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On the unitarizability of principal series representations of $\mathfrak{p}$-adic Chevalley groups

By

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Introduction

In this paper, we shall determine the unitarizability of unramified principal series representations of $\mathfrak{p}$-adic Chevalley groups of classical type. To be precise, let k be a non-archimedean local field and let $\mathbf{G}$ be a connected semi-simple algebraic group which is defined and splits over k. Let $\mathbf{T}$ be a maximal k-split torus and $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ defined over k which contains $\mathbf{T}$. Let $\mathbf{N}$ be the unipotent radical of $\mathbf{B}$. Let Σ be the root system and Δ be the set of simple roots determined by $(\mathbf{G}, \mathbf{B}, \mathbf{T})$. Let W be the Weyl group. Let G, T, B and N stand for the groups of k-rational points of $\mathbf{G}, \mathbf{T}, \mathbf{B}$ and $\mathbf{N}$ respectively. For a quasi-character χ of T, let $PS(\chi)$ denote the space of all locally constant functions φ on G which satisfy

$$\varphi(tng) = \delta_B(t)^{1/2}\chi(t)\varphi(g) \qquad \text{for every } t\in T,\ n\in N,\ g\in G.$$

Here δ_B denotes the modular function of B. Let $\pi(\chi)$ denote the admissible representation of G realized on $PS(\chi)$ by right translations. We call χ unramified if χ is trivial on the maximal compact subgroup of T. Let X denote the set of all unramified quasi-characters of T. If $\chi\in X$, $\pi(\chi)$ has the unique spherical constituent with respect to a standard maximal compact subgroup of G, which we denote by π_χ^1. Let $\mathbf{P}$ denote the set of all $\chi\in X$ for which π_χ^1 is unitarizable. It is well known that $\mathbf{P}$ is a W-stable compact subset of X. We call χ regular if $w\chi \neq \chi$ for every $w\in W$, $w \neq 1$. Set

$$X^r = \{\chi \mid \chi\in X \quad \text{and} \quad \chi \text{ is regular}\},$$

$$X^i = \{\chi \mid \chi\in X \quad \text{and} \quad \pi(\chi) \text{ is irreducible}\}.$$

In the notation above, our main results in this paper determine $\mathbf{P}\cap X^i$ when $\mathbf{G}$ is of classical type.

Let us explain main ideas of the proof. First we note the following fact (Lemma 6.2). Let $\mathbf{G}$ be of adjoint type and let $\psi\colon \tilde{\mathbf{G}} \to \mathbf{G}$ be a central isogeny defined over k. We define $\tilde{X}$ for $\tilde{\mathbf{G}}$ similarly as for $\mathbf{G}$. Then we have an surjective homomorphism $X \ni \chi \to \tilde{\chi} = \chi\circ\psi \in \tilde{X}$. We see that π_χ^1 is unitarizable if and only if $\pi_{\tilde{\chi}}^1$ is unitarizable. By this fact, we can freely move from the

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adjoint group to the simply connected group. For $w \in W$, set

$$X_w = \{\chi | \chi \in X, \ w\chi = \bar{\chi}^{-1}\}.$$

As is well known, if $\chi \in X^r$, there exists a non-zero intertwining operator T_w from $PS(\chi)$ to $PS(w\chi)$. First assume $\chi \in X^i \cap X^r$ and assume that $\pi(\chi)$ is unitarizable. Then we have $\chi \in X_w$ for some $w \in W$, $w^2 = 1$. We see that $\pi(\chi)$ is unitarizable if and only if

$$(1) \qquad (\varphi_1, \varphi_2) = c \int_{B \backslash G} (T_w(\varphi_1))(g) \overline{\varphi_2(g)} \, dg, \ \varphi_1, \varphi_2 \in PS(\chi)$$

is a positive definite hermitian form with a non-zero constant c. To determine directly the positive definiteness of (1) in general is very difficult however.

Let w_0 be the longest element of W and ω_0 be an element in the standard maximal compact subgroup of G which represents w_0. Since $B\omega_0 N$ is the big cell, we see easily that for every $\Phi \in C_c^\infty(N)$, there exists a unique $\varphi \in PS(\chi)$ such that $\Phi(n) = \varphi(\omega_0 n)$, $n \in N$. We put $\varphi = \iota_\chi(\Phi)$. Then

$$(2) \qquad T_{w,\chi}(\Phi) = T_w(\iota_\chi(\Phi))(\omega_0), \qquad \Phi \in C_c^\infty(N)$$

defines a distribution on N. Suggested by Godement [11], I. 19, we shall show that $cT_{w,\chi}$ is of positive type if $\pi(\chi)$ is unitarizable and the converse holds if χ is in the absolutely convergent domain for T_w (Lemma 5.4). Thus we are naturally led to the study of distributions of positive type.

In §2, we shall prove that the positivity of distributions is preserved under the direct image among nilpotent groups (Theorem 2.3). This result shall simplify our arguments considerably.

For a subset J of Δ, let Σ_J be the root system generated by J and W_J be the Weyl group attached to Σ_J. Let w_J denote the longest element of W_J. If $\pi(\chi)$ is unitarizable, then $\pi(w\chi)$ is unitarizable for every $w \in W$. By this fact, we may assume $\chi \in X_{w_J}$ for some $J \subseteq \Delta$. By Theorem 2.3, we can reduce the unitarizability problem to the case where $J = \Delta$ and $w_0 = w_\Delta$ acts on Δ by multiplication by -1 (cf. §8).

Let

$$X(\mathbf{T}) = \mathrm{Hom}(\mathbf{T}, \mathbf{G}_m), \qquad X_*(\mathbf{T}) = \mathrm{Hom}(\mathbf{G}_m, \mathbf{T}).$$

Set

$$V = X(\mathbf{T}) \otimes_{\mathbf{Z}} \mathbf{R}, \qquad V_* = X_*(\mathbf{T}) \otimes_{\mathbf{Z}} \mathbf{R}$$

and let $\langle \ , \ \rangle$ be the canonical pairing between V and V_*. Let $\check{\Sigma}$ be the inverse root system of Σ realized in V_*. Let $P(\Sigma)$ and $Q(\Sigma)$ be the lattices of weights and of root weights respectively. Assume that $\mathbf{G}$ is of adjoint type. Then we have $X(\mathbf{T}) = Q(\Sigma)$, $X_*(\mathbf{T}) = P(\check{\Sigma})$. Assume $w_0 = -1$ on Δ. Fix $z \in Q(\Sigma)$. For $v \in V$, we can define $\chi(v) \in X_{w_0}$ by

$$(3) \qquad \chi(v)(\beta(\varpi)) = (-1)^{\langle z, \beta \rangle} q^{\langle v, \beta \rangle}, \qquad \beta \in P(\check{\Sigma}),$$

where ϖ is a prime element and q is the module of k. Every quasi-character in X_{w_0} is of this form. Set

$$\check{\Sigma}_z = \{\alpha \in \check{\Sigma} | \langle v, \alpha \rangle \equiv 0 \mod 2\}.$$

Let $\mathfrak{H}_z$ be the family of hyperplanes in V defined by $\langle v, \alpha \rangle = \pm 1$ for $\alpha \in \check{\Sigma}_z$. The irreducibility criterion of Kato [13] says that $\pi(\chi(v))$ is irreducible if and only if $v \notin \mathfrak{H}_z$ (cf. Lemma 3.2). Let D be a connected component of $V - \mathfrak{H}_z$. A simple deformation argument shows that if $\pi(\chi(v_0))$ is unitarizable for a point $v_0 \in D$, then $\pi(\chi(v))$ is unitarizable for all $v \in D$. By Theorem 2.3, we see that if

$$\langle v_0, \alpha \rangle = 0 \qquad \text{for some } v_0 \in D,\ \alpha \in \check{\Sigma}_z, \tag{4}$$

then the unitarizability of $\pi(\chi(v_0))$ can be reduced to that of a lower rank group (Remark 8.3).

Suppose that we could obtain a sufficiently sharp estimate on $v \in V$ as a necessary condition for unitarizability. Then we can determine the unitarizability completely since, intuitively speaking, hyperplanes attached to $\alpha \in \check{\Sigma}_z$ are "crowded" in narrow domains around the origin of V so that (4) holds for such domains.

We shall realize this idea in §9 ~ §11 for groups of classical type. In fact, to prove (4) assuming a suitable estimate is rather easy. To obtain a sharp estimate for $v \in V$ such that $\pi(\chi(v))$ is unitarizable, we appeal to considerations on composition series of $\pi(\chi(v))$ for v on the boundary of D. A result of Tadič [26], Theorem 2.7 tells that if the points of D represent unitarizable representations, then all the composition factors of $\pi(\chi(v))$ for $v \in \bar{D}$, the closure of D, are unitarizable. By this fact, we can show that if a point on $\bar{D}$ satisfies certain conditions, then non-unitarizability follows (Lemmas 8.5 and 9.4). The existence of such points, for those D which are bounded and do not satisfy the estimate, shall be shown by delicate combinatorial considerations on the shape of D and by raising up the dimension of D.

Let $\{\varepsilon_i\}$ be the standard basis of V as in Bourbaki [7]. For the adjoint group of type B_ℓ, the final result is

Theorem B. *Let* $z \in Q(\Sigma)$ *and let* $v = \sum_{i=1}^{\ell} a_i \varepsilon_i \in V$. *Assume* $\pi(\chi(v))$ *is irreducible. Then* $\pi(\chi(v))$ *is unitarizable if and only if*

$$-1/2 < a_i < 1/2, \qquad 1 \le i \le \ell. \tag{5}$$

For the adjoint group of type C_ℓ, we obtain

Theorem C. *Let* $z = 0$ *and* $v = \sum_{i=1}^{\ell} a_i \varepsilon_i \in V$. *Assume* $\pi(\chi(v))$ *is irreducible. Then if* $\pi(\chi(v))$ *is unitarizable, we have*

$$-1 < a_i < 1, \qquad 1 \le i \le \ell. \tag{6}$$

Although the estimate (6) is not a sufficient condition for unitarizability, (6) guarantees that we can apply (4). Hence we can easily determine the unitarizability inductively. (For the case $z \neq 0$ and for type D_ℓ, see the text.)

We shall briefly explain the contents of each section. §1 reflects author's previous attempts to derive good estimate directly from the positivity of distributions. We have retained this section since it may be useful on future occasions. The reader who is interested only in the unitarizability should skip this section after confirming standard terminologies. In §2, we shall prove the functorial properties of distributions of positive type stated above. §3 and §4 are preparations on semi-simple groups and on intertwining operators. In §5, we shall show an explicit relation between the unitarizability of representations and the positivity of distributions. In particular, we shall prove that $T_{w_J,\chi}$ for $J \subsetneq \Delta$ can be written as a direct image. In §6, we shall study the relation of the unitarizability of π_χ^1 with the positive semi-definiteness of the hermitian form (1) in general context. After preparations on deformation in §7, we shall explain basic strategy for determining $\mathbf{P} \cap X^i$ in §8. In §9 ~ §11, we shall obtain the unitarizability conditions for groups of classical type. In §12, we shall reduce the unitarizability problem of π_χ^1 for simply connected groups of classical type to the case when χ is real valued. This result can be regarded as the first step toward the determination of $\mathbf{P}$.

The first version of this paper was completed in the spring of 1989. However it contained a serious error[1]. The results appeared in a short communication [23] are false except for Proposition 5, Theorem 6 in Case B and Theorem 7. I would like to express my sincere gratitude to Institute for advanced study for its hospitality and for providing abundant time to revise the previous draft. I should like to express my hearty thanks to Professor H. Hijikata for useful comments on semi-simple groups.

Notation and terminology

Let G be a locally compact Hausdorff topological group. By a Haar measure dx on G, we understand a left invariant Radon measure. We denote by δ_G the modular function of G. Symbolically we have $d(x^{-1}) = \delta_G(x)\,dx$. (This adapts modern convention used in [8], [10], [20]; it is the inverse of Weil's definition [22], p. 40.) If V is a compact subset of G, $\mathrm{vol}(V)$ denotes the volume of V measured by dx. We denote by $\hat{G}$ the set of the equivalence classes of all irreducible unitary representations of G.

By a t.d. group G, we understand a Hausdorff topological group which has contable open compact subsets as a basis of open subsets (cf. Silberger [20]). Such a G is locally compact. For a function (or a distribution) f on G, $\mathrm{supp}(f)$ denotes the support of f. Let T be a distribution on G and α be a function on G^{n+1} into $\mathbf{C}$. As in Schwartz [16], $T_t(\alpha(t, x_1, \dots, x_n))$ denotes the

[1] Dr. J-S. Li communicated the author some mistakes in the previous draft after the author had come to notice an error in the proof of Theorem 1, [23]. According to his communication, he has a complete determination of $\mathbf{P}$ in the case of type G_2. We can immediately obtain the unitarizability condition for $\pi(\chi)$ for $\chi \in X^i$ by our method also for type G_2. This is because the condition (4) is satisfied or $w\delta_B^{1/2} \in \bar{D}$ for some $w \in W$ if D is a bounded domain in the case G_2.

function obtained by taking the value of T for the function $\alpha(t, x_1, \ldots, x_n)$ of t with $(x_1, \ldots, x_n) \in G^n$ fixed, whenever this is well defined.

If R is a ring with unit, we denote by $R^\times$ the set of all invertible elements of R. If k is a commutative field, $M(\ell, k)$ denotes the ring of all $\ell \times \ell$-matrices with entries in k. The diagonal matrix with diagonal elements $a_1, a_2, \ldots, a_\ell$ is denoted by $\mathrm{diag}[a_1, a_2, \ldots, a_\ell]$. We set $GL(\ell, k) = M(\ell, k)^\times$.

From §3 on, we consider algebraic groups defined over a non-archimedean local field k. When an algebraic group is considered as an algebro-geometric object, we denote it by a bold face capital letter (by $\mathbf{G}$ for example), whereas the group of k-rational points is denoted by the corresponding Roman capital (by G for example).

