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On the canonical forms of 3×3 non-diagonalizable hyperbolic systems with real constant coefficients

By

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1. Introduction

Consider an $m\times m$ system of differential equations

$$\frac{\partial u}{\partial t}=\sum_{i=1}^{n}A_i\frac{\partial u}{\partial x_i} \tag{1.1}$$

where u is an m-vector and A_i are real constant $m\times m$ matrix coefficients. For simplicity, we further assume any nontrivial linear combination of A_i is not equal to the zero matrix or the identity. Otherwise, the system (1.1) can be reduced to the one with a smaller n. (See the comments between Definition 2.3 and 2.4.)

According to Gårding [1], (1.1) is called a hyperbolic system if the real linear combination of A_i:

$$\sum\xi_iA_i$$

has only real eigenvalues for any choice of $\xi_1, \xi_2, \cdots, \xi_n\in \boldsymbol{R}$. It is easy to see that some special classes of systems (1.1) satisfy this criterion. One example is the case when all A_i are simultaneously upper-triagular. Another example is the case when all A_i are simultaneously symmetric. However, few attempts have been made to find out all the canonical forms of hyperbolic systems (1.1). It is perhaps because the above criterion is stated in terms of the linear combinations of $A_1, A_2, \cdots, A_n$ and seems difficult to verify directly. The only exception is the case of $m=2$ (2×2 systems). In fact, Strang [5] proved that every hyperbolic 2×2 system can be reduced to either

$$\frac{\partial u}{\partial t}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\frac{\partial u}{\partial x_1}+\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\frac{\partial u}{\partial x_2}$$

or

$$\frac{\partial u}{\partial t}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\frac{\partial u}{\partial x_1}+\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\frac{\partial u}{\partial x_2}.$$

However, concerning the case $m\geqq3$, very little has been known.

In the previous paper [4], we fully studied a special subclass of 3×3 systems ($m=3$) where each $\sum\xi_iA_i$ is similar to a real diagonal matrix. The purpose of this paper is to classify the remaining subclass of 3×3 systems, that is, the one where

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some of $\sum \xi_i A_i$ are not diagonalizable.

All the results of this paper shall be summarized in the last section in terms of matrix families.

2. Definitions

Throughout this paper, we consider only real square (actually 3×3) matrices and their linear combinations with real coefficients. Although most of the definitions below are the same as in the previous paper [4], we write them for this paper to be self-contained.

Definition 2.1. The set of all linear combinations

$$A(\xi)=A(\xi_1, \xi_2, \cdots, \xi_n)=\sum_{j=1}^{n} \xi_j A_j \qquad (\xi_1, \xi_2, \cdots, \xi_n \in \boldsymbol{R})$$

of the $m\times m$ matrices $A_1, A_2, \cdots, A_n$ is said to be the matrix family spanned by $A_1, A_2, \cdots, A_n$ and is denoted by $\langle A_1, A_2, \cdots, A_n\rangle$.

We are now able to define the objects of our consideration.

Definition 2.2. A matrix family $\langle A_1, A_2, \cdots, A_n\rangle$ is said to have only real eigenvalues if each of its members has only real eigenvalues. In addition, Equation (1.1) with such A_i is called a hyperbolic system.

In the previous paper [4], we considered a special class of matrix families, namely, real-diagonalizable families defined just below. So, in this paper, we shall consider the remaining class, namely, nondiagonalizable matrix families with only real eigenvalues.

Definition 2.3. A matrix family $\langle A_1, A_2, \cdots, A_n\rangle$ is called real-diagonalizable if for every $A(\xi)\in\langle A_1, A_2, \cdots, A_n\rangle$, there exists a nonsingular matrix $S(\xi)$ (called a diagonalizer) such that

$$S(\xi)^{-1}A(\xi)S(\xi)$$

is a real diagonal matrix. Similarly $\langle A_1, A_2, \cdots, A_n\rangle$ is called non-diagonalizable if some $A(\xi)$ is not similar to any diagonal matrix.

Let us now consider what equivalence relation should be introduced for matrix families. It is easy to see the following three operations $\langle A_1, \cdots, A_n\rangle\to\langle B_1, \cdots, B_n\rangle$ do not affect the real-eigenvalue property of matrix families.

a) *Change of basis.*

$$\begin{aligned}
B_1&=m_{11}A_1+m_{12}A_2+\cdots+m_{1n}A_n\\
B_2&=m_{21}A_1+m_{22}A_2+\cdots+m_{2n}A_n\\
&\vdots\\
B_n&=m_{n1}A_1+m_{n2}A_2+\cdots+m_{nn}A_n
\end{aligned}$$

where $M=(m_{ij})$ is a nonsingular $n\times n$ real matrix.

b) *Addition of scalar multiples of identity.*

$$\begin{aligned} B_1 &= A_1 + \mu_1 I \\ B_2 &= A_2 + \mu_2 I \\ \vdots\ & \quad\ \vdots \qquad \vdots \\ B_n &= A_n + \mu_n I \end{aligned}$$

where I is the identity matrix and μ_i $(1 \leqq i \leqq n)$ are reals.

c) *Similarity transformation.*

$$\begin{aligned} B_1 &= T^{-1} A_1 T \\ B_2 &= T^{-1} A_2 T \\ \vdots\ & \qquad \vdots \\ B_n &= T^{-1} A_n T \end{aligned}$$

where T is a nonsingular $m \times m$ real matrix arbitrarily fixed.

It is perhaps worth noting how the above three operations tranform the original differential equation (1.1). First, a) corresponds to the change of space variables:

$$(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)^{\mathrm{T}} = M(x_1, x_2, \cdots, x_n)^{\mathrm{T}}.$$

Second, b) corresponds to the change of time-space variables of the type:

$$\tilde{x}_i = x_i - \mu_i t \qquad (1 \leqq i \leqq n).$$

Note that if some space variables disappear from (1.1) by these operations, they can be regarded as parameters for the solution of the *reduced* equation. Finally, c) corresponds to the change of unknowns:

$$(\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_m)^{\mathrm{T}} = T^{-1}(u_1, u_2, \cdots, u_m)^{\mathrm{T}}.$$

Combining the above a), b) and c), we are led to the following definition.

Definition 2.4. Matrix families $\langle A_1, A_2, \cdots, A_n \rangle$ and $\langle B_1, B_2, \cdots, B_{n'} \rangle$ are called equivalent if there exist a nonsingular matrix T and $\mu_j \in \boldsymbol{R}$ $(j=1, 2, \cdots, n)$ such that

$$\begin{aligned} &\langle T^{-1}A_1T - \mu_1 I,\ T^{-1}A_2T - \mu_2 I,\ \cdots,\ T^{-1}A_nT - \mu_n I \rangle \\ &= \langle B_1, B_2, \cdots, B_{n'} \rangle . \end{aligned}$$

And we denote this equivalence relation by

$$\langle A_1, A_2, \cdots, A_n \rangle \sim \langle B_1, B_2, \cdots, B_{n'} \rangle .$$

By using the above a) and b), it is easy to see that any matrix family is equivalent to some $\langle B_1, \cdots, B_n \rangle$ where $B_1, B_2, \cdots, B_n$ are linearly independent and none of their nonzero linear combinations is equal to any scalar multiple of identity. Let us define a word indicating this property.

Definition 2.5. A matrix family $\langle A_1, A_2, \cdots, A_n \rangle$ is called nondegenerate if $I, A_1, A_2, \cdots, A_n$ are linearly independent over reals.

Although the definitions above are valid also for families of matrices of arbitrary size, we shall consider only 3×3 matrix families from now on. Moreover, we shall treat our problem purely as that of matrix theory and refer the differential equation (1.1) no more.

3. Preliminaries

By straightforward calculations, it is easy to verify that the following (3.1), (3.1′), (3.2), (3.2′) have only real eigenvalues.

(3.1) $$\left\langle \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\right\rangle,$$

(3.1′) the transposition of (3.1),

(3.2) $$\left\langle \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}\right\rangle,$$

(3.2′) the transposition of (3.2).

Relating to this fact, we have the following lemma.

Lemma 3.1. *Let $\langle A_1, A_2, \cdots, A_n\rangle$ be a matrix family with only real eigenvalues. If all of A_i are 3×3 matrices whose (2, 1)- and (3, 1)-entries both vanish then $\langle A_1, A_2, \cdots, A_n\rangle$ is equivalent to a subfamily of (3.1) or (3.2). Similarly, if all of A_i are 3×3 matrices whose (1, 2)- and (1, 3)- entries both vanish then $\langle A_1, A_2, \cdots, A_n\rangle$ is equivalent to a subfamily of (3.1′) or (3.2′).*

Proof. We have only to prove the former half because the latter is merely the transposition of the former. From the assumption, all A_i have the form

$$\begin{bmatrix}*&*&*\\0&*&*\\0&*&*\end{bmatrix}$$

where each $*$ stands for a certain real number. So the right-lower 2×2 submatrices $\tilde{A}_i$ of A_i form a family of 2×2 matrix family which has only real eigenvalues. Proceeding as in Appendix of Strang [5], we know there exists a nonsingular 2×2 matrix $\tilde{T}$ such that

$$\tilde{T}^{-1}\tilde{A}_i\tilde{T}$$

are simultaneously either symmetric or upper-triangular. Thus by the similarity transformation with

$$T=\begin{bmatrix}1&0&0\\0&&\\0&\multicolumn{2}{c}{\tilde{T}}\end{bmatrix},$$

the family $\langle A_1, A_2, \cdots, A_n\rangle$ is equivalent to a subfamily of either (3.1) or (3.2). (Recall that we can add appropriate scalar multiples of identity to all A_i.) $\square$

Let us give *a characterization* of (3.1), (3.1′), (3.2) and (3.2′).

Lemma 3.2. *If* $\langle A_1, A_2, \cdots, A_n\rangle$ *is equivalent to a subfamily of either* (3.1) *or* (3.2) *then all members of* $\langle A_1, A_2, \cdots, A_n\rangle$ *have a common right eigenvector. Similarly, if* $\langle A_1, A_2, \cdots, A_n\rangle$ *is equivalent to a subfamily of* (3.1′) *or* (3.2′) *then all of its members have a common left eigenvector.*

Proof. We have only to prove the former half because the latter is merely the transposition of the former. Let us begin with the special case where $\langle A_1, A_2, \cdots, A_n\rangle$ is just a subfamily of either (3.1) or (3.2). Then

$$\begin{bmatrix}1\\0\\0\end{bmatrix}$$

is clearly the desired common right eigenvector. In the general case, there exist a certain similar transformation: $A\to T^{-1}AT$ which reduces $\langle A_1, A_2, \cdots, A_n\rangle$ to a subfamily of either (3.1) or (3.2). In this case the common right eigenvector becomes

$$T\begin{bmatrix}1\\0\\0\end{bmatrix}.$$

The proof is complete. $\square$

Let us gather here some properties of special kinds of cubic equations with a (real) parameter which will appear as characteristic equations of 3×3 matrix family.

