J. Math. Kyoto Univ. (JMKYAZ)
31-4 (1991) 1087-1094

Invariant connections for conformal and projective changes

Dedicated to Professor Doctor Makoto Matsumoto on the occasion of his seventieth birthday

By

Katsumi OKUBO

In the theory of conformal changes of Riemannian metrics, the Weyl conformal curvature tensor plays an essential role in case of dimension more than three. We consider, however, the theory from another standpoint. In the first section of the present paper we shall define a linear connection on a Riemannian space relative to a given Riemannian space, which is invariant under conformal change of metric. Thus the curvature tensor of the connection is a conformal invariant and the notion of relative conformal flatness is obtained.

The second section is devoted to the theory of projective changes of Finsler spaces in a similar way. Relative to a given Finsler space a projectively invariant nonlinear connection is defined. As a special case we have a Riemannian projective theory which will be developed in the following two sections.

§1. Conformal changes of a Riemannian space

Let M be an n-dimensional differential manifold and $T(M)$ its tangent bundle. A coordinate system $x=(x^i)$ in M induces a canonical coordinate system $(x, y)=(x^i, y^i)$ in $T(M)$. We put $\partial_i=\partial/\partial x^i$ and $\dot{\partial}_i=\partial/\partial y^i$.

Let us suppose that there is given on M a Riemannian metric tensor $a_{ij}(x)$. Putting $\alpha=(1/2)a_{ij}y^i y^j$, we denote the Riemannian structure by (M, α). The Christoffel symbols $\{{}_j{}^i{}_k\}$ constructed from a_{ij} are coefficients of the Riemannian connection.

We now consider another arbitrary Riemannian metric tensor $g_{ij}(x)$. Putting $L=(1/2)g_{ij}y^i y^j$, this Riemannina is denoted by (M, L). The Christoffel symbols constructed from g_{ij} are denoted by $\Gamma_j{}^i{}_k$. We put $a=\det(a_{ij})$, $g=\det(g_{ij})$ and

$$C\equiv\frac{1}{n}\log\frac{\sqrt{g}}{\sqrt{a}}, \qquad C_i\equiv\partial_i C. \tag{1}$$

C is a scalar function on M and consequently C_i is covariant vector field on M. Then we have a linear connection

$${}^c\Gamma_j{}^i{}_k\equiv\Gamma_j{}^i{}_k-C_j\delta^i{}_k-C_k\delta^i{}_j+C^i g_{jk}\,, \tag{2}$$

where $C^i\equiv g^{ij}C_j$. The connection ${}^c\Gamma$ is symmetric but not metrical. We shall call ${}^c\Gamma$ the C-connection relative to α. We have the following important theorem:

Communicated by Prof. H. Toda, March 29, 1990

Theorem 1. *The C-connection ${}^c\Gamma$ relative to α is a comformally invariant symmetric connection.*

Proof. Let $\bar{L}$ be a conformal change of L, i.e.,

$$\bar{g}_{ij}(x)=\exp(2\sigma)g_{ij}(x). \tag{3}$$

The Riemannian connections $\bar{\Gamma}$ and Γ are in the relation:

$$\bar{\Gamma}_j{}^i{}_k=\Gamma_j{}^i{}_k+\sigma_j\delta_k{}^i+\sigma_k\delta_j{}^i-g_{jk}\sigma^i, \tag{4}$$

where we put $\sigma_j=\partial_j\sigma$, $\sigma^i=g^{ij}\sigma_j$. Thus we get

$$\bar{\Gamma}_j{}^i{}_k-\{{}_j{}^i{}_k\}=\Gamma_j{}^i{}_k-\{{}_j{}^i{}_k\}+\sigma_j\delta_k{}^i+\sigma_k\delta_j{}^i-g_{jk}\sigma^i. \tag{5}$$

Since $\bar{C}$ is defined similarly to $\bar{L}$, by the contraction with respect to suffices (i, j), we get $n\bar{C}_k=nC_k+n\sigma_k$. Consequently, we have

$$\bar{C}_k=C_k+\sigma_k. \tag{6}$$

From the formulas (2), (3), (4) and (6), we get

$$\begin{aligned}{}^c\bar{\Gamma}_j{}^i{}_k&\equiv\bar{\Gamma}_j{}^i{}_k-\bar{C}_j\delta_k{}^i-\bar{C}_k\delta_j{}^i+\bar{C}^i\bar{g}_{jk}\\&=\Gamma_j{}^i{}_k-C_j\delta_k{}^i-C_k\delta_j{}^i+C^ig_{jk}\\&={}^c\Gamma_j{}^i{}_k\end{aligned}$$

(q. e. d.)