§1. Positive and bounded distributions on t.d.groups

In this section, we shall study certain classes of distributions on a t.d.group and generalize L. Schwartz's results on positive and bounded distributions. Let G be a t.d.group. By $C(G)$, $C_c(G)$, $C^\infty(G)$ and $C_c^\infty(G)$, we denote the space of all continuous functions, continuous functions with compact support, locally constant functions and locally constant functions with compact support with values in $\mathbf{C}$ respectively. For $\alpha \in C(G)$, we set

$$(1.1) \qquad \check{\alpha}(x) = \alpha(x^{-1}), \quad \tilde{\alpha}(x) = \overline{\alpha(x^{-1})}, \quad x \in G,$$

where — denotes the complex conjugation. A distribution on G is a linear functional on $C_c^\infty(G)$. Let $D(G)$ (resp. $D_c(G)$) denote the space of all distributions (resp. distributions with compact support) on G. If one of $T_1, T_2 \in D(G)$ is of compact support, we can define the convolution $T_1 * T_2 \in D(G)$ in the following way. (cf. Bernstein-Zelevinski [1], where only the case $T_1, T_2 \in D_c(G)$ is treated. But the generalization is straightforward.) First let G_1 and G_2 be t.d.groups. By setting

$$(\alpha_1 \otimes \alpha_2)(x_1, x_2) = \alpha_1(x_1)\alpha_2(x_2), \qquad x_1 \in G_1, x_2 \in G_2$$

for $\alpha_1 \in C_c^\infty(G_1)$, $\alpha_2 \in C_c^\infty(G_2)$, we have

$$C_c^\infty(G_1 \times G_2) \cong C_c^\infty(G_1) \otimes_{\mathbf{C}} C_c^\infty(G_2).$$

For $T_1 \in D(G_1)$, $T_2 \in D(G_2)$, we can define $T_1 \otimes T_2 \in D(G_1 \times G_2)$ by setting

$$(T_1 \otimes T_2)(\alpha_1 \otimes \alpha_2) = T_1(\alpha_1) T_2(\alpha_2), \qquad \alpha_i \in C_c^\infty(G_i),\ i = 1, 2.$$

Take $G_1 = G_2 = G$. Let $T_1, T_2 \in D(G)$ and assume one of T_1 and T_2 is of compact support. For $\alpha \in C_c^\infty(G)$, define $\beta \in C^\infty(G \times G)$ by $\beta(x_1, x_2) = \alpha(x_1 x_2)$. Since $\mathrm{supp}(T_1 \otimes T_2) = \mathrm{supp}(T_1) \times \mathrm{supp}(T_2)$ and $\mathrm{supp}(T_1 \otimes T_2) \cap \mathrm{supp}(\beta)$ is compact, we can define $T_1 * T_2$ by

$$(1.2) \qquad (T_1 * T_2)(\alpha) = (T_1 \otimes T_2)(\beta).$$

We can rewrite (1.2) more explicitly as

(1.3) $$(T_1 * T_2)(\alpha) = (T_1)_t((T_2)_x(\alpha(tx))), \qquad T_2 \in D_c(G),$$

(1.4) $$(T_1 * T_2)(\alpha) = (T_2)_x((T_1)_t(\alpha(tx))), \qquad T_1 \in D_c(G).$$

We see easily that $T_1 * T_2$ coincides with the usual convolution when T_1 and T_2 are functions in $C(G)$. We can verify easily the associativity:

(1.5) $$T_1 * (T_2 * T_3) = (T_1 * T_2) * T_3 \quad \text{if} \quad T_1, T_3 \in D_c(G), \; T_2 \in D(G).$$

Let $T \in D(G)$ and $\alpha \in C_c^\infty(G)$. Then we have

(1.6) $$(T * \alpha)(x) = T_t(\alpha(t^{-1}x)), \qquad x \in G,$$

(1.7) $$(\alpha * T)(x) = T_t(\alpha(xt^{-1})\delta_G(t)), \qquad x \in G.$$

Thus $T * \alpha$, $\alpha * T \in C^\infty(G)$. We set

(1.8) $$\check{T}(\alpha) = T_t(\check{\alpha}(t)\delta_G(t)),$$

(1.9) $$\bar{T}(\alpha) = \overline{T(\bar{\alpha})},$$

(1.10) $$\tilde{T} = \bar{\check{T}}.$$

When T is a function in $C(G)$, these definitions coincide with (1.1). By definition, we have

$$\check{\check{T}} = T, \; \bar{\bar{T}} = T, \; \tilde{\tilde{T}} = T, \; \tilde{T} = \check{\bar{T}}.$$

Let $T_1, T_2 \in D(G)$ and $\alpha \in C_c^\infty(G)$. We assume that one of T_1, T_2 is of compact support. We can also verify easily the following formulas.

(1.11) $$(T_1 \check{*} T_2) = \check{T}_2 * \check{T}_1,$$

(1.12) $$(T_1 * T_2)(\alpha) = (T_1)(\alpha * \check{T}_2),$$

(1.13) $$(T_1 * T_2)(\alpha) = (T_2)_x((\check{T}_1 * \alpha_1)(x)\delta_G(x)),$$

where $\alpha_1(x) = \alpha(x)\delta_G(x)^{-1}$, $x \in G$.

We are going to define certain classes of bounded distributions. For a closed subset U of G, let D_U denote the set of all distributions on G whose supports are contained in U. We set

$C_{L^1}^r(G) = \{\varphi \in C^\infty(G) \mid$ there exists an open compact subgroup $K = K(\varphi)$ of G such that $\varphi * T \in L^1(G)$ for any $T \in D_K\}$,

$C_{L^1}^l(G) = \{\varphi \in C^\infty(G) \mid$ there exists an open compact subgroup $K = K(\varphi)$ of G such that $T * \varphi \in L^1(G)$ for any $T \in D_K\}$,

$C_{L^1}^t(G) = \{\varphi \in C^\infty(G) \mid$ there exists an open compact subgroup $K = K(\varphi)$ of G such that $T_1 * \varphi * T_2 \in L^1(G)$ for any $T_1, T_2 \in D_K\}$.

We define a topology of $C^r_{L^1}(G)$ in the following way: A sequence $\{\varphi_n\}$ in $C^r_{L^1}(G)$ converges to 0 if and only if there exists an open compact subgroup K of G such that $\varphi_n * T \in L^1(G)$ and converges to 0 in $L^1(G)$ for any $T \in D_K$.
A topology of $C^l_{L^1}(G)$ is defined similarly.
A topology of $C^t_{L^1}(G)$ is defined by: A sequence $\{\varphi_n\}$ in $C^t_{L^1}(G)$ converges to 0 if and only if there exists an open compact subgroup K of G such that $T_1 * \varphi_n * T_2 \in L^1(G)$ and converges to 0 in $L^1(G)$ for any $T_1, T_2 \in D_K$.

Obviously $C^t_{L^1}(G) \subseteq C^r_{L^1}(G) \cap C^l_{L^1}(G)$ and the topology of $C^t_{L^1}(G)$ is finer than the induced topology from $C^r_{L^1}(G)$ or $C^l_{L^1}(G)$.

Lemma 1.1. *$C^\infty_c(G)$ is a dense subspace of $C^r_{L^1}(G)$, $C^l_{L^1}(G)$ and $C^t_{L^1}(G)$.*

Proof. It suffices to show that $C^\infty_c(G)$ is dense in $C^t_{L^1}(G)$. If $\alpha \in C^\infty_c(G)$ and $T_1, T_2 \in D_c(G)$, we have $T_1 * \alpha * T_2 \in C^\infty_c(G)$. Hence we have $C^\infty_c(G) \subseteq C^t_{L^1}(G)$. Take any $\varphi \in C^t_{L^1}(G)$. Let K be any open compact subgroup of G and let $G = \bigcup_{i=1}^\infty Kx_iK$. Set $X_n = \bigcup_{i=1}^n Kx_iK$,

$$\alpha_n(g) = \begin{cases} \varphi(g), & \text{for } g \in X_n \\ 0, & \text{for } g \notin X_n. \end{cases}$$

Then $\alpha_n \in C^\infty_c(G)$. Take any $T_1, T_2 \in D_K$. It suffices to show that $T_1 * \alpha_n * T_2$ converges to $T_1 * \varphi * T_2$ in $L^1(G)$ for $n \to \infty$. Considering the supports, we see easily that

$$T_1 * (\varphi - \alpha_n) * T_2 | X_n = 0,$$

$$T_1 * (\varphi - \alpha_n) * T_2 | G - X_n = T_1 * \varphi * T_2 | G - X_n.$$

Therefore $T_1 * (\varphi - \alpha_n) * T_2$ is obtained by restricting the support of $T_1 * \varphi * T_2 \in L^1(G)$ to $G - X_n$. Hence $T_1 * (\varphi - \alpha_n) * T_2$ converges to 0 in $L^1(G)$ for $n \to \infty$. This completes the proof.

Let $B^r(G)$, $B^l(G)$ and $B^t(G)$ be the space of continuous linear functionals on $C^r_{L^1}(G)$, $C^l_{L^1}(G)$ and $C^t_{L^1}(G)$ respectively. By Lemma 1.1, these spaces can be canonically regarded as subspaces of $D(G)$. Clearly we have

$$B^r(G),\ B^l(G) \subseteq B^t(G) \subseteq D(G).$$

We are going to examine the structure of D_K for an open compact subgroup K of G. Let ρ be an irreducible unitary representation of K and let $m^\rho_{ij}(x)$, $x \in K$ denote the matrix coefficients of ρ for $1 \le i, j \le \dim \rho$. We normalize a Haar measure dx of G so that $\mathrm{vol}(K) = 1$. We set

$$\langle m^\rho_{ij}, m^{\rho'}_{kl} \rangle = \int_K m^\rho_{ij}(x)\, \overline{m^{\rho'}_{kl}}\, dx,$$

where ρ' is an irreducible unitary representation of K. Then the following relations are well known (cf. Weil [22], p. 73).

(1.14) $$m^\rho_{ij} * m^{\rho'}_{kl} = 0 \quad \text{if } \rho \text{ is not equivalent to } \rho'.$$

(1.15) $$m_{ij}^{\rho} * m_{kl}^{\rho} = 0 \qquad \text{if } j \neq k.$$

(1.16) $$m_{ij}^{\rho} * m_{jl}^{\rho} = \frac{1}{\dim \rho} m_{il}^{\rho}.$$

(1.17) $$\langle m_{ij}^{\rho}, m_{kl}^{\rho'} \rangle = 0 \qquad \text{if } \rho \text{ is not equivalent to } \rho'.$$

(1.18) $$\langle m_{ij}^{\rho}, m_{kl}^{\rho} \rangle = 0 \qquad \text{if } i \neq k \text{ or } j \neq l.$$

(1.19) $$\langle m_{ij}^{\rho}, m_{ij}^{\rho} \rangle = \frac{1}{\dim \rho}.$$

Let $\hat{K}$ denote the set of equivalence classes of all irreducible unitary representations of K. Let φ be a locally constant function on K. We can find an open compact normal subgroup K_1 of K so that φ is left and right invariant under K_1. Then we can expand φ in the form

(1.20) $$\varphi(x) = \sum_{\rho \in \hat{K}} \sum_{i,j} a(\rho, i, j) m_{ij}^{\rho}(x), \qquad x \in K,$$

where $a(\rho, i, j) \in \mathbf{C}$ and if $a(\rho, i, j) \neq 0$, then ρ is trivial on K_1. The expansion (1.20) for φ, which is actually a finite sum, is unique. For $\rho \in \hat{K}$, $1 \leq i, j \leq \dim \rho$, define $T(\rho, i, j) \in D_K$ by

(1.21) $$T(\rho, i, j)(x) = (\dim \rho) \overline{m_{ij}^{\rho}(x)}, \qquad x \in K.$$

Then, by (1.17) ~ (1.19), we have

$$T(\rho, i, j)(\varphi) = a(\rho, i, j).$$

Therefore any distribution $T \in D_K$ can be expanded in the form

(1.22) $$T = \sum_{\rho \in \hat{K}} \sum_{i,j} c(\rho, i, j) T(\rho, i, j)$$

with $c(\rho, i, j) = T(m_{ij}^{\rho})$. In particular, the Dirac distribution δ supported on 1 is given by

(1.23) $$\delta = \sum_{\rho \in \hat{K}} \sum_{i} T(\rho, i, i).$$

Conversely any infinite series (1.22) with $c(\rho, i, j) \in \mathbf{C}$ defines a distribution $T \in D_K$. Since ρ is unitary, we have $\overline{m_{ij}^{\rho}(x)} = m_{ij}^{\rho}(x^{-1})$. Hence we get

(1.24) $$\check{m}_{ij}^{\rho} = \overline{m_{ji}^{\rho}}, \quad \widetilde{m_{ij}^{\rho}} = m_{ji}^{\rho}.$$

Let φ and T be given by (1.20) and (1.22) respectively. By (1.14) ~ (1.16), we obtain

(1.25) $$\varphi * \check{T} = \sum_{\rho} \frac{1}{\dim \rho} \sum_{k,j} (\sum_{i} a(\rho, k, i) c(\rho, j, i)) \check{T}(\rho, j, k),$$

(1.26) $$\check{T} * \varphi = \sum_{\rho} \frac{1}{\dim \rho} \sum_{i,k} (\sum_{j} a(\rho, j, k) c(\rho, j, i)) \check{T}(\rho, k, i).$$

The following Theorem is an analogue of Théorème XXV of L.Schwartz [16], p. 201.

Theorem 1.2. *Let $T \in D(G)$. The following three conditions are equivalent:*
(1) *For any open compact subgroup K of G, there exist a bounded continuous function f on G and $T_1 \in D_K$ such that $T = f * T_1$.*
(2) *$T \in B^r(G)$.*
(3) *For any $\alpha \in C_c^\infty(G)$, $T * \alpha \in C^\infty(G)$ is bounded on G.*

Proof. First we assume that (1) holds for some K. Suppose $\varphi_n \in C^r_{L^1}(G)$ converges to 0 for $n \to \infty$. By (1.12), we have

$$(f * T_1)(\varphi_n) = f(\varphi_n * \check{T}_1).$$

Since $\varphi_n * \check{T}_1$ converges to 0 in $L^1(G)$, $f(\varphi_n * \check{T}_1)$ converges to 0. Hence $T = f * T_1 \in B^r(G)$. Next we assume (2). Suppose $\varphi_n \in C_c^\infty(G)$ converges to 0 in $L^1(G)$. To prove (3), it suffices to show that $(T * \alpha)(\varphi_n)$ converges to 0. By (1.12), we have

$$(T * \alpha)(\varphi_n) = T(\varphi_n * \check{\alpha}).$$

Let K be any open compact subgroup of G and $T_1 \in D_K$. We see that $\varphi_n * \check{\alpha} * T_1$ converges to 0 in $L^1(G)$ since $\check{\alpha} * T_1 \in C_c^\infty(G)$. Therefore $T(\varphi_n * \check{\alpha})$ converges to 0 and we get (3).

Finally we assume (3) and shall prove (1). Put

$$B = \{\varphi \in C_c^\infty(G) \mid \|\varphi\|_{L^1} \leq 1\}.$$

Let $\alpha \in C_c^\infty(G)$, $\varphi \in B$. By (1.12), we have

$$(\varphi * T)(\alpha) = \varphi(\alpha * \check{T}).$$

Since $(\alpha * \check{T}) = T \check{*} \check{\alpha}$ by (1.11), $\alpha * \check{T} \in C^\infty(G)$ is bounded on G. Hence, when α is fixed, $(\varphi * T)(\alpha)$ is bounded for $\varphi \in B$. Let K be any open compact subgroup of G and we consider the restriction $\varphi * T | K \in D_K$. We may set

$$\varphi * T | K = \sum_{\rho \in \hat{K}} \sum_{i,j} c(\rho, i, j, \varphi) T(\rho, i, j).$$

Taking $\alpha = m^\rho_{ij}$, we see that $c(\rho, i, j, \varphi)$ is bounded for $\varphi \in B$, for every (ρ, i, j). Choose $c(\rho, i, j) > 0$ so that

$$|c(\rho, i, j, \varphi)| \leq c(\rho, i, j), \qquad \forall \varphi \in B.$$

By the first axiom of countability, $\hat{K}$ is a countable set. For (ρ, i, j), choose

$a(\rho, i, j) \in \mathbf{C}$ so that $\sum_{\rho}\sum_{i,j} a(\rho, i, j)$ and $\sum_{\rho}\sum_{i,j} a(\rho, i, j)c(\rho, i, j)$ are absolutely convergent and that $\dim \rho \times \dim \rho$-matrix $(a_{ij}) = (a(\rho, i, j))$ is non-singular for every ρ. Put

$$\alpha(x) = \begin{cases} \sum_{\rho}\sum_{i,j} a(\rho, i, j)\, m_{ij}^{\rho}(x), & \text{if } x \in K \\ 0, & \text{if } x \in G - K. \end{cases}$$

Then $\alpha \in C_c(G)$. Let $\hat{K} = \{\rho_n | n \in \mathbf{N}\}$. Let

$$\alpha_n = \sum_{k \leq n}\sum_{i,j} a(\rho_k, i, j)\, m_{ij}^{\rho_k} \text{ on } K \text{ and } \alpha_n = 0 \text{ on } G - K.$$

Then

$$(\varphi * T)(\alpha - \alpha_n) = \varphi((\alpha - \alpha_n) * \check{T})$$

is bounded for $\varphi \in B$ and this bound converges to 0 for $n \to \infty$. Therefore $\alpha * \check{T}$ is a continuous function as the uniformly convergent limit of $\alpha_n * \check{T} \in C^{\infty}(G)$. Since $\varphi(\alpha * \check{T})$ is bounded for $\varphi \in B$, $\alpha * \check{T}$ is bounded on G. Put $f = T * \check{\alpha} = (\alpha * \check{T})^{\vee}$. For every (ρ, i, j), we can choose $c_1(\rho, i, j)$ so that

$$\sum_{i} a(\rho, i, k) c_1(\rho, i, j) = \delta_{jk} \dim \rho$$

holds for every ρ, j, k, where δ_{jk} denotes Kronecter's δ. Set

$$T_1 = \sum_{\rho \in \hat{K}}\sum_{i,j} c_1(\rho, i, j)\, T(\rho, i, j) \in D_K.$$

Then, by (1.23) and (an obvious generalization of) (1.26), we get $\check{T}_1 * \alpha = \check{\delta} = \delta$. Hence we obtain

$$T = T * \delta = T * (\check{\alpha} * T_1) = (T * \check{\alpha}) * T_1 = f * T_1.$$

This completes the proof.

