S}$ is as in (1), then the numbers $n_1, \ldots, n_l$ of (1.19) are completely determined by $\mathcal{S}$. Indeed, if $\mu, \mu' \in \mathrm{Mul}(\mathcal{S})$ then by (1.6) there is a nonzero rational number q such that $q(\mu_0, \ldots, \mu_{k-1}) = (\mu'_0, \ldots, \mu'_{k-1})$.

Definition 1.22. We are now going to define a notation that we will use to avoid drawing pictures of local trees. We do this for practical reasons only and we suggest that the reader reconstructs all pictures whenever these notations are encountered. Let $\mathcal{T}$ be either a local tree or a weighted tree with a root (i.e., a distinguished vertex), let ρ be the weight of the root (with $\rho = *$ if $\mathcal{T}$ is a local tree) and suppose that, for each vertex v, the set of branches (of $\mathcal{T}$ at v) that don't contain the root has been totally ordered. In particular, let $\mathcal{B}_1, \ldots, \mathcal{B}_n$ $(n \geq 0)$ be the branches of $\mathcal{T}$ at the root. Then the tree $\mathcal{T}$ will be denoted by the symbols $([\mathcal{T}])$, where $[\mathcal{T}]$ is the sequence of symbols defined by

$$[\mathcal{T}] = \begin{cases} \rho & \text{if } n = 0 \\ \rho, [\mathcal{B}_1] & \text{if } n = 1 \\ \rho, ([\mathcal{B}_1]), \ldots, ([\mathcal{B}_n]) & \text{if } n > 1. \end{cases}$$

This makes sense, since each $\mathcal{B}_i$ is itself a weighted tree with a root (the root being the neighbour of the root of $\mathcal{T}$), with an ordering for each appropriate set of branches, etc. For instance, the local tree $\mathcal{T}$:



where $\circ$ is the root and $\mathscr{A}$, $\mathscr{B}$ are branches at v, is denoted by

$$(*, -1, (-2), (-2, -2, ([\mathscr{A}]), ([\mathscr{B}]))).$$

Abusing a bit, we write $\mathscr{T} = (*, -1, (-2), (-2, -2, ([\mathscr{A}]), ([\mathscr{B}])))$, which amounts to identify $\mathscr{T}$ and $([\mathscr{T}])$; doing the same thing with $\mathscr{A}$ and $\mathscr{B}$, i.e., writing $\mathscr{A} = ([\mathscr{A}])$ and $\mathscr{B} = ([\mathscr{B}])$, we get

$$\mathscr{T} = (*, -1, (-2), (-2, -2, \mathscr{A}, \mathscr{B})).$$

Remark. If $\omega \in \mathrm{N}$ the local tree (ω), defined before (1.15), is the same as $(*, \omega)$.

We are now ready to state the first significant result, in the theory of local trees.

Theorem 1.23. *Let ω and k be positive integers and let $\mathscr{S}: \mathscr{T}_0 \leftarrow \cdots \leftarrow \mathscr{T}_k$ be a sequence of local trees such that $\mathscr{T}_0 \geq (\omega)$, $\mathscr{T}_k$ has one principal vertex and that vertex is a branch point of $\mathscr{T}_k$. Then the following are equivalent:*
(a) *$\mathscr{T}_k$ contracts to a linear local tree,*
(b) *$\mathscr{S}$ is of type ω.*
Moreover, if these conditions hold then (see (1.9) *for definition of $j_0, \ldots, j_{l-1}$)*

1. *$\mathscr{T}_{j_0}, \ldots, \mathscr{T}_{j_{l-1}}$ contract to (ω);*
2. *$\mathscr{T}_k = (*, -1, (-n-1, -2, \ldots, -2), \mathscr{B})$, where "$-2$" occurs $\omega - 1$ times, n is the positive integer n_l of definition* (1.19), *$\mathscr{B}$ is a branch that the principal vertex (call it v) of $\mathscr{T}_k$ can absorb and v gets weight* 0 *after absorption of $\mathscr{B}$;*
3. *if $\mathscr{T}_k \leftarrow \mathscr{T}_{k+1} \leftarrow \cdots \leftarrow \mathscr{T}_{k+n}$ is the (unique) sequence such that $\mathscr{T}_{k+i}$ has one principal link $(0 \leq i \leq n)$, then $\mathscr{T}_{k+n} \geq (\omega)$.*

Since contractions do not change the number of principal links of a local tree, it certainly makes sense to assume, in (1.23), that $\mathscr{T}_k$ has one principal vertex. However, the assumption that that vertex is a branch point is there only to make the conclusion simpler; when we do have to cope with a sequence $\mathscr{S}$ such that the principal vertex of $\mathscr{T}_k$ is not a branch point, (2.23) gives a description of the nontrivial part of $\mathscr{S}$, say $\mathscr{T}_0 \leftarrow \cdots \leftarrow \mathscr{T}_{k_1}$, and $\mathscr{T}_{k_1} \leftarrow \cdots \leftarrow \mathscr{T}_k$ is trivial (i.e., every tree in it has one principal link).

Before we can prove the theorem, we need to introduce some notions and state some facts. The proofs are elementary and most of them are omitted; some can be found in [2].

Definition 1.24. A local tree $\mathcal{T}$ is *minimal* if it has no superfluous vertex.

Lemma 1.25. *Let $\mathcal{T}$ be a local tree that contracts to a linear local tree. If v is a branch point of $\mathcal{T}$ then there is a branch of $\mathcal{T}$ at v, not containing the root, which can be absorbed by v. If $\mathcal{M}$ is a minimal local tree such that $\mathcal{M} \leq \mathcal{T}$, then $\mathcal{M}$ is a linear local tree.*

Proof. Follows easily from (5.11) and (5.12) of [3].

Definition 1.26. A local tree $\mathcal{T}$ is *universally minimal* (write "$\mathcal{T}$ is UM") if for every sequence

$$\mathcal{T} = \mathcal{T}_0 \longleftarrow \cdots \longleftarrow \mathcal{T}_k \qquad (k \geq 0),$$

$\mathcal{T}_k$ is minimal. Observe that if $\mathcal{T}$ is UM then it is minimal, and $\mathcal{T}'$ is UM wherever $\mathcal{T} \leftarrow \mathcal{T}'$.

Lemma 1.27. *Let $\mathcal{T}$ be a local tree. Then the following are equivalent:*
1. *$\mathcal{T}$ is UM;*
2. *$\mathcal{T}$ is minimal and every linear principal vertex of $\mathcal{T}$ has negative weight.*

Lemma 1.28. *Suppose that $\mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k$ $(k \geq 1)$ and that $\mathcal{T}_0 \geq \mathcal{T}'_0$. Then there is a unique diagram*

$$\begin{array}{ccccccc} \mathcal{T}_0 & \longleftarrow & \mathcal{T}_1 & \longleftarrow & \cdots & \longleftarrow & \mathcal{T}_k \\ \vee\!\!/ & & \vee\!\!/ & & & & \vee\!\!/ \\ \mathcal{T}'_0 & \longleftarrow & \mathcal{T}'_1 & \longleftarrow & \cdots & \longleftarrow & \mathcal{T}'_k \end{array}$$

such that the underlying diagram of set maps is commutative. (By "unique", we mean unique up to isomorphisms commuting with all maps.)

Remark. Whenever we have a commutative diagram as in (1.28), where the first row is denoted by $\mathcal{S}$ and the second by $\mathcal{S}'$, we have $\mathrm{Mul}(\mathcal{S}) = \mathrm{Mul}(\mathcal{S}')$.

Lemma 1.29. *Let $\mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k$ $(k \geq 1)$ be such that $\mathcal{T}_{k-1}$ has more than one principal link and $\mathcal{T}_k$ contracts to a linear local tree.* If *$i < k$ then $\mathcal{T}_i$ can't contract to a UM tree.*

Proof. Let $i < k$ be such that $\mathcal{T}_i \geq \mathcal{U}$, where $\mathcal{U}$ is UM. Construct a commutative diagram as in (1.28):

$$\begin{array}{ccccc} \mathcal{T}_i & \longleftarrow & \cdots & \longleftarrow & \mathcal{T}_k \\ \vee\!\!/ & & & & \vee\!\!/ \\ \mathcal{U} = \mathcal{U}_i & \longleftarrow & \cdots & \longleftarrow & \mathcal{U}_k. \end{array}$$

Since $\mathcal{U}_k$ is minimal, $\mathcal{U}_k \leq \mathcal{T}_k$ and $\mathcal{T}_k$ contracts to a linear local tree, (1.25) implies that $\mathcal{U}_k$ is linear. Then clearly $\mathcal{U}_{k-1}$ is linear, which is absurd since $\mathcal{T}_{k-1}$ has more than one principal link and $\mathcal{T}_{k-1} \geq \mathcal{U}_{k-1}$.

Definition 1.30. 1. A local tree $\mathcal{T}$ is a *comb* if at every vertex v there are at most two branches that don't contain the root, and at most one of them is not a linear branch. (A linear branch is a branch which contains no branch point of $\mathscr{T}$: this means more than being linear as a graph.) In particular, the root is a linear vertex.

2. If $\mathscr{T}$ is a comb, a *tooth* of $\mathscr{T}$ is a linear branch $\mathscr{A}$ of $\mathscr{T}$, at either a branch point or the root, such that $\mathscr{A}$ doesn't contain the root. So every branch point has at least one tooth (one branch point has two teeth) and, if there are two principal links, the root has at least one tooth.
3. $\mathscr{T}$ is a *comb with negative teeth* if it is a comb such that
   (a) at every branch point there is at least one tooth $\mathscr{A}$ such that $\mathscr{A} < -1$;
   (b) if $\mathscr{T}$ has two principal vertices, then one of them, say v, has negative weight and belongs to a tooth $\mathscr{A}$ such that $\mathscr{A}\setminus\{v\} < -1$.

   (Recall that, if $\mathscr{G}$ is a weighted tree, $\mathscr{G} < -1$ means that every vertex of $\mathscr{G}$ has weight less than -1.)

Remark. Every linear local tree is a comb with negative teeth.

Lemma 1.31. *Suppose that either $\mathscr{T} \leftarrow \mathscr{T}'$ or $\mathscr{T} \geq \mathscr{T}'$. If $\mathscr{T}$ is a comb (resp. a comb with negative teeth) then so is $\mathscr{T}'$.*

Proof of (1.23). By (1.28), one can consider a diagram of the form

$$\begin{array}{ccccccc} \mathscr{T}_0 & \longleftarrow & \mathscr{T}_1 & \longleftarrow \cdots \longleftarrow & \mathscr{T}_k \\ \vee\!\!/ & & \vee\!\!/ & & \vee\!\!/ \\ (\omega) = \mathscr{T}'_0 & \longleftarrow & \mathscr{T}'_1 & \longleftarrow \cdots \longleftarrow & \mathscr{T}'_k. \end{array}$$

Then a little argument (which we leave to the reader) shows that we may assume that $\mathscr{T}_0 = (\omega)$. We will prove that (a) implies (b), (1) and (2); (2) $\Rightarrow$ (3) and (b) $\Rightarrow$ (a) are easily verified.

Suppose that (a) holds, i.e., $\mathscr{T}_k$ contracts to some linear local tree. Using the notation of (1.9), we write

$$\mathscr{J} = \mathscr{J}(\mathscr{S}) = \{j_0, \ldots, j_{l-1}\}$$
$$\mathscr{H} = \mathscr{H}(\mathscr{S}) = \{h_1, \ldots, h_l\}$$

where, clearly, $l \geq 1$ (for l is the number of branch points of $\mathscr{T}_k$). We proceed by induction on l.

Case $l = 1$. Then the principal vertex v of $\mathscr{T}_k$ is the only branch point of $\mathscr{T}_k$. Let L denote the principal vertex of $\mathscr{T}_0$. Since L is a free vertex of $\mathscr{T}_0$ it is a free vertex of $\mathscr{T}_k$. Thus $\mathscr{T}_k = (*, -1, \mathscr{A}, \mathscr{B})$ where $\mathscr{A}$ and $\mathscr{B}$ are linear branches at v and L is in $\mathscr{B}$ (say). Since $\mathscr{T}_k$ contracts to a linear tree and since, in $\mathscr{T}_k$, every vertex other than x_0, v, L has weight less than -1 we must have $\mathscr{B} = (-2, \ldots, -2, -1)$ by (1.25). Let $n > 0$ be the number of vertices of $\mathscr{B}$. Then one easily figures out that $\mathscr{S}$ begins with

$$\mathscr{T}_0 = (*, \omega) \longleftarrow (*, (-1), (\omega - 1)) \longleftarrow \cdots \longleftarrow (*, (-1, -2, \ldots, -2), (0)) = \mathscr{T}_\omega,$$

and continues

$$\mathcal{T}_{\omega+1} = (*, (-2, \ldots, -2), (-1, -1)) \longleftarrow \cdots$$
$$\cdots \longleftarrow (*, (-n, -2, \ldots, -2), (-1, -2, \ldots, -2, -1))$$
$$\longleftarrow (*, -1, (-n-1, -2, \ldots, -2), (-2, \ldots, -2, -1)) = \mathcal{T}_{\omega+n} = \mathcal{T}_k.$$

We leave it to the reader to check that $\mathcal{S}$ is of type ω and that $\mathcal{T}_k$ has the desired form (with, in particular, $n = n_l = n_1$).

Inductive step. Assume $l > 1$. For $1 \le v \le l$, let e_v be the branch point created in $\mathcal{T}_{h_v - 1} \leftarrow \mathcal{T}_{h_v}$. In particular, e_{l-1} is the principal vertex of $\mathcal{T}_{h_{l-1}} = (*, -1, \mathcal{A}', \mathcal{B}')$ where $\mathcal{A}'$ and $\mathcal{B}'$ are branches at e_{l-1}. We have $h_l = k$, so e_l is the principal vertex of $\mathcal{T}_k = (*, -1, \mathcal{A}, \mathcal{B})$, where $\mathcal{A}, \mathcal{B}$ are branches at e_l and $\mathcal{B} = (b_1, \ldots, b_s, \varepsilon, \mathcal{A}', \mathcal{B}')$ contains e_{l-1} (more precisely, $s \ge 0$, ε is the weight of e_{l-1} in $\mathcal{T}_k$ and the branches $\mathcal{A}', \mathcal{B}'$ at e_{l-1} are identical to what they were in $\mathcal{T}_{h_{l-1}}$).

Observe that, by (1.31) and the remark immediately before it, $\mathcal{T}_i$ is a comb with negative teeth $(0 \le i \le k)$; let $\mathcal{A}'$ be the tooth of $\mathcal{T}_k$ at e_{l-1} with $\mathcal{A}' < -1$. By (1.25), it follows that e_{l-1} can absorb $\mathcal{B}'$ in $\mathcal{T}_k$, hence in $\mathcal{T}_{h_{l-1}}$ as well. Thus

$\mathcal{T}_{h_{l-1}}$ *contracts to a linear local tree.*

Applying the inductive hypothesis to $\mathcal{S}_{l-1}: \mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_{h_{l-1}}$, we conclude that it is a sequence of type ω, $\mathcal{T}_{j_0}, \ldots, \mathcal{T}_{j_{l-2}}$ contract to (ω) and that $\mathcal{T}_{h_{l-1}}$ contracts to $\mathcal{T}'_{h_{l-1}} = (*, 0, -m-1, -2, \ldots, -2)$, where "$-2$" occurs $\omega - 1$ times and $m = n_{l-1}$. Construct the commutative diagram (1.28)

$$\begin{array}{ccc} \mathcal{T}_{h_{l-1}} & \longleftarrow \cdots \longleftarrow & \mathcal{T}_k \\ \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} \\ \mathcal{T}'_{h_{l-1}} & \longleftarrow \cdots \longleftarrow & \mathcal{T}'_k. \end{array}$$

Let $\alpha = j_{l-1} - h_{l-1} \ge 0$. If $\alpha = 0$ then $\mathcal{T}'_{h_{l-1}+1}$ is UM by (1.27); since $h_{l-1} + 1 = j_{l-1} + 1 < k$ (for $\mathcal{T}_{j_{l-1}+1}$ has two principal vertices by definition of j_{l-1}), this contradicts (1.29). Hence $\alpha > 0$. Note that

$$\mathcal{T}'_{j_{l-1}+1} = (*, (-1), (-2, \ldots, -2, -1, -m-1, -2, \ldots, -2)),$$

where the first sequence of "-2" contains α terms and the second has $\omega - 1$ terms. That contracts to $\mathcal{T}''_{j_{l-1}+1} = (*, (-1), (-1, \alpha - m - 1, -2, \ldots, -2))$ which can't be UM by (1.29). By (1.27), that tree is not minimal, and we have $\alpha = m$. We conclude that $\mathcal{T}'_{j_{l-1}} = (*, -1, -2, \ldots, -2, -1, -m-1, -2, \ldots, -2)$, where there are $\alpha - 1 = m - 1$ terms in the first sequence of "-2", and $\omega - 1$ in the second. Hence that tree contracts to (ω), and so does $\mathcal{T}_{j_{l-1}}$. Applying the inductive hypothesis (or the case $l = 1$) to $\mathcal{T}_{j_{l-1}} \leftarrow \cdots \leftarrow \mathcal{T}_k$, we see that it is of type ω and that $\mathcal{T}_k$ has the desired form. Since $\alpha = m = n_{l-1}$ and $\mathcal{S}_{l-1}$ is of type ω, one sees that $\mathcal{S}$ is of type ω. This completes the proof of (1.23).