Consequently, the curvature tensor ${}^cR$ of the C-connection ${}^c\Gamma$ defined by

$${}^cR_j{}^i{}_{kl}=\partial_l{}^c\Gamma_j{}^i{}_k+{}^c\Gamma_j{}^m{}_k{}^c\Gamma_m{}^i{}_l-\partial_k{}^c\Gamma_j{}^i{}_l-{}^c\Gamma_j{}^m{}_l{}^c\Gamma_m{}^i{}_k \tag{7}$$

is also conformally invariant.

We consider this confomally invariant curvature tensor ${}^cR$. This tensor is not coincident with the Weyl conformal curvature tensor.

Definition. A Riemannian space (M, L) is *conformally α-flat* or *a conformally flat space relative to α*, if ${}^cR$ vanishes.

For (M, α), ${}^c\Gamma_j{}^i{}_k=\Gamma_j{}^i{}_k=\{{}_j{}^i{}_k\}$ because $C=0$. Consequently, ${}^cR=R=S$, where R and S are curvature tensors of L and α respectively. Therefore we have

Theorem 2. *(M, α) is locally flat if and only if (M, α) is conformally α-flat.*

By the straightforward calculation, we get the following formulas:

$${}^cR_j{}^i{}_{kj}=R_j{}^i{}_{kl}+C_{jk}\delta_l{}^i+g_{jk}C_l{}^i-C_{jl}\delta_k{}^i-g_{jl}C_k{}^i, \tag{8}$$

where we put

$$C_{jk}=C_{j|k}-C_jC_k+\frac{1}{2}g_{jk}C^2,\quad C_j{}^i=g^{ik}C_{jk},\quad C^2=C_iC^i, \tag{9}$$

$$C_{j|k}=\partial_kC_j-C_l\Gamma_j{}^l{}_k. \tag{10}$$

It is okvious that C_{jk} is a symmetric tensor.

Theorem 3. *(M, L) is conformally flat if it is conformally α-flat and $n=\dim M\geqq 3$.*

Proof. From ${}^cR=0$, we have

(11) $$R_j{}^i{}_{kl}+C_{jk}\delta_l{}^i+g_{jk}C_l{}^i-C_{jl}\delta_k{}^i-g_{jl}C_k{}^i=0.$$

Contracting (11) by (i, l), we have

(12) $$R_{jk}+(n-2)C_{jk}+g_{jk}C^*=0, \qquad C^*\equiv C_i{}^i.$$

Transvecting (12) by g^{jk},

(13) $$R+2(n-1)C^*=0.$$

From (12) and (13), the equation (11) is rewritten in the form

(14) $$R_j{}^i{}_{kl}-\frac{1}{n-2}(R_{jk}\delta_l{}^i+g_{jk}R_l{}^i-R_{jl}\delta_k{}^i-g_{jl}R_k{}^i)$$
$$+\frac{1}{(n-1)(n-2)}R(g_{jk}\delta_l{}^i-g_{jl}\delta_k{}^i)=0.$$

The left hand side of the equation (14) is the same as the Weyl conformal curvature tensor. Consequently, the above theorem is proved. (q. e. d.)

Since ${}^cR$ is conformally invariant, we get

Theorem 4. *$(M, \bar{L})$ is conformally α-flat, if $\bar{L}$ is a conformal change of L and (M, L) is conformally α-flat.*

§2. Projective changes in a Finsler space

Let (M, α) be a fixed Finsler space and $L(x, y)$ be an arbitrary Finsler metric function on M. We put

(15) $$\alpha=\frac{1}{2}a_{ij}(x, y)y^iy^j, \qquad L=\frac{1}{2}g_{ij}(x, y)y^iy^j.$$

Remembering the definition of the Berwald connection, we introduce two sprays γ^i and G^i associated with α and L respectively:

(16) $$\gamma_i\equiv(\dot{\partial}_i\partial_j\alpha)y^j-\partial_i\alpha, \qquad \gamma^i\equiv a^{ij}\gamma_j,$$

(17) $$G_i\equiv(\dot{\partial}_i\partial_jL)y^j-\partial_iL, \qquad G^i\equiv g^{ij}G_j.$$