We see easily that $\varphi_n \in C_{L^1}^r(G)$ converges to 0 if and only if $\check{\varphi}_n(x)\delta_G(x) \in C_{L^1}^l(G)$ and converges to 0 in $C_{L^1}^l(G)$. Therefore $T \in B^r(G)$ if and only if $\check{T} \in B^l(G)$ by (1.8). Hence a similar result holds for $B^l(G)$. We give a statement of a Theorem on $B^t(G)$ which can be proved in a similar way without giving a proof.

Theorem 1.3. *Let $T \in D(G)$. The following three conditions are equivalent*:
(1) *For any open compact subgroup K of G, there exist a bounded continuous function f on G and $T_1, T_2 \in D_K$ such that $T = T_1 * f * T_2$.*
(2) $T \in B^t(G)$.
(3) *For any $\alpha, \beta \in C_c^{\infty}(G)$, $\alpha * T * \beta$ is bounded on G.*

Now we are going to study positive distributions on G. Hereafter we shall assume that G is unimodular. A continuous function Φ on G is called *of positive type* if

$$\int_G \int_G \Phi(y^{-1}z)\alpha(y)\overline{\alpha(z)}\,dydz \geq 0 \tag{1.27}$$

for any $\alpha \in C_c^\infty(G)$. It is well known that such Φ is bounded on G(cf. Weil [22]). A distribution T on G is called *of positive type* if

$$T(\alpha * \tilde{\alpha}) \geq 0 \qquad \text{for any } \alpha \in C_c^\infty(G). \tag{1.28}$$

By definition, we see easily that a continuous function is of positive type if and only if it defines a distribution of positive type. Let $P(G)$ denote the set of all positive distributions on G.

Lemma 1.4. *A distribution $T \in D(G)$ is of positive type if and only if $\tilde{\alpha} * T * \alpha \in C^\infty(G)$ is of positive type for any $\alpha \in C_c^\infty(G)$.*

Proof. For $\alpha \in C_c^\infty(G)$, we have

$$\check{T}(\alpha * \tilde{\alpha}) = T((\alpha \check{*} \tilde{\alpha})) = T(\bar{\alpha} * \tilde{\bar{\alpha}}).$$

Hence T is of positive type if and if $\check{T}$ is of positive type. For $f \in C(G)$, put

$$\mathrm{Tr}(f) = f(1).$$

We have, for $\alpha \in C_c^\infty(G)$,

$$\mathrm{Tr}(\tilde{\alpha} * \check{T} * \alpha) = \int_G \tilde{\alpha}(y)\,\check{T}_t(\alpha(t^{-1}y^{-1}))\,dy = T_t\left(\int_G (\tilde{\alpha}(y)\alpha(ty^{-1})\,dy\right)$$

$$= T_t\left(\int_G \tilde{\alpha}(y^{-1}t)\alpha(y)\,dy\right) = T(\alpha * \tilde{\alpha}).$$

Therefore T is of positive type if and only if $\mathrm{Tr}(\alpha * T * \tilde{\alpha}) \geq 0$ for any $\alpha \in C_c^\infty(G)$. If $\Phi \in C(G)$, we get

$$\mathrm{Tr}(\alpha * \Phi * \tilde{\alpha}) = \int_G \int_G \Phi(y^{-1}z)\alpha(y)\overline{\alpha(z)}\,dydz, \qquad \alpha \in C_c^\infty(G). \tag{1.29}$$

Now assume $T \in P(G)$ and take $\alpha \in C_c^\infty(G)$. Put $\Phi = \alpha * T * \tilde{\alpha}$. Then

$$\mathrm{Tr}(\beta * \Phi * \tilde{\beta}) = \mathrm{Tr}((\beta * \alpha) * T * (\widetilde{\beta * \alpha})) \geq 0$$

for any $\beta \in C_c^\infty(G)$. Hence by (1.29), we see that Φ is of positive type. Conversely assume $\tilde{\alpha} * T * \alpha$ is of positive type for any $\alpha \in C_c^\infty(G)$. Take any $\beta \in C_c^\infty(G)$. We can find $\alpha \in C_c^\infty(G)$ so that $\alpha * \beta = \beta$, by taking α as a suitable constant multiple of the characteristic function of a sufficiently small open compact subgroup of G. Since $\tilde{\alpha} * T * \alpha$ is of positive type, we have $\mathrm{Tr}(\tilde{\beta} * (\tilde{\alpha} * T * \alpha) * \beta) \geq 0$ by (1.29). Hence we obtain

$$\mathrm{Tr}((\widetilde{\alpha * \beta}) * T * (\alpha * \beta)) = \mathrm{Tr}(\tilde{\beta} * T * \beta) = \check{T}(\beta * \tilde{\beta}) \geq 0.$$

Therefore $\check{T}$ is of positive type. This completes the proof.

Lemma 1.5. *We assume $T \in P(G)$. Then $T = \tilde{T}$ and $\alpha * T * \beta$ is bounded on G for any $\alpha, \beta \in C_c^\infty(G)$.*

Proof. Let $\alpha, \beta \in C_c^\infty(G)$. We use the following formula which can be verified by direct computations.

$$4(\alpha * T * \beta) = (\alpha + \tilde{\beta}) * T * (\tilde{\alpha} + \beta) - (\alpha - \tilde{\beta}) * T * (\tilde{\alpha} - \beta) \\ + \sqrt{-1}(\alpha + \sqrt{-1}\tilde{\beta}) * T * (\tilde{\alpha} - \sqrt{-1}\beta) \\ - \sqrt{-1}(\alpha - \sqrt{-1}\tilde{\beta}) * T * (\tilde{\alpha} + \sqrt{-1}\beta). \tag{1.30}$$

By Lemma 1.4, $\tilde{\alpha} * T * \alpha$ is a bounded function on G for any $\alpha \in C_c^\infty(G)$. The second assertion follows by (1.30). We have

$$T_1(\alpha * \tilde{\alpha}) = \overline{\tilde{T}_1(\alpha * \tilde{\alpha})} \qquad \text{for } T_1 \in D(G),\ \alpha \in C_c^\infty(G).$$

Consider the formula obtaind from (1.30) by letting T as the Dirac distribution supported on 1. Taking the values of T and $\tilde{T}$ at this function, we see immediately that $T(\alpha * \beta) = \tilde{T}(\alpha * \beta)$, $\alpha, \beta \in C_c^\infty(G)$. Since for any $\alpha \in C_c^\infty(G)$, we can take $\beta \in C_c^\infty(G)$ so that $\alpha * \beta = \alpha$, we obtain $T = \tilde{T}$. This completes the proof.

By Theorem 1.3, we have $P(G) \subseteq B'(G)$. We shall prove an analogue of Theorem 1.2 for $P(G)$.

Theorem 1.6. *Let $T \in D(G)$. The following three conditions are equivalent:*
(1) *For any open compact subgroup K of G, there exist a continuous function f of positive type on G and $T_1 \in D_K$ such that $T = \tilde{T}_1 * f * T_1$.*
(2) *$T \in P(G)$.*
(3) *For any $\alpha \in C_c^\infty(G)$, $\tilde{\alpha} * T * \alpha$ is of positive type.*

Proof. We assume (1). For any $\alpha \in C_c^\infty(G)$, we have

$$\tilde{\alpha} * T * \alpha = \tilde{\alpha} * (\tilde{T}_1 * f * T_1) * \alpha = (\widetilde{T_1 * \alpha}) * f * (T_1 * \alpha).$$

Since $T_1 * \alpha \in C_c^\infty(G)$, $(\widetilde{T_1 * \alpha}) * f * (T_1 * \alpha)$ is a continuous function of positive type. Hence, by Lemma 1.4, we have $T \in P(G)$. The equivalence of (2) and (3) is proved as Lemma 1.4. We assume (2). By Lemma 1.5, $\alpha * T * \beta \in C^\infty(G)$ is bounded on G for any $\alpha, \beta \in C_c^\infty(G)$. Let B be defined as in the proof of Theorem 1.2. Let $\alpha, \beta \in C_c^\infty(G)$ and $\varphi \in B$. By (1.12) and (1.13), we have

$$(\alpha * T * \beta)(\varphi) = (T * \beta * \check{\varphi})(\check{\alpha}).$$

We see, when α and β are fixed, $(T * \beta * \check{\varphi})(\check{\alpha})$ is bounded for $\varphi \in B$. Let K be any open compact subgroup of G. For $\rho \in \hat{K}$, $1 \le i, j \le \dim\rho$, let $\beta = m_{ij}^\rho$ on K, $\beta = 0$ on $G - K$, and let us consider the restriction $T * \beta * \check{\varphi} | K \in D_K$. We may set

$$T * m_{ij}^\rho * \check{\varphi} | K = \sum_{\tau \in \hat{K}} \sum_{k,l} c(\rho, i, j, \tau, k, l, \varphi)\ \overline{T(\tau, k, l)}\ .$$

Taking $\alpha = \widetilde{m^{\tau}_{kl}}$, we see that $c(\rho, i, j, \tau, k, l, \varphi)$ is bounded for $\varphi \in B$. Choose $c(\rho, i, j, \tau, k, l) > 0$ so that

$$|c(\rho, i, j, \tau, k, l, \varphi)| \leq c(\rho, i, j, \tau, k, l), \qquad \forall \varphi \in B.$$

Since $\hat{K}$ is a countable set, we can choose $a(\rho, i, j) \in \mathbf{C}$ so that

$$\sum_{\rho,i,j} a(\rho, i, j) \quad \text{and} \quad \sum_{\tau,k,l} \sum_{\rho,i,j} c(\rho, i, j, \tau, k, l) a(\tau, k, l) a(\rho, i, j)$$

are absolutely convergent and that $(a_{ij}) = (a(\rho, i, j))$ is a non-singular matrix for every ρ. Let

$$\alpha = \sum_{\rho\in\hat{K}} \sum_{i,j} a(\rho, i, j) m^{\rho}_{ij} \quad \text{on} \quad K, \ \alpha = 0 \quad \text{on} \quad G - K.$$

Then $\alpha \in C_c(G)$. By (1.12) and (1.13), we have

$$(1.31) \qquad (\tilde{\alpha} * T * \alpha)(\varphi) = (T * \alpha * \check{\varphi})(\bar{\alpha}).$$

Similarly as in the proof of Theorem 1.2, we see that $\tilde{\alpha} * T * \alpha \in C(G)$ and is of positive type as the uniformly convergent limit of $\tilde{\alpha}_n * T * \alpha_n$ (α_n is the same as there). Put $f = \tilde{\alpha} * T * \alpha$. For every (ρ, i, j), we can choose $c_1(\rho, i, j)$ so that

$$\sum_i a(\rho, k, i) c_1(\rho, j, i) = \delta_{jk} \dim \rho$$

holds for every ρ, j, k. Set

$$T_1 = \sum_{\rho\in\hat{K}} \sum_{i,j} c_1(\rho, i, j) \check{T}(\rho, i, j).$$

Then, by (an obvious generalization of) (1.25), we get $\alpha * T_1 = \delta$, $\delta = \tilde{T}_1 * \tilde{\alpha}$. Hence we obtain

$$T = \delta * T * \delta = (\tilde{T}_1 * \tilde{\alpha}) * T * (\alpha * T_1) = \tilde{T}_1 * f * T_1.$$

This completes the proof.

Lemma 1.7. *Let $T \in D(G)$ and K be an open compact subgroup of G. Set*

$$T|K = \sum_{\rho\in\hat{K}} \sum_{i,j} c(\rho, i, j) T(\rho, i, j).$$

For $\rho \in \hat{K}$, let $C_\rho = (c(\rho, i, j))$ be the $\dim \rho \times \dim \rho$*-matrix. Then the following conditions are equivalent*:

(1) *$T(\alpha * \tilde{\alpha}) \geq 0$ for any $\alpha \in C_c^{\infty}(G)$ such that* $\operatorname{supp}(\alpha) \subseteq K$.

(2) *C_ρ is a positive (not necessarily definite) hermitian matrix for any $\rho \in \hat{K}$.*

Proof. Take any $\alpha \in C_c^{\infty}(G)$ such that $\operatorname{supp}(\alpha) \subseteq K$. We may set

$$\alpha = \sum_{\rho\in\hat{K}} \sum_{i,j} a(\rho, i, j) m^{\rho}_{ij}.$$

By (1.24), we have

$$\tilde{\alpha} = \sum_{\rho\in\hat{K}}\sum_{i,j} \overline{a(\rho, j, i)}\ m_{ij}^{\rho}.$$

Hence we get

$$\alpha * \tilde{\alpha} = \sum_{\rho\in\hat{K}} \frac{1}{\dim\rho} \sum_{i,j} (\sum_{k} a(\rho, i, k)\ \overline{a(\rho, j, k)})\, m_{ij}^{\rho}.$$

Therefore we obtain

$$T(\alpha * \tilde{\alpha}) = \sum_{\rho\in\hat{K}} \frac{1}{\dim\rho} \sum_{i,j} c(\rho, i, j)(\sum_{k} a(\rho, i, k)\ \overline{a(\rho, j, k)}). \tag{1.32}$$

The condition (1) is equivalent to

$$\sum_{i,j} c(\rho, i, j)(\sum_{k} a(\rho, i, j)\ \overline{a(\rho, j, k)}) \geq 0 \tag{1.33}$$

for any choice of $a(\rho, i, j)$, for every ρ. Let $A_\rho = (a(\rho, i, j))$ be the $\dim\rho \times \dim\rho$-matrix. Then (1.33) is equivalent to $\mathrm{Trace}(C_\rho \bar{A}_\rho^t A_\rho) \geq 0$. Therefore (1) is equivalent to $\mathrm{Trace}({}^t\bar{A}_\rho C_\rho A_\rho) \geq 0$ for any ρ and A_ρ. This is the case if and only if C_ρ is positive hermitian for any ρ. Hence the assertion follows.

We fix a Haar measure dx on G, not necessarily normalized so that $\mathrm{vol}(K) = 1$.