Lemma 3.3. *Let* $f(\lambda, \xi)$ *be a cubic polynomial of the form*

$$f(\lambda, \xi)\equiv\lambda^3+a_1\lambda^2+a_2\lambda+a_3+\xi(b_0\lambda^2+b_1\lambda+b_2)+c\xi^2$$

where $a_1, a_2, a_3, b_0, b_1, b_2, c\neq0$ *are real constants and* ξ *is a real parameter. Then the cubic equation*

$$f(\lambda, \xi)=0$$

has imaginary roots for some $\xi\in\boldsymbol{R}$.

Proof. Without losing the generality, we may assume $c>0$ because, otherwise, it suffices to consider $f(-\lambda, \xi)=0$ instead. Let us plot the graph of $f(\lambda, \xi)=0$ in the λ, ξ-plane. For this purpose, it is convenient to solve $f(\lambda, \xi)=0$ with respect to ξ.

$$\xi=\frac{1}{2c}[-b_0\lambda^2-b_1\lambda-b_2\pm\{(b_0\lambda^2+b_1\lambda+b_2)^2-4c(\lambda^3+a_1\lambda^2+a_2\lambda+a_3)\}^{1/2}].$$

From Fig. 1, we know that $f(\lambda, \xi)=0$ has only one real simple root and a pair of complex conjugate roots for ξ near $+\infty$ (resp. $-\infty$) when $b_0>0$ (resp. $b_0\leqq0$). $\square$

Lemma 3.4. *Let $f(\lambda, \xi)$ be a cubic polynomial of the form*

$$f(\lambda, \xi) \equiv \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 + \xi(b_1\lambda + b_2)$$

where a_1, a_2, a_3, b_1, b_2 are real constants satisfying $|b_1| + |b_2| > 0$ and ξ is a real parameter. Then the cubic equation

$$f(\lambda, \xi) = 0$$

has imaginary roots for some $\xi \in \boldsymbol{R}$.

Proof. We begin with the case $b_1 \neq 0$. Renaming $b_1\xi$ as ξ, we may assume $b_1 = 1$. Let us plot the graph of $f(\lambda, \xi) = 0$ by the form

$$\xi = -\frac{\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3}{\lambda + b_2}.$$

From Fig. 2, it is clear that $f(\lambda, \xi) = 0$ has only one real simple root and a pair of complex conjugate ones for ξ near $+\infty$.

We now go on to the case $b_1 = 0$. From the assumption of the present lemma, $b_2 \neq 0$. So the graph is reduced to that of

$$\xi = -\frac{1}{b_2}(\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3).$$

Therefore, it is clear that $f(\lambda, \xi) = 0$ has only one real simple root and a pair of complex conjugate ones for ξ near $\pm\infty$ (see Fig. 3). $\square$

Lemma 3.5. *Let $f(\lambda, \xi)$ be a cubic polynomial of the form*

$$f(\lambda, \xi) \equiv \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 + \xi(b_0\lambda^2 + b_1\lambda + b_2)$$

where a_1, a_2, a_3, $b_0 \neq 0$, b_1, b_2 are real constants and ξ is a real parameter. Then the cubic equation

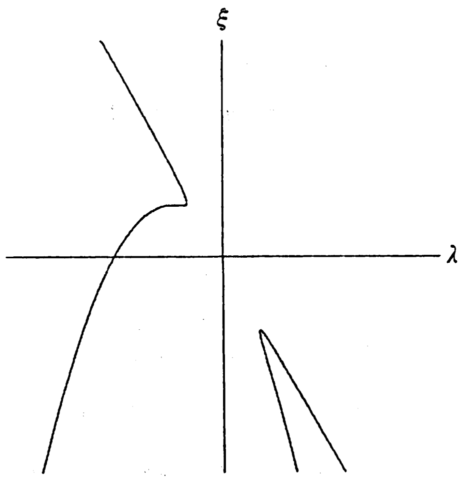

Fig. 1. $b_0 > 0$.

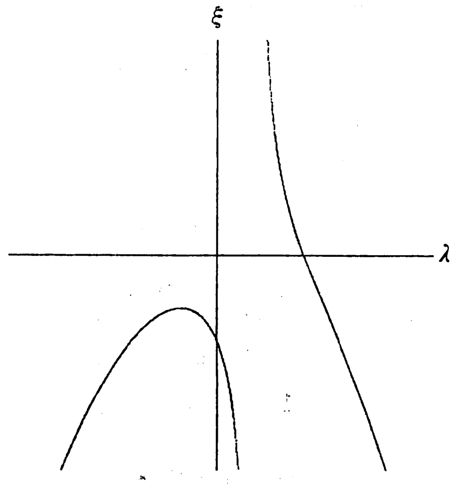

Fig. 2.

$$f(\lambda, \xi)=0$$

has only real roots for any $\xi\in\boldsymbol{R}$ *if and only if*

$$\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0$$

and

$$b_0\lambda^2+b_1\lambda+b_2=0$$

have only real roots, say, $\alpha_1\leqq\alpha_2\leqq\alpha_3$ *for the first equation and* $\beta_1\leqq\beta_2$ *for the second, and the inequality*

$$\alpha_1\leqq\beta_1\leqq\alpha_2\leqq\beta_2\leqq\alpha_3$$

holds.

Proof. We notice first that we may assume $b_0=1$ by renaming $b_0\xi$ as ξ. Now let us consider all possible cases one by one.

First we look into the case where

$$\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0$$

has imaginary roots. In this case, $f(\lambda, \xi)=0$ has imaginary roots for $\xi=0$.

Next we look into the case where

$$\lambda^2+b_1\lambda+b_2=0$$

has imaginary roots, or equivalently, the case where

$$\lambda^2+b_1\lambda+b_2>0 \qquad \text{for all } \xi\in\boldsymbol{R}.$$

Let us plot the graph of $f(\lambda, \xi)=0$ by the form

$$\xi=-\frac{\lambda^3+a_1\lambda^2+a_2\lambda+a_3}{\lambda^2+b_1\lambda+b_2}.$$

From Fig. 4, it is clear that $f(\lambda, \xi)=0$ has only one real simple root and a pair of

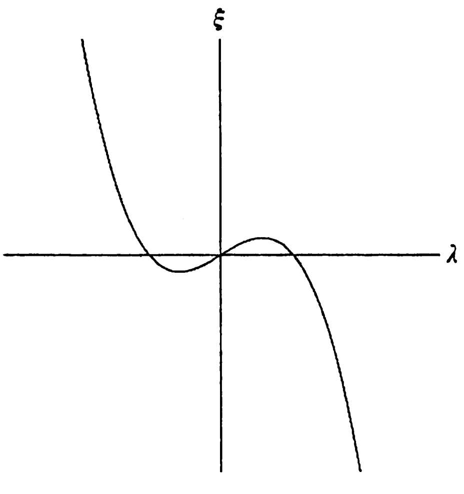

Fig. 3.

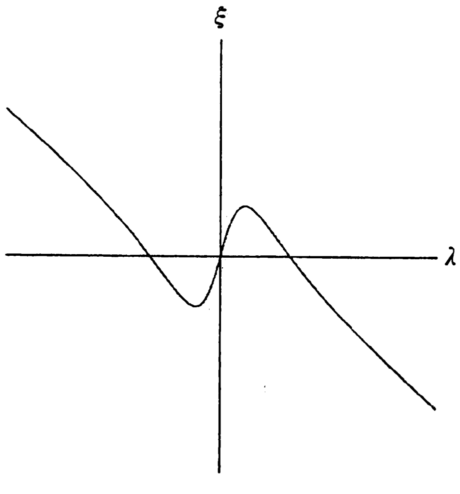

Fig. 4.

complex conjugate ones for ξ near $\pm\infty$.

We now go on to the case where both

$$\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0$$

and

$$b_0\lambda^2+b_1\lambda+b_2=0$$

has only real roots ($\alpha_1\leqq\alpha_2\leqq\alpha_3$ and $\beta_1\leqq\beta_2$). We will plot the graph of $f(\lambda, \xi)=0$ again by solving it with respect to ξ. For simplicity, we consider here only typical cases.

Let us look into the case $\alpha_1<\beta_1<\alpha_2<\beta_2<\alpha_3$. From Fig. 5, it is clear that $f(\lambda, \xi)=0$ has only real roots for any $\xi\in\boldsymbol{R}$.

Let us now consider the case $\beta_1<\alpha_1<\alpha_2<\beta_2<\alpha_3$. From Fig. 6, it follows that $f(\lambda, \xi)=0$ has only one real simple root and a pair of complex conjugate ones for ξ near $+\infty$.

Proceeding in this way, we can complete the proof. $\square$

Let us go back to our original problem to classify non-diagonalizable 3×3 matrix families with only real eigenvalues. By a change of basis, we may assume A_1 is a non-diagonalizable matrix with only real eigenvalues. By the addition of a scalar multiple of identity and by a similar tranformation, we may further assume A_1 is equal to either

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

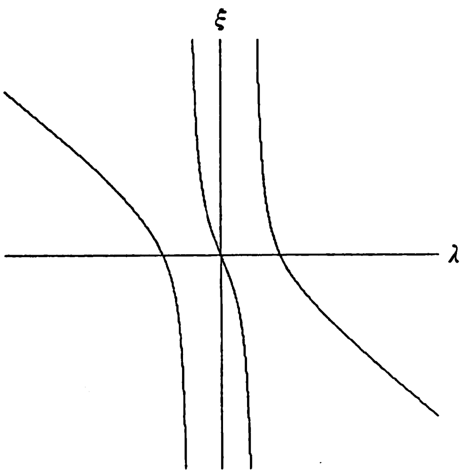

Fig. 5.

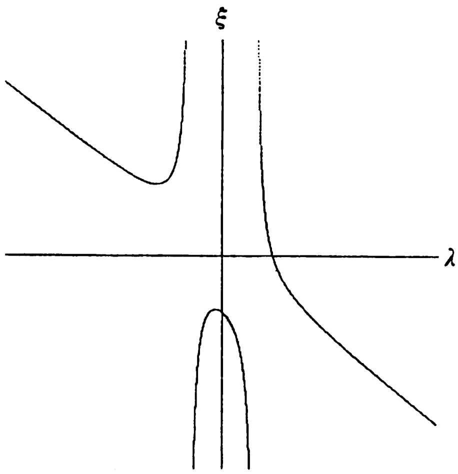

Fig. 6.

or

$$\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

where $a \neq 0$ is a certain real. In addition, by the similarity tranformation with

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix},$$

the third matrix becomes a scalar multiple of

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

In short, we may restrict ourselves to consider such matrix families $\langle A_1, A_2, \cdots, A_n \rangle$ where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

or

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

holds. We shall consider each case separately in the sequel.

4. Families with triple eigenvalues

In this section, we consider matrix families, say, $\langle A_1, A_2, \cdots, A_n \rangle$ with

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let us begin with the first case.