2. Coordinate lines in $\mathbf{A}^2$

We regard $\mathbf{A}^2$ as being equipped with a fixed coordinate system. In particular, it makes sense to speak of the degree of a curve in $\mathbf{A}^2$. An open immersion $\mathbf{A}^2 \subsetneq \mathbf{P}^2$ is said to be *standard* if it doesn't change the degrees of the curves; the standard immersions form an equivalence class. (Two open immersions $\mathbf{A}^2 \subsetneq \mathbf{P}^2$ are equivalent if they form a commutative diagram with some automorphism of $\mathbf{P}^2$.) Following several people, we adopt the following terminology for lines in the affine plane.

Definition 2.1. Let C be a curve in $\mathbf{A}^2$.
1. C is a *coordinate line* if there is an automorphism ϕ of $\mathbf{A}^2$ such that $\phi(C)$ has degree one. Equivalently, the polynomial $F \in \mathrm{k}[X, Y]$ determined by C is such that $\mathrm{k}[F, G] = \mathrm{k}[X, Y]$ for some $G \in \mathrm{k}[X, Y]$.
2. C is a *line* if $C \cong \mathbf{A}^1$ (abstractly). Equivalently, the polynomial F determined by C is such that $\mathrm{k}[X, Y]/(F)$ is a polynomial algebra in one indeterminate over k.

As is very well known (see [1]), all lines are coordinate lines if and only if $\mathrm{char}\,\mathrm{k} = 0$.

Definition 2.2. Let Γ be a curve in $\mathbf{A}^2$ with one place P at infinity. We say that Γ is *graph-theoretically linear* if there is an open immersion $\mathbf{A}^2 \subsetneq \mathbf{P}^2$ with the following property:

If $L = \mathbf{P}^2 \backslash \mathbf{A}^2$ *and* $\mathscr{T}_0 \leftarrow \cdots \leftarrow \mathscr{T}_k$ *is the sequence of local trees of* $(P, \Gamma, L, \mathbf{P}^2)$ *then* $\mathscr{T}_k$ *contracts to a linear local tree.*

Remarks. 1. Note that $\mathscr{T}_0 = (1)$, in (2.2). See (1.13) for the definition of the sequence of local trees of $(P, \Gamma, L, \mathbf{P}^2)$. Note that, in $\mu(P, \Gamma, L, \mathbf{P}^2)$, all blowings-up are of the third kind by (1.14.2).
2. It can be shown that if Γ is graph-theoretically linear then all open immersions $\mathbf{A}^2 \subsetneq \mathbf{P}^2$ satisfy the condition of (2.2).

Proposition 2.3. *Let* Γ *be a curve in* $\mathbf{A}^2$, *with one place at infinity. Then the following are equivalent*:
1. Γ *is graph-theoretically linear*,
2. Γ *is a coordinate line.*

Proof. (2)$\Rightarrow$(1) is trivial: Choose an open immersion $\mathbf{A}^2 \subsetneq \mathbf{P}^2$ such that the closure in $\mathbf{P}^2$ of Γ has degree one. Then $k(P, \Gamma, L, \mathbf{P}^2) = 0$, i.e., $\mathscr{T}_k = \mathscr{T}_0 = (1)$ which is already a linear local tree. Hence Γ is graph-theoretically linear.
(1)$\Rightarrow$(2) Let Γ be graph-theoretically linear and let $\mathbf{A}^2 \subsetneq \mathbf{P}^2$ be an open immersion satisfying the condition of (2.2). Let P be the place of Γ at infinity and $L = \mathbf{P}^2 \backslash \mathbf{A}^2$ the line at infinity. Then, as in (1.13), $(P, \Gamma, L, \mathbf{P}^2)$ determines an infinite sequence

of monoidal transformations and an infinite sequence of m-trees:

$$\begin{array}{ccccccc} \mathbf{A}^2 \hookrightarrow \mathbf{P}^2 = & S_0 & \longleftarrow \cdots \longleftarrow & S_k & \longleftarrow \cdots \\ & (\mathscr{T}_0, \mu_0) & \Longleftarrow \cdots \Longleftarrow & (\mathscr{T}_k, \mu_k) & \Longleftarrow \cdots \end{array}$$

where $k = k(P, \Gamma, L, \mathbf{P}^2)$. By definition, $\mathscr{T}_k$ contracts to a linear local tree. If $k = 0$ then $(\Gamma . L)_P = 1$ in $\mathbf{P}^2$, by definition of k; hence $\Gamma . L = 1$, Γ is a line in $\mathbf{P}^2$ and we are done. Assume $k > 0$. Then the hypothesis of (1.23) is satisfied and, by the last assertion of it, we see that $\mathscr{T}_{k+n} \geq (1)$ for some positive integer n. Since all blowings-up have centers *i.n.* $S_0 \backslash \mathbf{A}^2$, $\mathbf{A}^2$ is naturally embedded in S_{k+n} and, in fact, $S_{k+n} \backslash \mathbf{A}^2 = \operatorname{supp}(L^{k+n})$ and $(\mathscr{T}_{k+n}, \mu_{k+n})$ is the m-tree of $(P_{k+n+1}, \Gamma^{(k+n)}, L^{k+n}, S_{k+n})$-notation consistent with (1.13). By iterating the argument of (1.17), we see that the contraction $\mathscr{T}_{k+n} \geq (1)$ corresponds to a birational morphism $\rho: S_{k+n} \to S'$ which contracts all components of L^{k+n} except E_{k+n}. Let $P' = \rho(P_{k+n+1})$, $\Gamma' = \rho_*(\Gamma^{(k+n)})$ and $L' = \rho_*(E_{k+n})$; then by (1.17) the m-tree of (P', Γ', L', S') is $((1), \mu')$, where the multiplicity μ' of the principal link of (1) is equal to the multiplicity μ_{k+n} of the principal link of $\mathscr{T}_{k+n}$, i.e., it is 1. Hence $(\Gamma' . L')_{P'} = 1$ and since these two curves meet only at P', $\Gamma' . L' = 1$. Now we have an embedding of $\mathbf{A}^2$ in the nonsingular projective surface S', such that the complement of $\mathbf{A}^2$ is one curve L'. As is well known, S' must be a projective plane. Since $\Gamma' . L' = 1$, Γ' is a line in $S' = \mathbf{P}^2$ and we are done.

Our characterization of coordinate lines can be stated in terms of the multiplicity sequence at infinity.

Definition 2.4. Let Γ be an affine plane curve with one place P at infinity. Embed $\mathbf{A}^2$ in $\mathbf{P}^2$ the standard way. As noted in (1.13), an infinite sequence of monoidal transformations is uniquely determined,

$$\mathbf{P}^2 = S_0 \xleftarrow{\pi_1} S_1 \xleftarrow{\pi_2} S_2 \xleftarrow{\pi_3} \cdots.$$

Let P_i denote the center of $\pi_i: S_i \to S_{i-1}$ and $\Gamma^{(i)}$ the strict transform on S_i of the closure in $\mathbf{P}^2$ of Γ. The sequence $\mu(P_1, \Gamma^{(0)})$, $\mu(P_2, \Gamma^{(1)}), \ldots$ is called *the multiplicity sequence of Γ at infinity.* That sequence is completely determined by the "embedding" of Γ in $\mathbf{A}^2$, i.e., is independent of the choice of an embedding of $\mathbf{A}^2$ in $\mathbf{P}^2$–as long as that embedding is "standard".

Corollary 2.5. *Let Γ be a curve of degree d in $\mathbf{A}^2$, with one place at infinity. Let $(r_0, r_1, \ldots)$ be the multiplicity sequence of Γ at infinity. Then the following are equivalent:*

1. *Γ is a coordinate line.*
2. *Either $d = 1$ or there is a positive integer k such that $(r_0, \ldots, r_{k-1})$ is a sequence of type $(1, d, 1)$ (observe that, in the latter case, $d > 1$ and $r_j = 1$ if $j \geq k$).*

Proof. Clear from remark (2) after (2.2), together with (1.23).

3. Weak sequences

Recall that (1.23) is concerned with sequences of local trees, some members of which are contractible to linear local trees. This section is devoted to similar considerations, but the sequences of local trees are of a different type.

A *weighted tree with a root* is a pair $(\mathscr{G}, v_0)$ where $\mathscr{G}$ is a weighted tree and v_0 is a vertex of $\mathscr{G}$, called the root.

Definition 3.1. Let $\mathscr{T}$ be a local tree, and $\mathscr{G}$ a weighted tree with a root v_0.

1. If v a vertex of $\mathscr{T}$ other than the root then $\mathscr{T}^{v,\mathscr{G}}$ denotes the *local tree* obtained by taking the disjoint union of $\mathscr{T}$ and $\mathscr{G}$ and linking v to v_0. (For instance, let $\mathscr{T} = (*, 1, 2)$, let v be the principal vertex of $\mathscr{T}$ and let $\mathscr{G} = (0, -1)$, where the notation (1.22) is used. Then $\mathscr{T}^{v,\mathscr{G}} = (*, 1, (0, -1), (2))$.) If $\mathscr{G}$ consists of one vertex of weight $\alpha \in \mathbf{Z}$, we also write $\mathscr{T}^{v,\alpha} = \mathscr{T}^{v,\mathscr{G}}$. If $\mathscr{G}_1, \ldots, \mathscr{G}_p$ are weighted trees with roots, define $\mathscr{T}^{v,\mathscr{G}_1,\ldots,\mathscr{G}_p} = (\cdots(\mathscr{T}^{v,\mathscr{G}_1})\cdots)^{v,\mathscr{G}_p}$. Then $\mathscr{G}_1, \ldots, \mathscr{G}_p$ are branches of $\mathscr{T}^{v,\mathscr{G}_1,\ldots,\mathscr{G}_p}$ at v, called the *extra branches*. Clearly,

   (a) if $\mathscr{T} \longleftarrow \mathscr{T}'$ then $\mathscr{T}^{v,\mathscr{G}_1,\ldots,\mathscr{G}_p} \longleftarrow \mathscr{T}'^{v,\mathscr{G}_1,\ldots,\mathscr{G}_p}$;
   (b) if $\mathscr{T} \geq \mathscr{T}'$ and v is in $\mathscr{T}'$ then $\mathscr{T}^{v,\mathscr{G}_1,\ldots,\mathscr{G}_p} \geq \mathscr{T}'^{v,\mathscr{G}_1,\ldots,\mathscr{G}_p}$.

2. $\mathscr{T}[\mathscr{G}]$ denotes the *weighted tree* obtained by taking the disjoint union of $\mathscr{T}$ and $\mathscr{G}$ and identifying v_0 with the root of $\mathscr{T}$. (For instance, $\mathscr{T}[\mathscr{G}] = [-1, 0, 1, 2]$ if $\mathscr{T}$ and $\mathscr{G}$ are as above and if we use the notation of [3], (5.13) for linear weighted trees.) If $\mathscr{G}$ consists of one vertex of weight $\alpha \in \mathbf{Z}$, we also write $\mathscr{T}[\alpha] = \mathscr{T}[\mathscr{G}]$. We have the following properties:

   (a) if $\mathscr{T} \geq \mathscr{T}'$ then $\mathscr{T}[\mathscr{G}]$ contracts to $\mathscr{T}'[\mathscr{G}]$;
   (b) if $\mathscr{T} \leftarrow \mathscr{T}'$ and $|\mathscr{N}_{\mathscr{T}}(x_0)| \leq |\mathscr{N}_{\mathscr{T}'}(x_0)| = 1$ then $\mathscr{T}[\mathscr{G}] \sim \mathscr{T}'[\mathscr{G}']$, where $\mathscr{G}'$ is obtained from $\mathscr{G}$ by decreasing by 1 the weight of the root. In particular, $\mathscr{T}[\alpha] \sim \mathscr{T}'[\alpha - 1]$, $\alpha \in \mathbf{Z}$.

Definition 3.2. Given local trees $\mathscr{T}, \mathscr{T}'$, the symblol $\mathscr{T} \overset{+}{\longleftarrow} \mathscr{T}'$ indicates that we have chosen a map β', from the set of vertices of $\mathscr{T}$ to that of $\mathscr{T}'$, satisfying the following condition:

> *There exists a blowing-up $\mathscr{T} \leftarrow \mathscr{T}_1$ such that, if e is the vertex created in that blowing-up, then $\mathscr{T}' = \mathscr{T}_1^{e,\mathscr{G}_1,\ldots,\mathscr{G}_p}$ for some $\mathscr{G}_1, \ldots, \mathscr{G}_p$ $(p \geq 1)$, and β' is the composition of the identification map of $\mathscr{T} \leftarrow \mathscr{T}_1$ with the inclusion of $\mathscr{T}_1$ in $\mathscr{T}'$.*

Definition 3.3. A sequence $\mathscr{T}_0, \ldots, \mathscr{T}_k$ of local trees (with sets of vertices $T_0, \ldots, T_k$ respectively) is called a *weak sequence* if $k \geq 1$, $\mathscr{T}_k$ has one principal link and if, for $i = 1, \ldots, k$, there exists (and we have chosen) a map $\beta_i \colon T_{i-1} \to T_i$ such that either $\mathscr{T}_{i-1} \leftarrow \mathscr{T}_i$ or $\mathscr{T}_{i-1} \overset{+}{\longleftarrow} \mathscr{T}_i$. The sequence is said to be *weak at* $\mathscr{T}_i$ if $\mathscr{T}_{i-1} \overset{+}{\longleftarrow} \mathscr{T}_i$.

We now explain how the notion of weak sequence is related to Geometry.

Definition 3.4. Let S be a nonsingular projective surface, let $D \neq 0$ be a reduced effective divisor of S and P a place of D (i.e., a closed point of the nonsingular model of some irreducible component of D). We say that D *can be desingularized by blowing-up at* P if the following condition holds:

> *Let $S = S_0 \leftarrow S_1 \leftarrow \cdots$ be the infinite sequence of monoidal transformations determined by (P, D, S), let $s_i \in S_{i-1}$ be the center of $S_{i-1} \leftarrow S_i$ and let $F_i \in \mathrm{Div}(S_i)$ be the corresponding exceptional curve. For $G \in \mathrm{Div}(S)$, define*
>
> $$\begin{cases} G^0 = G \in \mathrm{Div}(S_0), \\ G^i = (\textit{strict transform of } G^{i-1}) + F_i \in \mathrm{Div}(S_i), \qquad i \geq 1. \end{cases}$$
>
> *Then D^i has s.n.c., for some i.*

Observe that, by Lemma (5.19) of [3], if $S \backslash \mathrm{supp}(D) \cong \mathbf{A}^2$ then D can be desingularized by blowing-up at some place P of D.