Lemma 1.8. *Let $T \in P(G)$ and K be an open compact subgroup of G. Let K_0 be an open compact subgroup of G contained in K and α be the characteristic function of K_0. Set $v = \mathrm{vol}(K)$, $v_0 = \mathrm{vol}(K_0)$. For $\rho \in \hat{K}$, let χ_ρ be the character of ρ and consider χ_ρ as an element of $C_c^\infty(G)$ by setting 0 outside of K. Let $\hat{K}(1)$ denote the finite set of all $\rho \in \hat{K}$ which occur in $\mathrm{Ind}_{K_0}^{K} 1$. Then we have*

$$|(T * \alpha)(x)| \leq \frac{v_0}{v} \sum_{\rho\in\hat{K}(1)} (\dim\rho) T(\chi_\rho), \qquad x \in K.$$

Proof. Set

$$T|K = \sum_{\rho\in\hat{K}}\sum_{i,j} c(\rho, i, j) T(\rho, i, j),$$

where $T(\rho, i, j) = (\dim\rho)\overline{m_{ij}^{\rho}}$ as before. We have $c(\rho, i, j) = v^{-1} T(m_{ij}^{\rho})$. Let $\rho \in \hat{K}$ and A be a $\dim\rho \times \dim\rho$ unitary matrix. Consider the unitary representation $\rho'(x) = (m_{ij}^{\rho'}(x)) = A^{-1}(m_{ij}^{\rho}(x))A$, $x \in K$, which is equivalent to ρ. Let $C_\rho = (c(\rho, i, j))$. If we use $m_{ij}^{\rho'}$ instead of m_{ij}^{ρ}, the matrix C_ρ changes to $A^{-1}C_\rho A$. Since C_ρ is hermitian, we can choose A so that $A^{-1}C_\rho A$ is a diagonal matrix. Thus we may assume that every $\rho \in \hat{K}$ is chosen so that C_ρ is a diagonal matrix. By Lemma 1.7, every diagonal entry of C_ρ is a non-negative real number. We may set

(1.34)
$$\alpha = \sum_{\rho \in \hat{K}(1)} \sum_{i,j} a(\rho, i, j) m_{ij}^{\rho}.$$

Since $\check{\alpha} = \alpha$, we obtain

$$\alpha = \sum_{\rho \in \hat{K}(1)} \sum_{i,j} a(\rho, j, i) \overline{m_{ij}^{\rho}},$$

by (1.24). Then we obtain

$$(T * \alpha)(x) = v \sum_{\rho \in \hat{K}(1)} \sum_{i,j} (\sum_{k} c(\rho, i, k) a(\rho, j, k)) \overline{m_{ij}^{\rho}(x)}, \qquad x \in K.$$

Since C_ρ is diagonal, we get

(1.35)
$$(T * \alpha)(x) = v \sum_{\rho \in \hat{K}(1)} \sum_{i} c(\rho, i, i) (\sum_{j} a(\rho, j, i)) \overline{m_{ij}^{\rho}(x)}, \qquad x \in K.$$

By (1.17) ~ (1.19), we have

$$\int_{K_0} m_{ij}^{\rho}(x)\, dx = \int_{K} \check{\alpha}(x) m_{ij}^{\rho}(x)\, dx = v(\dim \rho)^{-1} a(\rho, j, i).$$

Hence we have

$$\sum_{i} a(\rho, j, i) \overline{m_{ij}^{\rho}(x)} = v^{-1} (\dim \rho) \int_{K_0} (\sum_{j} m_{ij}^{\rho}(y) \overline{m_{ij}^{\rho}(x)}) dy, \qquad x \in K.$$

Since ρ is unitary, we have

$$\sum_{j} |m_{ij}^{\rho}(x)|^2 = 1, \qquad x \in K,$$

and by the Schwarz inequality, we get

$$|\sum_{j} m_{ij}^{\rho}(y) \overline{m_{ij}^{\rho}(x)}| \leq 1, \qquad x, y \in K.$$

Therefore we obtain

$$|\sum_{j} a(\rho, j, i) \overline{m_{ij}^{\rho}(x)}| \leq v^{-1} v_0 (\dim \rho), \qquad x \in K.$$

Since $c(\rho, i, i) \geq 0$, we get

$$|(T * \alpha)(x)| \leq v_0 \sum_{\rho \in \hat{K}(1)} \dim \rho \sum_{i} c(\rho, i, i), \qquad x \in K,$$

by (1.35). Since $\chi_\rho = \sum_i m_{ii}^{\rho}$, we have $\sum_i c(\rho, i, i) = v^{-1} T(\chi_\rho)$. Therefore we obtain the estimate

$$|(T * \alpha)(x)| \leq v^{-1} v_0 \sum_{\rho \in \hat{K}(1)} (\dim \rho)\, T(\chi_\rho),$$

and complete the proof.

Lemma 1.8 has some applications. We conclude this section by:

Proposition 1.9. *Let $T \in P(G)$ and K be an open compact subgroup of G. Let K_0 be an open compact subgroup of G contained in K and α be the characteristic function of K_0. Let Z be the center of G. We assume that there exists an increasing sequence Z_n of open compact subgroups of Z such that $Z = \bigcup_{n=1}^{\infty} Z_n$. Then $T * \alpha$ is bounded on $K \cdot Z$.*

Proof. Put $K_n = K \cdot Z_n$ and set

$$M_n = \mathrm{vol}(K_n)^{-1} \sum_{\rho \in \hat{K}_n(1)} (\dim \rho)\, T(\chi_\rho).$$

By Lemma 1.8, it suffices to show $M_{n+1} = M_n$ for $n \geq 1$. We may assume that $K_{n+1} \triangleright K_n$ and that K_{n+1}/K_n is a cyclic group of prime order p_n for $n \geq 1$. For every $\tau \in K_{n+1}/K_n$ and $\rho \in \hat{K}_n$, we have $\rho^\tau \cong \rho$ where $\rho^\tau \in \hat{K}_n$ is defined by $\rho^\tau(k) = \rho(\tau^{-1} k \tau)$, $k \in K_n$. Put $p = p_n$. We can show without difficulty that

$$\mathrm{Ind}_{K_n}^{K_{n+1}} \rho \cong \bigoplus_{i=0}^{p-1} (\sigma \otimes \eta^i)$$

for some $\sigma \in \hat{K}_{n+1}$, where $\mathrm{Ind}_{K_n}^{K_{n+1}} \rho$ denotes the induced representation from ρ and η denotes a generator of $\widehat{K_{n+1}/K_n}$. If $\rho \in \hat{K}_n(1)$, then $\sigma \otimes \eta^i \in \hat{K}_{n+1}(1)$ for all i, $0 \leq i \leq p-1$. By the formula of induced characters, we have $\sum_{i=0}^{p-1} \chi_{\sigma \otimes \eta^i} = p\chi_\rho$ as functions in $C_c^\infty(G)$. Hence we get

$$\sum_{i=0}^{p-1} (\dim \sigma \otimes \eta^i)\, T(\chi_{\sigma \otimes \eta^i}) = p(\dim \rho)\, T(\chi_\rho).$$

Since all $\sigma \in \hat{K}_{n+1}(1)$ occurs in $\mathrm{Ind}_{K_n}^{K_{n+1}} \rho$ for some $\rho \in \hat{K}_n(1)$ and $\mathrm{vol}(K_{n+1}) = p\,\mathrm{vol}(K_n)$, we obtain $M_{n+1} = M_n$. This completes the proof.

§2. Preservation of positivity under the direct image

Let G be a unimodular t.d. group and H be a closed subgroup of G. For a distribution T on H, we define the direct image distribution $\iota_* T$ on G by

(2.1) $$(\iota_* T)(\alpha) = T(\alpha | H), \qquad \alpha \in C_c^\infty(G).$$

Lemma 2.1. *We assume that H is unimodular. If $\iota_* T$ is of positive type, then T is also of positive type.*

Proof. Take any $\alpha \in C_c^\infty(H)$. Let U be an open compact subset of H such that $\mathrm{supp}(\alpha) \subseteq U$. Let K_0 be an open compact subgroup of H such that α is right invariant under K_0. Take an open comact subgroup K of G so that $K \cap H \subseteq K_0$. Define a function β on G by

(2.2) $$\beta(g) = \begin{cases} \alpha(h) & \text{if } g = hk \in HK,\ h \in H,\ k \in K, \\ 0 & \text{if } g \notin HK. \end{cases}$$

Then $\beta \in C_c^\infty(G)$ is well defined, $\operatorname{supp}(\beta) \subseteq UK$ and β is right invariant under K. Let dg, dh and dk be Haar measures on G, H and on K respectively. Choosing dk suitably, we may assume $dg = dhdk$ on HK. Let $x \in H$. Then we have

$$\begin{aligned}(\beta * \tilde{\beta})(x) &= \int_G \beta(g)\,\overline{\beta(x^{-1}g)}\,dg = \int_{UK} \beta(g)\,\overline{\beta(x^{-1}g)}\,dg \\ &= \int_U \int_K \beta(hk)\,\overline{\beta(x^{-1}hk)}\,dhdk = \operatorname{vol}(K)\int_U \beta(h)\,\overline{\beta(x^{-1}h)}\,dh \\ &= \operatorname{vol}(K)\int_U \alpha(h)\,\overline{\alpha(x^{-1}h)}\,dh = (\alpha * \tilde{\alpha})(x).\end{aligned}$$

Therefore we obtain

$$(\iota_* T)(\beta * \tilde{\beta}) = \operatorname{vol}(K)\,T(\alpha * \tilde{\alpha}). \tag{2.3}$$

Hence the assertion follows.

The converse of this Lemma holds under some additional conditions.

Lemma 2.2. *In addition to the assumptions of Lemma* 2.1, *we assume that there exists an increasing sequence of open compact subgroups V_n of G such that $G = \bigcup_n V_n$. Then if T is of positive type, $\iota_* T$ is also of positive type.*

Proof. Take any $\beta \in C_c^\infty(G)$. By the assumption, there exists an open compact subgroup V of G such that $\operatorname{supp}(\beta) \subseteq V$. We can find an open compact subgroup K of G so that β is right invariant under K and that $K \subseteq V$. Put $U = V \cap H$. Then U is an open compact subgroup of H. Let $\{x_i\}$ be a complete set of representatives of the finite set $U \backslash V / K$. For every x_i, define $\alpha_i \in C_c^\infty(H)$ by

$$\alpha_i(h) = \begin{cases} \beta(hx_i) & \text{if } h \in U, \\ 0 & \text{if } h \notin U. \end{cases} \tag{2.4}$$

Let dg, dh and dk be the same as in the proof of Lemma 2.1. On Hx_iK, we have

$$dg = c_i dhdk,\ g = hx_ik,\ h \in H,\ k \in K$$

with some positive constant c_i. For $x \in U$, we have

$$\begin{aligned}(\beta * \tilde{\beta})(x) &= \int_V \beta(g)\,\overline{\beta(x^{-1}g)}\,dg = \sum_i \int_{Ux_iK} \beta(g)\,\overline{\beta(x^{-1}g)}\,dg \\ &= \sum_i c_i \int_U \int_K \beta(hx_ik)\,\overline{\beta(x^{-1}hx_ik)}\,dhdk = \sum_i c_i \operatorname{vol}(K)\int_U \beta(hx_i)\,\overline{\beta(x^{-1}hx_i)}\,dh \\ &= \operatorname{vol}(K)\sum_i c_i \int_U \alpha_i(h)\,\overline{\alpha_i(x^{-1}h)}\,dh = \operatorname{vol}(K)\sum_i c_i(\alpha_i * \tilde{\alpha}_i)(x).\end{aligned}$$

Since

$$\operatorname{supp}(\beta * \tilde{\beta}) \cap H \subseteq V \cap H = U, \qquad \operatorname{supp}(\alpha_i * \tilde{\alpha}_i) \subseteq U,$$

the above computation yields

$$(\beta * \tilde{\beta})(x) = \operatorname{vol}(K) \sum_i c_i (\alpha_i * \tilde{\alpha})(x), \qquad x \in H. \tag{2.5}$$

If T is of positive type, we have

$$(\iota_* T)(\beta * \tilde{\beta}) = \operatorname{vol}(K) \sum_i c_i (\alpha_i * \tilde{\alpha}_i) \geq 0.$$

This completes the proof.

We summarize the obtained results by the following Theorem.

Theorem 2.3. *Let G be a t.d. group and H be a closed subgroup of G. We assume that G and H are unimodular. We further assume that there exists an increasing sequence of open compact subgroups V_n of G such that $G = \bigcup_n V_n$. Let T be a distribution on H. Then the direct image $\iota_* T$ is of positive type if and only if T is of positive type.*

§3. Semi-simple groups and quasi-characters of T

Let k be a non-archimedean local field and $|\ |$ be the absolute value of k. Let $\mathfrak{O}$ be the maximal compact subring and ϖ be a prime element of k. Let $q = |\varpi|^{-1}$ be the module of k.

Let $\mathbf{G}$ be a connected semi-simple algebraic group defined over k. We assume that $\mathbf{G}$ splits over k. Let $\tilde{\mathbf{G}}$ be the universal covering group of $\mathbf{G}$ and ψ be the central isogeny of $\tilde{\mathbf{G}}$ onto $\mathbf{G}$. Let $\tilde{\mathbf{T}}$ be a maximal torus of $\tilde{\mathbf{G}}$ which splits over k and $\tilde{\mathbf{B}}$ be a Borel subgroup of $\tilde{\mathbf{G}}$ which contains $\tilde{\mathbf{T}}$. Set

$$\mathbf{T} = \psi(\tilde{\mathbf{T}}), \qquad \mathbf{B} = \psi(\tilde{\mathbf{B}}).$$

Then $\mathbf{T}$ is a k-split maximal torus and $\mathbf{B}$ is a Borel subgroup of $\mathbf{G}$. Set

$$X(\tilde{\mathbf{T}}) = \operatorname{Hom}(\tilde{\mathbf{T}}, \mathbf{G}_m), \qquad X(\mathbf{T}) = \operatorname{Hom}(\mathbf{T}, \mathbf{G}_m),$$
$$X_*(\tilde{\mathbf{T}}) = \operatorname{Hom}(\mathbf{G}_m, \tilde{\mathbf{T}}), \qquad X_*(\mathbf{T}) = \operatorname{Hom}(\mathbf{G}_m, \mathbf{T}).$$

We have $X(\mathbf{T}) \subseteq X(\tilde{\mathbf{T}})$, $X_*(\tilde{\mathbf{T}}) \subseteq X_*(\mathbf{T})$ canonically. Set

$$V = X(\tilde{\mathbf{T}}) \otimes_{\mathbf{Z}} \mathbf{R}, \qquad V_* = X_*(\tilde{\mathbf{T}}) \otimes_{\mathbf{Z}} \mathbf{R}$$

and let $\langle\ ,\ \rangle$ be the canonical pairing between V and V_*. Let Σ be the root system realized in V and Δ be the set of simple roots determined by $(\tilde{\mathbf{G}}, \tilde{\mathbf{B}}, \tilde{\mathbf{T}})$. Let ℓ be the rank of $\mathbf{G}$ and set $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_\ell\}$. Let W be the Weyl group and l denote the length function on W. We can endow V a W-invariant positive definite inner product $(\ ,\ )$ so that Σ is a root system in V with respect to $(\ ,\ )$. Let $\check{\Sigma}$ be the inverse root system of Σ realized in V_*. Let

$P(\Sigma)$ and $Q(\Sigma)$ denote the lattices of weights and of root weights in V respectively. Then we have

$$X(\tilde{\mathbf{T}}) = P(\Sigma) \supseteq X(\mathbf{T}) \supseteq Q(\Sigma),$$

$$X_*(\tilde{\mathbf{T}}) = Q(\check{\Sigma}) \subseteq X_*(\mathbf{T}) \subseteq P(\check{\Sigma}),$$

(3.1) $$X(\mathbf{T}) = Q(\Sigma),\ X_*(\mathbf{T}) = P(\check{\Sigma}) \qquad \text{if } \mathbf{G} \text{ is of adjoint type.}$$

For $\alpha \in \Sigma$, Let $\check{\alpha} \in \check{\Sigma}$ be the co-root of α. Put

(3.2) $$a_\alpha = \check{\alpha}(\varpi) \in T.$$

We have

(3.3) $$\langle \alpha, \check{\beta} \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)}, \qquad \alpha, \beta \in \Sigma,$$

(3.4) $$w\check{\alpha}(t)w^{-1} = (w\check{\alpha})(t), \qquad w \in W,\ t \in k.$$

Let $\sigma_\alpha \in W$ denote the reflexion defined by α. We set $\sigma_i = \sigma_{\alpha_i}$ for $\alpha_i \in \Delta$, $1 \leq i \leq \ell$.

Let x_α be the isomorphism of $\mathbf{G}_a$ onto the root subgroup of $\mathbf{G}$ corresponding to $\alpha \in \Sigma$. By definition, we have

(3.5) $$tx_\alpha(u)t^{-1} = x_\alpha(\alpha(t)u), \qquad t \in T,\ u \in k.$$

Let Σ^+ be the set of positive roots in Σ and let $\mathbf{N}$ (resp. $\mathbf{N}^-$) be the maximal unipotent subgroup of $\mathbf{G}$ generated by $x_\alpha(\ )$, $\alpha \in \Sigma^+$ (resp. $\alpha \in \Sigma^-$). Then $\mathbf{B} = \mathbf{TN}$, $B = TN$. The modular function δ_B of B is given by

(3.6) $$\delta_B(tn) = |\prod_{\alpha \in \Sigma^+} \alpha(t)|, \qquad t \in T,\ n \in N.$$

For $\beta \in X_*(\mathbf{T})$, we have $\delta_B(\beta(t)) = |t|^{n_\beta}$, $t \in k^\times$ where $n_\beta = \sum_{\alpha \in \Sigma^+} \langle \alpha, \beta \rangle$. If $\beta = \check{\beta}_0$ for $\beta_0 \in \Delta$, we have $n_\beta = 2$ (cf. [7], p. 169). Hence we obtain

(3.7) $$\delta_B(a_\alpha) = q^{-2} \qquad \text{for } \alpha \in \Delta.$$

Let L be a non-negative integer. Let K_L be the open compact subgroup of G generated by all $x_\alpha(t)$, $t \in \varpi^L \mathfrak{O}$, $\alpha \in \Sigma$, $\beta(t)$, $t \in 1 + \varpi^L \mathfrak{O}$, $\beta \in X_*(\mathbf{T})$. (If $L = 0$, we understand $1 + \varpi^L \mathfrak{O} = \mathfrak{O}^\times$.) Then K_L forms a fundamental system of open neighbourhoods of $1 \in G$. Set $K = K_0$. Then K is a maximal compact subgroup of G and we have the Iwasawa decomosition $G = BK$. Let U_L^+ (resp. U_L^-) denote the open compact subgroup of N (resp. N^-) generated by all $x_\alpha(t)$ (resp. $x_{-\alpha}(t)$), $\alpha \in \Sigma^+$, $t \in \varpi^L \mathfrak{O}$. Let T_L denote the open compact subgroup of T generated by all $\beta(t)$, $\beta \in X_*(\mathbf{T})$, $t \in 1 + \varpi^L \mathfrak{O}$.