Lemma 4.1. *Suppose that the matrix family* $\langle A_1, A_2, \cdots, A_n \rangle$ *with*

$$A_1=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has only real eigenvalues. Then either of the following holds.

a) *The* (2, 1)- *and* (3, 1)-*entries of all* A_i *simultaneously vanish.*

b) *The* (2, 1)- *and* (2, 3)-*entries of all* A_i *simultaneously vanish.*

Proof. Let

$$B=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

be an arbitrary linear combination of $A_2, A_3, \cdots, A_n$. Let us first show

$$b_{21}=0, \qquad b_{23}b_{31}=0. \tag{4.1}$$

Because ξA_1+B has only real eigenvalues with all $\xi\in\boldsymbol{R}$, we can apply Lemma 3.4 to the characteristic equation

$$\det(\xi A_1+B-\lambda I)=0,$$

$$\det(B-\lambda I)+\xi\{b_{21}(\lambda-b_{33})+b_{23}b_{31}\}=0.$$

In this way, we must have (4.1).

From the fact that (4.1) holds for any linear combination of $A_2, A_3, \cdots, A_n$, follows the claim of the present lemma. □

From the last lemma, we obtain the following proposition.

Proposition 4.2. *Suppose that the matrix family* $\langle A_1, A_2, \cdots, A_n\rangle$ *satisfying*

$$A_1=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has only real eigenvalues. Then the family $\langle A_1, A_2, \cdots, A_n\rangle$ *is equivalent to a subfamily of* (3.1), (3.1′), (3.2) *or* (3.2′).

Proof. We can apply the preceding Lemma 4.1. When a) of Lemma 4.1 occurs, the conclusion follows directly from Lemma 3.1. When b) of Lemma 4.1 occurs, we use the similarity transformation with

$$T=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

to $\langle A_1, A_2, \cdots, A_n\rangle$. Then Lemma 3.1 becomes applicable again and we obtain the conclusion. $\square$

Let us now consider another type of matrix families with triple eigenvalues.

Lemma 4.3. *Suppose that the matrix family* $\langle A, B\rangle$ *spanned by*

$$A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ B=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \neq 0$$

has only real eigenvalues. Then either of the following a), b) *holds.*

a) $b_{31}=b_{21}=b_{32}=0$.

b) $b_{31}=0, \quad b_{21}=-b_{32}\neq 0 \quad b_{11}=b_{33}$.

Proof. Let us consider under what condition $\xi A+B$ has only real eigenvalue with any $\xi\in \boldsymbol{R}$. So we can apply Lemma 3.3 and Lemma 3.4 successively to the characteristic equation:

$$\det(\xi A+B-\lambda I)=0,$$

$$\det(B-\lambda I)+\xi\{b_{21}(\lambda-b_{33})+b_{32}(\lambda-b_{11})+b_{31}(b_{12}+b_{23})\}+b_{31}\xi^2=0.$$

Therefore we have

$$b_{31}=0,$$

$$b_{21}+b_{32}=0,$$

$$-b_{21}b_{33}-b_{32}b_{11}+b_{31}(b_{12}+b_{23})=0$$

which are equivalent to

$$b_{31}=0, \quad b_{32}=-b_{21}, \quad b_{21}(b_{11}-b_{33})=0,$$

From this, the conclusion directly follows. $\square$

Let us investigate the case where

$$A_1=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Lemma 4.4. *Suppose that* $\langle A_1, A_2, \cdots, A_n\rangle$ *with*

$$A_1=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

has only real eigenvalues. Suppose also that at least one of $A_2, \cdots, A_n$ *is not upper-triangular. Then each* A_i $(i=2, \cdots, n)$ *satisfies*

a) $[A_i]_{31}=0$,

b) $[A_i]_{32}=-[A_i]_{21}$,

c) $[A_i]_{11}=[A_i]_{33}$

where $[A_i]_{kl}$ *denotes the* (k, l)*-entry of* A_i.

Proof. Without loss of generality, we may assume A_2 is not upper-triagular. Then, choosing $\varepsilon>0$ small enough, none of

$$A_2+\varepsilon A_i \qquad (i\geqq 3)$$

is upper-triangular. Notice that $\langle A_1, A_2\rangle$, $\langle A_1, A_2+\varepsilon A_i\rangle$ $(i\geqq 3)$ have only real eigenvalues as subfamilies of $\langle A_1, A_2, \cdots, A_n\rangle$. From this fact and the above Lemma 4.3, we obtain

$$\begin{aligned} &[A_2]_{31}=0, \\ &[A_2]_{32}+[A_2]_{21}=0, \\ &[A_2]_{11}-[A_2]_{33}=0 \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &[A_2+\varepsilon A_i]_{31}=[A_2]_{31}+\varepsilon[A_i]_{31}=0, \\ &[A_2+\varepsilon A_i]_{32}+[A_2+\varepsilon A_i]_{21}=[A_2]_{32}+[A_2]_{21} \\ &\qquad\qquad +\varepsilon([A_i]_{32}+[A_i]_{21})=0, \\ &[A_2+\varepsilon A_i]_{11}-[A_2+\varepsilon A_i]_{33}=[A_2]_{11}-[A_2]_{33} \\ &\qquad\qquad +\varepsilon([A_i]_{11}-[A_i]_{33})=0 \end{aligned} \tag{4.3}$$

for $i\geqq 3$. Subtracting (4.2) from (4.3), we have

$$\begin{aligned} &[A_i]_{31}=0, \\ &[A_i]_{32}+[A_i]_{21}=0, \\ &[A_i]_{11}-[A_i]_{33}=0 \end{aligned}$$

for $i\geqq 3$. These and (4.2) are the desired equalities. □

From this lemma and by a certain change of basis, we may assume

$$[A_2]_{21}=-1, \quad [A_2]_{32}=1,$$

$$[A_i]_{21}=[A_i]_{32}=0 \quad (i\geqq 3)$$

and

$$[A_i]_{31}=0, \quad [A_i]_{23}=0 \quad (i\geqq 2),$$

$$[A_i]_{11}=[A_i]_{33} \quad (i\geqq 2).$$

By an appropriate addition of scalar multiples of identity, the last equality becomes

$$[A_i]_{11}=[A_i]_{33}=0 \quad (i\geqq 2).$$

Proposition 4.5. *The matrix family $\langle A_1, A_2\rangle$ spanned by*

$$A_1=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ A_2=\begin{bmatrix} 0 & a_2 & a_3 \\ -1 & a_1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

has only real eigenualues if and only if there exist three reals, $\alpha_1\leqq\alpha_2\leqq\alpha_3$ such that

$$a_1=\alpha_1+\alpha_2+\alpha_3,$$

$$a_2=\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1,$$

$$a_3=-\alpha_1\alpha_2\alpha_3.$$

Proof. Since A_1 has only real eigenvalues, it suffices to obtain the condition for ξA_1+A_2 to have only real eigenvalues for all $\xi\in\boldsymbol{R}$. And the characteristic equation of ξA_1+A_2 turns out to be

$$\det(\xi A_1+A_2-\lambda I)=0,$$

$$-\lambda^3+a_1\lambda^2-a_2\lambda-a_3=0.$$

Denoting by α_1, α_2, α_3 the three roots of this characteristic equation, we have

$$a_1=\alpha_1+\alpha_2+\alpha_3,$$

$$a_2=\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1,$$

$$a_3=-\alpha_1\alpha_2\alpha_3.$$

Thus the conclusion immediately follows. □

By the last lemma, we can specify A_2 of the matrix family $\langle A_1, A_2, A_3\rangle$ spanned by three matrices.

Proposition 4.6. *Suppose that the matrix family $\langle A_1, A_2, A_3\rangle$ is spanned by*

$$A_1=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_2=\begin{bmatrix} 0 & \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 & -\alpha_1\alpha_2\alpha_3 \\ -1 & \alpha_1+\alpha_2+\alpha_3 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$A_3=\begin{bmatrix} 0 & b_1 & b_2 \\ 0 & b_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

where $\alpha_1 \leqq \alpha_2 \leqq \alpha_3$, b_0, b_1, b_2 *are real constants. Then* $\langle A_1, A_2, A_3 \rangle$ *has only real eigenvalues if and only if*

$$b_0 \neq 0$$

and there exist two reals $\beta_1 \leqq \beta_2$ *such that*

$$\alpha_1 \leqq \beta_1 \leqq \alpha_2 \leqq \beta_2 \leqq \alpha_3,$$

$$b_1 = b_0(\beta_1 + \beta_2),$$

$$b_2 = -b_0 \beta_1 \beta_2 .$$

Proof. Since any linear combination of A_1 and A_3 has only real eigenvalues, it suffices to investigate the condition for

$$\xi A_1 + A_2 + \eta A_3$$

to have only real eigenvalues for all ξ, $\eta \in \boldsymbol{R}$. Calculating the characteristic equation,

$$\det(\xi A_1 + A_2 + \eta A_3 - \lambda I) = 0,$$

$$-(\lambda - \alpha_1)(\lambda - \alpha_2)(\lambda - \alpha_3) + \eta(b_0 \lambda^2 - b_1 \lambda - b_2) = 0.$$

Regarding η as a parameter, Lemma 3.4 and Lemma 3.5 are applicable. And we have

$$b_0 \neq 0,$$

$$b_1 = b_0(\beta_1 + \beta_2),$$

$$b_2 = -b_0 \beta_1 \beta_2$$

where β_1, β_2 are two roots of

$$b_0 \lambda^2 - b_1 \lambda - b_2 = 0.$$

Thus the conclusion immediately follows. □

Remark. Plotting a graph of $\det(A_2 + \eta A_3 - \lambda I) = 0$ on λ, η-plane, we see easily that the middle eigenvalue of $A_2 + \eta A_3$ is equal to $(\beta_1 + \beta_2)/2$ for some $\eta \in \boldsymbol{R}$.

Let us also consider the matrix families spanned by four or more matrices.

Proposition 4.7. *Suppose that a nondegenerate* 3×3 *matrix family* $\langle A_1, A_2, \cdots, A_n \rangle$ $(n \geqq 4)$ *with*

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

has only real eigenvalues. Then $\langle A_1, A_2, \cdots, A_n \rangle$ *is equivalent to a subfamily of* (3.1), (3.1′), (3.2) *or* (3.2′).

Proof. Assume the contrary. Note that we may also assume

$$A_2=\begin{bmatrix} 0 & \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 & -\alpha_1\alpha_2\alpha_3 \\ -1 & \alpha_1+\alpha_2+\alpha_3 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$A_3=\begin{bmatrix} 0 & \beta_1+\beta_2 & -\beta_1\beta_2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

where α_1, α_2, α_3, β_1, β_2 are real. (See Propositions 4.5 and 4.6). By a change of basis, we may further assume

$$A_4=\begin{bmatrix} 0 & c_1 & c_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\neq 0.$$

However $\langle A_1, A_2, A_4\rangle\subset\langle A_1, A_2, \cdots, A_n\rangle$ cannot have only real eigenvalues from Proposition 4.6. We are thus led to a contradiction. □

Before concluding this section, we write down another fact that the above matrix families are completely of different nature from (3.1), (3.1′), (3.2) and (3.2′).

Proposition 4.8. *Let*

$$A_1=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_2=\begin{bmatrix} 0 & \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 & -\alpha_1\alpha_2\alpha_3 \\ -1 & \alpha_1+\alpha_2+\alpha_3 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$A_3=\begin{bmatrix} 0 & \beta_1+\beta_2 & -\beta_1\beta_2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then neither $\langle A_1, A_2\rangle$ *nor* $\langle A_1, A_2, A_3\rangle$ *is equivalent to any subfamily of* (3.1), (3.1′), (3.2) *or* (3.2′).