Definition 3.5. We say that a 4-tuple (P, C, L, S) *satisfies the conditions of* (3.5) if

- S is a nonsingular projective surface;
- C, L are connected, effective divisors of S such that $C + L$ is reduced, L has *s.n.c.* but $C + L$ doesn't;
- the members of $\mathscr{G}[U]$ are trees, where $U = S \backslash \mathrm{supp}(C + L)$;
- P is a place of C and $C + L$ can be desingularized by blowing-up at P;
- if $L \neq 0$ then the image of P on S belongs to $\mathrm{supp}(L) \cap \mathrm{supp}(C)$.

Now suppose (P, C, L, S) satisfies the conditions of (3.5), let $D = C + L$ and consider the sequence of monoidal transformations $S = S_0 \leftarrow S_1 \leftarrow \cdots$ determined by (P, D, S), with notations s_i, F_i, G^i as in (3.4). Let $\mathscr{T}_0 \leftarrow \cdots \leftarrow \mathscr{T}_k$ be the sequence of local trees of (P, C, L, S). The notion of *weak sequence of* (P, C, L, S), which we will soon define, is motivated by the question

How can we obtain the weighted tree $\mathscr{G}(S_k, D^k)$ from the local tree $\mathscr{T}_k$?

To make the notation simpler, let's denote a divisor of some S_i and its strict transform in $S_j (j > i)$ by the same symbol. If only one irreducible component C_* of C contains s_1 then the above question has a simple answer: $\mathscr{G}(S_k, D^k) = \mathscr{T}_k[\mathscr{G}(S_k, C)]$, where $\mathscr{G}(S_k, C)$ is regarded as a weighted tree with a root, the root being C_*. From now-on, assume that s_1 belongs to more than one irreducible component of C–this is case that requires the notion of weak sequence. Write $C = \Sigma \Gamma_\nu \in \mathrm{Div}(S)$, where each Γ_ν is a connected, reduced effective divisor having exactly one irreducible component C_ν containing s_1. Observe that there is a unique ν_* such that $s_{k+1} \in C_{\nu_*}$. If $\nu \neq \nu_*$ then for some $i \leq k$ the blowing-up $S_{i-1} \leftarrow S_i$ "gets C_ν away from P", i.e.,

$$s_i \in C_\nu \quad \text{in } S_{i-1}$$

$$s_{i+1} \notin C_\nu \quad \text{in } S_i.$$

Let $\alpha_1 < \cdots < \alpha_m$ be the indices i such that $S_{i-1} \leftarrow S_i$ gets some C_ν's away from P; for $r = 1, \ldots, m$, say that $S_{\alpha_r - 1} \leftarrow S_{\alpha_r}$ gets $C_{\nu_{r1}}, \ldots, C_{\nu_{rp_r}}$ away from P. Then we have for each $r = 1, \ldots, m$:

- $\Gamma_{\nu_{rj}} . L^{\alpha_r} = C_{\nu_{rj}} . F_{\alpha_r} = 1$ $(1 \leq j \leq p_r)$;
- $\Gamma_{\nu_{r1}}, \ldots, \Gamma_{\nu_{rp_r}}$ meet F_{α_r} at distinct points and s_{α_r+1} is not one of those points;
- $(\Gamma_{\nu_{r1}} + \cdots + \Gamma_{\nu_{rp_r}}) + L^{\alpha_r} \in \operatorname{Div}(S_{\alpha_r})$ has *s.n.c.*.

Let $\mathcal{G}_{rj} = \mathcal{G}(S_{\alpha_r}, \Gamma_{\nu_{rj}})$ and $\mathcal{G}_* = \mathcal{G}(S_k, \Gamma_{\nu_*})$ and regard $C_{\nu_{rj}}$ (resp. C_{ν_*}) as the root of $\mathcal{G}_{rj}$ (resp. $\mathcal{G}_*$). Define

$$\mathcal{W}_i = \begin{cases} \mathcal{T}_i & \text{if } 0 \leq i < \alpha_1, \\ (\cdots(\mathcal{T}_i^{F_{\alpha_1}, \mathcal{G}_{11}, \ldots, \mathcal{G}_{1p_1}})\cdots)^{F_{\alpha_r}, \mathcal{G}_{r1}, \ldots, \mathcal{G}_{rp_r}} & \text{if } \textit{either } 1 \leq r < m \text{ and} \\ & \alpha_r \leq i < \alpha_{r+1} \textit{ or} \\ & r = m \text{ and } \alpha_m \leq i \leq k. \end{cases}$$

Then $\mathcal{W}_0, \ldots, \mathcal{W}_k$ is a weak sequence; it is weak at $\mathcal{W}_{\alpha_1}, \ldots, \mathcal{W}_{\alpha_m}$. Moreover, the question raised after (3.5) is answered by $\mathcal{G}(S_k, D^k) = \mathcal{W}_k[\mathcal{G}_*]$.

Definition 3.6. Let (P, C, L, S) be a 4-tuple which satisfies the conditions of (3.5). *The weak sequence of* (P, C, L, S) is the sequence $\mathcal{W}_0, \ldots, \mathcal{W}_k$, as defined in the above discussion.

To be precise, if only one irreducible component of C contains s_1 (i.e., $m = 0$) we define the weak sequence of (P, C, L, S) by $(\mathcal{W}_0, \ldots, \mathcal{W}_k) = (\mathcal{T}_0, \ldots, \mathcal{T}_k)$.

In any case, since we start with a $D \in \operatorname{Div}(S)$ which doesn't have *s.n.c.*, we have $k > 0$, D^{k-1} doesn't have *s.n.c.* and, by (5.19) of [3], F_k is a branch point of $\mathcal{G}(S_k, D^k)$, i.e.,

the principal vertex of $\mathcal{W}_k$ *is a branch point.*

Let P_{rj} denote the place of $C_{\nu_{rj}}$ which corresponds to the point $C_{\nu_{rj}} \cap F_{\alpha_r}$ in S_{α_r}. Now fix $j \in \{1, \ldots, p_1\}$ and let $\mathcal{T} = \mathcal{T}(P_{1j}, L^{\alpha_1}, S_{\alpha_1})$: note that $\mathcal{T}$ has only one principal link. Clearly, the sequence $\mathcal{W}_0 \leftarrow \cdots \leftarrow \mathcal{W}_{\alpha_1 - 1} \leftarrow \mathcal{T}$ consists of the sequence of local trees of $(P_{1j}, C_{\nu_{1j}}, L, S)$, followed by a (possibly empty) sequence of blowings-up in which every tree has exactly one principal link. Thus $\mathcal{T}$ carries some information about the curve $C_{\nu_{1j}}$ and its embedding in S. On the other hand, $\mathcal{T}$ is related to the weak sequence $\mathcal{W}_0, \ldots, \mathcal{W}_k$ in the following manner:

$$\begin{array}{ccccccc} & & & & \mathcal{T} & & \\ & & & & \downarrow & & \\ \mathcal{W}_0 & \longleftarrow & \cdots & \longleftarrow & \mathcal{W}_{\alpha_1 - 1} & \overset{+}{\longleftarrow} & \mathcal{W}_{\alpha_1} \cdots \mathcal{W}_k \end{array}$$

and $\mathscr{W}_k[\mathscr{G}_*] \in \mathscr{G}[U]$ by above. If, for instance, $U \cong \mathbf{A}^2$, then $\mathscr{W}_k[\mathscr{G}_*] \sim [1]$ and it might be possible to say something about $\mathscr{T}$, hence about $C_{v_{1j}}$, by investigating the graph-theoretic situation described by the above diagram. These ideas, and in particular the question whether $\mathscr{T}$ is contractible to a linear local tree, underly the rest of this section–which is purely graph-theoretic. Actual applications to geometry are given in the last section.

Theorem 3.7. *Let $\mathscr{W}_0, \ldots, \mathscr{W}_k$ be a weak sequence of local trees, weak at $\mathscr{W}_1$ and possibly at other places. Let $\mathscr{W}_0 \leftarrow \mathscr{T}$ be the blowing-up such that $\mathscr{T}$ has one principal link. Assume that $\mathscr{T}$ does not contract to a linear local tree and that there exist a linear weighted tree $\mathscr{L}$ and a weighted tree with a root $\mathscr{G}$ such that*

$$\mathscr{W}_k[\mathscr{G}] \sim \mathscr{L}.$$

Then every extra branch created in $\mathscr{W}_0 \overset{+}{\longleftarrow} \mathscr{W}_1$ can be absorbed by the vertex to which it is attached, the principal vertex of $\mathscr{T}$ is a branch point, $\mathscr{T}$ contracts to a local tree whose only branch point is its principal vertex and, given a weighted tree $\mathscr{G}'$ with a root, $\mathscr{T}[\mathscr{G}']$ is equivalent to a linear weighted tree iff $\mathscr{G}'$ can be absorbed by the principal vertex of $\mathscr{T}$.

Moreover, if $\langle \mathscr{L} \rangle \leq 1$ then $\mathscr{W}_0$ can't contract to a local tree containing a nonprincipal vertex of nonnegative weight.
(The nonnegative integer $\langle \mathscr{L} \rangle$ is defined in (5.8) of [3].)

Proof. Suppose the principal vertex of $\mathscr{W}_k$ is not a branch point. Then $k-1 \geq 1$, $\mathscr{W}_{k-1}$ has one principal link and, if $\mathscr{G}'$ is the weighted tree with a root obtained from $\mathscr{G}$ by increasing by 1 the weight of the root, then $\mathscr{W}_{k-1}[\mathscr{G}'] \sim \mathscr{W}_k[\mathscr{G}]$ (for $\mathscr{W}_{k-1} \leftarrow \mathscr{W}_k$). Hence it's enough to prove the theorem for the weak sequence $\mathscr{W}_0, \ldots, \mathscr{W}_{k-1}$ and $\mathscr{W}_0 \leftarrow \mathscr{T}$, i.e., k can be decreased. Therefore we may assume that

(∗) *the principal vertex of $\mathscr{W}_k$ is a branch point (of weight -1); consequently, it survives to any contraction of $\mathscr{W}_k[\mathscr{G}]$ to a linear weighted tree.*

Now let's prove the last assertion of the theorem. Suppose $\mathscr{W}_0 \geq \mathscr{W}'_0$, for some local tree $\mathscr{W}'_0$ having a nonprincipal vertex of nonnegative weight. Observe that, by the two assertions included in the first part of (3.1), (1.28) can be generalised to weak sequences in such a way that the upper sequence is weak at some tree iff the lower sequence is weak at the corresponding tree. So we may form the "commutative diagram"

$$\begin{array}{ccccc} \mathscr{W}_0 & \overset{+}{\longleftarrow} & \mathscr{W}_1 & \cdots & \mathscr{W}_k \\ \vee\!\!/ & & \vee\!\!/ & & \vee\!\!/ \\ \mathscr{W}'_0 & \overset{+}{\longleftarrow} & \mathscr{W}'_1 & \cdots & \mathscr{W}'_k. \end{array}$$

Thus $\mathscr{W}'_k$ has a vertex v of nonnegative weight, such that v is not a neighbour of the principal vertex–call it u. Since u has been created in the blowing-up involved in the "passage" from $\mathcal{W}'_{k-1}$ to $\mathcal{W}'_k$, it has weight -1. So its weight is not increased by the contraction $\mathcal{W}_k \geq \mathcal{W}'_k$ and it follows from (*) that u is a branch point of $\mathcal{W}'_k$. Since $\mathcal{W}'_k[\mathcal{G}] \sim \mathcal{W}_k[\mathcal{G}] \sim \mathcal{L}$, u can absorb a branch of $\mathcal{W}'_k[\mathcal{G}]$ by (5.11) of [3]; that branch doesn't contain v (for v has nonnegative weight) so $\mathcal{W}'_k[\mathcal{G}]$ contracts to a weighted tree $\mathcal{G}^+$ which contains vertices u, v with nonnegative weights and not neighbours of each other. By [3], (5.9), $1 < \langle \mathcal{G}^+ \rangle = \langle \mathcal{L} \rangle$ and the last assertion is proved–and so is (3.8), below.

The assertion about $\mathcal{T}[\mathcal{G}']$ is an immediate consequence of the preceding one.

Let $\mathcal{W}_{i_1}, \ldots, \mathcal{W}_{i_q}$, $1 = i_1 < \cdots < i_q \leq k$, be the trees at which the sequence is weak. For the rest of the proof, we proceed by induction on q. The case $q = 1$ will be proved after the

Inductive Step. Suppose $q > 1$ and let $\mathcal{W}_{i_2 - 1} \leftarrow \mathcal{T}'$ be the blowing-up such that $\mathcal{T}'$ has one principal link. We claim that $\mathcal{T}'[-1]$ is equivalent to a linear tree. Indeed, this is clear if $\mathcal{T}'$ contracts to a linear local tree; if $\mathcal{T}'$ doesn't contract to a linear local tree, the claim follows from the inductive hypothesis applied to $\mathcal{W}_{i_2-1} \leftarrow \mathcal{T}'$ and the weak sequence $\mathcal{W}_{i_2-1}, \mathcal{W}_{i_2}, \ldots, \mathcal{W}_k$. Then the inductive step follows by applying the inductive hypothesis to $\mathcal{W}_0 \leftarrow \mathcal{T}$ and the weak sequence $\mathcal{W}_0, \ldots, \mathcal{W}_{i_2-1}, \mathcal{T}'$.

Case $q = 1$. If $\mathcal{T}$ has a superfluous vertex u that is not a neighbour of the principal vertex, then u is a superfluous vertex of $\mathcal{W}_0$. Let $\mathcal{W}'_0$ be the elementary contraction of $\mathcal{W}_0$ at u and form the commutative diagram:

$$\begin{array}{ccccccccc}
\mathcal{T} & \longrightarrow & \mathcal{W}_0 & \overset{+}{\longleftarrow} & \mathcal{W}_1 & \longleftarrow & \cdots & \longleftarrow & \mathcal{W}_k \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & & & \rotatebox{90}{$\leq$} \\
\mathcal{T}' & \longrightarrow & \mathcal{W}'_0 & \overset{+}{\longleftarrow} & \mathcal{W}'_1 & \longleftarrow & \cdots & \longleftarrow & \mathcal{W}'_k .
\end{array}$$

Since $\mathcal{T}'$ doesn't contract to a linear local tree and $\mathcal{W}_k[\mathcal{G}] \sim \mathcal{W}'_k[\mathcal{G}]$, it's enough to prove this case for the weak sequence $\mathcal{W}'_0, \ldots, \mathcal{W}'_k$ and $\mathcal{W}'_0 \leftarrow \mathcal{T}'$. In other words, we may assume that

(**) *all superfluous vertices of $\mathcal{T}$ are neighbours of the principal vertex.*

Consider the blowings-up

$$\mathcal{T} \longrightarrow \mathcal{W}_0 = \mathcal{T}_0 \longleftarrow \mathcal{T}_1 \longleftarrow \cdots \longleftarrow \mathcal{T}_k$$

such that if e is the vertex created in $\mathcal{T}_0 \leftarrow \mathcal{T}_1$ then for some $\mathcal{G}_1, \ldots, \mathcal{G}_p$ $(p \geq 1)$ we have $\mathcal{W}_i = \mathcal{T}_i^{e, \mathcal{G}_1, \ldots, \mathcal{G}_p}$ $(i = 1, \ldots, k)$. Before we continue the proof, let us state a definition and a lemma:

Definition. Let $\mathcal{T} = (T, x, R, \Omega)$ and $\mathcal{T}_i = (T_i, x_i, R_i, \Omega_i)$ $(i = 0, 1)$ be local trees and suppose that $\mathcal{T}$ has one principal link and that $\mathcal{T} \to \mathcal{T}_0 \leftarrow \mathcal{T}_1$. Let e (resp. e') be the vertex created in $\mathcal{T}_0 \leftarrow \mathcal{T}_1$ (resp. $\mathcal{T}_0 \leftarrow \mathcal{T}$). We define an injective set map $T \backslash \{x\} \to T_1$ by

$$\begin{cases} e' \longmapsto e, \\ t \longmapsto \beta_1(\beta^{-1}(t)), \qquad t \in T\setminus\{e', x\}, \end{cases}$$

where $\beta_1 : T_0 \to T_1$ and $\beta : T_0 \to T$ are the identification maps. That map should be thought of as a natural embedding of $\mathcal{T}$ in $\mathcal{T}_1$ (or in $\mathcal{T}_1^{e,\mathcal{G}_1,\dots,\mathcal{G}_p}$, for arbitrary $\mathcal{G}_1,\dots,\mathcal{G}_p$). Observe that the root of $\mathcal{T}$ is not embedded in these trees.