Moreover, we put

(18) $$\gamma_j{}^i=\dot{\partial}_j\Big(\frac{1}{2}\gamma^i\Big), \qquad \gamma_j{}^i{}_k=\dot{\partial}_k\gamma_j{}^i,$$

(19) $$G_j{}^i=\dot{\partial}_j\Big(\frac{1}{2}G^i\Big), \qquad G_j{}^i{}_k=\dot{\partial}_kG_j{}^i.$$

(20) $$B=\frac{1}{n+1}(G_i{}^i-\gamma_i{}^i), \quad B_i=\dot{\partial}_iB, \quad B_{ij}=\dot{\partial}_jB_i.$$

Then $B(x, y)$ is a scalar function and $B_{ij}=B_{ji}$. Further we define a new spray ${}^{p}N^{i}$ which is called *the p-spray relative to α*:

$$
{}^{p}N^{i}=G^{i}-2By^{i}. \tag{21}
$$

A Finsler space $(M, \bar{L})$ is called projective to another Finsler space (M, L) if the extremals of $(M, \bar{L})$ are all coincident with those of (M, L).

We have a well-known theorem ([3]): $(M, \bar{L})$ is projective to (M, L), if and only if $\bar{G}^{i}=G^{i}+2py^{i}$, where $p(x, y)$ is a scalar funtion.

Now we will prove the following theorem:

Theorem 5. *The p-spray ${}^{p}N^{i}$ is projectively invariant.*

Proof. Let $(M, \bar{L})$ be projective to (M, L), i.e.,

$$
\bar{G}^{i}=G^{i}+2py^{i}. \tag{22}
$$

Differentiating (22) by y^{j}, we have

$$
\bar{G}_{j}{}^{i}\equiv\dot{\partial}_{j}\Big(\frac{1}{2}\bar{G}^{i}\Big)=G_{j}{}^{i}+p\delta_{j}{}^{i}+p_{j}y^{i}, \qquad p_{j}\equiv\dot{\partial}_{j}p. \tag{23}
$$

Thus we get

$$
\bar{G}_{j}{}^{i}-\gamma_{j}{}^{i}=G_{j}{}^{i}-\gamma_{j}{}^{i}+p\delta_{j}{}^{i}+p_{j}y^{i}. \tag{24}
$$

Contracting (24) with respect to (i, j), we get $(n+1)\bar{B}=(n+1)B+(n+1)p$, because p is positively homogeneous of degree 1. Therefore, we get

$$
\bar{B}=B+p. \tag{25}
$$

From (21), (22) and (25), it follows that

$$
{}^{p}\bar{N}^{i}\equiv\bar{G}^{i}-2\bar{B}y^{i}=G^{i}+2py^{i}-2(B+p)y^{i}=G^{i}-2By^{i}\equiv{}^{p}N^{i}. \qquad \text{(q.e.d.)}
$$

Conversely, we get

Theorem 6. *A Finsler spce $(M, \bar{L})$ is projective to another Finsler space (M, L), if ${}^{p}\bar{N}^{i}={}^{p}N^{i}$.*

Proof. ${}^{p}\bar{N}^{i}\equiv\bar{G}^{i}-2\bar{B}y^{i}$ and ${}^{p}N^{i}\equiv G^{i}-2By^{i}$ by the definition. By the condition, $\bar{G}^{i}=G^{i}+2(\bar{B}-B)y^{i}=G^{i}+2py^{i}$. Consequently, $(M, \bar{L})$ is projective to (M, L).
(q. e. d).

From the p-spray ${}^{p}N^{i}$ we get a nonlinear connection

$$
{}^{p}N_{j}{}^{i}\equiv\dot{\partial}_{j}\Big(\frac{1}{2}{}^{p}N^{i}\Big)=G_{j}{}^{i}-B_{j}y^{i}-B\delta_{j}{}^{i}, \tag{26}
$$

and an h-connection

$$
{}^{p}\Gamma_{j}{}^{i}{}_{k}\equiv\partial_{k}{}^{p}N_{j}{}^{i}=G_{j}{}^{i}{}_{k}-B_{j}\delta_{k}{}^{i}-B_{k}\delta_{j}{}^{i}-B_{jk}y^{i}. \tag{27}
$$