Proof. It suffices to consider the case of $\langle A_1, A_2\rangle$. As is easily verified, there are no common eigenvectors for A_1 and A_2. Therefore Lemma 3.2 is applicable and the conclusion follows. □

5. Families with double eigenvalues

Throughout this section, we consider the matrix families which contain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and have only real eigenvalues. We denote this specific matrix by

$$A_1 \quad \text{or} \quad A$$

until the end of this section. Since we have fully discussed the families with triple eigenvalues in the previous section, we chiefly consider families with at most double eigenvalues.

Lemma 5.1. *Suppose that a nondegenerate matrix family* $\langle A_1, A_2, \cdots, A_n\rangle$ $(n \geqq 2)$ *satisfies*

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and has only real eigenvalues. Then for any $B=(b_{ij}) \in \langle A_2, \cdots, A_n\rangle$, *we have*

$$b_{32}=0$$

and the quadratic equation

$$(\lambda - b_{22})(\lambda - b_{33}) + b_{12}b_{31} = 0$$

has only real roots.

Proof. Let $B=(b_{ij}) \in \langle A_2, \cdots, A_n\rangle$ be arbitrary. Then the characteristic equation for $\xi A_1 + B$ turns out to be

$$\text{(5.1)} \quad \begin{aligned} &\det(-\lambda + \xi A_1 + B) = 0, \\ &\det(-\lambda I + B) + \xi\{-b_{32}(b_{11}+b_{23}) + (\lambda - b_{22})(\lambda - b_{33}) + b_{12}b_{31}\} - b_{32}\xi^2 = 0. \end{aligned}$$

Applying Lemma 3.3, we have

$$b_{32}=0.$$

Hence, from Lemma 3.5, we see

$$(\lambda - b_{22})(\lambda - b_{33}) + b_{12}b_{31} = 0$$

has only real roots. Q.E.D. □

Lemma 5.2. *Let the assumptions be the same as in Lemma* 5.1. *Then either of the following* 1), 2) *holds.*

1) $\langle A_1, A_2, \cdots, A_n\rangle$ *is equivalent to a subfamily of* (3.1), (3.1′), (3.2) *or* (3.2′).

2) *There exists* $B=(b_{ij}) \in \langle A_2, \cdots, A_n\rangle$ *such that*

$$b_{12}b_{31} \neq 0.$$

And in the case 2), $\langle A_1, A_2, \cdots, A_n\rangle$ *cannot be equivalent to any subfamily of* (3.1), (3.1′), (3.2) *or* (3.2′).

Proof. Let us begin by proving that 1) occurs if 2) does not. So let us assume

$$b_{12}b_{31}=0 \qquad \text{for all } (b_{ij})\in\langle A_2, \cdots, A_n\rangle.$$

This condition means either

$$b_{12}=0 \qquad \text{for all } (b_{ij})\in\langle A_2, \cdots, A_n\rangle$$

or

$$b_{31}=0 \qquad \text{for all } (b_{ij})\in\langle A_2, \cdots, A_n\rangle.$$

From Lemma 5.1, we have

$$b_{32}=0 \qquad \text{for all } (b_{ij})$$

in both cases. In the first case where

$$b_{12}=b_{32}=0 \qquad \text{for all } (b_{ij})\in\langle A_2, \cdots, A_n\rangle,$$

the similarity transformation with

$$T=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

reduces $\langle A_1, A_2, \cdots, A_n\rangle$ to a subfamily of (3.1) or (3.2). In the second case where

$$b_{31}=b_{32}=0 \qquad \text{for all } (b_{ij})\in\langle A_2, \cdots, A_n\rangle,$$

the similarity transformation with

$$T=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

reduces the matrix family to a subfamily of (3.1′) or (3.2′).

Let us now prove that if

$$b_{12}b_{31}\neq 0$$

for some

$$B=(b_{ij})\in\langle A_1, A_2, \cdots, A_n\rangle$$

then this matrix family is not equivalent to any subfamily of (3.1), (3.1′), (3.2) or (3.2′). By Lemma 3.2, it suffices to show no (left or right) eigenvector of A_1 coincides with any of B. This can be done by a straightforward caculation. Now the proof is complete. □

Let us rewrite Lemma 3.5 in more convenient forms.

Lemma 5.3. *Given a matrix family* $\langle A, B\rangle$ *spanned by*

$$A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

where

$$b_{32}=0.$$

Let the polynomials $c(\lambda)$ *and* $q(\lambda)$ *be as*

$$c(\lambda)\equiv\det(-\lambda I+B),$$

$$q(\lambda)\equiv(\lambda-b_{22})(\lambda-b_{33})+b_{12}b_{31}.$$

Suppose that the quadratic equation

$$q(\lambda)=0$$

has real distinct roots $\beta_1<\beta_2$. *Then the following three conditions are equivalent.*

1) *The matrix family* $\langle A, B\rangle$ *has only real eigenvalues at most double.*
2) *The matrix family* $\langle A, B\rangle$ *has only real eigenvalues.*
3) $c(\beta_1)\leqq 0$, $c(\beta_2)\geqq 0$.

Proof. 1)$\Rightarrow$2) is clear. Let us prove 2)$\Rightarrow$3). In this case, Lemma 3.5 is applicable for the characteristic equation of $\xi A+B$:

$$\det(-\lambda I+\xi A+B)=0,$$

$$c(\lambda)+\xi q(\lambda)=0.$$

Thus we obtain

$$c(\lambda)=0$$

has three real roots $\alpha_1\leqq\alpha_2\leqq\alpha_3$ satisfying

$$\alpha_1\leqq\beta_1\leqq\alpha_2\leqq\beta_2\leqq\alpha_3.$$

From this, $c(\lambda)\equiv-(\lambda-\alpha_1)(\lambda-\alpha_2)(\lambda-\alpha_3)$ and

$$c(\beta_1)\equiv-(\beta_1-\alpha_1)(\beta_1-\alpha_2)(\beta_1-\alpha_3)\leqq 0,$$

$$c(\beta_2)\equiv-(\beta_2-\alpha_1)(\beta_2-\alpha_2)(\beta_2-\alpha_3)\geqq 0.$$

Let us now show 3)$\Rightarrow$1). For this purpose, we need only plot the graph of

$$\det(-\lambda I+\xi A+B)\equiv c(\lambda)+\xi q(\lambda)=0$$

just as in the proof of Lemma 3.5. Note that A itself has clearly only real eigenvalues at most double. $\square$

Lemma 5.4. *Given a matrix family* $\langle A, B\rangle$ *spanned by*

$$A=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix},\ B=\begin{bmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{bmatrix}$$

where

$$b_{32}=0.$$

Let the polynomials $c(\lambda)$ *and* $q(\lambda)$ *be as*

$$c(\lambda)\equiv\det(-\lambda I+B),$$
$$q(\lambda)\equiv(\lambda-b_{22})(\lambda-b_{33})+b_{12}b_{31}.$$

Suppose that the quadratic equation

$$q(\lambda)=0$$

has a real repeated root β. *Then* $\langle A, B\rangle$ *has only real eigenvalues at most double if and only if*

$$c(\beta)=0, \quad c'(\beta)>0.$$

Proof. We begin with the necessity. We can apply Lemma 3.5 to

$$\det(-\lambda I+\xi A+B)\equiv c(\lambda)+\xi q(\lambda)=0.$$

Thus we know

$$c(\lambda)=0$$

has three real roots $\alpha_1\leqq\alpha_2\leqq\alpha_3$ with $\alpha_2=\beta$ because

$$\alpha_1\leqq\beta\leqq\alpha_2\leqq\beta\leqq\alpha_3.$$

We also have $\alpha_1<\beta<\alpha_3$ because otherwise ($\alpha_1=\beta$ or $\alpha_3=\beta$)

$$c(\lambda)+\xi q(\lambda)=0$$

would have triple roots for some $\xi\in\boldsymbol{R}$ as an easy calculation shows. From these, we obtain

$$c(\lambda)\equiv-(\lambda-\alpha_1)(\lambda-\beta)(\lambda-\alpha_3).$$

Therefore

$$c(\beta)=0,$$
$$c'(\beta)\equiv-(\beta-\alpha_1)(\beta-\alpha_3)>0.$$

As for the sufficiency, we need only plot the graph of $c(\lambda)+\xi q(\lambda)=0$ as usual. $\square$

From the above Lemma 5.2, we may assume $[A_2]_{12}[A_2]_{31}\neq0$ in the sequel. Let us now reduce A_2.

Lemma 5.5. *Suppose that a nondegenerate family* $\langle A_1, A_2, \cdots, A_n\rangle$ $(n\geqq2)$ *satisfies*

1) $A_1=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$,

2) $[A_i]_{32}=0$ *for all* $i=2, \cdots, n$,

3) $[A_2]_{12}\neq0$, $[A_2]_{31}\neq0$.

Then there exists a similarity transformation with nonsingular T *of the form*

$$T=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & k\\ 0 & 0 & 1\end{bmatrix} \quad (k: \textit{a certain real})$$

such that

$$T^{-1}A_1T=A_1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\frac{[T^{-1}A_2T]_{13}}{[T^{-1}A_2T]_{12}}=\frac{[T^{-1}A_2T]_{21}}{[T^{-1}A_2T]_{31}}.$$

Proof. We put

$$A_2=B=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

From the assumptions,

$$b_{12}\neq 0, \quad b_{31}\neq 0.$$

So we can find two numbers c_1, c_2 such that

$$b_{13}=b_{12}(c_1-c_2),$$
$$b_{21}=b_{31}(c_1+c_2).$$

Then

$$T=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{bmatrix}$$

is the desired matrix. □

Let us investigate families spanned by two matrices.

Proposition 5.6. *A given nondegenerate matrix family* $\langle A_1, A_2\rangle$ *with*

$$A_1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

has only real eigenvalues at most double if and only if it is equivalent to either

1) $$\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \gamma & \alpha+1 & (\alpha+1)\beta \\ (\alpha-1)\beta & \alpha & 0 \\ \alpha-1 & 0 & -\alpha \end{bmatrix} \right\rangle$$

where

$$|\alpha|\neq 1, \quad (\alpha^2-1)(2\beta+1)\geqq 0, \quad |\gamma|\leqq|2\beta+1|$$

or

2) $$\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \beta \\ \beta & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \right\rangle$$

where

$$2\beta+1>0$$

or

3) *a subfamily of* (3.1), (3.1′), (3.2) *or* (3.2′).