Lemma. *Consider local trees $\mathcal{T} \to \mathcal{T}_0 \leftarrow \mathcal{T}_1$, where $\mathcal{T}$ has one principal link. Let e be vertex created in $\mathcal{T}_0 \leftarrow \mathcal{T}_1$, let $\mathcal{G}_1,\dots,\mathcal{G}_p$ be weighted trees with roots and embed $\mathcal{T}$ in $\mathcal{W}_1 = \mathcal{T}_1^{e,\mathcal{G}_1,\dots,\mathcal{G}_p}$ as in the above definition. Let b be a vertex of $\mathcal{T}$, other than the root; then b has same weight in $\mathcal{W}_1$ as in $\mathcal{T}$. Let $\mathcal{B}_1,\dots,\mathcal{B}_n$ $(n \geq 0)$ be the branches of $\mathcal{T}$ at b, not containing the root. Then the following hold:*

1. *If b is not the principal vertex of $\mathcal{T}$ then the branches of $\mathcal{W}_1$ at b, not containing the root, are $\mathcal{B}_1,\dots,\mathcal{B}_n$. $\mathcal{W}_1$ has one more branch $\mathcal{B}_*$ at b: $\mathcal{B}_*$ contains the root, all principal vertices, the extra branches $\mathcal{G}_1,\dots,\mathcal{G}_p$ and possibly other vertices.*
2. *If b is the principal vertex of $\mathcal{T}$ then b is just e in $\mathcal{W}_1$, so $\mathcal{G}_1,\dots,\mathcal{G}_p$ are branches of $\mathcal{W}_1$ at b. Moreover:*
   (a) *If $\mathcal{W}_1$ has one principal link then its other branches at b, not containing the root, are $\mathcal{B}_1,\dots,\mathcal{B}_n$, and $\mathcal{W}_1$ has one more branch $\mathcal{B}_*$ at b: $\mathcal{B}_*$ is just the root.*
   (b) *If $\mathcal{W}_1$ has two principal links then its other branches at b, not containing the root, are $\mathcal{B}_1,\dots,\mathcal{B}_{n-1}$ (if $\mathcal{B}_1,\dots,\mathcal{B}_n$ are suitably labelled); $\mathcal{W}_1$ has one more branch $\mathcal{B}_*$ at b: $\mathcal{B}_*$ consists of the root and $\mathcal{B}_n$.*

We now return to the proof of (3.7). Since $\mathcal{T}$ doesn't contract to a linear local tree, it is not a linear local tree; so $\mathcal{T}$ must have a branch point. Let b be a branch point of $\mathcal{T}$, and let $\mathcal{B}_1,\dots,\mathcal{B}_n$ $(n \geq 2)$ be the branches of $\mathcal{T}$ at b, not containing the root. Embed $\mathcal{T}$ in $\mathcal{W}_1$ as in the above definition.

If b is not the principal vertex of $\mathcal{T}$ then by the above lemma the branches of $\mathcal{W}_k$ at b are $\mathcal{B}_1,\dots,\mathcal{B}_n$ and $\mathcal{B}_*$, where $\mathcal{B}_*$ contains, in particular, the principal vertex of $\mathcal{W}_k$. Since $\mathcal{W}_k[\mathcal{G}]$ contracts to a linear weighted tree, b must "absorb" $n-1$ of the $n+1$ branches (of $\mathcal{W}_k[\mathcal{G}]$ at b) *so by* (*) it must absorb some $\mathcal{B}_i$. This is impossible, because by (**) $\mathcal{B}_i$ contains no superfluous vertices (for b is not the principal vertex of $\mathcal{T}$).

So, not only does $\mathcal{T}$ contract to a local tree whose only branch point is its principal vertex, but $\mathcal{T}$ itself is such a tree (this is because of assumption (**)). Now let b be the principal vertex of $\mathcal{T}$; then $\mathcal{B}_1,\dots,\mathcal{B}_n$ are linear branches. To finish the proof, there are two cases to consider–and in both cases we have $b = e$ in $\mathcal{W}_i$ $(i \geq 1)$.

Case 1. $\mathcal{W}_1$ has one principal link.
By the above lemma, the branches of $\mathcal{W}_k$ at b are $\mathcal{B}_1,\dots,\mathcal{B}_n$, $\mathcal{G}_1,\dots,\mathcal{G}_p$ and $\mathcal{B}_*$, where $\mathcal{B}_*$ contains the root of $\mathcal{W}_k$ (but $\mathcal{B}_*$ may not contain the principal vertex

of $\mathcal{W}_k$ since b might be that vertex). For each i, if b can absorb $\mathcal{B}_i$ in $\mathcal{W}_k$ then b can absorb $\mathcal{B}_i$ in $\mathcal{T}$. Since $\mathcal{T}$ doesn't contract to a linear local tree, at least two $\mathcal{B}_i$'s can't be absorbed (in $\mathcal{T}$, hence in $\mathcal{W}_k[\mathcal{G}]$). Thus b must absorb every other branch (in $\mathcal{W}_k[\mathcal{G}]$) and, in particular, $\mathcal{G}_1, \ldots, \mathcal{G}_p$.

Case 2. $\mathcal{W}_1$ has two principal links.
By the above lemma, if $\mathcal{B}_1, \ldots, \mathcal{B}_n$ are suitably labelled then the branches of $\mathcal{W}_k$ at b are $\mathcal{B}_1, \ldots, \mathcal{B}_{n-1}, \mathcal{G}_1, \ldots, \mathcal{G}_p$ and $\mathcal{B}_*$ where, now, $\mathcal{B}_*$ does contain the principal vertex of $\mathcal{W}_k$ (because $\mathcal{W}_1$ has two principal links and $\mathcal{W}_k$ has only one $\Rightarrow k > 1$ and b is distinct from the principal vertex of $\mathcal{W}_k$). By (*), b can't absorb $\mathcal{B}_*[\mathcal{G}]$. Hence at most one branch in $\mathcal{B}_1, \ldots, \mathcal{B}_{n-1}, \mathcal{G}_1, \ldots, \mathcal{G}_p$ can't be absorbed by b. Since $\mathcal{T}$ doesn't contract to a linear local tree, some $\mathcal{B}_i (i < n)$ can't be absorbed, so b absorbs $\mathcal{G}_1, \ldots, \mathcal{G}_p$.

We point out the following fact, which was established in the above proof.

Lemma 3.8. *Let $\mathcal{W}_0, \ldots, \mathcal{W}_k$ be a weak sequence such that the principal vertex of $\mathcal{W}_k$ is a branch point. If there exists a weighted tree with a root $\mathcal{G}$ such that $\mathcal{W}_k[\mathcal{G}]$ is equivalent to a linear weighted tree $\mathcal{L}$ with $\langle \mathcal{L} \rangle \leq 1$, then no $\mathcal{W}_i$ with $i < k$ can contract to a local tree having a nonprincipal vertex of nonnegative weight.*

Theorem 3.9. *Let ω, k be positive integers and let*

$$\mathcal{S} : \mathcal{T}_0 \longleftarrow \cdots \longleftarrow \mathcal{T}_k$$

be such that $\mathcal{T}_0 \geq (\omega)$, $\mathcal{T}_k$ has one principal link and its principal vertex is a branch point. Suppose that $\mathcal{T}_k$ does not contract to a linear local tree and that there exist a linear weighted tree $\mathcal{L}$ and a weighted tree with a root $\mathcal{G}$ such that $\mathcal{T}_k[\mathcal{G}] \sim \mathcal{L}$. Finally, suppose that $\mathcal{T}_{k-1}$ can't contract to a local tree having a nonprincipal vertex of nonnegative weight (and note that by (3.8) *this condition holds whenever $\langle \mathcal{L} \rangle \leq 1$).*

Then $\mathcal{T}_k$ contracts to a local tree whose only branch point is its principal vertex and $\mathcal{G}$ can be absorbed by the principal vertex of $\mathcal{T}_k$.
Let $\mu = (\mu_0, \ldots, \mu_k)$ be the unique element of $\operatorname{Mul}(\mathcal{S})$ *such that $\mu_k(a_k) = 1$, where a_j is the principal link of $\mathcal{T}_j$ whenever $\mathcal{T}_j$ has only one principal link, and write $i = \mu_0(a_0)$ and $r_j = \mu_j(x_0)$, $0 \leq j < k$. Define integers w and p by*

$$w - \sum_{j=0}^{k-1} r_j^2 = -1 \qquad \textit{and} \qquad p - \sum_{j=0}^{k-1} \frac{r_j(r_j - 1)}{2} = 0.$$

Then the following conditions hold, where we use the notations of (1.9) *determined by $\mathcal{S}$ and μ:*
(a) *If* $\gcd(i, r_0) = 1$ *(i.e., $l = 1$) then*

$$w = ir_0 - 1 \qquad \textit{and} \qquad p = \frac{(i-1)(r_0 - 1)}{2}.$$

Moreover, if $\mathcal{L} = [1]$ then $\omega > 2$, $i = \omega - 1$, $r_0 = 1$ and the principal vertex of $\mathcal{T}_k$ gets weight 0 after absorption of $\mathcal{G}$.

(b) *If* $\gcd(i, r_0) \neq 1$ *(i.e.* $l > 1$*) then* $\mathscr{S}_h: \mathscr{T}_0 \leftarrow \cdots \leftarrow \mathscr{T}_h$ *is of type* ω, *where* $h = h_{l-1}$; *thus* $n_{l-1} = m_{l-2}/i_{l-1}$ *is a positive integer. Writing* $\delta = j_{l-1} - h \geq 0$, *we have* $n_{l-1} \geq \delta$ *and*

$$i^2 - w = (\omega - 1)\sum_{j=0}^{h-1} r_j^2 + [\omega n_{l-1} + 1 - \delta]\, i_{l-1}^2 - i_{l-1}m_{l-1} + 1.$$

$$\left(1 + \frac{2}{\omega}\right)i + 2p - w - 2 = [n_{l-1} - \delta + 2/\omega]\, i_{l-1} - m_{l-1}.$$

Moreover, if $\mathscr{L} = [1]$ *then* $n_{l-1} = \delta$, $\omega > 2$, $i_{l-1} = \omega - 1$, $m_{l-1} = 1$ *and the principal vertex of* $\mathscr{T}_k$ *gets weight* 0 *after absorption of* $\mathscr{G}$.

(c) *If* $\omega \leq 2$ *or* $\mathscr{L} = [1]$ *or* $\mathscr{T}_{j_{l-1}} \not\geq (\omega)$ *then*

$$\left(1 + \frac{2}{\omega}\right)i + 2p - w - 2 > 0.$$

Before we prove (3.9) we state some numerical lemmas, the verification of which we leave to the reader. But first, let us introduce the notation

$$f(x) = \frac{x(x-1)}{2}, \qquad x \in \mathbf{Z}.$$

Lemma 3.10. *Let* $i \geq \rho_0 \geq i' > 0$ *be integers such that* $\gcd(i, \rho_0) = i'$. *If the corresponding euclidean algorithm is written as*

$$\begin{aligned} i &= \alpha_0\rho_0 + \rho_1 \\ \rho_0 &= \alpha_1\rho_1 + \rho_2 \\ &\vdots \\ \rho_{s-1} &= \alpha_s\rho_s \qquad (\text{where } \rho_s = i'), \end{aligned}$$

then $\alpha_0\rho_0^2 + \cdots + \alpha_s\rho_s^2 = i\rho_0$ *and* $\alpha_0\rho_0 + \cdots + \alpha_s\rho_s = i + \rho_0 - i'$.

Together with (1.8), this gives

Corollary 3.11. *Let* $(\mathscr{T}_0, \mu_0) \Leftarrow \cdots \Leftarrow (\mathscr{T}_k, \mu_k)$ $(k \geq 1)$ *be such that* $\mathscr{T}_v$ *has one principal link iff* $v \in \{0, k\}$. *Let* a *(resp.* a'*) be the principal link of* $\mathscr{T}_0$ *(resp.* $\mathscr{T}_k$*) and write* $i = \mu_0(a)$, $i' = \mu_k(a')$. *Let* $r_v = \mu_v(x_0)$, $0 \leq v \leq k-1$. *Then*

$$\sum_{j=0}^{k-1} r_j^2 = ir_0, \quad \sum_{j=0}^{k-1} r_j = i + r_0 - i' \quad \text{and} \quad \sum_{j=0}^{k-1} f(r_j) = \frac{ir_0 - i - r_0 + i'}{2}.$$

Lemma 3.12. *Let* ω, i, i' *be positive integers and let* $(r_0, \ldots, r_{k-1}) = (m_0, \ldots, i_l)$ *be a sequence of type* (ω, i, i'), *with notation as in* (1.19). *If* $m = m_0 + \cdots + m_{l-1}$, *then*

1. $i = \omega m + i'$.

2. $\sum_{j=0}^{k-1} r_j = (\omega + 2)m - m_{l-1}$,

3. $i^2 = \omega \sum_{j=0}^{k-1} r_j^2 + (\omega n_l + 1)i'^2$,

4. $f(i - 1) = \omega \sum_{j=0}^{k-1} f(r_j) + f(\omega)m + (\omega n_l + 1)f(i') - i' + 1$.

Proof of (3.9). Let the notations of (1.9) be in force, i.e., we consider $\mathscr{J}(\mathscr{S}) = \{j_0, \dots, j_{l-1}\}$, $\mathscr{H}(\mathscr{S}) = \{h_1, \dots, h_l\}$, and the numbers $i_0, \dots, i_l$ and $m_0, \dots, m_{l-1}$. For $1 \le v \le l$, let e_v be the branch point created in $\mathscr{T}_{h_v - 1} \leftarrow \mathscr{T}_{h_v}$.

As in (1.23), we may assume that $\mathscr{T}_0 = (\omega)$. Then $\mathscr{T}_v$ is a comb with negative teeth, $0 \le v \le k$, by (1.31). Also, $\mathscr{T}_k = (*, -1, \mathscr{A}, \mathscr{B})$ where $\mathscr{A}$, $\mathscr{B}$ are the branches at e_l, not containing the root, and $\mathscr{A} < -1$ is a linear branch. So the branches of $\mathscr{T}_k[\mathscr{G}]$ at e_l are $\mathscr{A}, \mathscr{B}$ and $\mathscr{G}$. Now e_l can't absorb $\mathscr{A} < -1$, and e_l can't absorb $\mathscr{B}$ (for $\mathscr{T}_k$ doesn't contract to a linear local tree). However, $\mathscr{T}_k[\mathscr{G}] \sim \mathscr{L}$ so e_l must absorb some branch, in $\mathscr{T}_k[\mathscr{G}]$, by (5.11) of [3]. Hence e_l absorbs $\mathscr{G}$.

If $l = 1$ then $\mathscr{T}_k$ is already a local tree whose only branch point is its principal vertex. If $l > 1$, let's prove that $\mathscr{T}_k$ contracts to such a local tree. $\mathscr{T}_k$ has three branches at e_{l-1}, say $\mathscr{A}'$, $\mathscr{B}'$ and $\mathscr{B}'_*$ where $\mathscr{A}' < -1$ is a linear branch and $\mathscr{B}'_*$ contains, in particular, e_l, which is a branch point of weight -1; hence e_{l-1} can absorb neither $\mathscr{A}'$ nor $\mathscr{B}'_*[\mathscr{G}]$ in $\mathscr{T}_k[\mathscr{G}]$. Since $\mathscr{T}_k[\mathscr{G}]$ contracts to a linear tree, e_{l-1} must absorb one of the three branches. So it absorbs $\mathscr{B}'$ and $\mathscr{T}_k$ contracts as specified.