Lemma 3.1. *If L is a positive integer, then we have*

$$K_L = T_L U_L^+ U_L^- = T_L U_L^- U_L^+.$$

Proof. $T_L U_L^+ U_L^-$ is a compact subset of G. Hence it is closed. Therefore we have

$$\bigcap_{M=1}^{\infty} T_L U_L^+ U_L^- K_M = T_L U_L^+ U_L^- . \tag{3.8}$$

We can easily verify the relation

$$x_\alpha(t)x_{-\alpha}(u) = \check{\alpha}(1+tu)x_{-\alpha}(u(1+tu))x_\alpha(t/(1+tu)), \tag{3.9}$$

where $\alpha \in \Sigma$, t, $u \in k$, $1 + tu \neq 0$. If α, $\beta \in \Sigma$ and $\alpha + \beta \neq 0$, we have the basic relation (cf. Steinberg [21], p. 30)

$$x_\alpha(t)x_\beta(u)x_\alpha(t)^{-1}x_\beta(u)^{-1} = \prod_{i,j\in \mathbf{N}} x_{i\alpha+j\beta}(c_{ij}t^i u^j), \qquad t, u \in k, \tag{3.10}$$

where the product is taken over i, j such that $i\alpha + j\beta \in \Sigma$ in some fixed order of roots (say increasing), and $c_{ij} \in \mathbf{Z}$ does not depend on t, u.

By (3.5), (3.9) and (3.10), we see easily that $K_{L'}$ is a normal subgroup of K_L if $L' \geq L$. Let M be a positive integer. By (3.10), we get

$$x_\alpha(t)x_\beta(u) \equiv x_\beta(u)x_\alpha(t) \qquad \mathrm{mod}\, K_{L+M}$$

if $\alpha + \beta \neq 0$, $t \in \varpi^L \mathfrak{O}$, $u \in \varpi^M \mathfrak{O}$. Taking account of (3.9), we obtain

$$U_L^+ U_M^- \subseteq U_M^- U_L^+ K_{L+M}, \tag{3.11}$$

$$U_L^- U_M^+ \subseteq U_M^+ U_L^- K_{L+M}. \tag{3.12}$$

By repeated application of (3.12), we obtain

$$K_M \subseteq T_M U_M^+ U_M^- K_{2M}. \tag{3.13}$$

We shall show

$$K_L \subseteq T_L U_L^+ U_L^- K_{iL}, \qquad i \geq 2 \tag{3.14}$$

by induction on i. We have

$$\begin{aligned} T_L U_L^+ U_L^- K_{iL} &\subseteq T_L U_L^+ U_L^- T_{iL} U_{iL}^+ U_{iL}^- K_{2iL} && \text{by (3.13)} \\ = T_L U_L^+ U_L^- U_{iL}^+ U_{iL}^- K_{2iL} &\subseteq T_L U_L^+ U_{iL}^+ U_L^- K_{(i+1)L} U_{iL}^- K_{2iL} && \text{by (3.12)} \\ &= T_L U_L^+ U_L^- K_{(i+1)L}. \end{aligned}$$

Hence we get (3.14). By (3.8), we have $K_L \subseteq T_L U_L^+ U_L^-$ and by definition, we obtain $K_L = T_L U_L^+ U_L^-$. Taking inverse, we get $K_L = T_L U_L^- U_L^+$. This completes the proof.

Let χ be a quasi-character of T. We call χ *unramified* if χ is trivial on $T \cap K = \langle \beta(t) | \beta \in X_*(\mathbf{T}),\ t \in \mathfrak{O}^\times \rangle$. Let X be the group of all unramified quasicharacters of T. We can extend $\langle\ ,\ \rangle$ to the pairing between $V \otimes_{\mathbf{R}} \mathbf{C}$ and $V_* \otimes_{\mathbf{R}} \mathbf{C}$. For $x \in V \oplus \sqrt{-1}\, V = V \otimes_{\mathbf{R}} \mathbf{C}$, we can define $\chi_x \in X$ by

$$\chi_x(\beta(\varpi)) = \exp(2\pi \langle x, \beta \rangle), \qquad \beta \in X_*(\mathbf{T}), \tag{3.15}$$

since T is generated over $T \cap K$ by $\beta(\varpi)$, $\beta \in X_*(\mathbf{T})$. From

$$X(\mathbf{T}) = \{x \in V \mid \langle x, \beta \rangle \in \mathbf{Z} \quad \text{for all } \beta \in X_*(\mathbf{T})\},$$

we obtain

$$X \cong V \oplus \sqrt{-1}\,(V/X(\mathbf{T})). \tag{3.16}$$

By (3.16), we can endow X the structure of complex Lie group inherited from $V \otimes_{\mathbf{R}} \mathbf{C}$. The Weyl group W acts on X on the left by

$$(w\chi(t) = \chi(w^{-1}tw), \qquad t \in T. \tag{3.17}$$

Let $\chi \in X$. We have

$$(\sigma_\alpha \chi)(\beta(t)) = \chi(\check{\alpha}(t))^{-\langle \alpha, \beta \rangle} \chi(\beta(t)) \tag{3.18}$$

for $\alpha \in \Sigma$, $\beta \in X_*(\mathbf{T})$, $t \in k^\times$, since $\sigma_\alpha(\beta) = \beta - \langle \alpha, \beta \rangle \check{\alpha}$. Following S.Kato [13], we set

$$W_\chi = \{w \in W \mid w\chi = \chi\}, \; W_{(\chi)} = \langle \sigma_\alpha \mid \alpha \in \Sigma, \; \chi(a_\alpha) = 1 \rangle.$$

By (3.18), we see that $W_{(\chi)}$ is a normal subgroup of W_χ.

Lemma 3.2. *Let* $\mathbf{G}$ *be of adjoint type and* $\chi \in X$. *Then we have* $W_\chi = W_{(\chi)}$.

Proof. For $y_0, z_0 \in V$, we set

$$W(y_0, z_0) = \{w \in W \mid wy_0 = y_0, \; wz_0 - z_0 \in Q(\Sigma)\}.$$

We have $X(\mathbf{T}) = Q(\Sigma)$, $X_*(\mathbf{T}) = P(\check{\Sigma})$ since $\mathbf{G}$ is of adjoint type. We take $x \in V \oplus \sqrt{-1}\,V$ so that

$$\chi_x(\beta(\varpi)) = \exp(2\pi \langle x, \beta \rangle) \qquad \text{for every } \beta \in P(\check{\Sigma}).$$

By (3.16), we have

$$W_\chi = \{w \in W \mid wx - x \in \sqrt{-1}\,Q(\Sigma)\}.$$

Put $x = y + \sqrt{-1}\,z$ with $y, z \in V$. Then we have $W_\chi = W(y, z)$. First we shall show that $W(y, z)$ is generated by reflexions $\sigma_\alpha \in W(y, z)$, $\alpha \in \Sigma$. This assertion for the case $y = 0$ is given in Bourbaki [7], p. 227 as exercise 1) and can be proved easily in the way suggested there. Obviously we have $W(y, z) \subseteq W(0, z)$. Put

$$\Sigma^* = \{\alpha \in \Sigma \mid \sigma_\alpha \in W(0, z)\}.$$

Let $\mathfrak{H}$ be the family of hyperplanes in V defined by $\alpha \in \Sigma^*$. If $w \in W(0, z)$, $\alpha \in \Sigma^*$, then we have $w(\alpha) \in \Sigma^*$ since $w\sigma_\alpha w^{-1} = \sigma_{w(\alpha)} \in W(0, z)$. Therefore $w(H) \in \mathfrak{H}$ if $w \in W(0, z)$, $H \in \mathfrak{H}$. Thus the condition (D1) in [7], p. 72 is satisfied and (D'2) is satisfied obviously. Let $w \in W(y, z)$. Considering a chamber C with respect to $\mathfrak{H}$ such that $y \in \bar{C}$, we can apply the assertion (I) of [7], p. 75. We see that there exist $\alpha_1, \ldots, \alpha_n \in \Sigma^*$ such that

$$w = \sigma_{\alpha_1} \cdots \sigma_{\alpha_n}, \ \sigma_{\alpha_i} y = y \qquad \text{for } 1 \le i \le n.$$

This proves that $W(y, z)$ is generated by $\sigma_\alpha \in W(y, z)$, $\alpha \in \Sigma$.

Now let $\sigma_\alpha \in W(y, z)$, $\alpha \in \Sigma$. By (3.18), we obtain

$$\chi(\check{\alpha}(t))^{\langle \alpha, \beta \rangle} = 1 \qquad \text{for every } \beta \in P(\check{\Sigma}), \ t \in k^\times.$$

The group generated by $\langle \alpha, \beta \rangle$, $\beta \in P(\check{\Sigma})$ coincides with $\mathbf{Z}$ for fixed α. Therefore we obtain $\chi(a_\alpha) = 1$, i.e., $\sigma_\alpha \in W_{(\chi)}$. This completes the proof.

Lemma 3.3. *Assume* $\mathbf{G}$ *is of adjoint type and let* $\chi \in X$. *Assume* $w\chi = \bar{\chi}^{-1}$ *for some* $w \in W$. *Then there exists a* $w_1 \in W$ *such that* $w_1^2 = 1$, $w_1 \chi = \bar{\chi}^{-1}$.

Proof. Take $x \in V \oplus \sqrt{-1}\, V$ such that

$$\chi(\beta(\varpi)) = \exp(2\pi \langle x, \beta \rangle) \qquad \text{for every } \beta \in P(\check{\Sigma}).$$

Then we have

$$\bar{\chi}^{-1}(\beta(\varpi)) = \exp(2\pi \langle -\bar{x}, \beta \rangle) \qquad \text{for every } \beta \in P(\check{\Sigma}).$$

Put $x = y + \sqrt{-1}\, z$ with $y, z \in V$ and set

$$W'(y, z) = \{ w' \in W \mid w'y = -y, \ w'x - z \in Q(\Sigma) \}.$$

Then, for $w' \in W$, $w'\chi = \bar{\chi}^{-1}$ if and only if $w' \in W'(y, z)$.

Now let $W(0, z)$, Σ^* be the same as in the proof of Lemma 3.2. Note that $W(0, z)$ is generated by the reflexions obtained from Σ^*. If $y = 0$, we have $W'(0, z) = W(0, z)$ and the assertion is obvious. Assume $y \neq 0$ and let V' be the one dimensional subspace of V spanned by y. The restriction of w to V' is of order 2. We apply [7], p. 128, exercise 4). It follows that there exists $w_1 \in W(0, z)$ such that $w_1^2 = 1$, w_1 leaves V' stable and that $w_1 | V' = w | V'$. Then we have $w_1 \in W'(y, z)$. Hence the assertion follows.

Let $w \in W$. If $\chi \in X$ and $w\chi = \bar{\chi}^{-1}$, we call χ *w-hermitian.* Let X_w denote the group of all unramified w-hermitian quasi-characters of T.

Lemma 3.4. *Let* $w \in W$, $w^2 = 1$. *Then* X_w *is a real analytic Lie subgroup of dimension* ℓ *of the complex Lie group* X. *For every* $w' \in W$, $w' \neq 1$, $\{\chi \in X_w \mid w'\chi = \chi\}$ *is a proper submanifold of* X_w. *For every* $\beta \in X_*(\mathbf{T})$, $\beta \neq 0$, $\{\chi \in X_w \mid \chi(\beta(\varpi)) = 1\}$ *is a proper submanifold of* X_w.

Proof. For $x \in V \oplus \sqrt{-1}\, V$, define $\chi_x \in X$ by (3.15). Then we see immediately that

$$X_w \cong \{ x = y + \sqrt{-1}\, z \mid y, z \in V, \ wy = -y, \ wz - z \in X(\mathbf{T}) \} / \sqrt{-1}\, X(\mathbf{T}) \tag{3.19}$$

under the isomorphism (3.16). Put

$$V^+ = \{ v \in V \mid (w-1)v = 0 \}, \ V^- = \{ v \in V \mid (w+1)v = 0 \}.$$

Since $w^2 = 1$, we have

(3.20) $$V = V^+ \oplus V^-.$$

Let R denote a complete set of representatives of $(X(\mathbf{T}) \cap (w-1)V)/(w-1)X(\mathbf{T})$. Since $X(\mathbf{T})$ is a lattice in V, R is a finite set. For each $a \in R$, take $z_a \in V$ so that $a = (w-1)z_a$. Then if $wz - z = a$, we have $(w-1)(z - z_a) = 0$, i.e., $z \in z_a + V^+$. If

$$(z_a + z_1) - (z_b + z_2) \in X(\mathbf{T}), \quad z_1, z_2 \in V^+, \quad a, b \in R,$$

we get $(w-1)(z_a - z_b) = a - b \in (w-1)X(\mathbf{T})$, i.e., $a = b$. Hence we have

$$\{z \in V \mid wz - z \in X(\mathbf{T})\} = \textstyle\bigcup_{a \in R}(z_a + V^+).$$

Therefore, by (3.19), we obtain

(3.21) $$X_w \cong V^- \oplus \textstyle\bigcup_{a \in R}\sqrt{-1}\,(z_a + V^+/(X(\mathbf{T}) \cap V^+)).$$

In view of (3.20), this proves the first assertion.

Let $w' \in W$. We have

$$\{\chi \in X_w \mid w'\chi = \chi\} \cong$$
$$\{y \in V^- \mid w'y = y\} \oplus \textstyle\bigcup_{a \in R}\sqrt{-1}\,(\{z_a + z \mid z \in V^+,\ (w'-1)(z_a + z) \in X(\mathbf{T})\}/(X(\mathbf{T}) \cap V^+)).$$

We may assume

(3.22) $$w'y = y \qquad \text{for all } y \in V^-.$$

Since $V^+ \to V^+/(X(\mathbf{T}) \cap V^+)$ is a local homeomorphism, it suffices to show that

$$V_1 = \{z \in V^+ \mid (w'-1)(z_a + z) \in X(\mathbf{T})\}$$

is a proper submanifold of V^+ for every $a \in R$. If $(w'-1)(z_a + z_1) \in X(\mathbf{T})$ for some $z_1 \in V^+$, we have

$$V_1 = z_1 + \{z \in V^+ \mid (w'-1)z \in X(\mathbf{T})\}.$$

Clearly this defines a proper submanifold of V^+ except for the case

(3.23) $$w'z = z \qquad \text{for all } z \in V^+.$$

If (3.22) and (3.23) are satisfied, we get $w' = 1$ by (3.20). This proves the second assertion.

Let $\beta \in X_*(\mathbf{T})$, $\beta \neq 0$. By (3.21), we have

$$\{\chi \in X_w \mid \chi(\beta(\varpi)) = 1\} \cong$$
$$\{y \in V^- \mid \langle y, \beta\rangle = 0\} \oplus \textstyle\bigcup_{a \in R}\sqrt{-1}\,(\{z_a + z \mid z \in V^+,\ \langle z_a + z, \beta\rangle \in \mathbf{Z}\}/(X(\mathbf{T}) \cap V^+)).$$

In the similar way as above, we see that this defines a proper submanifold of X_w. This completes the proof.

For a subset J of Δ, let $\Sigma_J = \mathbf{Z}\cdot J \cap \Sigma$ be the root system generated by J and let W_J be the Coxeter group generated by the reflexions σ_α, $\alpha \in J$. Let w_J

be the longest element of W_J. The following Lemma slightly sharpens the result given in [7], p. 225.