Proof. From Lemma 5.1, 5.2 and 5.5, we may assume A_2 is as follows:

$$A_2=\begin{bmatrix} \gamma & \alpha_2 & \alpha_2\beta \\ \alpha_1\beta & \alpha & \delta \\ \alpha_1 & 0 & -\alpha \end{bmatrix} \tag{5.2}$$

where $\alpha_1\neq 0$, $\alpha_2\neq 0$. Again from Lemma 5.1, the quadratic equation

$$(\lambda-\alpha)(\lambda+\alpha)+\alpha_1\alpha_2=0, \tag{5.3}$$

$$\lambda^2-\alpha^2+\alpha_1\alpha_2=0$$

has only real roots, that is, real distinct roots or a real repeated root. In the former situation, replacing A_2 by its appropriate scalar multiple, we may assume these roots are ± 1. So we obtain from (5.3),

$$\alpha_1\alpha_2=(\alpha-1)(\alpha+1).$$

Now applying a similarity transformation with T of the form

$$T=\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (k\neq 0),$$

$\alpha_1\neq 0$ and $\alpha_2\neq 0$ become

$$\alpha_1=\alpha-1, \quad \alpha_2=\alpha+1. \tag{5.4}$$

On the other hand, in the situation where (5.3) has a repeated root,

$$\alpha_1\alpha_2=\alpha^2.$$

So replacing A_2 by its appropriate scalar multiple, we may assume $\alpha=1$ (recall $\alpha_1\neq 0$, $\alpha_2\neq 0$ mean $\alpha\neq 0$). Now applying a similarity transformation with T of the form

$$T=\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (k\neq 0),$$

$\alpha_1\neq 0$, $\alpha_2\neq 0$ become to $\alpha=1$:

$$\alpha_1=\alpha_2=\alpha=1. \tag{5.5}$$

So we may assume either (5.4) with $\alpha\neq\pm 1$ or (5.5) for the matrix (5.2). We may also assume

$$\delta=0$$

in (5.2) after using a change of basis of $\langle A_1, A_2\rangle$.

Let us first consider the case (5.4) with $\alpha\neq\pm 1$. Then we can apply Lemma 5.3 to $\langle A_1, A_2\rangle$ and we obtain

$$(\alpha^2-1)(-\gamma+2\beta+1)\geqq 0,$$

$$(\alpha^2-1)(-\gamma-2\beta-1)\leqq 0.$$

Combining these, we have

$$-(\alpha^2-1)(2\beta+1)\leqq(\alpha^2-1)\gamma\leqq(\alpha^2-1)(2\beta+1).$$

From this, we obtain the first of the desired families.

Let us now consider the case (5.5). In this case, Lemma 5.4 is applicable. Thus we obtain

$$\gamma=0,\quad 2\beta+1>0.$$

The proof is completed. □

Let us now investigate families spanned by three matrices. For this purpose, we prepare some lemmas.

Lemma 5.7. *Suppose that a matrix family $\langle A, B\rangle$ is spanned by*

$$A=\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix},\ B=\begin{bmatrix}\sigma&\beta&\mu\\\nu&0&\tau\\\beta&\upsilon&0\end{bmatrix}$$

where β, μ, ν, σ, τ, υ are real constants. Suppose also that $\langle A, B\rangle$ has only real eigenvalues. Then $-\sigma A+B$ has zero as a triple eigenvalue.

Proof. From Lemma 5.1, we see

$$\lambda^2+\beta^2=0$$

has only real eigenvalues. This implies

$$\beta=0.$$

Again from Lemma 5.1, we also have

$$\upsilon=0.$$

Now it is easy to show $-\sigma A+B$ has zero as a triple eigenvalue. □

We can specify our matrix families by the following lemma.

Lemma 5.8. *Suppose that a nondegenerate matrix family $\langle A_1, A_2, A_3\rangle$ with*

$$A_1=\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}$$

has only real eigenvalues at most double. Suppose also that it is not equivalent to any subfamily of (3.1), (3.1′), (3.2) *or* (3.2′). *Then* $\langle A_1, A_2, A_3\rangle$ *is equivalent to a family of the form*

$$\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \rho & 1 & -\alpha \\ \alpha & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \sigma & \beta & \mu \\ \nu & 1 & 0 \\ \beta & 0 & -1 \end{bmatrix} \right\rangle$$

where α, β, μ, ν, ρ, σ *are real constants satisfying*

$$0 \leqq \beta \leqq 1 .$$

Remark. Without loss of generality, we may put

$$\mu=\gamma+\delta, \quad \nu=\gamma-\delta .$$

And the calculation below becomes easier if we further put

$$\gamma=\gamma'-\alpha\beta .$$

Proof. By adding approriate scalar multiple of identity to A_2 and A_3, we may assume

(5.6) $$[N]_{22}+[N]_{33}=0$$

for any $N\in\langle A_1, A_2, A_3\rangle$. Using a change of basis, we may further assume

(5.7) $$[A_2]_{22}=[A_2]_{33}=0$$

and

$$[A_3]_{22}=-[A_3]_{33}\neq 0 \quad \text{or} \quad =0 .$$

On the other hand, from the assumption that $\langle A_1, A_2, A_3\rangle$ is not equivalent to any subfamily of (3.1), (3.1′), (3.2) or (3.2′), and from Lemma 5.2, there is a member of $\langle A_1, A_2, A_3\rangle$ whose (1, 2)- and (3, 1)-entries do not vanish. We denote one of such matrices by

(5.8) $$M=\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \qquad (m_{12},\ m_{31}\neq 0,\ m_{33}=-m_{22}).$$

Fixing η arbitrarily, the matrix family

$$\langle A_1, \eta A_2+M\rangle \quad (\subset\langle A_1, A_2, A_3\rangle)$$

has only real eigenvalues. Applying Lemma 5.1 to this family, we know

$$\lambda^2-m_{22}{}^2+(\eta[A_2]_{12}+m_{12})(\eta[A_2]_{31}+m_{31})=0$$

has only real roots. So

$$-m_{22}{}^2+(\eta[A_2]_{12}+m_{12})(\eta[A_2]_{31}+m_{31})\leqq 0$$

where $m_{12}\neq 0$, $m_{31}\neq 0$ (see (5.8)). Because the last inequality holds for an arbitarily fixed $\eta\in\boldsymbol{R}$, we obtain

$$[A_2]_{12}[A_2]_{31}<0$$

or

$$[A_2]_{12}=[A_2]_{31}=0.$$

However, the second case cannot occur because of (5.7) and Lemma 5.7. So multiplying A_2 by a scalar and then using a similarity transformation with T of the form

$$T=\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (k\neq 0),$$

we have

(5.9) $$[A_2]_{12}=1, \quad [A_2]_{31}=-1.$$

Then using Lemma 5.5, we may further assume

(5.10) $$[A_2]_{13}=-\alpha, \quad [A_2]_{21}=\alpha$$

for some real α. From Lemma 5.1,

(5.11) $$[A_2]_{32}=0.$$

Replacing A_2 by A_2+cA_1 with appropriate c, we may also have

(5.12) $$[A_2]_{23}=0.$$

From (5.7), (5.9), (5.10), (5.11) and (5.12), we may specify A_2 as

(5.13) $$A_2=\begin{bmatrix} \rho & 1 & -\alpha \\ \alpha & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

where α, ρ are real constants.

Let us now consider A_3. Replacing A_3 by $A_3+c_1A_1+c_2A_2$ with appropriate c_1 and c_2, we may assume

(5.14) $$[A_3]_{12}=[A_3]_{31}, \quad [A_3]_{23}=0.$$

From Lemma 5.1, we also have

(5.15) $$[A_3]_{32}=0.$$

From (5.14) and (5.15) and from (5.6) which holds also for A_3, we may specify A_3 as

$$A_3=\begin{bmatrix} \sigma & \beta & \mu \\ \nu & \theta & 0 \\ \beta & 0 & -\theta \end{bmatrix}.$$

Here $\theta\neq 0$ holds because $\theta=0$ would imply that $-\sigma A_1+A_3$ must have a triple eigenvalue by virtue of Lemma 5.7. So multiplying A_3 by a scalar, θ becomes 1:

$$A_3=\begin{bmatrix} \sigma & \beta & \mu \\ \nu & 1 & 0 \\ \beta & 0 & -1 \end{bmatrix}.$$

We may assume

$$\beta \geqq 0$$

in the last matrix because otherwise we need only consider

$$\langle T^{-1}A_1T,\ T^{-1}A_2T,\ T^{-1}A_3T\rangle$$

with

$$T=\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally, applying Lemma 5.1 to this $\langle A_1, A_3\rangle$, we see

$$\lambda^2+\beta^2-1=0$$

has real roots where $\beta \geqq 0$. This implies

$$0 \leqq \beta \leqq 1 .$$

We have thus completed the proof. □

In order to investigate the matrix family indicated in the preceding Lemma 5.8, the next lemma is also a convenient tool.

Lemma 5.9. *Let*

$$A_1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ A_2=\begin{bmatrix} \rho & 1 & -\alpha \\ \alpha & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},\ A_3=\begin{bmatrix} \sigma & \beta & \gamma-\delta \\ \gamma+\delta & 1 & 0 \\ \beta & 0 & -1 \end{bmatrix}$$

where α, β, γ, δ, ρ, σ are real constants satisfying

$$0 \leqq \beta \leqq 1 .$$

Then $\langle A_1, A_2, A_3\rangle$ has only real eigenvalues at most double if and only if any $\langle A_1, \eta A_2+A_3\rangle$ with arbitrarily fixed $\eta \in \boldsymbol{R}$ has the same property.

Proof. The necessity is clear. For the proof of the sufficiency, it is enough to prove

$$\xi A_1+A_2$$

with any $\xi \in \boldsymbol{R}$ has only real eigenvalues at most double. From the assumption.

$$\xi A_1+\frac{1}{\eta}(\eta A_2+A_3)$$

has only real eigenvalues for any $\eta \in \boldsymbol{R}$. Taking $\eta \to \infty$, we see that

$$\xi A_1+A_2$$

has only real eigenvalues. The multiplicity of its eigenvalues turns out to be at most double by virtue of Lemma 5.3. Q.E.D. □

Let us split the case into $0\leqq\beta<1$ and $\beta=1$. Here β is the one in A_3 of our matrix family $\langle A_1, A_2, A_3\rangle$ mentioned in Lemma 5.9.

Lemma 5.10. *Let*

$$A_1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A_2=\begin{bmatrix} \rho & 1 & -\alpha \\ \alpha & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, A_3=\begin{bmatrix} \sigma & \beta & \gamma-\delta \\ \gamma+\delta & 1 & 0 \\ \beta & 0 & -1 \end{bmatrix}$$

where

$$0\leqq\beta<1.$$

Then $\langle A_1, A_2, A_3\rangle$ has only real eigenvalues at most double if and only if

$$P(\eta)\equiv(\eta^2-\beta^2+1)\{(2\alpha-1)\eta^2+2\delta\eta+\beta^2+2\beta\gamma\}^2$$
$$-\{(\eta^2-\beta^2)(\rho\eta+\sigma)+2(\alpha\beta+\gamma)\eta+2\beta\delta\}^2\geqq0$$

and

$$Q(\eta)\equiv(2\alpha-1)\eta^2+2\delta\eta+\beta^2+2\beta\gamma\geqq0$$

hold for any $\eta\in\boldsymbol{R}$.