Before we prove that conditions (a)–(c) hold, let us explain why $l = 1$ is equivalent to $\gcd(i, r_0) = 1$, as asserted in (a) and (b). We claim that $\mathscr{T}_1$ has two principal links. If not, then $\mathscr{T}_1 = (*, -1, \omega - 1)$ has a nonprincipal vertex with nonnegative weight, and so do $\mathscr{T}_2, \dots, \mathscr{T}_{k-1}$, so one of the hypotheses of the theorem is violated. Hence:

(1) $\mathscr{T}_1$ *has two principal links.*

Clearly, $\mathscr{T}_{k-1}$ has two principal links, since the principal vertex e_l of $\mathscr{T}_k$ is a branch point. Thus it is clear that $l = 1$ is equivalent to: $\mathscr{T}_v$ *has one principal link iff* $v \in \{0, k\}$, and by (1.8), $l = 1 \Leftrightarrow \gcd(i, r_0) = \mu(a_k) = 1$.

CONDITION (a). The two equations follow immediately from (3.11). The other assertions follow from the proof of condition (b), from (15) to the end: Put $l = 1$ and observe that $\mathscr{T}_{j_{l-1}} = \mathscr{T}_0 = (\omega)$, *i.e.*, the two rows of diagram (15) are the same.

CONDITION (b). Suppose $l > 1$. Consider the integer $h = h_{l-1} > 0$; the branch point e_{l-1}, which can absorb the branch $\mathscr{B}'$ of $\mathscr{T}_k$, was created in $\mathscr{T}_{h-1} \leftarrow \mathscr{T}_h$; in fact, $\mathscr{T}_h = (*, -1, \mathscr{A}', \mathscr{B}')$, so $\mathscr{B}'$ can be absorbed in $\mathscr{T}_h$ as well. Hence

(2) *$\mathscr{T}_h$ contracts to a linear local tree.*

So we consider the sequence $\mathscr{S}_h$: $(\omega) = \mathscr{T}_0 \leftarrow \cdots \leftarrow \mathscr{T}_h$. By (1.23),

(3) *$\mathscr{S}_h$ is of type ω, $\mathscr{T}_h = (*, -1, (-n-1, -2, \ldots, -2), \mathscr{B}')$ where $n = n_{l-1}$ and where "-2" occurs $\omega - 1$ times, and the absorption of $\mathscr{B}'$ increases by 1 the weight of e_{l-1}.*

Observe that $\mu_h(a_h) = i_{l-1}$ by definition. By (3),

(4) *$(r_0, \ldots, r_{h-1})$ is of type (ω, i, i_{l-1}).*

Applying (3.12) to $\mathscr{S}_h$, we deduce (where $m = m_0 + \cdots + m_{l-2}$):

(5) $$i^2 = \omega \sum_{j=0}^{h-1} r_j^2 + [\omega n + 1] i_{l-1}^2,$$

(6) $$f(i-1) = \omega \sum_{j=0}^{h-1} f(r_j) + f(\omega) m + [\omega n + 1] f(i_{l-1}) - i_{l-1} + 1,$$

(7) $$\sum_{j=0}^{h-1} r_j = (\omega + 2) m - m_{l-2},$$

(8) $$i = \omega m + i_{l-1}.$$

By definition of j_{l-1} and h, $\mathscr{T}_j$ has one principal link whenever $h \le j \le j_{l-1}$, so

(9) $$h \le j < j_{l-1} \Longrightarrow r_j = i_{l-1}.$$

If we define $\delta = j_{l-1} - h \ge 0$, then

$$\mathscr{T}_{j_{l-1}} = (*, -1, -2, \ldots, -2, (-n-1, -2, \ldots, -2), \mathscr{B}'),$$

where the first sequence of "-2" contains δ terms and the second $\omega - 1$ terms. So $\mathscr{T}_{j_{l-1}}$ contracts to the following linear local tree:

$$\mathscr{L}_{j_{l-1}} = (*, 0, \delta - n - 1, -2, \ldots, -2),$$

where "-2" occurs $\omega - 1$ times. We claim that $\delta \le n$. In fact, if $\delta > n$ then $\mathscr{L}_{j_{l-1}}$ has a nonprincipal vertex with nonnegative weight. Since by definition $j_{l-1} < k$, one can consider the commutative diagram (1.28) determined by $\mathscr{L}_{j_{l-1}} \le \mathscr{T}_{j_{l-1}} \leftarrow \cdots \leftarrow \mathscr{T}_{k-1}$ and deduce that $\mathscr{T}_{k-1}$ contracts to a local tree which contains a nonprincipal vertex with nonnegative weight. This contradicts one of the assumptions. So,

(10) $$\delta \le n.$$

On the other hand, we have $\mu_{j_{l-1}}(x_0) = m_{l-1}$ and $\mu_{j_{l-1}}(a_{j_{l-1}}) = i_{l-1}$ by definition, and $(i_{l-1}, m_{l-1}) = i_l = 1$ by our choice of $\mu \in \mathrm{Mul}(\mathscr{S})$. By (3.11),

(11) $$\sum_{j=j_{l-1}}^{k-1} r_j^2 = i_{l-1} m_{l-1}.$$

$$\sum_{j=j_{l-1}}^{k-1} r_j = i_{l-1} + m_{l-1} - 1, \tag{12}$$

$$\sum_{j=j_{l-1}}^{k-1} f(r_j) = \frac{(i_{l-1} - 1)(m_{l-1} - 1)}{2}. \tag{13}$$

We can now check that the two equations of condition (b) hold.

$$\begin{aligned} i^2 - w &= (i^2 - \sum_{j=0}^{h-1} r_j^2) - \sum_{j=h}^{j_{l-1}-1} r_j^2 - \sum_{j=j_{l-1}}^{k-1} r_j^2 + 1 \\ &= (\omega - 1)\sum_{j=0}^{h-1} r_j^2 + (\omega n + 1) i_{l-1}^2 - \delta i_{l-1}^2 - i_{l-1} m_{l-1} + 1, \end{aligned}$$

by (5), (9) and (11). So

$$i^2 - w = (\omega - 1)\sum_{j=0}^{h-1} r_j^2 + (\omega n + 1 - \delta) i_{l-1}^2 - i_{l-1} m_{l-1} + 1, \tag{14}$$

which is the first equation, since $n = n_{l-1}$ by definition–see (3). For the second equation, observe that

$$\begin{aligned} f(i-1) - p &= (f(i-1) - \sum_{j=0}^{h-1} f(r_j)) - \sum_{j=h}^{j_{l-1}-1} f(r_j) - \sum_{j=j_{l-1}}^{k-1} f(r_j) \\ &= (\omega - 1)\sum_{j=0}^{h-1} f(r_j) + f(\omega)m + (\omega n + 1 - \delta) f(i_{l-1}) - i_{l-1} \\ &\quad + 1 - \frac{(i_{l-1} - 1)(m_{l-1} - 1)}{2}, \end{aligned}$$

by (6), (9) and (13). By multiplying that equation by 2 we obtain

$$\begin{aligned} i^2 - 3i + 2 - 2p &= (\omega - 1)\sum_{j=0}^{h-1} r_j^2 - (\omega - 1)\sum_{j=0}^{h-1} r_j + 2f(\omega)m \\ &\quad + (\omega n + 1 - \delta)(i_{l-1}^2 - i_{l-1}) - 2i_{l-1} + 2 - i_{l-1} m_{l-1} \\ &\quad + i_{l-1} + m_{l-1} - 1 \\ &= (i^2 - w) - (\omega - 1)\sum_{j=0}^{h-1} r_j + 2f(\omega)m \\ &\quad - (\omega n + 2 - \delta) i_{l-1} + m_{l-1} \end{aligned}$$

by (14). Therefore,

$$\begin{aligned} 3i + 2p - w - 2 &= (\omega - 1)\sum_{j=0}^{h-1} r_j - 2f(\omega)m + (\omega n + 2 - \delta) i_{l-1} - m_{l-1} \\ &= (\omega - 1)[(\omega + 2)m - m_{l-2}] - \omega(\omega - 1)m \\ &\quad + (\omega n + 2 - \delta) i_{l-1} - m_{l-1} \end{aligned}$$

$$\begin{aligned} &= 2(\omega - 1)m - (\omega - 1)m_{l-2} + (\omega n + 2 - \delta)i_{l-1} - m_{l-1} \\ &= 2(\omega - 1)m - (\omega - 1)m_{l-2} + (\omega - 1)ni_{l-1} \\ &\quad + (n + 2 - \delta)i_{l-1} - m_{l-1} \\ &= 2(\omega - 1)m + (n + 2 - \delta)i_{l-1} - m_{l-1} \end{aligned}$$

by (7) and the fact that $-m_{l-2} + ni_{l-1} = 0$, which follows from (4). Since $m = (i - i_{l-1})/\omega$ by (8), we find

$$3i + 2p - w - 2 = 2\left(1 - \frac{1}{\omega}\right)(i - i_{l-1}) + (n + 2 - \delta)i_{l-1} - m_{l-1},$$

from which the desired equation follows.

Next, consider the diagram

$$\begin{array}{ccc} \mathcal{T}_{j_{l-1}} & \longleftarrow \cdots \longleftarrow & \mathcal{T}_k \\ \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} \\ \mathcal{L}_{j_{l-1}} = \mathcal{T}^*_{j_{l-1}} & \longleftarrow \cdots \longleftarrow & \mathcal{T}^*_k . \end{array}$$

If $\delta < n$ then, by the description of $\mathcal{L}_{j_{l-1}}$ given above, between (9) and (10), we conclude that $\mathcal{T}^*_k = (*, -1, \mathcal{A}^*, \mathcal{B}^*)$ where $\mathcal{A}^* = \mathcal{A} < -1$ and $\mathcal{B}^* < -1$ are linear branches. Recall that the principal vertex of $\mathcal{T}_k$ can absorb $\mathcal{G}$ (in $\mathcal{T}_k[\mathcal{G}]$). Thus the principal vertex of $\mathcal{T}^*_k$ can absorb $\mathcal{G}$ (in $\mathcal{T}^*_k[\mathcal{G}]$) and

$$\mathcal{L} \sim \mathcal{T}_k[\mathcal{G}] \sim \mathcal{T}^*_k[\mathcal{G}] \sim [A^*, \alpha, B^*],$$

where A^* and B^* are nonempty sequences of integers less than -1 and α is some nonnegative integer. So $\mathcal{L} \not\sim [1]$ by, say, (5.16) of [3]. Thus $\mathcal{L} \sim [1] \Rightarrow \delta = n$, as asserted.

Now assume that $\mathcal{L} \sim [1]$. Then $\delta = n$ and, by our knowledge of $\mathcal{L}_{j_{l-1}}$, $\mathcal{T}_{j_{l-1}} \geq \mathcal{L}_{j_{l-1}} \geq (\omega)$. Consider the local tree $\mathcal{T}$ defined by

$$(15) \qquad \begin{array}{ccc} \mathcal{T}_{j_{l-1}} & \longleftarrow \cdots \longleftarrow & \mathcal{T}_k \\ \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} \\ (\omega) & \longleftarrow \cdots \longleftarrow & \mathcal{T} . \end{array}$$

In that diagram, each tree in the lower row has the same number of principal links as the corresponding tree in the upper row; hence, in the lower row, only (ω) and $\mathcal{T}$ have one principal link (all others have two). Thus $\mathcal{T} = (*, -1, \mathcal{A}, (b_1, \ldots, b_v, \omega'))$, where $\mathcal{A} < -1$ is a linear branch, $\omega' < \omega$ is the weight of the vertex which *was* the principal vertex of (ω), $v \geq 0$ and $b_i < -1$ for $1 \leq i \leq v$. Since $\mathcal{T}_k$ doesn't contract to a linear tree, $\mathcal{T}$ doesn't contract to a linear tree, i.e.,

$$(16) \qquad \omega' \neq -1 \quad \textit{or} \quad \exists_i b_i < -2.$$

In the notation of (1.22), write $\mathcal{A} = (\alpha_1, \ldots, \alpha_q)$. Since e_l absorbs $\mathcal{G}$, $\mathcal{T}[\mathcal{G}]$ contracts to the linear weighted tree

$$\mathscr{H} = [\alpha_q, \dots, \alpha_1, \alpha, b_1, \dots, b_v, \omega'],$$

where $\alpha \geq 0$. Now $\mathscr{H} \sim \mathscr{T}[\mathscr{G}] \sim \mathscr{T}_k[\mathscr{G}] \sim [1]$, so $\mathscr{H}$ must be minimal. Indeed, if $\mathscr{H}$ is not minimal then $\omega' = -1$ and by (16) it contracts to a minimal weighted tree $\mathscr{H}' = [\alpha_q, \dots, \alpha_1, \alpha, b_1, \dots, b_{i-1}, b_i + 1]$ which has more than two vertices but only one nonnegative weight. Such a tree can't be equivalent to [1], again by [3], (5.16).

So $\mathscr{H}$ is minimal. Since $|\mathscr{H}| > 2$, (5.16) of [3] implies that $v = 0$, $\alpha = 0$ and $\omega' > 0$. Since $v = 0$, every vertex of $\mathscr{A}$ has weight -2 by definition of $\mathscr{T}$, thus $\omega' = 1$ by [3], (5.16) again. Now $v = 0$ implies that i_{l-1} is a multiple of m_{l-1}, because of the relation (see (1.8)) between the euclidean algorithm of (i_{l-1}, m_{l-1}) and the sequence of multiplicities of the roots in $((\omega), \mu_{j_{l-1}}) \Leftarrow \cdots \Leftarrow (\mathscr{T}, \mu_k)$. Thus $m_{l-1} = i_l = 1$ and $1 = \omega' = \omega - i_{l-1}$. Since $i_{l-1} > m_{l-1}$ by definition, we get $\omega = 1 + i_{l-1} > 2$. This proves condition (b). (Note that the principal vertex of $\mathscr{T}_k$ gets weight $\alpha = 0$ after absorption of $\mathscr{G}$.)

CONDITION (c). If $\gcd(i, r_0) = 1$ then

$$\left(1 + \frac{2}{\omega}\right)i + 2p - w - 2 = \frac{2}{\omega}i - r_0$$

by condition (a). Clearly, $\frac{2}{\omega}i - r_0 > 0$ if $\omega \leq 2$; if $\mathscr{L} = [1]$ then $\frac{2}{\omega}i - r_0 = 1 - \frac{2}{\omega} > 0$ by condition (a) again. Note that $j_{l-1} = j_0 = 0$, so it is not possible that $\mathscr{T}_{j_{l-1}} \ngeq (\omega)$.

Now suppose that $\gcd(i, r_0) \neq 1$. By condition (b), it is enough to prove $\left(n_{l-1} - \delta + \frac{2}{\omega}\right)i_{l-1} - m_{l-1} > 0$. This is certainly the case if $\omega \leq 2$ $\left(\text{for } n_{l-1} - \delta + \frac{2}{\omega} \geq \frac{2}{\omega} \geq 1\right)$. By (b), if $\mathscr{L} = [1]$ then $\delta = n_{l-1}$, $m_{l-1} = 1$, $i_{l-1} = \omega - 1$ and $\omega > 2$, so

$$\left(n_{l-1} - \delta + \frac{2}{\omega}\right)i_{l-1} - m_{l-1} = \frac{2}{\omega}(\omega - 1) - 1 = 1 - \frac{2}{\omega} > 0.$$

By our knowledge of $\mathscr{L}_{j_{l-1}}$, it is clear that the condition $\mathscr{T}_{j_{l-1}} \ngeq (\omega)$ is equivalent to $\delta < n_{l-1}$, which implies

$$\left(n_{l-1} - \delta + \frac{2}{\omega}\right)i_{l-1} - m_{l-1} > i_{l-1} - m_{l-1} > 0.$$

This completes the proof of the theorem.