Lemma 3.5. *Let $w \in W$, $w^2 = 1$. Let w_1 be an element of minimal length in the conjugacy class of w. Then there exists a subset J of Δ such that $w_1 = w_J$. Furthermore we have $w_J(\alpha) = -\alpha$ for every $\alpha \in J$.*

Proof. Take any $\alpha \in \Delta$. Assume $w_1\alpha < 0$. Then we have $l(\sigma_\alpha w_1) = l(w_1) - 1$ (cf. [21], p. 269) since $w_1^2 = 1$. From the minimality of $l(w_1)$, we have $l(\sigma_\alpha w_1 \sigma_\alpha) \geq l(w_1)$. Hence we must have $l(\sigma_\alpha w_1 \sigma_\alpha) = l(\sigma_\alpha w_1) + 1$. Therefore we get $(\sigma_\alpha w_1)(\alpha) > 0$. Since $w_1\alpha < 0$, $\sigma_\alpha(w_1\alpha) > 0$, we must have $w_1\alpha = -\alpha$. Thus we have shown:

(3.24) $$\text{For every } \alpha \in \Delta, \qquad w_1\alpha > 0 \quad \text{or} \quad w_1\alpha = -\alpha.$$

In particular, we have $w_1(\Delta) \subseteq \Sigma^+ \cup (-\Delta)$. By [7], p. 225, exercise 17), a), we have $w_1 = w_J$ for some $J \subseteq \Delta$. Since $w_J\alpha < 0$ for $\alpha \in J$, we have $w_J\alpha = -\alpha$ for every $\alpha \in J$ by (3.24). This completes the proof.

§ 4. Intertwining operators

Let $\chi \in X$. We denote by $PS(\chi)$ the space of all locally constant functions φ on G which satisfy

(4.1) $$\varphi(tng) = \delta_B(t)^{1/2}\chi(t)\varphi(g) \qquad \text{for all } t \in T,\ n \in N,\ g \in G.$$

Let $\varphi_{K,\chi} \in PS(\chi)$ denote the function which takes constant value 1 on K. Let $\pi(\chi)$ denote the admissible representation of G realized on $PS(\chi)$ by right translations. $\pi(\chi)$ is of finite length and has a unique K-spherical constituent which we denote by π_χ^1. It is well known (cf. Kato [13]) that $\pi(\chi)$ is irreducible if and only if

(4.2) $$\chi(a_\alpha) \neq q \qquad \text{for every } \alpha \in \Sigma,$$

(4.3) $$W_\chi = W_{(\chi)}$$

are satisfied. $PS(\chi)$ is generated by $\varphi_{K,\chi}$ if and only if (4.3) and

(4.4) $$\chi(a_\alpha) \neq q \qquad \text{for every } \alpha \in \Sigma^+$$

are satisfied. Let X^i denote the set of all $\chi \in X$ such that $\pi(\chi)$ is irreducible. We call $\chi \in X$ *regular* if $w\chi \neq \chi$ for every $w \in W$, $w \neq 1$. Let X^r denote the set of all $\chi \in X$ which are regular.

Let S denote the space of all locally constant functions on K which are left $B \cap K$-invariant. By $G = BK$, it is clear that the restriction map

$$R(\chi)\colon PS(\chi) \ni \varphi \longrightarrow \varphi | K \in S$$

defines an isomorphism of vector spaces. For an open subgroup U of K, we have

(4.5) $$R(\chi)(PS(\chi)^U) = S^U$$

where $PS(\chi)^U$ (resp. S^U) denotes the space of all vectors fixed under U in $PS(\chi)$ (resp. S^U). For $w \in W$, set

$$\textstyle\sum_w^+ = \{\alpha \in \sum^+ \mid w^{-1}\alpha < 0\},\ \sum_w^- = \{\alpha \in \sum^+ \mid w^{-1}\alpha > 0\},\ \Psi_w^+ = \{\alpha \in \sum^+ \mid w\alpha < 0\},$$
$$\textstyle N_w = \langle x_\alpha(t) \mid \alpha \in \sum_w^+,\ t \in k\rangle,\ N_w^- = \langle x_\alpha(t) \mid \alpha \in \sum_w^-,\ t \in k\rangle.$$

It is obvious that

$$\Psi_{w_1 w_2}^+ \subseteq w_2^{-1}\Psi_{w_1}^+ \cup \Psi_{w_2}^+, \qquad w_1, w_2 \in W. \tag{4.6}$$

If $l(w_1 w_2) = l(w_1) + l(w_2)$, then

$$\Psi_{w_1 w_2}^+ = w_2^{-1}\Psi_{w_1}^+ \cup \Psi_{w_2}^+ \qquad \text{(disjoint union)}. \tag{4.7}$$

(cf. [7], p. 158, Cor. 2.) Since Σ_w^+ and Σ_w^- are closed sets of roots, N_w and N_w^- are subgroups of N. It is well known that

$$N = N_w^- N_w, \qquad N_w^- \cap N_w = \{1\}. \tag{4.8}$$

For each $w \in W$, we choose $x_w \in K \cap N_G(T)$ which represents w. We consider an intertwining operator

$$(T_w(\chi)\varphi)(g) = \int_{wNw^{-1} \cap N \backslash N} \varphi(x_w^{-1} ng)\, dn, \qquad \varphi \in PS(\chi),\ g \in G, \tag{4.9}$$

with the invariant measure normalized so that $\mathrm{vol}(wNw^{-1} \cap N \backslash (wNw^{-1} \cap N) U_0^+) = 1$. This definition does not depend on the choice of x_w. It is well known that the integral (4.9) converges absolutely when χ satisfies certain conditions (see below), and can be meromorphically continued to the whole X (cf. Casselman [10], Shahidi [18]). For later use, let us study this integral more closely.

By (4.8), we have

$$(T_w(\chi)\varphi)(g) = \int_{N_w} \varphi(x_w^{-1} ng)\, dn \tag{4.10}$$

when the integral converges absolutely. Let $\alpha \in \Sigma$ and put $\omega_\alpha = x_\alpha(1) x_{-\alpha}(-1) x_\alpha(1)$. Then we have a relation

$$\omega_\alpha^{-1} x_\alpha(t) = x_\alpha(-t^{-1}) \check{\alpha}(t^{-1}) x_{-\alpha}(t^{-1}), \qquad t \in k^\times. \tag{4.11}$$

(cf. Steinberg [21].) Now assume $w = \sigma_\alpha$, $\alpha \in \Delta$. We can take $x_w = \omega_\alpha$. Then we have

$$(T_{\sigma_\alpha}(\chi)\varphi)(g) = \int_{N_w} \varphi(\omega_\alpha^{-1} ng)\, dn = \int_k \varphi(\omega_\alpha^{-1} x_\alpha(u) g)\, du, \tag{4.12}$$

where du is normalized so that $\mathrm{vol}(\mathfrak{O}) = 1$. By (3.7), we have

$$\varphi(\check{\alpha}(t) g) = q^{-n} \chi(a_\alpha)^n \varphi(g), \qquad t \in \varpi^n \mathfrak{O}^\times,\ g \in G. \tag{4.13}$$

There is a positive integer L such that $\varphi | K$ is left invariant under K_L. Let

$g \in K$. By (4.11) and (4.13), we get, for $n > 0$,

$$\int_{\varpi^{-n}\mathfrak{O}^\times} \varphi(\omega_\alpha^{-1} x_\alpha(u) g)\, du = q^{-n}\chi(a_\alpha)^n \int_{\varpi^{-n}\mathfrak{O}^\times} \varphi(x_{-\alpha}(u^{-1})g)\, du.$$

Therefore

$$\sum_{n=L}^{\infty} \int_{\varpi^{-n}\mathfrak{O}^\times} \varphi(\omega_\alpha^{-1} x_\alpha(u) g)\, du = \sum_{n=L}^{\infty} q^{-n}\chi(a_\alpha)^n \operatorname{vol}(\varpi^{-n}\mathfrak{O}^\times)\varphi(g) = \frac{q-1}{q}\frac{\chi(a_\alpha)^L}{1-\chi(a_\alpha)}\varphi(g),$$

the sum being absolutely convergent if $|\chi(a_\alpha)| < 1$. Hence we obtain

$$\begin{aligned}(T_{\sigma_\alpha}(\chi)\varphi)(g) &= \frac{1}{q^L}\sum_{u \in \mathfrak{O} \bmod \varpi^L} \varphi(\omega_\alpha^{-1} x_\alpha(u) g) \\ &+ \frac{1}{q^L}\sum_{n=1}^{L-1} q^n \chi(a_\alpha)^n \sum_{u \in \varpi^n \mathfrak{O}^\times \bmod \varpi^L} \varphi(x_{-\alpha}(u) g) + \frac{q-1}{q}\frac{\chi(a_\alpha)^L}{1-\chi(a_\alpha)}\varphi(g).\end{aligned} \tag{4.14}$$

Thus we have shown that the integral (4.12) converges absolutely if $|\chi(a_\alpha)| < 1$ (The case $g \notin K$ can be easily reduced to the case $g \in K$); T_{σ_α} has meromorphic continuation to X and holomorphic at χ if $\chi(a_\alpha) \neq 1$. Here we understand "meromorphic" in the following sense. Put $T'_w(\chi) = R(\chi)T_w(\chi)R(\chi)^{-1}$, which is an operator in $\operatorname{End}(S)$; T_w is meromorphic (resp. holomorphic) at χ if $(T'_w(\chi)f)(k)$ is a complex valued meromorphic (resp. holomorphic) function at $\chi \in X$ for every fixed $f \in S$, $k \in K$.

It is well known that

$$T_{w_1 w_2}(\chi) = T_{w_1}(w_2\chi)\, T_{w_2}(\chi) \tag{4.15}$$

if $w_1, w_2 \in W$, $l(w_1 w_2) = l(w_1) l(w_2)$ and χ is regular (cf. Casselman [10]). By analytic continuation, (4.7) and by the above result, we see easily that T_w is holomorphic at χ if $\chi(a_\alpha) \neq 1$ for every $\alpha \in \Psi_w^+$ and then it gives an intertwining operator from $PS(\chi)$ to $PS(w\chi)$. Furthermore we see that the integral (4.9) converges absolutely if $|\chi(a_\alpha)| < 1$ for every $\alpha \in \Psi_w^+$. The relation (4.15) holds if $\chi(a_\alpha) \neq 1$ for every $\alpha \in \Psi_{w_1 w_2}^+$, $l(w_1 w_2) = l(w_1) + l(w_2)$. We put

$$c_\alpha(\chi) = \frac{1 - q^{-1}\chi(a_\alpha)}{1 - \chi(a_\alpha)}, \qquad \alpha \in \Sigma, \tag{4.16}$$

$$c_w(\chi) = \prod_{\alpha \in \Psi_w^+} c_\alpha(\chi), \qquad w \in W. \tag{4.17}$$

Since

$$1 + \frac{q-1}{q}\frac{\chi(a_\alpha)}{1-\chi(a_\alpha)} = c_\alpha(\chi),$$

we get

$$T_{\sigma_\alpha}(\chi)\varphi_{K,\chi} = c_\alpha(\chi)\varphi_{K,\sigma_\alpha\chi}, \qquad \alpha \in \Delta$$

by (4.14). By (4.7) and (4.15), we obtain

$$T_w(\chi)\varphi_{K,\chi} = c_w(\chi)\varphi_{K,w\chi}, \qquad w \in W \tag{4.18}$$

if $\chi(a_\alpha) \neq 1$ for every $\alpha \in \Psi_w^+$. Thus we recover a result of Casselman ([9], Theorem 3.1) in the case of Chevalley groups.

Lemma 4.1. *Let $\alpha \in \Delta$, $f \in S$, $T'_{\sigma_\alpha}(\chi) = R(\chi)T_{\sigma_\alpha}(\chi)R(\chi)^{-1} \in \mathrm{End}(S)$. Then we have*

$$c_\alpha(\chi)^{-1}(T'_{\sigma_\alpha}(\chi)f)(k) = f(k) + O(|1 - \chi(a_\alpha)|), \qquad k \in K,$$

for $\chi(a_\alpha) \to 1$, where O-term is uniform for $k \in K$.

This Lemma is an obvious consequence of (4.14).

Lemma 4.2. *Let $w \in W$, $w^2 = 1$, $\chi \in X$. If $w\chi = \bar{\chi}^{-1}$ and if $\chi(a_\alpha) \neq 1$ for every $\alpha \in \Psi_w^+$, then we have $c_w(\chi) \in \mathbf{R}$.*

Proof. Since $w^2 = 1$, we have $\Psi_w^+ = -w\Psi_w^+$. By definition (4.17), we obtain

$$\begin{aligned}\overline{c_w(\chi)} &= \prod_{\alpha \in \Psi_w^+} c_\alpha(\bar{\chi}) = \prod_{\alpha \in \Psi_w^+} c_\alpha((w\chi)^{-1}) \\ &= \prod_{\alpha \in -\Psi_w^+} c_\alpha(w\chi) = \prod_{\alpha \in -w\Psi_w^+} c_\alpha(\chi) = c_w(\chi).\end{aligned}$$

Hence the assertion follows.

Lemma 4.3. *For $w = w_1 w_2$, $w, w_1, w_2 \in W$, we have*

$$c_w(\chi)^{-1} T_w(\chi) = c_{w_1}(w_2\chi)^{-1} T_{w_1}(\chi) c_{w_2}(\chi)^{-1} T_{w_2}(\chi) \tag{4.19}$$

as meromorphic functions on X.

Proof. By the principle of analytic continuation, it suffices to prove (4.19) when χ is regular and $\chi(a_\alpha) \neq q, q^{-1}$ for every $\alpha \in \Sigma$. Now both sides of (4.19) give intertwining operators from $PS(\chi)$ to $PS(w\chi)$. Since $PS(\chi)$ and $PS(w\chi)$ are irreducible, they are different only by a scalar factor. By (4.18), we find that this scalar factor is 1. This completes the proof.

Lemma 4.4. *Let $w \in W$ and $w = \sigma_1\sigma_2 \cdots \sigma_n$ be a reduced expression of w. Put $\theta_i = (\sigma_n\sigma_{n-1} \cdots \sigma_{i+1})(\alpha_i)$, where $1 \leq i \leq n-1$, $\sigma_i = \sigma_{\alpha_i}$, $\alpha_i \in \Delta$, and $\theta_n = \alpha_n$. Then we have $\Psi_w^+ = \{\theta_1, \theta_2, \ldots, \theta_n\}$. Let $\chi \in X$. If $\chi(a_{\theta_i}) = 1$, we have*

$$w\chi = (\sigma_1 \cdots \sigma_{i-1}\sigma_{i+1} \cdots \sigma_n)\chi.$$

Proof. The first assertion follows from (4.7) (cf. [7], p. 158, Cor. 2). If $\chi(a_{\theta_i}) = 1$, we have $\sigma_{\theta_i}\chi = \chi$ by (3.18). Since

$$\sigma_{\theta_i} = (\sigma_n \cdots \sigma_{i+1})\sigma_i(\sigma_n \cdots \sigma_{i+1})^{-1},$$

we get

$$(\sigma_n\cdots\sigma_{i+1})^{-1}\sigma_{\theta_i}\chi=(\sigma_{i+1}\cdots\sigma_n)\chi=(\sigma_i\sigma_{i+1}\cdots\sigma_n)\chi.$$

Hence the assertion follows.

Lemma 4.5. *Let $\chi\in X$ and $w\in W$. Set $A=\{w_1\in W\mid w_1\chi=w\chi\}$. If w' is an element of minimal length in A, then $T_{w'}$ is holomorphic at χ.*

Proof. We may assume $w'\neq 1$. Let $w'=\sigma_1\sigma_2\cdots\sigma_n$ be a reduced expression of w' and define $\theta_i\in\Psi_{w'}$ as in Lemma 4.4. Assume that $T_{w'}$ is not holomorphic at χ. Then we have $\chi(a_{\theta_i})=1$ for some θ_i and we get

$$(\sigma_1\cdots\sigma_{i-1}\sigma_{i+1}\cdots\sigma_n)\chi=w'\chi=w\chi.$$

This contradicts the minimality of $l(w')$ and completes the proof.