Further we get the h-curvature tensor

$$ {}^{p}R_{j}{}^{i}{}_{kl} \equiv \delta_{l}{}^{p}\Gamma_{j}{}^{i}{}_{k} + {}^{p}\Gamma_{j}{}^{m}{}_{k}{}^{p}\Gamma_{m}{}^{i}{}_{l} - \delta_{k}{}^{p}\Gamma_{j}{}^{i}{}_{l} - {}^{p}\Gamma_{j}{}^{m}{}_{l}{}^{p}\Gamma_{m}{}^{i}{}_{k}, \tag{28} $$

where we put $\delta_i = \partial_i - {}^{p}N_i{}^j \dot{\partial}_j$. From Theorem 6, it follows that the non-linear connection ${}^{p}N_j{}^i$, h-connection ${}^{p}\Gamma_j{}^i{}_k$ and h-curvature ${}^{p}R_j{}^i{}_{kl}$ are all projectively invariant with respect to α.

Example. Let (M, α) be a Riemannian space and $\{{}_j{}^i{}_k\}$ be the Christoffel symbols constructed from α or $a_{ij}(x)$.

A Randers space (M, L) is a manifold equipped with a metric function L as follows ([1], [2], [4])

$$ L \equiv \frac{1}{2}(\sqrt{a_{ij}y^i y^j} + b_i(x)y^i)^2. \tag{29} $$

We put

$$ \lambda = \sqrt{a_{ij}(x)y^i y^j}, \qquad \beta = b_i(x)y^i, \tag{30} $$

$$ E_{ji} = b_{i;j} - b_{j;i}, \quad b_{i;j} = \partial_j b_i - b_k\{{}_i{}^k{}_j\}, \quad E_j{}^i = a^{ik}E_{jk}. \tag{31} $$

By the straightforward calculation, we have

$$ {}^{p}N^i = \gamma^i + \lambda E_0{}^i, \quad E_0{}^i \equiv y^j E_j{}^i, \qquad \text{where } \gamma^i \text{ are defined by (16).} \tag{32} $$

For another Randers space $(M, \bar{L})$, where $\bar{L} \equiv (1/2)(\lambda + \bar{\beta})^2$ and $\bar{\beta} \equiv \bar{b}_i(x)y^i$, we have

$$ {}^{p}\bar{N}^i = \gamma^i + \lambda \bar{E}_0{}^i. \tag{33} $$

Consequently, if $(M, \bar{L})$ is projective to (M, L), we have $\bar{E}_0{}^i = E_0{}^i$ from Theorem 5, (32) and (33). Therefore it is obvious that $\bar{E}_j{}^i = E_j{}^i$

Conversely, $\bar{E}_j{}^i = E_j{}^i$ implies ${}^{p}\bar{N}^i = {}^{p}N^i$ and so $(M, \bar{L})$ is projective to (M, L), from Theorem 6. Consequently,

Theorem 7. *A Randers space $(M, \bar{L})$ is projective of another Randers space (M, L) if and only if $\bar{E}_j{}^i = E_j{}^i$.*

Corollary. *A Randers space (M, L) is projective to the Riemannian space (M, α) if and only if $E_j{}^i = 0$.*

Proof. We suppose $\bar{L} = \alpha$. Then $\bar{E}_j{}^i = 0$ and it is obvious that Corollary is gotten (q. e. d.)

Though these results are already known ([1], [4]), it is interesting that our theory gives them as an application.

§ 3. Projective changes of a Riemannian space

We deal with projective chages of a Riemannian space as a special case of a Finsler space in § 2.

Let (M, α) be a given Riemannian space and (M, L) be an arbitrary Riemannian space with the same underlying manifold M. We use the same notations as those of

§ 1 and § 2.

If we put $B^{*}=(1/(n+1))\log(\sqrt{g}/\sqrt{a})$, from (20) we get

$$B_{i}\equiv\dot{\partial}_{i}B=\partial_{i}B^{*}\equiv B_{i}^{*} \tag{34}$$

Consequently, $B_{i}=B_{i}(x)$ and $B_{jk}=0$. Therefore, (27) gives

$${}^{p}\Gamma_{j}{}^{i}{}_{k}=G_{j}{}^{i}{}_{k}-B_{i}^{*}\delta_{k}{}^{i}-B_{k}^{*}\delta_{j}{}^{i}. \tag{35}$$