Proof. From Lemma 5.9, it suffices to consider the condition where $\langle A_1, \eta A_2+A_3\rangle$ has only real eigenvalues at most double. Now Lemma 5.3 is applicable because in the present situation,

$$q_\eta(\lambda)\equiv\lambda^2-\eta^2+\beta^2-1,$$
$$c_\eta(\lambda)\equiv\det(-\lambda I+\eta A_2+A_3)$$

and the quadratic equation $q_\eta(\lambda)=0$ has real distinct roots $\pm(\eta^2-\beta^2+1)^{1/2}$ because of $0\leqq\beta<1$. So we obtain the condition:

$$c_\eta((\eta^2-\beta^2+1)^{1/2})\geqq0$$
$$c_\eta(-(\eta^2-\beta^2+1)^{1/2})\leqq0$$

hold for any aribtrarily fixed $\eta\in\boldsymbol{R}$. These inequalities are, in turn, equivalent to

$$P(\eta)\equiv-c_\eta((\eta^2-\beta^2+1)^{1/2})c_\eta(-(\eta^2-\beta^2+1)^{1/2})\geqq0,$$
$$Q(\eta)\equiv\frac{c_\eta((\eta^2-\beta^2+1)^{1/2})-c_\eta(-(\eta^2-\beta^2+1)^{1/2})}{(\eta^2-\beta^2+1)^{1/2}}\geqq0.$$

Here $P(\eta)$ is the polynomial of sixth order and $Q(\eta)$ is the quadratic polynomial. They are what we have been looking for. □

We can consider the case $\beta=1$ similarly.

Lemma 5.11. *Let*

$$A_1=\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix},\ A_2=\begin{bmatrix}\rho&1&-\alpha\\\alpha&0&0\\-1&0&0\end{bmatrix},\ A_3=\begin{bmatrix}\sigma&1&\gamma-\delta\\\gamma+\delta&1&0\\1&0&-1\end{bmatrix}.$$

Let also

$$c_\eta(\lambda)\equiv\det(-\lambda I+\eta A_2+A_3).$$

Then $\langle A_1, A_2, A_3\rangle$ *has only real eigenvalues at most double if and only if*

$$c_\eta(\eta)\geqq 0,\quad c_\eta(-\eta)\leqq 0 \qquad \textit{for any } \eta>0,$$

$$c_\eta(\eta)\leqq 0,\quad c_\eta(-\eta)\geqq 0 \qquad \textit{for any } \eta<0,$$

and

$$c_0'(0)>0.$$

Proof. From Lemma 5.9, it suffices to consider the condition where $\langle A_1, \eta A_2+A_3\rangle$ has only real eigenvalues at most double.

If $\eta\neq 0$, Lemma 5.3 is applicable because in the present situation,

$$q_\eta(\lambda)\equiv\lambda^2-\eta^2,$$

$$c_\eta(\lambda)\equiv\det(-\lambda I+\eta A_2+A_3)$$

and the quadratic equation $q_\eta(\lambda)=0$ has real distinct roots $\pm|\eta|$. So we obtain the condition:

$$c_\eta(|\eta|)\geqq 0$$

$$c_\eta(-|\eta|)\leqq 0$$

for any $\eta\neq 0$.

If $\eta=0$, Lemma 5.4 is applicable because the quadratic equation $q_0(\lambda)=0$ has zero as a repeated root. So we have

$$c_0'(0)>0.$$

The proof is complete. □

Let us first settle down the case $\beta=1$ by using Lemma 5.11 just obtained.

Proposition 5.12. *Let the assumptions be the same as in Lemma* 5.11. *Then* $\langle A_1, A_2, A_3\rangle$ *has only real eigenvalues at most double if and only if it is equivalent to*

$$\left\langle\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix},\begin{bmatrix}2\alpha-1&1&-\alpha\\\alpha&0&0\\-1&0&0\end{bmatrix},\begin{bmatrix}2\delta&1&\gamma-\delta\\\gamma+\delta&1&0\\1&0&-1\end{bmatrix}\right\rangle$$

where

$$2\alpha-1\geqq 0,$$

$$2\gamma+1>0,$$

$$\delta^2 \leqq (2\alpha-1)(2\gamma+1).$$

Proof. In this case, we obtain

$$c_\eta(\eta) \equiv (\rho+2\alpha-1)\eta^3+(\sigma+2\delta)\eta^2+(-\rho+2\alpha+4\gamma+1)\eta-(\sigma-2\delta) \tag{5.16}$$

$$c_\eta(-\eta) \equiv (\eta^2-1)\{(\rho-2\alpha+1)\eta+\sigma-2\delta\} \tag{5.17}$$

$$c_0'(0)=2\gamma+1 \tag{5.18}$$

Because $c_0'(0)>0$ (see Lemma 5.11), we have

$$2\gamma+1>0. \tag{5.19}$$

From (5.17) and the property of $c_\eta(-\eta)$ mentioned in Lemma 5.11, we also have

$$\rho-2\alpha+1=0, \quad \sigma-2\delta=0. \tag{5.20}$$

Substituting $\rho=2\alpha-1$, $\sigma=2\delta$ in (5.16),

$$c_\eta(\eta) \equiv 2\eta\{(2\alpha-1)\eta^2+2\delta\eta+2\gamma+1\}.$$

From the property of $c_\eta(\eta)$ (see Lemma 5.11), we have

$$2\alpha-1 \geqq 0, \quad \delta^2 \leqq (2\alpha-1)(2\gamma+1). \tag{5.21}$$

Combining (5.19), (5.20) and (5.21), we obtain the conclusion. □

Let us now settle down the case $\beta=0$.

Proposition 5.13. *Let*

$$A_1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2=\begin{bmatrix} \rho & 1 & -\alpha \\ \alpha & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad A_3=\begin{bmatrix} \sigma & 0 & \gamma-\delta \\ \gamma+\delta & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then $\langle A_1, A_2, A_3\rangle$ has only real eigenvalues at most double if and only if it is equivalent to

$$\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \rho & 1 & -\alpha \\ \alpha & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\rangle$$

where

$$2\alpha-1 \geqq 0,$$

$$\rho^2+\sigma^2 \leqq (2\alpha-1)^2.$$

Proof We can apply Lemma 5.10 with $\beta=0$ and we have

$$P(\eta) \equiv \eta^2(\eta^2+1)\{(2\alpha-1)\eta+2\delta\}^2-\eta^2\{\eta(\rho\eta+\sigma)+2\gamma\}^2 \geqq 0, \tag{5.22}$$

$$Q(\eta) \equiv (2\alpha-1)\eta^2+2\delta\eta \geqq 0 \tag{5.23}$$

for any $\eta \in \boldsymbol{R}$. Because (5.23) holds for any η, we must have

(5.24) $$2\alpha-1\geqq 0,$$

(5.25) $$\delta=0.$$

Substituting $\delta=0$ in (5.22), we see

$$P(\eta)\equiv(2\alpha-1)^2\eta^4(\eta^2+1)-\eta^2\{\eta(\rho\eta+\sigma)+2\gamma\}^2\geqq 0$$

holds for any η. But this is untrue for small η if $\gamma\neq 0$. So we must have

(5.26) $$\gamma=0$$

and

$$P(\eta)\equiv\eta^4\{(2\alpha-1)^2(\eta^2+1)-(\rho\eta+\sigma)^2\}\geqq 0$$

for any η. Considering the discriminant of the quadratic polynomial in the brackets, we obtain

(5.27) $$\rho^2+\sigma^2\leqq(2\alpha-1)^2.$$

Combining (5.24), (5.25), (5.26) and (5.27), we obtain the conclusion. □

Now let us work on the case $0<\beta<1$.

Lemma 5.14. *Suppose that the matrix family* $\langle A_1, A_2, A_3\rangle$ *spanned by*

$$A_1=\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix},\ A_2=\begin{bmatrix}\rho&1&-\alpha\\\alpha&0&0\\-1&0&0\end{bmatrix},\ A_3=\begin{bmatrix}\sigma&\beta&\gamma-\delta\\\gamma+\delta&1&0\\\beta&0&-1\end{bmatrix}$$

where

$$0<\beta<1$$

has only real eigenvalues at most double. Then one of the following 1), 2), 3), 4) *holds.*

1) $\rho=\left(1-\frac{1}{\beta^2}\right)\gamma+\alpha\beta+\frac{\alpha}{\beta}-\frac{1}{\beta},\quad \sigma=\left(\beta+\frac{1}{\beta}\right)\delta.$

2) $\delta=\alpha\beta+\gamma,\quad \sigma=2\alpha\beta^2+2\alpha+2\beta\gamma-\beta\rho-1.$

3) $\delta=-\alpha\beta-\gamma,\quad \sigma=-2\alpha\beta^2-2\alpha-2\beta\gamma+\beta\rho+1.$

4) $\gamma=-\alpha\beta,\quad \delta=0.$

Proof. All we have to do is to sharpen Lemma 5.10. By a straightforward calculation, we know $P(\eta)$ is divisible by $\eta^2-\beta^2$. So there exists a polynomial $\tilde{P}(\eta)$ of fourth order such that

$$P(\eta)\equiv(\eta^2-\beta^2)\tilde{P}(\eta).$$

Because $P(\eta)\geqq 0$ for all $\eta\in\boldsymbol{R}$, $\tilde{P}(\eta)$ must also be divisible by $(\eta^2-\beta^2)$, that is,

$$\tilde{P}(\beta)=\tilde{P}(-\beta)=0$$

must hold. These equalities turns out to be

$$\tilde{P}(\beta)=4(\gamma+\delta+\alpha\beta)\{-\beta^2\rho-\beta\sigma+(\beta^2-1)\gamma+(\beta^2+1)\delta+\alpha\beta^3+\alpha\beta-\beta\}=0,$$

$$\tilde{P}(-\beta)=4(\gamma-\delta+\alpha\beta)\{-\beta^2\rho+\beta\sigma+(\beta^2-1)\gamma-(\beta^2+1)\delta+\alpha\beta^3+\alpha\beta-\beta\}=0.$$

Hence we obtain 1) of the present lemma from

$$-\beta^2\rho-\beta\sigma+(\beta^2-1)\gamma+(\beta^2+1)\delta+\alpha\beta^3+\alpha\beta-\beta=0$$

and

$$-\beta^2\rho+\beta\sigma+(\beta^2-1)\gamma-(\beta^2+1)\delta+\alpha\beta^3+\alpha\beta-\beta=0,$$

2) from

$$-\beta^2\rho-\beta\sigma+(\beta^2-1)\gamma+(\beta^2+1)\delta+\alpha\beta^3+\alpha\beta-\beta=0$$

and

$$\gamma-\delta+\alpha\beta=0,$$

3) from

$$\gamma+\delta+\alpha\beta=0$$

and

$$-\beta^2\rho+\beta\sigma+(\beta^2-1)\gamma-(\beta^2+1)\delta+\alpha\beta^3+\alpha\beta-\beta=0,$$

finally 4) from

$$\gamma+\delta+\alpha\beta=0 \quad \text{and} \quad \gamma-\delta+\alpha\beta=0.$$

Thus the proof is complete. □

Let us consider 1), 2), 3), 4) of Lemma 5.14 separately.