Corollary 3.13. *Let* $\mathscr{S}: \mathscr{T}_0 \leftarrow \cdots \leftarrow \mathscr{T}_k$ *satisfy the hypothesis of theorem* (3.9) *for some* ω *and assume, in addition, that*

$$\gcd(i, r_0) = 1 \quad or \quad \omega \leq 2 \quad or \quad \mathscr{L} = [1] \quad or \quad \mathscr{T}_{j_{l-1}} \ngeq (\omega).$$

Then no triple (d, u, v) *of nonnegative numbers can satisfy one of the following two conditions*:

(a) $u + v \le d,\ i = d,\ w = d^2 - u^2 - v^2,$

$$\text{and } p = \frac{(d-1)(d-2)}{2} - \frac{u(u-1)}{2} - \frac{v(v-1)}{2};$$

(b) $v + v + r_0 \le d,\ i = d + (\omega - 1)r_0,\ w = d^2 - u^2 - v^2 + (\omega - 1)r_0^2,$

$$\text{and } p = \frac{(d-1)(d-2)}{2} - \frac{u(u-1)}{2} - \frac{v(v-1)}{2} + (\omega - 1)\frac{r_0(r_0-1)}{2}.$$

Proof. Assume that $\gcd(i, r_0) = 1$. Then by (3.9), $w = ir_0 - 1$ and $p = (i-1)(r_0 - 1)/2$. If (d, u, v) satisfies (a), then $d^2 - u^2 - v^2 = dr_0 - 1$ and $d^2 - 3d + 2 - u^2 - v^2 + u + v = (d-1)(r_0 - 1)$. These two equations imply that $(d - u - v) + (d - r_0) = 0$, whence $r_0 \ge d = i$, which is absurd. (Note that whenever the hypothesis of (3.9) is satisfied we have $i > r_0$, because of (1) in the proof of (3.9).) If (d, u, v) satisfies (b), then $d^2 - u^2 - v^2 + (\omega - 1)r_0^2 = dr_0 + (\omega - 1)r_0^2 - 1$ and $(d-1)(d-2) - u(u-1) - v(v-1) + (\omega - 1)r_0(r_0 - 1) = (d + (\omega - 1)r_0 - 1)(r_0 - 1)$. From these two equations, we find $(d - u - v - r_0) + d = 0$, whence $d \le 0$, contradiction. That proves the case $\gcd(i, r_0) = 1$.

Now assume that $\gcd(i, r_0) \ne 1$. Then $\omega \le 2$ or $\mathscr{L} = [1]$ or $\mathscr{T}_{j_{l-1}} \not\ge (\omega)$, so condition (c) of theorem (3.9) says that $B > 0$, where we define

$$A = i^2 - w - 1, \qquad B = \left(1 + \frac{2}{\omega}\right)i + 2p - w - 2.$$

Now a little calculation gives

$$(1) \qquad A = \begin{cases} u^2 + v^2 - 1, & \text{if (a) holds,} \\ 2d(\omega - 1)r_0 + (\omega - 1)(\omega - 2)r_0^2 + u^2 + v^2 - 1, & \text{if (b) holds,} \end{cases}$$

$$(2) \qquad B = \begin{cases} (-2 + 2/\omega)d + u + v, & \text{if (a) holds,} \\ (-2 + 2/\omega)(d - r_0) + u + v, & \text{if (b) holds.} \end{cases}$$

If $\omega \ge 2$ then $-2 + \dfrac{2}{\omega} \le -1$, so

$$0 < B \le \begin{cases} -d + u + v \le 0, & \text{if (a) holds,} \\ -(d - r_0) + u + v \le 0, & \text{if (b) holds,} \end{cases}$$

and this is absurd. If $\omega = 1$ then, by (3.9),

$$A = (n_{l-1} + 1 - \delta)i_{l-1}^2 - i_{l-1}m_{l-1} = i_{l-1}((n_{l-1} + 1 - \delta)i_{l-1} - m_{l-1}) = xy$$

$$B = (n_{l-1} - \delta + 2)i_{l-1} - m_{l-1} = i_{l-1} + (n_{l-1} - \delta + 1)i_{l-1} - m_{l-1} = x + y$$

where we define $x = i_{l-1}$ and $y = (n_{l-1} - \delta + 1)i_{l-1} - m_{l-1}$. Thus x and y are integers, $x \ge 2$ and $y \ge 1$. Whence $B^2 - 2A = x^2 + y^2 \ge 5$. On the other hand, we find from (1) and (2) that if either (a) or (b) holds then $B^2 - 2A = 2 - (u-v)^2 \le 2$, which is absurd.

Corollary 3.14. *Let $\mathscr{S}: \mathscr{T}_0 \leftarrow \cdots \leftarrow \mathscr{T}_k$ satisfy the hypothesis of theorem* (3.9) *for some ω. If there exists a triple (d, u, v) of nonnegative integers satisfying*

$$d > v,\ d \ge u + r_0,\ i = d + (\omega - 1)r_0,\ w = d^2 - u^2 - v^2 + (\omega - 1)r_0^2$$

$$\text{and } p = \frac{(d-1)(d-2)}{2} - \frac{u(u-1)}{2} - \frac{v(v-1)}{2} + (\omega - 1)\frac{r_0(r_0 - 1)}{2},$$

then the integer l of (3.9) *is at least* 3 *and $\mathscr{T}_{j_1} \ge (\omega)$.*

Proof. Note that if $l \ge 3$ then $\mathscr{T}_0 \leftarrow \cdots \leftarrow \mathscr{T}_h$ is of type ω by part (b) of (3.9), where $h = h_{l-1} \ge h_2 > j_1$; hence $\mathscr{T}_{j_1} \ge (\omega)$. It remains to prove that $l \ne 1$ and $l \ne 2$. If $l = 1$ then, as in the proof of (3.13), we find $0 = d - u - v - r_0 + d = (d - u - r_0) + (d - v) > 0$, a contradiction.

Suppose $l = 2$. Define A and B as in the proof of (3.13) then

$$A = 2d(\omega - 1)r_0 + (\omega - 1)(\omega - 2)r_0^2 + u^2 + v^2 - 1 \tag{1}$$

$$B = (-2 + 2/\omega)(d - r_0) + u + v. \tag{2}$$

On the other hand, we have by part (b) of (3.9)

$$A = (\omega - 1)\sum_{j=0}^{h_1 - 1} r_j^2 + [\omega n_1 + 1 - \delta]\, i_1^2 - i_1 m_1,$$

$$B = [n_1 - \delta + 2/\omega]\, i_1 - m_1,$$

$$r_0 = n_1 i_1,\ i_1 = i - \omega r_0 = d - r_0,$$

$$\sum_{j=0}^{h_1 - 1} r_j^2 = i r_0 = d r_0 + (\omega - 1) r_0^2,$$

the last equation by (3.11) and the third line by parts (2) and (3) of (1.19). So

$$\begin{aligned} A &= (\omega - 1)(dr_0 + (\omega - 1)r_0^2) \\ &\quad + [((\omega - 1)n_1 + 1 - 2/\omega) + (n_1 - \delta + 2/\omega)]\, i_1^2 - i_1 m_1 \\ &= (\omega - 1)(dr_0 + (\omega - 1)r_0^2) + [(\omega - 1)n_1 + 1 - 2/\omega]\, i_1^2 + B i_1 \\ &= 2(\omega - 1)dr_0 + (\omega - 1)(\omega - 2)r_0^2 + (1 - 2/\omega)\, i_1^2 + B i_1. \end{aligned} \tag{3}$$

We get $u^2 + v^2 - 1 = (1 - 2/\omega)i_1^2 + Bi_1 = (u + v)i_1 - i_1^2$ by equations (1), (2) and (3), i.e.,

$$v^2 - i_1 v + (i_1^2 - i_1 u + u^2 - 1) = 0.$$

That quadratic equation in v has discriminant

$$\Delta = -3i_1^2 + 4ui_1 - 4(u^2 - 1) = -3\left[\left(i_1 - \frac{2}{3}u\right)^2 + \frac{4}{9}(2u^2 - 3)\right]$$

which is negative whenever $u \geq 2$. Thus $u \leq 1$ and $i_1 - \frac{2}{3}u \geq \frac{4}{3}$, since $l > 1$ implies $i_1 \geq 2$. Hence $\Delta < 0$ in any case, a contradiction.

4. Birational morphisms $\mathbf{A}^2 \to \mathbf{A}^2$

We studied birational morphisms of non-complete surfaces in [3], with special attention to the case $\mathbf{A}^2 \to \mathbf{A}^2$. In particular, we considered the problem of classifying the irreducible birational endomorphisms of $\mathbf{A}^2$ ($f: \mathbf{A}^2 \to \mathbf{A}^2$ is *irreducible* if it is not an automorphism of $\mathbf{A}^2$ and if, whenever it is factored as a composition $f = h \circ g$ of birational endomorphisms of $\mathbf{A}^2$, g or h is an automorphism of $\mathbf{A}^2$). This section shows how the machinery of local trees and weak sequences can be used to investigate that classification problem.

We use the notations and terminologies of [3], and in particular see (1.2) for the notions of minimal decomposition, fundamental point, missing curve, contracting curve, for the set J and for the numbers $n(f)$, $c(f)$, $q_0(f)$, $j(f)$ and $\delta(f)$; see (2.8) for the matrices μ, $\mathscr{E}$, ε and ε' determied by a minimal decomposition of f.

The following result can be found in [3] with a somewhat fancy proof (see (4.10) of [3]). We reprove it here by using the methods of this paper.

Theorem 4.1. *Let f be a birational endomorphism of $\mathbf{A}^2$, with $n(f) = 1$. Then f is a simple affine contraction.*

Proof. By [3], (4.3), f has one missing curve and that curve is rational with one place P at infinity; we have to prove that that curve is a coordinate line (2.1). Embed $\mathbf{A}^2$ in $\mathbf{P}^2$ the standard way, let $L = \mathbf{P}^2 \backslash \mathbf{A}^2$, let $\widetilde{\mathbf{P}^2} \to \mathbf{P}^2$ be the blowing-up of $\mathbf{P}^2$ at the fundamental point P_1 and let C denote the closure in $\mathbf{P}^2$ of the missing curve of f and also its strict transform in $\widetilde{\mathbf{P}^2}$. Note that both $(P, C, L, \widetilde{\mathbf{P}^2})$ and $(P, C, L, \mathbf{P}^2)$ satisfy the conditions of (1.13). Using the notation of (1.13), write $S_0 = \widetilde{\mathbf{P}^2}$, etc., and consider the sequence of m-trees of $(P, C, L, \widetilde{\mathbf{P}^2})$:

$$\begin{array}{ccc} S_0 & \longleftarrow \cdots \longleftarrow & S_k \\ \mu(P, C, L, \widetilde{\mathbf{P}^2}): (\mathscr{T}_0, \mu_0) & \Longleftarrow \cdots \Longleftarrow & (\mathscr{T}_k, \mu_k). \end{array}$$

If $k = 0$ then $C.L = \mu_0(\{P, L\}) = \mu_k(\{P, L\}) = 1$, so C is a line in $\mathbf{P}^2$ and we are done.

Assume $k > 0$. Let $d = \deg C$, $u = \mu(P_1, C)$(i.e., $u = 1$, but we don't need to know that) and

$$\alpha = d^2 - u^2 - \sum_{j=0}^{k-1} (\mu_j(x_0))^2.$$

We see that $\mathscr{T}_k[\alpha] \cong \mathscr{G}(S_k, C^{(k)} + L^k) \sim [1]$ and that the principal vertex of $\mathscr{T}_k$ is a branch point. If $\mathscr{T}_k$ does not contract to a linear local tree the hypothesis of (3.9) is satisfied (with $\omega = 1$, $\mathcal{G} = [\alpha]$, $\mathcal{L} = [1]$) and consequently $\alpha = -1$; it follows that the triple $(d, u, 0)$ satisfies condition (a) of (3.13), which is absurd. Consequently, $\mathcal{T}_k$ does contract to a linear local tree. Since the sequences $\mu(P, C, L, \tilde{\mathbf{P}}^2)$ and $\mu(P, C, L, \mathbf{P}^2)$ are identical, the missing curve is graph-theoretically linear, i.e., it is a coordinate line by (2.3).

For the rest of this section, we shall study those birational endomorphisms of $\mathbf{A}^2$ having the property that *each missing curve is blown-up at most twice*, i.e., those endomorphisms for which every column of the matrix μ has at most two nonzero entries. We begin by stating the results: the first one says that we are in fact restricting ourselves to the case $n(f) \leq 2$.

Theorem 4.2. *Let f be an irreducible birational endomorphism of $\mathbf{A}^2$, with $n(f) \geq 2$. If every missing curve of f is blown-up at most twice then $n(f) = 2$.*

Note that, if f is such that every missing curve is blown-up at most twice, then so is h whenever $f = h \circ g$. Hence (4.2) can be rephrased as:

4.3. *Let f be a birational endomorphism of $\mathbf{A}^2$ each of whose missing curve is blown-up at most twice. If $n(f) > 0$ then $f = h \circ g$ for some birational endomorphisms g, h of $\mathbf{A}^2$ such that $1 \leq n(h) \leq 2$.*

The next two results give a complete classification in the case $n(f) = 2$.

Theorem 4.4. *Let f be an irreducible birational endomorphism of $\mathbf{A}^2$, with $n(f) = 2$. Then $q(f) = 2$ and there is a coordinate system on $\mathbf{A}^2$ such that the closures of the missing curves meet the line at infinity at distinct points, when $\mathbf{A}^2$ is embedded in $\mathbf{P}^2$ the standard way. Moreover, that coordinate system is unique, up to affine automorphism of $\mathbf{A}^2$ (i.e., linear automorphism + translation), and has the following property: If the missing curves C_1, C_2 and the fundamental points P_1, P_2 are suitably labelled (where P_2 may be i.n. P_1) then there is a positive integer b such that:*

1. *C_2 is a rational curve of degree $2b + 1$, with one place at infinity;*
2. *$\mu(P_1, C_2) = b + 1$, $\mu(P_2, C_2) = b$;*
3. *C_1 is the line (of degree one) through P_1 and P_2;*
4. *the multiplicity sequence of C_2 at infinity begins with a sequence of type $(2, 2b + 1, 1)$ and continues $1, 1, \ldots$.*

Theorem 4.5. *Let C_1, C_2, P_1, P_2 be curves and points in $\mathbf{A}^2$ satisfying the four conditions listed in (4.4), for some positive integer b. Then there exists an irreducible birational endomorphism $f\colon \mathbf{A}^2 \to \mathbf{A}^2$, with $n(f) = 2$, having C_1, C_2 as missing curves and P_1, P_2 as fundamental points. Moreover, that endomorphism is unique, up to equivalence.*

We mention the following related fact: *Given a positive integer b and a sequence $\mathcal{S}$ of type $(2, 2b + 1, 1)$, there exist C_1, C_2, P_1, P_2 satisfying conditions (1–3) of (4.4) and such that the multiplicity sequence of C_2 at infinity begins with $\mathcal{S}$.*

To avoid repeating long parts of arguments, (4.2) and (4.4) are proved together. The following two lemmas are needed.

Lemma 4.6. *If f is a birational endomorphism of $\mathbf{A}^2$ with $n(f) > 0$ and such that some column of the matrix μ has less that two nonzero entries then $f = h \circ g$ where h is a simple affine contraction in $\mathbf{A}^2$.*

Proof. Suppose that $\mu(P_i, C_1) = 0$ if $i \neq 1$. Then every entry of the first column is divisible by $\mu(P_1, C_1)$, whence $\mu(P_1, C_1) = 1$ by (4.3b) of [3]. Let W be the surface obtained by blowing-up $\mathbf{A}^2$ at P_1 and removing the strict transform of C_1, and let $h: W \to \mathbf{A}^2$ be the birational morphism so obtained. Clearly, $f = h \circ g$ for some birational morphism $g: \mathbf{A}^2 \to W$. By (4.4) of [3] we have $W \cong \mathbf{A}^2$, and h is a simple affine contraction by (4.1).