Since (M, L) is Riemannian, we have $G_{j}{}^{i}{}_{k}=\Gamma_{j}{}^{i}{}_{k}$. Therefore,

$${}^{p}\Gamma_{j}{}^{i}{}_{k}=\Gamma_{j}{}^{i}{}_{k}-B_{j}^{*}\delta_{k}{}^{i}-B_{k}^{*}\delta_{j}{}^{i}. \tag{36}$$

From (28), we have

$${}^{p}R_{j}{}^{i}{}_{kl}=\partial_{l}{}^{p}\Gamma_{j}{}^{i}{}_{k}+{}^{p}\Gamma_{j}{}^{m}{}_{k}{}^{p}\Gamma_{m}{}^{i}{}_{l}-\partial_{k}{}^{p}\Gamma_{j}{}^{i}{}_{l}-{}^{p}\Gamma_{j}{}^{m}{}_{l}{}^{p}\Gamma_{m}{}^{i}{}_{k}. \tag{37}$$

Immediately, we get the following theorem:

Theorem 8. *The symmetric linear connection ${}^{p}\Gamma_{j}{}^{i}{}_{k}(x)$ is projectively invariant.*

Now, we introduce the following definition:

Definition. A Riemannian space (M, L) is called *projectively α-flat* or *a projectively flat space relative to α*, if the projectively invariant cuvature tensor ${}^{p}R$ vanishes.

In case of $L=\alpha$, we have $B^{*}=0$ and $B_{i}^{*}=0$. Therefore, ${}^{p}\Gamma_{j}{}^{i}{}_{k}=\Gamma_{j}{}^{i}{}_{k}$ and ${}^{p}R_{j}{}^{i}{}_{kl}=R_{j}{}^{i}{}_{kl}$. Consequently, we get

Theorem 9. *A Riemannian space (M, α) is locally flat if and only if it is projectively α-flat.*

Moreover, get the following two theorems.

Theorem 10. *A Riemannian space (M, L) is projectively flat and consequently it is of constant curvature, if it is projectively α-flat and* $n=\dim M\geqq 3$.

Proof. By the straightforward calculation, we have

$${}^{p}R_{j}{}^{i}{}_{kl}=R_{j}{}^{i}{}_{kl}+B_{jk}^{*}\delta_{l}{}^{i}-B_{jl}^{*}\delta_{k}{}^{i}, \tag{38}$$

$$B_{jk}^{*}\equiv B_{j|k}^{*}+B_{j}^{*}B_{k}^{*},\qquad B_{j|k}^{*}\equiv\partial_{k}B_{j}^{*}-B_{i}^{*p}\Gamma_{j}{}^{i}{}_{k}. \tag{39}$$

The condition ${}^{p}R_{j}{}^{i}{}_{kl}=0$ implies

$$R_{j}{}^{i}{}_{kl}+B_{jk}^{*}\delta_{l}{}^{i}-B_{jl}^{*}\delta_{k}{}^{i}=0. \tag{40}$$

Contracting (40) with respect to (i, l), we have

$$R_{jk}=(n-1)B_{jk}^{*}. \tag{41}$$

From (40) and (41), we get

$$R_{j}{}^{i}{}_{kl}-\frac{1}{n-1}(R_{jk}\delta_{l}{}^{i}-R_{jl}\delta_{k}{}^{i})=0. \tag{42}$$

The left hand side of (42) is coincident with the Weyl projective curvature tensor. Therefore, (M, L) is projectively flat and from the well-known theorem, it is of constant curvature. (q. e. d.)

Theorem 11. *A Riemannian space $(M, \bar{L})$ is projectively α-flat if $\bar{L}$ is a projective change of L and (M, L) is projectively α-flat.*

Proof. Siuce ${}^pR_j{}^i{}_{kl}$ is projectively invariant, the theorem is obvious. (q. e. d.)

§4. Projective parameters

Finally we consider the projective parameters of extremals. We use the same notations as §3.

Let (M, α) be a given Riemannian space. For an arbitrary Riemannian space (M, L), we defined the projectively invariant symmetric linear connection ${}^p\Gamma$, i.e.,

$${}^p\Gamma_j{}^i{}_k=\Gamma_j{}^i{}_k-B_j^*\delta_k{}^i-B_k^*\delta_j{}^i\,, \qquad B_j^*\equiv\partial_jB^*\,.$$

From this relation, it is concluded that the path with respect to Γ is coincident with that of ${}^p\Gamma$.