Lemma 4.6. *Let $\chi\in X$ and $V_1,\ldots,V_n$ be G-submodules of $PS(\chi)$. We assume*

$$V_i\neq\{0\},\ 1\leq i\leq n,\ (\sum_{i\neq j}V_i)\cap V_j=\{0\},\ 1\leq j\leq n.$$

Then we have $n\leq|W_\chi|$.

Proof. Let $V\neq\{0\}$ be a G-submodule of $PS(\chi)$. By the Frobenius reciprocity, we have

$$\mathrm{Hom}_G(V,\ PS(\chi))=\mathrm{Hom}_T(V_N,\ \mathbf{C}_{\delta_B^{1/2}\chi}),$$

where V_N denotes the Jacquet module of V (cf. Cartier [8], Theorem 3.4, Borel-Wallach [6], p. 304). Hence a T-module isomorphic to $\mathbf{C}_{\delta_B^{1/2}\chi}$ occurs in V_N. By the assumption, we have

$$PS(\chi)_N\supseteq(V_1)_N\oplus\cdots\oplus(V_n)_N.$$

Hence a T-submodule isomorphic to $\mathbf{C}_{\delta_B^{1/2}\chi}$ occurs at least with multiplicity n in $PS(\chi)_N$. On the other hand, the semi-simplification of $PS(\chi)_N$ as T-module is isomorphic to $\bigoplus_{w\in W}\mathbf{C}_{\delta_B^{1/2}w\chi}$. Therefore $\mathbf{C}_{\delta_B^{1/2}\chi}$ must appear at least n-times in $\bigoplus_{w\in W}\mathbf{C}_{\delta_B^{1/2}w\chi}$. Hence we obtain $n\leq|W_\chi|$. This completes the proof.

§5. Unitarizability and positivity of distributions

Let G be a t.d. group. Let π be an irreducible admissible representation of G on a vector space V over $\mathbf{C}$. If there exists a non-degenerate hermitian form $(\ ,\ )$ on V which is invariant, i.e.,

$$(5.1)\qquad (\pi(g)u,\ \pi(g)v)=(u,\ v)\qquad\text{for every } g\in G,\ u,\ v\in V,$$

we call π *hermitian.* Let $\bar{\pi}$ be the complex conjugate representation of π on the vector space $\bar{V}$ and $\tilde{\bar{\pi}}$ be the contragredient of $\bar{\pi}$ realized on $\tilde{\bar{V}}$. Let $\langle\ ,\ \rangle$ be the canonical pairing between $\tilde{\bar{V}}$ and $\bar{V}$. For any $u\in V$, there exists unique $u_1\in\tilde{\bar{V}}$ such that

(5.2) $$(u, v) = \langle u_1, \bar{v} \rangle \qquad \text{for every } v \in V.$$

Put

(5.3) $$I(u) = u_1, \qquad u \in V.$$

Then we can verify easily that I defines an equivalence of π and $\tilde{\bar{\pi}}$. By Schur's Lemma, I is unique up to a scalar multiple. Conversely, if $\pi \cong \tilde{\bar{\pi}}$, then we can obtain a non-degenerate invariant hermitian form on V by (5.2) and (5.3) when we choose an isomorphism I from π to $\tilde{\bar{\pi}}$ suitably; any non-degenerate invariant hermitian form is of this form. Thus we see that π is hermitian if and only if $\pi \cong \tilde{\bar{\pi}}$. If there exists an invariant hermitian form on V which is positive definite, we call π *unitarizable*.

Now we go back to the case where G is the group of k-rational points of a Chevalley group. Let $\chi \in X$. We have

(5.4) $$\widetilde{\pi(\chi)} \cong \pi(\chi^{-1}), \qquad \overline{\pi(\chi)} \cong \pi(\bar{\chi}).$$

The pairing between $\pi(\chi)$ and $\pi(\chi^{-1})$ is given by (cf. Casselman [10], 3.1.2)

(5.5) $$\langle \varphi_1, \varphi_2 \rangle = \int_{B\backslash G} \varphi_1(g)\varphi_2(g)dg, \quad \varphi_1 \in PS(\chi),\ \varphi_2 \in PS(\chi^{-1}).$$

Assume π_χ^1 is hermitian. Then we have $\pi_\chi^1 \cong \tilde{\bar{\pi}}_\chi^1 \cong \pi_{\bar{\chi}^{-1}}^1$. Hence there must exists a $w \in W$ such that $w\chi = \bar{\chi}^{-1}$ (cf. Cartier [8]). Lemma 4.5 guarantees that we can choose w so that T_w is holomorphic at χ.

First we assume that $\chi \in X$ is regular and that $\pi(\chi)$ is irreducible hermitian. We have $w\chi = \bar{\chi}^{-1}$ with a unique $w \in W$. Since $w^2\chi = \chi$, we get $w^2 = 1$. By the discussion above, we see that

(5.6) $$(\varphi_1, \varphi_2) = c\int_{B\backslash G} (T_w(\chi)\varphi_1)(g)\,\overline{\varphi_2(g)}\,dg, \qquad \varphi_1, \varphi_2 \in PS(\chi),$$

defines an invariant hermitian form on $PS(\chi)$, where c is a non-zero constant. By $G = BK$, (5.6) equals (cf. [10], 3.1.3)

(5.7) $$(\varphi_1, \varphi_2) = c\int_K (T_w(\chi)\varphi_1)(g)\,\overline{\varphi_2(g)}\,dg, \qquad \varphi_1, \varphi_2 \in PS(\chi).$$

By (4.18), we may take $c = c_w(\chi)^{-1}$. By Lemma 4.2, we may take $c = \pm 1$ and we see that (5.6) and (5.7) are positive definite with $c = \pm 1$ if and only if $\pi(\chi)$ is unitarizable. Here $c = \pm 1$ has the same signature as $c_w(\chi)$.

Lemma 5.1. *Let $w \in W$, $w^2 = 1$. Let $\chi \in X_w$ and assume that T_w is holomorphic at χ. Then*

$$(\varphi_1, \varphi_2) = \int_{B\backslash G} (T_w(\chi)\varphi_1)(g)\,\overline{\varphi_2(g)}\,dg, \qquad \varphi_1, \varphi_2 \in PS(\chi)$$

defines an invariant hermitian form on $PS(\chi)$.

Proof. What we must show is $(\varphi_1, \varphi_2) = \overline{(\varphi_2, \varphi_1)}$ or equivalently

$$\int_K (T'_w(\chi)f_1)(k)\overline{f_2(k)}\,dk = \int_K (T'_w(\chi)f_2)(k)\overline{f_1(k)}\,dk, \qquad f_1, f_2 \in S, \tag{5.8}$$

where $T'_w(\chi) = R(\chi)T_w(\chi)R(\chi)^{-1}$ as before. In view of the proof of Lemma 3.4, we can find a sequence $\chi_n \in X_w$ such that χ_n converges to χ, χ_n is regular and that $\pi(\chi_n)$ is irreducible. Then

$$\int_K (T'_w(\chi_n)f_1)(k)\overline{f_2(k)}\,dk, \qquad f_1, f_2 \in S,$$

defines a hermitian form on S. Since $T'_w(\chi)f_i$ is locally constant on K, we see that $T'_w(\chi)f_i$ converges to $T'_w(\chi)f_i$ uniformly on K, $i = 1, 2$. Therefore (5.8) holds. This completes the proof.

Let w_0 be the longest element of W and $\omega_0 \in K \cap N_G(T)$ be an element which represents w_0.

Lemma 5.2. *Let* $PS(\delta_B^{1/2})$ *be the space of all locally constant functions on* G *which satisfy*

$$f(bg) = \delta_B(b)f(g) \qquad \textit{for every } b \in B,\ g \in G.$$

When invariant measures are suitably normalized, we have

$$\int_{B\backslash G} f(g)\,dg = \int_N f(\omega_0 n)\,dn \qquad \textit{for every } f \in PS(\delta_B^{1/2}).$$

Proof. Let $d\dot{g}$ denote a right invariant measure on $B\backslash G$ and dg be the Haar measure on G. We have

$$\int_G \varphi(g)\,dg = \int_{B\backslash G}\left(\int_B \varphi(bg)\,db\right)d\dot{g} \qquad \text{for every } \varphi \in C_c^\infty(G),$$

when a Haar measure db on B is suitably normalized. It is easy to see that on the open dense subset $B\omega_0 N$ of G, dg is given by $dg = db\,dn$, $g = b\omega_0 n$, $b \in B$, $n \in N$, when a Haar measure dn on N is suitably normalized. Hence we have

$$\int_G \varphi(g)\,dg = \int_N\left(\int_B \varphi(b\omega_0 n)\,db\right)dn \qquad \text{for every } \varphi \in C_c^\infty(G).$$

Since the map

$$C_c^\infty(G) \ni \varphi(g) \longrightarrow f(g) = \int_B \varphi(bg)\,db \in PS(\delta_B^{1/2})$$

is surjective (cf. Weil [22], p. 43; we need a slight modification), the assertion follows.

By this Lemma, we have

$$(5.9)\qquad \int_{B\backslash G} (T_w(\chi)\varphi_1)(g)\,\overline{\varphi_2(g)}\,dg = \int_N (T_w(\chi)\varphi_1)(\omega_0 n)\,\overline{\varphi_2(\omega_0 n)}\,dn.$$

Lemma 5.3. *Let $\chi \in X$. For $\varphi \in PS(\chi)$, define a locally constant function Φ on N by $\Phi(n) = \varphi(\omega_0 n)$, $n \in N$. Then the space of Φ contains $C_c^\infty(N)$ when φ extends over all functions in $PS(\chi)$.*

Proof. For $\Phi \in C_c^\infty(N)$, define a function φ on G by

$$(5.10)\qquad \begin{cases} \varphi(b\omega_0 n) = \delta_B(b)^{1/2}\chi(b)\Phi(n), & b \in B,\ n \in N, \\ \varphi(g) = 0 & \text{if } g \notin B\omega_0 N. \end{cases}$$

Clearly this is well defined and $\varphi(g)$ is locally constant at $g \in B\omega_0 N$. What we must show is that $\varphi(g)$ is locally constant at $g \notin B\omega_0 N$. Since $\mathbf{N}$ is unipotent, we can find an open compact subgroup U of N which contains the support of Φ. It is sufficient to find an open compact subgroup V of G such that

$$B\omega_0 UV \subseteq B\omega_0 U.$$

Since $B\omega_0 U$ is open in G, there exists an open compact subgroup V_1 of G such that $V_1 \subseteq \omega_0^{-1} B\omega_0 U$. Set

$$V = \bigcap_{n\in U} n^{-1} V_1 n = \bigcap_{n \in V_1 \cap U\backslash U} n^{-1} V_1 n.$$

Clearly V is an open compact subgroup of G and we have $nVn^{-1} \subseteq V_1$ for every $n \in U$. Hence $UV \subseteq V_1 U$. Then we obtain

$$B\omega_0 UV \subseteq B\omega_0 V_1 U \subseteq B\omega_0\omega_0^{-1} B\omega_0 UU = B\omega_0 U.$$

This completes the proof.

We note that if the function Φ is defined by $\Phi(n) = \varphi(\omega_0 n)$, $n \in N$ with $\varphi \in PS(\chi)$, φ is uniquely determined by Φ since $B\omega_0 N$ is dense in G. For $\Phi \in C_c^\infty(N)$, let $\iota_\chi(\Phi)$ denote the $\varphi \in PS(\chi)$ defined by (5.10). We set

$$(5.11)\qquad T_{w,\chi}(\Phi) = (T_w(\chi)(\iota_\chi(\Phi)))(\omega_0), \qquad \Phi \in C_c^\infty(N).$$

Then $T_{w,\chi}$ defines a distribution on N for $\chi \in X$, $w \in W$ whenever $\chi(a_\alpha) \neq 1$ for every $\alpha \in \Psi_w^+$.

Lemma 5.4. *Let $w \in W$, $\chi \in X$ and assume $\chi(a_\alpha) \neq 1$ for all $\alpha \in \Psi_w^+$. Let $c \in \mathbf{C}^\times$ and assume*

$$(\varphi_1, \varphi_2) = c\int_{B\backslash G} (T_w(\chi)\varphi_1)(g)\,\overline{\varphi_2(g)}\,dg, \qquad \varphi_1, \varphi_2 \in PS(\chi)$$

defines an invariant hermitian form on $PS(\chi)$. If $(\ ,\ )$ is positive semi-definite, then $cT_{w,\chi}$ is of positive type. Conversely if $cT_{w,\chi}$ is of positive type and if $|\chi(a_\alpha)| < 1$ for all $\alpha \in \Psi_w^+$, then $(\ ,\ )$ is positive semi-definite.

Proof. Let $\Phi \in C_c^\infty(N)$ and put $\varphi = \iota_\chi(\Phi)$, $T = T_{w,\chi}$. By (1.6) and (1.8), we have

$$(\check{T} * \Phi)(n) = \check{T}_t(\Phi(t^{-1}n)) = T_t(\Phi(tn)) = (T_w(\chi)\varphi)(\omega_0 n), \qquad n \in N,$$

since $T_w(\chi)$ is a G-homomorphism. By (5.9), (1.11) and (1.12), we get

$$(\varphi, \varphi) = c\int_N (\check{T} * \Phi)(n)\bar{\Phi}(n)\,dn = c(\check{T} * \Phi)(\bar{\Phi}) = c\check{T}(\bar{\Phi} * \check{\Phi}) = cT(\check{\bar{\Phi}} * \check{\Phi}).$$

Hence we have

$$(5.12) \qquad (\varphi, \varphi) = cT(\Phi * \tilde{\Phi}).$$

If (,) is positive semi-definite, we have $cT(\Phi * \tilde{\Phi}) \geq 0$ for every $\Phi \in C_c^\infty(N)$. This shows that cT is of positive type.

Conversely we assume that $cT_{w,\chi}$ is of positive type and that $|\chi(a_\alpha)| < 1$ for every $\alpha \in \Psi_w^+$. We can take an increasing sequence $\{N_i\}$ of open compact subgroups so that $N = \bigcup_{i=1}^\infty N_i$. Take any $\varphi \in PS(\chi)$. Define $\Phi_i \in C_c^\infty(N)$ by

$$\Phi_i(n) = \begin{cases} \varphi(\omega_0 n) & \text{if } n \in N_i, \\ 0 & \text{if } n \notin N_i. \end{cases}$$

It suffices to show

$$(5.13) \qquad (\varphi, \varphi) = \lim_{i \to \infty} (\iota_\chi(\Phi_i), \iota_\chi(\Phi_i))$$

since $(\iota_\chi(\Phi_i), \iota_\chi(\Phi_i)) \geq 0$ by (5.12). We have

$$(\varphi, \iota_\chi(\Phi_i)) = c\int_K (T_w(\chi)\varphi)(k)\,\overline{\iota_\chi(\Phi_i)(k)}\,dk = c\int_{K \cap B\omega_0 N} (T_w(\chi)\varphi)(k)\,\overline{\iota_\chi(\Phi_i)(k)}\,dk$$

since $K - (K \cap B\omega_0 N)$ is of measure 0. By definition of Φ_i, this integral equals

$$c\int_{K \cap B\omega_0 N_i} (T_w(\chi)\varphi)(k)\,\overline{\varphi(k)}\,dk.$$

Hence we obtain immediately that

$$\lim_{i \to \infty} (\varphi, \iota_\chi(\Phi_i)) = (\varphi, \varphi) = \lim_{i \to \infty} \overline{(\iota_\chi(\Phi_i), \varphi)}.$$

Therefore (5.13) is reduced to

$$(5.14) \qquad \lim_{i \to \infty} (\iota_\chi(\Phi_i),\ \varphi - \iota_\chi(\Phi_i)) = 0.$$

For $f \in PS(\chi)$, let $\|f\|_{L^2(K)}$ denote the L^2-norm of $f|K$. Clearly we have

$$\lim_{i \to \infty} \|\varphi - \iota_\chi(\Phi_i)\|_{L^2(K)} = 0.$$

Since

$$|(\iota_\chi(\Phi_i),\ \varphi - \iota_\chi(\Phi_i)| \le |c|\, \| T_w(\chi)\iota_\chi(\Phi_i)\|_{L^2(K)} \| \varphi - \iota_\chi(\Phi_i)\|_{L^2(K)}.$$

it suffices to show that $\| T_w(\chi)\iota_\chi(\Phi_i)\|_{L^2(K)}$ is bounded for $i \to \infty$. Take any $f \in PS(\chi)$. Let L be a positive integer such that $f|K$ is left invariant under K_L. Let $\alpha \in \Delta$. By (4.14), we find

$$\| T_{\sigma_\alpha}(\chi) f \|_{L^2(K)} \le \mu_\alpha(\chi) \| f \|_{L^2(K)}$$

with a positive constant $\mu_\alpha(\chi)$ which does not depend on f and L if $|\chi(a_\alpha)| < 1$. Therefore we have

$$\| T_w(\chi) f \|_{L^2(K)} \le \mu_w(\chi) \| f \|_{L^2(K)}$$

with a positive constant $\mu_w(\chi)$ which does not depend on f. Since $\| \iota_\chi(\Phi_i)\|_{L^2(K)} \le \| \varphi \|_{L^2(K)}$, the boundedness of $\| T_w(\chi)(\iota_\chi(\Phi_i))\|_{L^2(K)}$ follows. This completes the proof.