Lemma 4.7. *Let f be a birational endomorphism of $\mathbf{A}^2$ all of whose missing curves have degree one (with respect to some coordinate system on $\mathbf{A}^2$). If $n(f) > 0$ then $f = h \circ g$ where h is a simple affine contraction in $\mathbf{A}^2$.*

Proof. By (4.6), we may assume that each missing curve is blown-up at least twice. We now show that $q(f) = 1$, so that the result follows from (4.11) of [3].

Suppose $q(f) \geq 2$. Choose a minimal decomposition for f, with notation as in (1.2h) of [3], embed $Y_0 = \operatorname{codom}(f) = \mathbf{A}^2$ in $\bar{Y}_0 = \mathbf{P}^2$ the standard way and consider the corresponding diagram

$$\begin{array}{ccccccc} \mathbf{A}^2 & \hookrightarrow & Y_n & \hookrightarrow & \bar{Y}_n & & \\ & & \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \bar{\pi}_n} & & \\ \downarrow{\scriptstyle f} & & \vdots & & \vdots & & \\ & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \bar{\pi}_1} & & \\ \mathbf{A}^2 & = & Y_0 & \hookrightarrow & \bar{Y}_0 & = & \mathbf{P}^2 \end{array}$$

where $\bar{\pi}_i$ is the blowing-up of $\bar{Y}_{i-1}$ at $P_i (1 \leq i \leq n)$. Let L be the line at infinity and $D = C_1 + \cdots + C_q + \sum_{j \in J} E_j + L \in \operatorname{Div}(\bar{Y}_n)$. By [3], (2.17), it is clear that D has *s.n.c.* iff the missing curves meet L at distinct points.

If D has *s.n.c.* then the weighted graph $\mathcal{G}(\bar{Y}_n, D)$ contracts to a minimal weighted tree $\mathcal{G}$ which has exactly one vertex of nonnegative weight–that vertex is L and its weight is positive. Moreover, if L is the only vertex of $\mathcal{G}$ then the weight of L is greater than one. So $\mathcal{G} \not\to [1]$ by [3], (5.16), and this is absurd.

If D does not have *s.n.c.* then by [3], (5.19), there is a monoidal transformation $S_1 \to S_0 = \bar{Y}_n$ with center s_1 and exceptional curve F_1 such that $D + F_1 \in \operatorname{Div}(S_1)$ has *s.n.c.*. Let $\mathcal{G} = \mathcal{G}(S_1, D + F_1)$. Since the point s_1 belongs to L and to at least two C_i's, F_1 is a branch point of weight -1 (in $\mathcal{G}$) and also a "special vertex", which contradicts [3], (5.18).

Proof of (4.2) *and* (4.4). Let f be an irreducible birational endomorphism of $\mathbf{A}^2$ such that $n(f) \geq 2$ and such that every missing curve is blown-up at most twice. By (4.6) every missing curve is blown-up *exactly* twice and by (4.11) of [3] we have $q(f) \geq 2$. Given any open immersion $\iota: \mathbf{A}^2 = \text{codom}(f) \subsetneq \mathbf{P}^2$, let L be the line at infinity and consider the following condition on ι:

(*) *All missing curves of f meet L at the same point, in* $\mathbf{P}^2$.

As is well known, the embeddings that don't satisfy (*) form a finite number of equivalence classes (two embeddings $\mathbf{A}^2 \subsetneq \mathbf{P}^2$ are equivalent if they form a commutative diagram with some automorphism of $\mathbf{P}^2$). The first part of the argument consists in the construction of an embedding that doesn't satisfy (*). From the existence of such an embedding we will then deduce that $n(f) = 2$ (which will settle (4.2)) and prove the rest of (4.4).

CLAIM 1. *Some missing curve is a coordinate line.*

Proof. Choose a minimal decomposition for f, with notation as in (1.2h) of [3], choose an embedding ι that satisfies (*) and consider the diagram in the proof of (4.7).

Let $S = \overline{Y}_n$, $C = C_1 + \cdots + C_q + \sum_{j \in J} E_j \in \text{Div}(S)$ and $D = C + L \in \text{Div}(S)$. Clearly, $S \setminus \mathbf{A}^2 = \text{supp}(D)$. By (*), C is connected and D does not have *s.n.c.*. By [3], (5.19), there exists a place P of one of the C_v's such that "D can be desingularized by blowing-up at P" (see (3.4)) and it follows that (P, C, L, S) satisfies the conditions of (3.5).

Consider the weak sequence $\mathscr{W}_0, \ldots, \mathscr{W}_k$ of (P, C, L, S). We refer to the discussion preceding and following (3.6) for the definition of the numbers α_1 and p_1, the curves $C_{v_{1j}}$, the places $P_{v_{1j}}$, the graph $\mathscr{G}_*$, etc., and for the meaning of the sentence "the blowing-up $S_{i-1} \leftarrow S_i$ gets C_v away from P".

Let's show that, given $j \in \{1, \ldots, p_1\}$, C_{v_j} is a coordinate line. Let $\mathscr{T} = \mathscr{T}(P_{1j}, L^{\alpha_1}, S_{\alpha_1})$. Clearly, the sequence $\mathscr{W}_0 \leftarrow \cdots \leftarrow \mathscr{W}_{\alpha_1 - 1} \leftarrow \mathscr{T}$ consists of the sequence of local trees of $(P_{1j}, C_{v_{1j}}, L, S)$ followed by a (possibly empty) sequence of blowings-up in which every tree has exactly one principal link. We are now going to show that $\mathscr{T}$ contracts to a linear local tree. Since the sequence of local trees of $(P_{1j}, C_{v_{1j}}, L, S)$ is identical to that of $(P_1, C_{v_{1j}}, L, \overline{Y}_0)$, it will then follow that $C_{v_{1j}}$ is graph-theoretically linear, and hence a coordinate line by (2.3).

Suppose $\mathscr{T}$ doesn't contract to a linear local tree. Since $\mathscr{W}_k[\mathscr{G}_*] \sim [1]$, we may apply (3.7) to the weak sequence $\mathscr{W}_{\alpha_1 - 1}, \ldots, \mathscr{W}_k$. We conclude that the principal vertex of $\mathscr{T}$ is a branch point, $\mathscr{T}[-1]$ is equivalent to a linear weighted tree and $\mathscr{W}_{\alpha_1 - 1}$ can't contract to a local tree having a nonprincipal vertex of nonnegative weight. Hence the sequence

$$(1) = \mathscr{W}_0 \longleftarrow \cdots \longleftarrow \mathscr{W}_{\alpha_1 - 1} \longleftarrow \mathscr{T}$$

satisfies the hypothesis of (3.13). Now (3.7) also says that every extra branch created in $\mathscr{W}_{\alpha_1 - 1} \overset{+}{\longleftarrow} \mathscr{W}_{\alpha_1}$ (in particular, $\mathscr{G}_{1j}$) can be absorbed by the vertex F_{α_1}, which implies that $C_{v_{1j}}^2 = -1$ in S_{α_1}, since every other vertex of $\mathscr{G}_{1j}$ has weight less than -1 (by, say, (1.2i) of [3]). Therefore (d, u, v) satisfies the condition (a) of (3.13) (which is absurd), where d is the degree of $C_{v_{1j}}$ in $\bar{Y}_0$ and u, v are the two nonzero entries of the v_{1j}th column of the matrix μ. That proves claim 1.

By replacing if necessary the minimal decomposition of f by another one, we may assume that C_1 is a coordinate line and that $\mu(P_1, C_1) = \mu(P_2, C_1) = 1$. These assumptions will be in force until the end of the proof of claim 6, below. Consider the following condition on the embedding ι:

(**) *C_1 has degree one in $\bar{Y}_0$.*

Since (**) is satisfied by infinitely many nonequivalent embeddings, there are embeddings satisfying both (*) and (**).

CLAIM 2. *If ι satisfies both (*) and (**) then $k > 1$ and the center of the blowing-up $S_2 \to S_1$ is a point of L. Moreover, C_1 is the only missing curve of degree one and the blowing-up $S_1 \to S_0$ gets C_1, and no other missing curve, away from P.*

Proof. Suppose $k = 1$. Then $D^1 \in \mathrm{Div}(S_1)$ has *s.n.c.*, which implies that $C_v . L = 1$ in S_0 (hence in $\bar{Y}_0$) for $1 \le v \le q$, i.e., all missing curves are lines in $\bar{Y}_0$. Since that contradicts (4.7), we see that $k > 1$.

Now suppose that the center of $S_2 \to S_1$ in not on L. Then, in $\mathscr{G}(S_k, D^k) \in \mathscr{G}[\mathbf{A}^2]$, L has weight 0 and is not a neighbour of F_k, which is a *branch point* of weight -1 (by, say, (5.19) of [3]). As is shown in the proof of (3.7), such a tree contracts to a weighted tree $\mathscr{G}^+$ which contains vertices of nonnegative weights and not neighbours of each other, so that $\langle \mathscr{G}(S_k, D^k) \rangle = \langle \mathscr{G}^+ \rangle > 1$, a contradiction.

Clearly, the first blowing-up $S_0 \leftarrow S_1$ gets a missing curve away from P iff that curve has degree one in $\bar{Y}_0$, i.e., $\alpha_1 = 1$ and $C_{v_{11}}, \ldots, C_{v_{1p_1}}$ are those missing curves which are lines in $\bar{Y}_0$. Thus each $C_{v_{1j}}$ has weight -2 in $\mathscr{W}_1, \ldots, \mathscr{W}_k$. Hence the branches of $\mathscr{G}(S_k, D^k) = \mathscr{W}_k[\mathscr{G}_*]$ at F_1 are $\mathscr{G}_{11}, \ldots, \mathscr{G}_{1p_1}, \mathscr{B}$, where $\mathscr{B}$ is the one that contains F_k and where $\mathscr{G}_{1j} < -1$ for each j ([3], (5.17)). Since F_k is a branch point of weight -1, $\mathscr{B}$ can't be absorbed by F_1. Hence no branch can be absorbed by F_1 and, by [3], (5.11), F_1 must be a linear vertex of $\mathscr{G}(S_k, D^k)$, i.e., $p_1 = 1$ and claim 2 is proved.

CLAIM 3. *If $v \ne 1$ then $\mu(P_1, C_v) \ne 0$ or $\mu(P_2, C_v) \ne 0$.*

Proof. Choose an ι satisfying (*) and (**) and let d be the degree of C_v in $\bar{Y}_0$. By previous claim, $d > 1$ and $C_1 \cap C_v = \emptyset$ in S_1. Therefore

$$\mu(s_1, C_v) = C_1 . C_v (\text{in } S_0) = d - \mu(P_1, C_v) - \mu(P_2, C_v).$$

Let s_1' denote the point $C_v \cap L$ of $\bar{Y}_0$. Then we must have $\mu(s_1, C_v) = \mu(s_1', C_v) < d$, which proves the claim.

CLAIM 4. *Suppose $j(f) > 0$. Then for each $v \in \{2, \ldots, q\}$ there is a unique $\xi(v) \in \{1, 2\}$ such that $\mu(P_{\xi(v)}, C_v) \neq 0$ and a unique $\zeta(v) \in \{3, \ldots, n\}$ such that $\mu(P_{\zeta(v)}, C_v) \neq 0$. Moreover, the numbers $\zeta(2), \ldots, \zeta(q)$ are distinct.*

Proof. By [3], (4.12), $j(f) > 0$ implies that $\delta(f) < j(f)$. Since μ has exactly $\delta(f)$ zero rows, μ has at least $q + 1$ nonzero rows. Since the upper two entries of the first column are nonzero and since each column has exactly two nonzero entries, the result follows from claim 3.

CLAIM 5. *Suppose $j(f) > 0$. If $v \neq 1$ and either C_v doesn't meet $E_{\xi(v)}$ in Y_n or $\mu(P_{\xi(v)}, C_v) = 1$ then the ring of functions $\mathrm{k}[X, Y]$ of $Y_0 = \mathbf{A}^2$ is generated, as an algebra over* k, *by the irreducible polynomials G_1, G_v that correspond to C_1, C_v (i.e., some coordinate system on $\mathbf{A}^2$ has C_1 and C_v as coordinate axes).*

Proof. Let $u = \mu(P_{\xi(v)}, C_v)$. If C_v doesn't meet $E_{\xi(v)}$ in Y_n then $\mu(P_{\zeta(v)}, C_v) = u$ too, so the v^{th} column of μ is a multiple of u. By [3], (4.3b), it follows that $u = 1$. Since $\{i \mid P_i \in C_1 \cap C_v\} = \{\xi(v)\}$ and since C_1 and C_v are disjoint in Y_n, the intersection number of C_1 and C_v at finite distance is $u = 1$, i.e., the k-vector space $\mathrm{k}[X, Y]/(G_1, G_v)$ has dimension 1; since missing curves have one place at infinity, the assertion follows from a known result–see for instance [4], (1.17).

CLAIM 6. *Let $\mathscr{C}$ be the connected component of $Y_n \setminus \mathbf{A}^2$ containing C_1.*

1. *If $\mathscr{C}$ contains some E_i then some coordinate system on $\mathbf{A}^2$ has C_1 and some other missing curve as coordinate axes.*
2. *If $\mathscr{C}$ doesn't contain any E_i then, for some coordinate system on $\mathbf{A}^2$, C_1 and some other missing curve C' meet the line at infinity at distinct points (when $\mathbf{A}^2$ is embedded in $\mathbf{P}^2$ the standard way) and, if $n(f) = 2$, the multiplicity sequence of C' at infinity begins with a sequence of type $(2, d', 1)$ where $d' = \deg C'$.*

Proof. For the first assertion, let's proceed by contradiction and assume that $\mathscr{C}$ contains some E_i and that no coordinate system on $\mathbf{A}^2$ has C_1 and another missing curve as coordinate axes. Clearly, either E_1 or E_2 is in $\mathscr{C}$ and, in fact, E_2 is in $\mathscr{C}$ whenever P_2 is *i.n.* P_1. So in any case we may assume that $\mathscr{C}$ contains E_2. Note that $j(f) > 0$ and $n(f) > 2$.

Our assumptions and claim 5 imply that C_v meets $E_{\xi(v)}$ in Y_n, for $v = 2, \ldots, q$. In particular, $E_{\xi(v)}$ can't be in $\mathscr{C}$, so $\xi(2) = \cdots = \xi(q) = 1$ and the missing curves are already disjoint in Y_1. Consequently, no element of $\{P_2, P_{\zeta(2)}, \ldots, P_{\zeta(q)}\}$ is *i.n.* another element of that set and, if we define

$$M(v) = \begin{cases} \max\{i > 2 \mid P_i \text{ i.n. } P_2\}, & \text{if } v = 1, \\ \max\{i \geq \zeta(v) \mid P_i \text{ i.n. } P_{\zeta(v)}\}, & v = 2, \ldots, q, \end{cases}$$

then $M(1), \ldots, M(q)$ are distinct elements of $\{3, \ldots, n\}$ (and in particular $n(f) > 3$). Since $E_{M(v)}^2 = -1$ in $Y_n (v = 1, \ldots, q)$ no $M(v)$ is in J by (1.2i) of [3]. Whence

$$J = \{1,\ldots,n\}\backslash\{M(1),\ldots,M(q)\} \qquad \text{and } 1 \in J.$$

Recall that C_v meets E_1 in Y_1 in Y_n if $2 \leq v \leq q$. By (2.17) of [3], it therefore follows that $q = 2$, that E_1 doesn't meet C_1 in Y_n (which implies $P_2 \in E_1$) and that C_2 meets E_1 transversally in Y_n. Define $u = \mu(P_1, C_2)$; since $u > 1$ by claim 5, we see that $P_{\zeta(2)} \in E_1$ and that $\mu(P_{\zeta(2)}, C_2) = u - 1$. We may assume that $\zeta(2) = 3$. Since $\mu(P_i, C_2) = 0$ for $i > 3$ and since E_1 and E_3 meet C_2 in Y_3, E_1 and E_3 meet C_2 in Y_n; so $3 \notin J$ by (2.17) of [3], i.e., $M(2) = 3$ and $\{1,\ldots,n\}\backslash J = \{3, n\}$. Moreover, $P_i \notin E_3$ $(4 \leq i \leq n)$ and, if $n(f) > 4$, $P_i \in E_{i-1}$ $(5 \leq i \leq n)$ for 3 and n are the only i's such that $E_i^2 = -1$ in Y_n. Finally, $P_4 \in E_1 \cap E_2$ since these two curves are disjoint in Y_n. By what has been said,

$$\mathscr{E} = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(see [3], (2.8), for definition of $\mathscr{E}$, ε', μ) and therefore

$$\varepsilon'\mu = \begin{bmatrix} 1 & 0 & 1 & \cdots \\ a & b & 0 & \cdots \end{bmatrix} \begin{bmatrix} 1 & u \\ 1 & 0 \\ 0 & u-1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2u-1 \\ a+b & au \end{bmatrix}$$

where a, b are positive integers. Then one sees that the determinant of $\varepsilon'\mu$ is neither 1 nor -1, which contradicts [3], (4.3b).