Lemma 5.15. *Let the matrices A_1, A_2, A_3 be as*

$$A_1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ A_2=\begin{bmatrix} \rho & 1 & -\alpha \\ \alpha & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},\ A_3=\begin{bmatrix} \sigma & \beta & \gamma-\delta \\ \gamma+\delta & 1 & 0 \\ \beta & 0 & -1 \end{bmatrix}$$

where

$$0<\beta<1$$

and

$$\rho=\left(1-\frac{1}{\beta^2}\right)\gamma+\alpha\beta+\frac{\alpha}{\beta}-\frac{1}{\beta}, \quad \sigma=\left(\beta+\frac{1}{\beta}\right)\delta.$$

Then $\langle A_1, A_2, A_3\rangle$ has only real eigenvalues at most double if and only if

$$2\alpha-1\geqq 0,$$

$$\frac{-\alpha\beta^2+\alpha\beta-\beta}{1+\beta}\leqq\gamma\leqq\frac{\alpha\beta^2+\alpha\beta-\beta}{1-\beta},$$

$$\delta^2\leqq\frac{1-\beta^2}{\beta^2}\left\{\gamma-\frac{-\alpha\beta^2+\alpha\beta-\beta}{1+\beta}\right\}\left\{-\gamma+\frac{\alpha\beta^2+\alpha\beta-\beta}{1-\beta}\right\}.$$

Proof. Substituting the given ρ, σ in $P(\eta)$ of Lemma 5.10, we have

$$P(\eta)\equiv\frac{1-\beta^2}{\beta^4}(\eta^2-\beta^2)^2\hat{P}(\eta)\geqq 0$$

or equivalently

$$\hat{P}(\eta)\geqq 0 \tag{5.28}$$

where

$$\hat{P}(\eta)\equiv(1-\beta^2)\left\{\gamma-\frac{-\alpha\beta^2+\alpha\beta-\beta}{1+\beta}\right\}\left\{-\gamma+\frac{\alpha\beta^2+\alpha\beta-\beta}{1-\beta}\right\}\eta^2$$
$$+2\beta\delta\{(\beta^2+1)\gamma+\alpha\beta^3-\alpha\beta+\beta\}\eta$$
$$+\beta^2\{-(1-\beta^2)\delta^2+(\beta+2\gamma)^2\}.$$

We also have (see Lemma 5.10)

$$Q(\eta)\equiv(2\alpha-1)\eta^2+2\delta\eta+\beta(\beta+2\gamma)\geqq 0. \tag{5.29}$$

The discriminants of $\hat{P}(\eta)$ and $Q(\eta)$ are

$$D_{\hat{P}}\equiv 4\beta^4(\beta+2\gamma)^2\left[\delta^2-\frac{1-\beta^2}{\beta^2}\left\{\gamma-\frac{-\alpha\beta^2+\alpha\beta-\beta}{1+\beta}\right\}\left\{-\gamma+\frac{\alpha\beta^2+\alpha\beta-\beta}{1-\beta}\right\}\right],$$
$$D_Q\equiv 4\{\delta^2-\beta(2\alpha-1)(2\gamma+\beta)\}.$$

Note that the following inequality holds between the expressions inside the brackets of the discriminants.

$$\frac{1-\beta^2}{\beta^2}\left\{\gamma-\frac{-\alpha\beta^2+\alpha\beta-\beta}{1+\beta}\right\}\left\{-\gamma+\frac{\alpha\beta^2+\alpha\beta-\beta}{1-\beta}\right\}$$
$$=\beta(2\alpha-1)(2\gamma+\beta)-\frac{1-\beta^2}{\beta^2}(\gamma-\alpha\beta+\beta)^2$$
$$\leqq\beta(2\alpha-1)(2\gamma+\beta).$$

Recall that $\hat{P}(\eta)$ is always nonnegative if and only if it is either a nonnegative constant or has a nonpositive discriminant and a positive coefficient for the quadratic term. The same holds also for $Q(\eta)$. Using these, we obtain the required result. □

The cases 2), 3), 4) of Lemma 5.14 can be reduced to special subcases of 1) as follows.

Lemma 5.16. *Suppose that the matrix family* $\langle A_1, A_2, A_3\rangle$ *is spanned by*

$$A_1=\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix},\ A_2=\begin{bmatrix}\rho&1&-\alpha\\\alpha&0&0\\-1&0&0\end{bmatrix},\ A_3=\begin{bmatrix}\sigma&\beta&\gamma-\delta\\\gamma+\delta&1&0\\\beta&0&-1\end{bmatrix}$$

where

$$0<\beta<1,$$
$$\delta=\alpha\beta+\gamma,$$
$$\sigma=2\alpha\beta^2+2\alpha+2\beta\gamma-\beta\rho-1.$$

Suppose also that $\langle A_1, A_2, A_3\rangle$ has only real eigenvalues at most double. Then

$$\rho=\left(1-\frac{1}{\beta^2}\right)\gamma+\alpha\beta+\frac{\alpha}{\beta}-\frac{1}{\beta},\quad \sigma=\left(\beta+\frac{1}{\beta}\right)\delta$$

are satisfied.

Proof. Substituting

$$\delta=\alpha\beta+\gamma,\quad \sigma=2\alpha\beta^2+2\alpha+2\beta\gamma-\beta\rho-1$$

in $P(\eta)$ in Lemma 5.10, we obtain

$$P(\eta)\equiv(\eta^2-\beta^2)^2\hat{P}(\eta)\geqq 0$$

or equivalently

$$\hat{P}(\eta)\geqq 0$$

where

$$\begin{aligned}\hat{P}(\eta)\equiv&\{-\rho^2+(2\alpha-1)^2\}\eta^2+2(\rho-2\alpha\beta-2\gamma)(\beta\rho-2\alpha+1)\eta\\&-\beta^2(\rho-2\alpha\beta-2\gamma)^2-2(\beta+2\gamma)(\rho-2\alpha\beta-2\gamma)\\&-(\beta+2\gamma)^2.\end{aligned}$$

Similarly, we have also

$$Q(\eta)\equiv(2\alpha-1)\eta^2+2(\gamma+\alpha\beta)\eta+\beta(2\gamma+\beta)\geqq 0.$$

The discriminants of $\hat{P}(\eta)$ and $Q(\eta)$ are

$$\begin{aligned}&D_{\hat{P}}=4(\rho-2\alpha\beta+\beta)^2\{-\rho^2+(2\alpha-1)^2+(\rho-2\gamma-2\alpha\beta)^2\},\\&D_Q=4(\gamma-\alpha\beta+\beta)^2.\end{aligned}$$

Because $D_{\hat{P}}\leqq 0$, we get either

$$\rho=(2\alpha-1)\beta$$

or

$$\rho=\pm(2\alpha-1)=2\alpha\beta+2\gamma.$$

Here we have used the fact that the coefficient of η^2 of $P(\eta)$, i.e.,

$$-\rho^2+(2\alpha-1)^2$$

is nonnegative. Furthermore, because $D_Q\leqq 0$, we get also

$$\gamma=(\alpha-1)\beta.$$

From these and the nonnegativity of $\hat{P}(\eta)$, we obtain

$$\alpha\geqq\frac{1}{2},\quad 0<\beta<1,\quad \gamma=(\alpha-1)\beta,\quad \rho=(2\alpha-1)\beta$$

and hence

$$\delta=(2\alpha-1)\beta,\quad \sigma=(2\alpha-1)(\beta^2+1).$$

Now, it is easy to verify

$$\rho=\left(1-\frac{1}{\beta^2}\right)\gamma+\alpha\beta+\frac{\alpha}{\beta}-\frac{1}{\beta}, \quad \sigma=\left(\beta+\frac{1}{\beta}\right)\delta$$

are satisfied. □

Lemma 5.17. *Suppose that the matrix family $\langle A_1, A_2, A_3\rangle$ is spanned by*

$$A_1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A_2=\begin{bmatrix} \rho & 1 & -\alpha \\ \alpha & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, A_3=\begin{bmatrix} \sigma & \beta & \gamma-\delta \\ \gamma+\delta & 1 & 0 \\ \beta & 0 & -1 \end{bmatrix}$$

where

$$0<\beta<1,$$
$$\delta=-(\alpha\beta+\gamma),$$
$$\sigma=-(2\alpha\beta^2+2\alpha+2\beta\gamma-\beta\rho-1).$$

Suppose also that $\langle A_1, A_2, A_3\rangle$ has only real eigenvalues at most double. Then

$$\rho=\left(1-\frac{1}{\beta^2}\right)\gamma+\alpha\beta+\frac{\alpha}{\beta}-\frac{1}{\beta}, \quad \sigma=\left(\beta+\frac{1}{\beta}\right)\delta$$

are satisfied.

Proof. Substituting

$$\delta=-(\alpha\beta+\gamma), \quad \sigma=-(2\alpha\beta^2+2\alpha+2\beta\gamma-\beta\rho-1)$$

in $P(\eta)$ in Lemma 5.10, we obtain

$$P(\eta)\equiv(\eta^2-\beta^2)^2\hat{P}(\eta)\geqq 0$$

or equivalently

$$\hat{P}(\eta)\geqq 0$$

where

$$\begin{aligned}\hat{P}(\eta)\equiv&\{-\rho^2+(2\alpha-1)^2\}\eta^2-2(\rho-2\alpha\beta-2\gamma)(\beta\rho-2\alpha+1)\eta\\ &-\beta^2\rho^2-\{4(1-\beta^2)\gamma-4\alpha\beta^3+2\beta\}\rho\\ &+4(1-\beta^2)\gamma^2+8\alpha\beta(1-\beta^2)\gamma-\beta^2(4\alpha^2\beta^2-4\alpha+1).\end{aligned}$$

Similarly, we have also

$$Q(\eta)\equiv(2\alpha-1)\eta^2-2(\gamma+\alpha\beta)\eta+\beta(2\gamma+\beta)\geqq 0.$$

Furthermore $\hat{P}(-\eta)$ and $Q(-\eta)$ turn out to be $\hat{P}(\eta)$ and $Q(\eta)$ in the proof of Lemma 5.16. So the same calculation is valid and we have

$$\alpha\geqq\frac{1}{2}, \quad 0<\beta<1, \quad \gamma=(\alpha-1)\beta, \quad \rho=(2\alpha-1)\beta$$

and hence

$$\delta=-(2\alpha-1)\beta, \quad \sigma=-(2\alpha-1)(\beta^2+1).$$

Now, it is easy to verify

$$\rho=\left(1-\frac{1}{\beta^2}\right)\gamma+\alpha\beta+\frac{\alpha}{\beta}-\frac{1}{\beta}, \quad \sigma=\left(\beta+\frac{1}{\beta}\right)\delta$$

are satisfied. □

Lemma 5.18. *Suppose that the matrix family* $\langle A_1, A_2, A_3\rangle$ *is spanned by*

$$A_1=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}, A_2=\begin{bmatrix}\rho & 1 & -\alpha\\ \alpha & 0 & 0\\ -1 & 0 & 0\end{bmatrix}, A_3=\begin{bmatrix}\sigma & \beta & \gamma-\delta\\ \gamma+\delta & 1 & 0\\ \beta & 0 & -1\end{bmatrix}$$

where

$$0<\beta<1,$$

$$\gamma=-\alpha\beta, \quad \delta=0.$$

Suppose also that $\langle A_1, A_2, A_3\rangle$ *has only real eigenvalues at most double. Then*

$$\rho=\left(1-\frac{1}{\beta^2}\right)\gamma+\alpha\beta+\frac{\alpha}{\beta}-\frac{1}{\beta}, \quad \sigma=\left(\beta+\frac{1}{\beta}\right)\delta$$

are satisfied.