Now, we consider an extremal C of (M, L), i.e., it is a path with respect to Γ. The curve C is also a path with respect to ${}^p\Gamma$. Let s be an affine parameter of C with respect to Γ and u be an affine parameter of the same curve C with respect to ${}^p\Gamma$. The notion of the Schwartzian derivative is defined by

$$\{n, s\}=\left[((d_s)^3u)d_su-\frac{3}{2}((d_s)^2u)^2\right](d_su)^{-2}\,, \tag{43}$$

where we put $d_s=d/ds$.

Since the parameters s and u are affine parameters of the path C, we have the equations of the path C as follow

$$(d_s)^2x^i+\Gamma_j{}^i{}_kd_sx^jd_sx^k=0 \quad\text{and}\quad (d_u)^2x^i+{}^p\Gamma_j{}^i{}_kd_ux^k=0 \tag{44}$$

respeɔtively, where we put $d_u=d/du$. From these equations, we get

$$\{u, s\}=\frac{2}{n-1}(R_{jk}-{}^pR_{jk})d_sx^jd_sx^k\,. \tag{45}$$

Consequently, we get

Theorem 12. *The relation* (45) *is obtained if u and s are affine parameters of a path with respect to Γ and ${}^p\Gamma$ respectively.*

Here, we write various formulas with respect to the Schwartzian derivative:

$$\{u, s\}=0 \Longleftrightarrow u=(cs+d)(as+b)^{-1} \tag{46}$$

$$\{t, s\}=\{u, s\} \Longleftrightarrow \{t, u\}=0\,, \tag{47}$$

$$\{u, s\}=-(d_us)^{-2}\{s, u\}\,, \tag{48}$$

$$\{t, s\}=\{t, u\}(d_su)^2+\{u, s\}\,. \tag{49}$$

Now, we continue to discuss the parameters of a path with respect to Γ. Let t be an arbitrary parameter of an extremal C. From (49) and Theorem 12, we get

Theorem 13. *If u and s are affine parameters of a path C with respect to Γ and ${}^p\Gamma$ respectively and t is an arbitary parameter of the same curve C, then we have*

$$\{t, s\}-\frac{2}{n-1}R_{jk}d_sx^jd_sx^k=\left(\{t, u\}-\frac{2}{n-1}{}^pR_{jk}d_ux^jd_ux^k\right)(d_su)^2. \tag{50}$$

A projective parameter t of a path C with respect to Γ is defined by

$$\{t, s\}-\frac{2}{n-1}R_{jk}d_sx^jd_sx^k=0. \tag{51}$$

From (50) the parameter t is a projective parameter of the curve C with respect to ${}^p\Gamma$. Therefore, we get

Theorem 14. *Let $(M, \bar{L})$ be projective to (M, L). If $\bar{t}$ and t are projective parameters of a common extremal C of $(M, \bar{L})$ and (M, L) respectively, then we have $\bar{t}=(ct+d)(at+b)^{-1}$*

Proof. Since $(M, \bar{L})$ is projective to (M, L) and ${}^p\Gamma$ is projectively invariant, we have

$${}^p\bar{R}_{jk}={}^pR_{jk}. \tag{52}$$

Consequently, $\bar{t}$ and t are both projective parameters of the path C with respect to ${}^p\Gamma$, which is equal to ${}^p\bar{\Gamma}$. Therefore, from Theorem 13 we get

$$\{\bar{t}, u\}=\{t, u\}. \tag{53}$$

Thus (46) and (45) lead us to the conclusion. (q. e. d.)

Though Theorem 14 is a well-known theorem, it is interesting that it is proved as an application of our theory.

FACULTY OF EDUCATION,
SHIGA UNIVERSITY

References

[1] M. Hashiguchi and Y. Ichijyo, Randers spaces with rectilinear geodesics, Rep. Fac. Sci. Kagoshima Univ., **13** (1980).
[2] M. Matsumoto, Foundations of Finsler geometry and special Finsler spaces, Kaiseisha Press, Japan, 1986.
[3] M. Matsumoto, Projective changes of Finsler metrics and projectively flat Finsler spaces, Tensor (N. S.), **34** (1980).
[4] M. Matsumoto, Projectively flat Finsler spaces of dimension two and an example of projective change, Proc. fifth Natl. Sem. Finsler and Lagrange Spaces, Braşov, Romania, 1988.