To prove the second part of claim 6, we assume that $\mathscr{C}$ contains no E_i and we choose an embedding ι that satisfies both (*) and (**). If $q > 2$ then we may assume that $C_{v_{21}} = C_2$(i.e., $v_{21} = 2$). In any case, the curve C_2 in Y_0 has one place P' at infinity. Let

$$\bar{Y}_0 = S'_0 \longleftarrow S'_1 \longleftarrow S'_2 \longleftarrow \cdots$$

be the infinite sequence of monoidal transformations determined by $(P', C_2, \bar{Y}_0)$ and denote by F'_j the exceptional curve created in $S'_{j-1} \leftarrow S'_j$. Let

$$(\mathscr{T}'_0, \mu_0) \Longleftarrow (\mathscr{T}'_1, u_1) \Longleftarrow \cdots$$

be the infinite sequence of m-trees of $(P', C_2, L, \bar{Y}_0)$: let r_j be the multiplicity of the root of $\mathscr{T}'_j$(i.e., $(r_0, r_1, \ldots)$ is the multiplicity sequence of C_2 at infinity), let d be the degree of C_2 in $\bar{Y}_0$ and let u, v be the two nonzero entries of the second column of μ where, say, v occupies a lower position than u does. We now proceed to prove the following fact:

CLAIM 6.1. *Let $n_1 = r_0/r_1$; then $n_1 \in \mathbf{Z}$ and $d - r_0 = r_1 = \cdots = r_{2n_1}$. Moreover, if $n(f) = 2$ then $(r_0, r_0, r_1, r_2, r_3, \ldots)$ begins with a sequence of type $(2, d + r_0, 1)$.*

Proof. Define

$$\lambda = \begin{cases} k, & \text{if } q = 2, \\ \alpha_2, & \text{if } q > 2 \end{cases}$$

and note that $\lambda \geq k(P', C_2, L, \bar{Y}_0)$—see (1.13) for the definition of $k(P', C_2, L, \bar{Y}_0)$. Consider the sequence of m-trees

$$(\mathscr{V}_{-1}, v_{-1}) \Longleftarrow (\mathscr{V}_0, v_0) \Longleftarrow (\mathscr{V}_1, \mu_1) \Longleftarrow \cdots \Longleftarrow (\mathscr{V}_\lambda, \mu_\lambda)$$

where, in the notation of (1.22),

- $\mathscr{V}_{-1} = (*, 2)$, $v_{-1}(x_0) = r_0$ and the value of v_{-1} at the principal link is $d + r_0$;
- $\mathscr{V}_0 = (*, (-1), (1))$, $v_0(x_0) = r_0$ and the value of v_0 at the principal link that contains the vertex of weight -1 (resp. 1) is r_0 (resp. d);
- $\mathscr{V}_j = \mathscr{W}_j$ $(1 \leq j < \lambda)$ and

$$\mathscr{V}_\lambda = \begin{cases} \mathscr{W}_k, & \text{if } q = 2, \\ \mathscr{T}(P', L^{\alpha_2}, S_{\alpha_2}), & \text{if } q > 2. \end{cases}$$

(Observe how $\mathscr{V}_0 \leftarrow \mathscr{V}_1$ is a consequence of the assumption "C contains no E_i".) Note that if $\mathscr{V}_\lambda$ contracts to a linear local tree then

$$(2) = \mathscr{V}_{-1} \longleftarrow \cdots \longleftarrow \mathscr{V}_{k(P', C_2, L, \bar{Y}_0)}$$

is a sequence of type 2 by (1.23), thus $(r_0, r_0, r_1, r_2, \ldots, r_{k(P', C_2, L, \bar{Y}_0) - 1})$ is a sequence of type $(2, d + r_0, 1)$ and claim 6.1 follows. Let's assume that $\mathscr{V}_\lambda$ does not contract to a linear local tree. We now show that $n(f) > 2$ and that claim 6.1 holds in this case too. Let $\mathscr{S}$ be the sequence $\mathscr{V}_{-1} \leftarrow \cdots \leftarrow \mathscr{V}_\lambda$.

Case $q(f) = 2$. $\mathscr{V}_\lambda = \mathscr{W}_k$, so the principal vertex of $\mathscr{V}_\lambda$ is a branch point and $\mathscr{V}_\lambda[\mathscr{G}_*] \sim [1]$. Hence the sequence $\mathscr{S}$ satisfies the hypotheses of (3.9) and (3.13), with $\omega = 2$. By (3.9), $\mathscr{G}_*$ can be absorbed by the principal vertex of $\mathscr{V}_\lambda$ (which implies $C_2^2 = -1$ in $S_k = S_\lambda$). Hence the triple (d, u, v) satisfies condition (b) of (3.13) except, perhaps, for the inequality $u + v + r_0 \leq d$. Thus $u + v + r_0 > d$ (otherwise we contradict (3.13)) and since

$$d = \begin{cases} r_0 + u + v, & \text{if } n = 2, \\ r_0 + u, & \text{if } n > 2, \end{cases}$$

we see that $n > 2$. The assertion is then a consequence of (3.14). (That result asserts that there is a j such that $1 < j < \lambda$ and $\mathscr{V}_j \geq (2)$; then $\mathscr{V}_i$ contracts to a linear local tree, where $i = \max(\mathscr{H}(\mathscr{S}) \cap \{-1, \ldots, j\})$, hence $\mathscr{V}_{-1} \leftarrow \cdots \leftarrow \mathscr{V}_i$ is of type 2 by (1.23), etc.)

Case $q(f) > 2$. By (3.7) applied to the weak sequence

$$\mathscr{W}_{\alpha_2 - 1} \overset{+}{\longleftarrow} \mathscr{W}_{\alpha_2} \cdots \mathscr{W}_k,$$

the principal vertex of $\mathscr{V}_\lambda$ is a branch point, $\mathscr{V}_\lambda[-1]$ is equivalent to a linear weighted tree and $\mathscr{V}_{\lambda-1}$ can't contract to a local tree having a nonprincipal vertex of nonnegative weight. Hence the sequence $\mathscr{S}$ satisfies the hypothesis of (3.9), with $\omega = 2$. Moreover, (3.7) says that every extra branch created in $\mathscr{W}_{\alpha_2 - 1} \overset{+}{\longleftarrow} \mathscr{W}_{\alpha_2}$ can be absorbed by the vertex to which it is attached (which implies that $C_2^2 = -1$ in $S_{\alpha_2} = S_\lambda$). Hence the triple (d, u, v) satisfies the condition of (3.14) and, as in the case $q(f) = 2$, claim 6.1 follows.

From claim 6.1 we see that

$$\mathscr{T}'_{2n_1+1} = (*, -1, -2, \dots, -2, (-n_1 - 1), (-2, \dots, -2, -1))$$

where the first sequence of "-2" has n_1 terms. Hence $\mathscr{T}'_{2n_1+1} \geq (1)$ and to that contraction corresponds a birational morphism $\rho: S'_{2n_1+1} \to S'$ that contracts L, $F'_2, F'_3, \dots, F'_{2n_1}, F'_1$, in that order. So we get an embedding $\mathbf{A}^2 \subsetneq S'$ such that the complement of $\mathbf{A}^2$ is $L' = \rho(F'_{2n_1+1})$, which is a curve of self-intersection 1. Hence $S' = \mathbf{P}^2$. Moreover, if x denotes the point $F'_{2n_1} \cap F'_{2n_1+1}$ of S'_{2n_1+1}, then C_1 meets L' at the point $\rho(x)$, which is distinct from the point at which C_2 meets L'. So we have constructed an embedding $\mathbf{A}^2 \subsetneq \mathbf{P}^2$ such that C_1 and C_2 meet the line at infinity at distinct points; note that the multiplicity sequence of C_2 at infinity is now $(r_j)_{j > 2n_1}$, which begins with a sequence of type $(2, d - r_0, 1)$ whenever $n(f) = 2$ (by claim 6.1); one easily sees that $d - r_0 = \deg C_2$, so the proof of claim 6 is complete.

By claims 1 and 6, there exists an embedding $\imath: \mathbf{A}^2 \subsetneq \mathbf{P}^2$ that doesn't satisfy $(*)$; we choose such an embedding. Then one of the missing curves, say C_1, has degree one and doesn't meet any other missing curve at infinity, by [3], (4.3c). Choose a minimal decomposition such that $\mu(P_1, C_1) = 1 = \mu(P_2, C_1)$. For each $j \in \{2, \dots, q\}$, let $\xi(j)$ and $\zeta(j)$ be the two elements of $\{1, \dots, n\}$ such that $\mu(P_{\xi(j)}, C_j) \neq 0$ and $\mu(P_{\zeta(j)}, C_j) \neq 0$, with $\xi(j) < \zeta(j)$. The minimal decomposition and the immersion $\imath$ determine a commutative diagram as the one displayed in the proof of (4.7).

Claim 7. $n(f) = 2$.

Proof. Assume $n(f) > 2$. Let $j \in \{2, \dots, q\}$. Since C_1 and C_j don't meet at infinity they must meet at finite distance, whence $\xi(j) \leq 2$. We claim that $\zeta(j) > 2$. Indeed, if $\zeta(j) \leq 2$ and $j(f) = 0$ then f is reducible by (4.5) of [3]; if $\zeta(j) \leq 2$ and $j(f) > 0$ then μ has at least $n(f) - q(f) = j(f)$ zero rows, i.e., $\delta(f) \geq j(f) > 0$ and f is reducible by (4.3a) of [3]. Hence $\zeta(j) > 2$, which implies that C_1 and C_j are already disjoint in $\bar{Y}_{\xi(j)}$, i.e.,

$$\mu(P_{\xi(j)}, C_j) = C_1 . C_j = \deg C_j \text{ in } \bar{Y}_0.$$

So we must have $\deg C_j = 1$, i.e., all missing curves have degree one. This contradicts (4.7), hence $n(f) = 2$ and claim 7 is proved.

Note that the proof of (4.2) is now complete; let's now finish that of (4.4). By changing if necessary the minimal decomposition, we may assume that $\mu(P_1, C_2) \geq \mu(P_2, C_2) = b \in \mathbf{N}$. Since the determinant of μ is ± 1, we have $\mu(P_1, C_2) = b + 1$. Since C_1 and C_2 are disjoint in $\bar{Y}_2$, the degree of C_2 in $\bar{Y}_0$ must be $\mu(P_1, C_2) + \mu(P_2, C_2) = 2b + 1$. Finally, it is well known that there can be at most one coordinate system that doesn't satisfy (*) (up to affine automorphism of $\mathbf{A}^2$). Hence the coordinate system which we are considering right now is essentially the same as the one that is given by the second part of claim 6. Consequently, the multiplicity sequence of C_2 at infinity begins with a sequence of type $(2, d', 1)$ where d′ is the degree $2b + 1$ of C_2. So (4.4) is proved.

Proof of (4.5). Embed $\mathbf{A}^2$ in $\mathbf{P}^2$ the standard way and let L be the line at infinity. Blow-up $\mathbf{P}^2$ at P_1 and P_2, then C_1 and C_2 are disjoint and C_1 is exceptional of the first kind. It's enough to show that the complement of $\operatorname{supp}(C_1 + C_2 + L)$ is $\mathbf{A}^2$. Equivalently, if S is the complete nonsingular surface obtained by contracting C_1 and if $U = S \setminus \operatorname{supp}(C_2 + L)$, we have to show that $U \cong \mathbf{A}^2$. By [3], (4.1), it's enough to prove that $[1] \in \mathscr{G}[U]$.

Let P be the place of C_2 which corresponds to the point $C_2 \cap L$ of S. Let $S = S_0 \leftarrow S_1 \leftarrow \cdots$ be the sequence of monoidal transformations determined by the triple (P, C_2, S) and let

$$(\mathscr{T}_0, \mu_0) \Longleftarrow \cdots \Longleftarrow (\mathscr{T}_k, \mu_k)$$

be the sequence of m-trees of (P, C_2, L, S). If D is the reduced effective divisor of S_k such that $S_k \setminus U = \operatorname{supp}(D)$ then, clearly, D has *s.n.c.* iff C_2 is nonsingular in S_k. Now the last condition of (5.3) says that

$$(r_0, \ldots, r_{k-1}) = (\mu_j(x_0))_{j=0,\ldots,k-1} \text{ is of type } (2, 2b+1, 1).$$

Let's use the notation $f(x) = x(x-1)/2$, $x \in \mathbf{Z}$, as in the numerical lemma (3.12). Using parts (1) and (4) of that lemma (with $\omega = 2$, $i = 2b + 1$, $i' = 1$) we find that the arithmetic genus of C_2 in S_k is

$$(f(2b) - f(b+1) - f(b)) - \sum_{j=0}^{k-1} f(r_j) = b(b-1) - b(b-1) = 0,$$

so D has *s.n.c.*. Now the dual graph $\mathscr{G}(S_k, D)$ is just $\mathscr{T}_k[\beta]$, where $\beta = C_2^2$ in S_k. By part (3) of (3.12) we find $\beta = n_l$, where n_l is determined by $(r_0, \ldots, r_{k-1})$ as in (1.19). Since $\mathscr{T}_0 \leftarrow \cdots \leftarrow \mathscr{T}_k$ is of type 2, $\mathscr{T}_k \geq (*, 0, -n_l - 1, -2)$ by (1.23). Hence

$$\mathscr{G}[U] \ni \mathscr{T}_k[\beta] = \mathscr{T}_k[n_l] \sim [n_l, 0, -n_l - 1, -2] \sim [1],$$

as desired.

One of the consequences of (4.3) and (4.4) is that if f is a birational endomorphism of $\mathbf{A}^2$ with the special property that every missing curve is blown-up at most twice, then one of the missing curves is a coordinate line. The following example shows that this fails in general.

Example 4.8 (Russell). In $\mathbf{A}^2$, let $P_1 = (0, 0)$, $P_2 = (0, 1)$, $P_3 = (-1, -1)$, $P_4 = (1, 2)$ and let C_1, C_2, C_3 and C_4 be the curves given by the polynomials

$$F_1 = Y^3 + 8X^2 - 6XY - Y^2$$

$$F_2 = Y^4 + 32X^3 - 48X^2Y + 20XY^2 - 2Y^3 + 20X^2 - 20XY + Y^2$$

$$F_3 = Y^4 - 32X^3 + 48X^2Y - 20XY^2 - 2Y^3 - 28X^2 + 20XY + Y^2$$

$$F_4 = Y^5 + 128X^4 - 288X^3Y + 224X^2Y^2 - 60XY^3 - 2Y^4 + 96X^3 - 156X^2Y + 60XY^2 + Y^3$$

respectively. Blow-up $\mathbf{A}^2$ at P_1, P_2, P_3 and P_4, and remove from the surface so obtained the strict transforms of C_1, C_2, C_3 and C_4. The resulting open set is isomorphic to $\mathbf{A}^2$, so an equivalence class of endomorphisms $f: \mathbf{A}^2 \to \mathbf{A}^2$ is determined. Note that

$$\mu = \begin{bmatrix} 2 & 2 & 2 & 3 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix},$$

so the endomorphism is irreducible by [3], (4.5). Note that all missing curves are singular.

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