Proof. Substituting

$$\gamma=-\alpha\beta, \quad \delta=0$$

in $P(\eta)$ in Lemma 5.10, we obtain

$$P(\eta)\equiv(\eta^2-\beta^2)^2\hat{P}(\eta)\geqq 0$$

or equivalently

$$\hat{P}(\eta)\equiv\{-\rho^2+(2\alpha-1)^2\}\eta^2-2\rho\sigma\eta-\sigma^2+(2\alpha-1)^2(1-\beta^2)\geqq 0.$$

We also have

$$Q(\eta)\equiv(2\alpha-1)(\eta^2-\beta^2)\geqq 0.$$

Because $Q(\eta)\geqq 0$ $(0<\beta<1)$, we get

$$\alpha=\frac{1}{2}.$$

So the inequality $\hat{P}(\eta)\geqq 0$ turns out to be

$$\hat{P}(\eta)\equiv-(\rho\eta+\sigma)^2\geqq 0$$

and we must have

$$\rho=\sigma=0.$$

Summing up, we have obtained (recall $\alpha=\frac{1}{2}$, $\gamma=-\alpha\beta$ and $\delta=0$)

$$\alpha=\frac{1}{2}, \quad 0<\beta<1, \quad \gamma=-\frac{1}{2}\beta, \quad \delta=0, \quad \rho=0, \quad \sigma=0.$$

Therefore

$$\rho=\Big(1-\frac{1}{\beta^2}\Big)\gamma+\alpha\beta+\frac{\alpha}{\beta}-\frac{1}{\beta}, \quad \sigma=\Big(\beta+\frac{1}{\beta}\Big)\delta$$

are satisfied. □

Combining Lemma 5.14 to 5.18, we obtain the following Proposition.

Proposition 5.19. *Let the matrices* A_1, A_2, A_3 *be as*

$$A_1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2=\begin{bmatrix} \rho & 1 & -\alpha \\ \alpha & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad A_3=\begin{bmatrix} \sigma & \beta & \gamma-\delta \\ \gamma+\delta & 1 & 0 \\ \beta & 0 & -1 \end{bmatrix}$$

where

$$0<\beta<1.$$

Then $\langle A_1, A_2, A_3\rangle$ *has only real eigenvalues at most double if and only if*

$$2\alpha-1\geqq 0,$$

$$\frac{-\alpha\beta^2+\alpha\beta-\beta}{1+\beta}\leqq\gamma\leqq\frac{\alpha\beta^2+\alpha\beta-\beta}{1-\beta},$$

$$\delta^2\leqq\frac{1-\beta^2}{\beta^2}\Big\{\gamma-\frac{-\alpha\beta^2+\alpha\beta-\beta}{1+\beta}\Big\}\Big\{-\gamma+\frac{\alpha\beta^2+\alpha\beta-\beta}{1-\beta}\Big\},$$

$$\rho=\Big(1-\frac{1}{\beta^2}\Big)\gamma+\alpha\beta+\frac{\alpha}{\beta}-\frac{1}{\beta},$$

$$\sigma=\Big(\beta+\frac{1}{\beta}\Big)\delta.$$

Proof. We begin with the necessity. Applying Lemma 5.14, we know one of the four mentioned cases 1), 2), 3), 4) occurs. However, by virtue of Lemma 5.16, 5.17, 5.18, the last three of these cases, 2), 3), 4) turn out to be special subcases of the first case 1), that is,

$$\rho=\Big(1-\frac{1}{\beta^2}\Big)\gamma+\alpha\beta+\frac{\alpha}{\beta}-\frac{1}{\beta}, \quad \sigma=\Big(\beta+\frac{1}{\beta}\Big)\delta.$$

Therefore Lemma 5.15 is always applicable and the proof of the necessity is complete. The sufficiency is clear again from Lemma 5.15. □

Let us now investigate families spanned by four or more matrices.

Proposition 5.20. *Suppose that a nondegenerate matrix family* $\langle A_1, A_2, \cdots, A_n\rangle$ $(n\geqq 4)$ *with*

$$A_1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

has only real eigenvalues. Then $\langle A_1, A_2, \cdots, A_n\rangle$ *is equivalent to a subfamily of* (3.1), (3.1′), (3.2) *or* (3.2′).

Proof. By addition of scalar multiple of identity and by change of the basis, we may assume

$$[A_2]_{22}=[A_2]_{33}=0, \tag{5.30}$$

$$[A_3]_{22}=[A_3]_{33}=0 \tag{5.31}$$

while

$$[A_4]_{22}=-[A_4]_{33}\neq 0 \quad \text{or} \quad =0.$$

Let us show by contradiction that

$$\begin{aligned}[\xi A_2+\eta A_3]_{12}&=\xi[A_2]_{12}+\eta[A_3]_{12}=0,\\ [\xi A_2+\eta A_3]_{31}&=\xi[A_2]_{31}+\eta[A_3]_{31}=0\end{aligned} \tag{5.32}$$

for some $(\xi, \eta)\in \boldsymbol{R}^2\setminus\{(0, 0)\}$. Assume the contrary. Then there would exist (ξ, η) such that

$$\begin{aligned}[\xi A_2+\eta A_3]_{12}&=\xi[A_2]_{12}+\eta[A_3]_{12}=1,\\ [\xi A_2+\eta A_3]_{31}&=\xi[A_2]_{31}+\eta[A_3]_{31}=1.\end{aligned}$$

These equalities together with (5.30) and (5.31) leads to a contradiction, applying Lemma 5.1 to $\langle A_1, \xi A_2+\eta A_3\rangle$.

Renaming $\xi A_2+\eta A_3$ satisfying (5.32) as A_2, we may assume also

$$[A_2]_{12}=[A_2]_{31}=0. \tag{5.33}$$

In addition, again from Lemma 5.1,

$$[A_2]_{32}=0. \tag{5.34}$$

Summing up, we may assume A_2 has the following form.

$$A_2=\begin{bmatrix} * & 0 & * \\ * & 0 & * \\ 0 & 0 & 0 \end{bmatrix}.$$

By addition of an appropriate scalar mulitple of A_1, A_2 may be further assumed to be as

$$A_2=\begin{bmatrix} 0 & 0 & * \\ * & 0 & * \\ 0 & 0 & 0 \end{bmatrix}.$$

This A_2 has zero as a triple eigenvalue and is similar to either

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the first case, the claim follows from Proposition 4.2. And in the second case, the claim follows from Proposition 4.7. Thus the proof is complete. □

6. Summary

Let us summarize all the facts proved in this paper. For convenience, we reproduce (3.1), (3.1′), (3.2) and (3.2′) as (6.1), $\cdots$, etc.

$$(6.1)\qquad \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\rangle,$$

(6.1′) transposition of (6.1),

$$(6.2)\qquad \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle$$

(6.2′) transposition of (6.2).

First of all, combining Propositions 4.2, 4.7, 5.20, we obtain the following theorem.

Theorem 6.1. *Suppose that a nondegenerate non-diagonalizable* 3×3 *matrix family* $\langle A_1, A_2, \cdots, A_n\rangle$ $(n\geqq 4)$ *has only real eigenvalues. Then it is equivalent to a subfamily of* (6.1), (6.1′), (6.2) *or* (6.2′).

Combining Propositions 4.2, 4.5, 4.6, we have the following theorem.

Theorem 6.2. *Given a nondegenerate non-diagonalizable* 3×3 *matrix family with only real eigenvalues. Suppose that one of its nonzero members has a triple eigenvalue. Then it is equivalent to either*

$$\left\langle \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 & -\alpha_1\alpha_2\alpha_3 \\ -1 & \alpha_1+\alpha_2+\alpha_3 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle$$

with

$$\alpha_1\leqq\alpha_2\leqq\alpha_3$$

or

$$\left\langle \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 & -\alpha_1\alpha_2\alpha_3 \\ -1 & \alpha_1+\alpha_2+\alpha_3 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \beta_1+\beta_2 & -\beta_1\beta_2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\rangle$$

with

$$\alpha_1 \leqq \beta_1 \leqq \alpha_2 \leqq \beta_2 \leqq \alpha_3$$

unless it is equivalent to a subfamily of (6.1), (6.1′), (6.2) *or* (6.2′).

Finally, combining Propositions 5.6, 5.12, 5.13, 5.19, we obtain the following theorem.

Theorem 6.3. *Given a nondegenerate non-diagonalizable* 3×3 *matrix family with only real eigenvalues. Suppose that each of its nonzero members has eigenvalues at most double. Then it is equivalent to one of the following* 1), 2), 3), 4), 5) *unless it is equivalent to a subfamily of* (6.1), (6.1′), (6.2) *or* (6.2′).

$$1)\quad \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \gamma & \alpha+1 & (\alpha+1)\beta \\ (\alpha-1)\beta & \alpha & 0 \\ \alpha-1 & 0 & -\alpha \end{bmatrix} \right\rangle$$

where

$$|\alpha| \neq 1, \quad (\alpha^2-1)(2\beta+1) \geqq 0, \quad |\gamma| \leqq |2\beta+1|$$

are satisfied.

$$2)\quad \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \beta \\ \beta & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \right\rangle$$

where

$$2\beta+1>0$$

is satisfied.

$$3)\quad \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2\alpha-1 & 1 & -\alpha \\ \alpha & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2\delta & 1 & \gamma-\delta \\ \gamma+\delta & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \right\rangle$$

where

$$2\alpha-1 \geqq 0,$$

$$2\gamma+1>0,$$

$$\delta^2 \leqq (2\alpha-1)(2\gamma+1)$$

are satisfied.

$$4)\quad \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \rho & 1 & -\alpha \\ \alpha & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\rangle$$

where

$$2\alpha-1 \geqq 0,$$

$$\rho^2+\sigma^2 \leqq (2\alpha-1)^2$$

are satisfied.

$$5)\ \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \rho & 1 & -\alpha \\ \alpha & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \sigma & \beta & \gamma-\delta \\ \gamma+\delta & 0 & 0 \\ \beta & 0 & -1 \end{bmatrix} \right\rangle$$

where

$$2\alpha-1\geqq 0,$$

$$0<\beta<1,$$

$$\frac{-\alpha\beta^2+\alpha\beta-\beta}{1+\beta}\leqq\gamma\leqq\frac{\alpha\beta^2+\alpha\beta-\beta}{1-\beta},$$

$$\delta^2\leqq\frac{1-\beta^2}{\beta^2}\left\{\gamma-\frac{-\alpha\beta^2+\alpha\beta-\beta}{1+\beta}\right\}\left\{-\gamma+\frac{\alpha\beta^2+\alpha\beta-\beta}{1-\beta}\right\},$$

$$\rho=\left(1-\frac{1}{\beta^2}\right)\gamma+\alpha\beta+\frac{\alpha}{\beta}-\frac{1}{\beta},$$

$$\sigma=\left(\beta+\frac{1}{\beta}\right)\delta$$

are satisfied.

Thus we have listed up all the canonical forms of non-diagonalizable 3×3 matrix families with only real eigenvalues.

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