Further elaboration on the converse part of Lemma 5.4 shall be given in Lemma 12.1.

Lemma 5.5. *Let $\Phi \in C_c^\infty(N)$. There exists a positive integer L such that $\iota_\chi(\Phi)$ is right invariant under K_L for all $\chi \in X$.*

Proof. Let U be an open compact subgroup of N such that $U \supseteq \mathrm{supp}(\Phi)$. We can find a positive integer M so that Φ is right and left invariant under the translations of U_M^+ and that $U_M^+ \subseteq U$. Since $\bigcap_{n \in U_M \backslash U} n^{-1} K_M n$ is open in G, we can find a positive integer L so that $n K_L n^{-1} \subseteq K_M$ for every $n \in U$. In view of the proof of Lemma 5.3, we may also assume $B\omega_0 U K_L \subseteq B\omega_0 U$. Take $\chi \in X$ and put $\varphi = \iota_\chi(\Phi)$. Let $g = b\omega_0 n$ with $b \in B$, $n \in U$. Then it is sufficient to show that $\varphi(g) = \varphi(gk)$ for every $k \in K_L$. We have $gk = b\omega_0 nk = b\omega_0 k_1 n$ with some $k_1 \in K_M$. By Lemma 3.1, we have $k_1 = tu^- u^+$ with $t \in T_M$, $u^- \in U_M^-$, $u^+ \in U_M^+$. Then we get

$$gk = b(\omega_0 t \omega_0^{-1})(\omega_0 u^- \omega_0^{-1})\omega_0 u^+ n,\ \omega_0 t \omega_0^{-1} \in T \cap K,\ \omega_0 u^- \omega_0^{-1} \in U_M^+.$$

Hence we obtain

$$\varphi(gk) = \delta_B(b)^{1/2} \chi(b) \Phi(u^+ n) = \delta_B(b)^{1/2} \chi(b) \Phi(n) = \varphi(g).$$

This completes the proof.

Lemma 5.6. *Let $\Phi \in C_c^\infty(N)$. Then $T_{w,\chi}(\Phi)$ is meromorphic on X and holomorphic at $\chi \in X$ if $\chi(a_\alpha) \ne 1$ for every $\alpha \in \Psi_w^+$.*

Proof. Let L be a positive integer as in Lemma 5.5. We can take a double coset decomposition $K = \bigcup_x (B \cap K) x K_L$ so that $x \in B\omega_0 N$. Put $x = b_x \omega_0 n_x$ with $b_x \in B$, $n_x \in N$. Define $f_w \in S$ by

$$f_x = \Phi(n_x) \times \text{ the characteristic function of } (B \cap K) x K_L.$$

Let $\chi \in X$ and set $\varphi = \iota_\chi(\Phi)$. By (5.10), we have $R(\chi)(\varphi) = \sum_x \chi(b_x) f_x$. Let $T'_w(\chi) = R(\chi) T_w(\chi) R(\chi)^{-1} \in \mathrm{End}(S)$ as before. Then, for every $k \in K$, $(T'_w(\chi) f_x)(k)$ is meromorphic on X and holomorphic at χ if $\chi(a_\alpha) \neq 1$ for every $\alpha \in \Psi_w^+$. Since

$$T_{w,\chi}(\Phi) = (T_w(\chi)\varphi)(\omega_0) = T'_w(\chi)(R(\chi)\varphi))(\omega_0) = \sum_x \chi(b_x)(T'_w(\chi) f_x)(\omega_0),$$

the assertion follows.

Let ψ be the central isogeny from $\tilde{\mathbf{G}}$ to $\mathbf{G}$ as in §3. Then ψ induces an isomorphism $\tilde{N} \cong N$. We can choose ω_0 and x_w from $\psi(\tilde{K})$ for $w \in W$. Take $\chi \in X$ and set $\tilde{\chi} = \chi \circ \psi$. For $\varphi \in PS(\chi)$, put $\tilde{\varphi}(g) = \varphi(\psi(g))$, $g \in \tilde{G}$. Then we have $\tilde{\varphi} \in PS(\tilde{\chi})$. By definition of the intertwining operator, we see that

$$(T_w(\tilde{\chi})\tilde{\varphi})(g) = (T_w(\chi)\varphi)(\psi(g)), \qquad g \in \tilde{G} \tag{5.15}$$

if $|\chi(a_\alpha)| < 1$ for every $\alpha \in \Psi_w^+$. Take $\Phi \in C_c^\infty(N) = C_c^\infty(\tilde{N})$ identifying N with $\tilde{N}$. Then we have $\iota_{\tilde{\chi}}(\Phi) = \widetilde{\iota_\chi(\Phi)}$. Hence, by (5.15), we have

$$(T_w(\tilde{\chi})\iota_{\tilde{\chi}}(\Phi))(g) = (T_w(\chi)\iota_\chi(\Phi))(\psi(g)), \qquad g \in \tilde{G},$$

if $|\chi(a_\alpha)| < 1$ for every $\alpha \in \Psi_w^+$. Since both terms are meromorphic functions of χ when Φ and g are fixed, we obtain the following Lemma.

Lemma 5.7. *Let $\chi \in X$, $w \in W$ and assume $\chi(a_\alpha) \neq 1$ for every $\alpha \in \Psi_w^+$. Then we have $T_{w,\tilde{\chi}} = T_{w,\chi}$ when $\tilde{N}$ and N are identified by the isomorphism ψ and ω_0, x_w are chosen from $\psi(K)$.*

We assume that $\mathbf{G}$ is simply connected. Let J be a subset of Δ. Let Σ_J, W_J and w_J be the same as in §3. Let G_J be the universal Chevalley group over k generated by $x_\alpha(t)$, $\alpha \in \Sigma_J$, $t \in k$. As for G, we define the corresponding objects B_J, T_J, N_J, K_J and X_J.

Lemma 5.8. *Let $\chi \in X$, $w \in W_J$. We take representatives ω and ω_J of w and w_J respectively from K_J. We assume $\chi(a_\alpha) \neq 1$ for every $\alpha \in \Psi_w^+$. Set $\eta = \chi | T_J \in X_J$. Let $T^J_{w,\eta}$ be the distribution on N_J defined as in* (5.11), *i.e.,*

$$T^J_{w,\eta}(\Phi_1) = (T_w(\eta)\iota_\eta(\Phi_1))(\omega_J), \qquad \Phi_1 \in C_c^\infty(N_J).$$

Let $\omega' = \omega_J^{-1}\omega_0$. Then $\omega'^{-1} N_J \omega' \subseteq N$. Put

$$\Phi'(n) = \Phi(\omega'^{-1} n \omega'), \qquad n \in N_J$$

for every $\Phi \in C_c^\infty(N)$. Then we have

$$T_{w,\chi}(\Phi) = T^J_{w,\eta}(\Phi').$$

In other word, $T_{w,\chi}$ is the direct image of the distribution on $\omega'^{-1} N_J \omega'$ which is obtained from $T^J_{w,\eta}$ by the isomorphism $N_J \cong \omega'^{-1} N_J \omega'$.

Proof. Since $w \in W_J$, it is clear that $N_w \subseteq N_J$. The group $\omega'^{-1} N_J \omega'$ is generated by $x_\alpha(t)$, $\alpha \in (w_J w_0)^{-1}\Sigma_J^+$, $t \in k$. We have

$$w_0 w_J \Sigma_J^+ = w_0(-\Sigma_J^+) \subseteq \Sigma^+.$$

Hence we obtain $\omega'^{-1} N_J \omega' \subseteq N$.

Let $\Phi \in C_c^\infty(N)$, $\varphi = \iota_\chi(\Phi)$. Then we have $\varphi \in PS(\chi)$, $\Phi(n) = \varphi(\omega_0 n)$, $n \in N$. Put

$$\varphi'(g) = \varphi(g\omega'),\ g \in G_J, \qquad \Phi'(n) = \varphi'(\omega_J n),\ n \in N_J.$$

Since $\delta_{B_J} = \delta_B | B_J$ (cf. (3.7)), we have $\varphi' \in PS(\eta)$. For $n \in N_J$, we have

$$\Phi'(n) = \varphi'(\omega_J n) = \varphi(\omega_J n \omega') = \varphi(\omega_0 \omega'^{-1} n \omega') = \Phi(\omega'^{-1} n \omega').$$

Hence $\Phi' \in C_c^\infty(N_J)$ and $\varphi' = \iota_\eta(\Phi')$.

First assume $|\chi(a_\alpha)| < 1$ for every $\alpha \in \Psi_w^+$. Then we get

$$T_{w,\chi}(\Phi) = \int_{N_w} \varphi(\omega^{-1} n \omega_0)\, dn = \int_{N_w} \varphi(\omega^{-1} n \omega_J \omega')\, dn = \int_{N_w} \varphi'(\omega^{-1} n \omega_J)\, dn = T_{w,\eta}^J(\Phi').$$

By analytic continuation (cf. Lemma 5.6), $T_{w,\chi}(\Phi) = T_{w,\eta}^J(\Phi')$ holds whenever $\chi(a_\alpha) \neq 1$ for every $\alpha \in \Psi_w^+$. This completes the proof.

It is well known that

$$\mathbf{B} \times \mathbf{N} \ni (b, n) \longrightarrow b\omega_0 n \in \mathbf{B}\omega_0\mathbf{N} (\subset \mathbf{G})$$

is a biregular mapping defined over k and $\mathbf{B}\omega_0\mathbf{N}$ is open in $\mathbf{G}$ for k-Zariski topology. Hence $\mathbf{B} \times \mathbf{N}$ is birationally equivalent to $\mathbf{G}$ over k. Therefore there is a rational mapping $\mathbf{b}_0$ (resp. $\mathbf{n}_0$) of $\mathbf{G}$ into $\mathbf{B}$ (resp. $\mathbf{N}$) defined over k which is regular on $\mathbf{B}\omega_0\mathbf{N}$ such that $g = \mathbf{b}_0(g)\omega_0\mathbf{n}_0(g)$ for $g \in \mathbf{B}\omega_0\mathbf{N}$. We have $\omega_0^{-1} n_1 \omega_0 \in \mathbf{B}\omega_0\mathbf{N}$ for a generic point n_1 of $\mathbf{N}$ over k. Hence we can define a rational mapping $\mathbf{b}$ (resp. $\mathbf{n}$) of $\mathbf{N}$ into $\mathbf{B}$ (resp. $\mathbf{N}$) defined over k by

$$\mathbf{b}(n_1) = \mathbf{b}_0(\omega_0 n_1 \omega_0^{-1}),\ \mathbf{n}(n_1) = \mathbf{n}_0(\omega_0 n_1 \omega_0^{-1}), \qquad n_1 \in \mathbf{N}$$

Then $\mathbf{b}$ and $\mathbf{n}$ are regular on $\mathbf{N} \cap \omega_0^{-1}\mathbf{B}\omega_0\mathbf{N}\omega_0$, and we have

$$\omega_0 n_1 \omega_0^{-1} = \mathbf{b}(n_1)\omega_0\mathbf{n}(n_1),\ n_1 \in \mathbf{N} \cap \omega_0^{-1}\mathbf{B}\omega_0\mathbf{N}\omega_0. \tag{5.16}$$

We see easily that $\mathbf{n}$ is a birational mapping of $\mathbf{N}$ into $\mathbf{N}$, and that $\mathbf{n}$ gives a biregular mapping of $\mathbf{N} \cap \omega_0^{-1}\mathbf{B}\omega_0\mathbf{N}\omega_0$ onto itself.

Lemma 5.9. *For $n_1 \in N$, let $n = \mathbf{n}(n_1)$. Then we have $dn_1 = \delta_B(\mathbf{b}(n))\, dn$.*

Proof. Let $dn_1 = c(n)\, dn$. We see that $c(n)$ is a continuous function on $N \cap \omega_0^{-1} B\omega_0 N\omega_0$, which is an open dense subset of N. Take any $\varphi \in PS(\delta_B^{1/2})$. By Lemma 5.2, we have

$$\int_{B\backslash G} \varphi(g)\, dg = \int_N \varphi(\omega_0 n)\, dn.$$

Hence we obtain

$$\int_{B\backslash G} \varphi(g)\,dg = \int_{B\backslash G} \varphi(g\omega_0^{-1})\,dg = \int_N \varphi(\omega_0 n_1 \omega_0^{-1})\,dn_1$$
$$= \int_N \delta_B(\mathbf{b}(n_1))c(n)\varphi(\omega_0 n)\,dn.$$

Since $w_0^2 = 1$, we have $\omega_0^2 \in T \cap K$. By this fact, we see easily that $\delta_B(\mathbf{b}(n_1)) = \delta_B(\mathbf{b}(n))^{-1}$. Therefore

$$\int_N \varphi(\omega_0 n)\,dn = \int_N \delta_B(\mathbf{b}(n))^{-1} c(n)\varphi(\omega_0 n)\,dn$$

holds for any $\varphi \in PS(\delta_B^{1/2})$. Hence the assertion follows from Lemma 5.3.

The following Lemma gives an explicit form of the distribution $T_{w,\chi}$ in the case $w = w_0$.

Lemma 5.10. *Let $w = w_0$, $\chi \in X$ and assume $|\chi(a_\alpha)| < 1$ for any $\alpha \in \Sigma^+$. Then the distribution $T_{w,\chi}$ is given by a locally integrable function $\delta_B(\mathbf{b}(n))^{1/2}\chi(\mathbf{b}(n))^{-1}$, $n \in N$.*

Proof. Take any $\Phi \in C_c^\infty(G)$ and set $\varphi = \iota_\chi(\Phi) \in PS(\chi)$. By definition, we have

$$T_\chi(\Phi) = (T_{w_0}(\chi)\varphi)(\omega_0) = \int_N \varphi(\omega_0^{-1} n_1 \omega_0)\,dn_1 = \int_N \varphi(\omega_0 n_1 \omega_0^{-1})\,dn_1,$$

with absolutely convergent integrals. We change the variables by $n = \mathbf{n}(n_1)$. We have

$$\chi(\mathbf{b}(n_1)) = \chi(\mathbf{b}(n))^{-1}, \qquad \delta_B(\mathbf{b}(n_1)) = \delta_B(\mathbf{b}(n))^{-1}.$$

By Lemma 5.9, the above integral is equal to

$$\int_N \delta_B(\mathbf{b}(n))^{-1/2}\chi(\mathbf{b}(n))^{-1}\varphi(\omega_0 n)\delta_B(\mathbf{b}(n))\,dn = \int_N \delta_B(\mathbf{b}(n))^{1/2}\chi(\mathbf{b}(n))^{-1}\Phi(n)\,dn,$$

and we see easily that this integral is absolutely convergent. Hence the assertion follows.

For general $\chi \in X$, $T_{w,\chi}$ can be given by analytic continuation (cf. Lemma 5.6).

Example 5.11. Let $\mathbf{G}$ be the universal Chevalley group of type C_ℓ. We may set

$$\mathbf{G} = \{g \in GL(2\ell) | {}^t g J g = J\}, \qquad J = \begin{pmatrix} 0_\ell & 1_\ell \\ -1_\ell & 0_\ell \end{pmatrix},$$

$$\mathbf{T} = \left\{ \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_\ell & & & \\ & & & t_1^{-1} & & \\ & & & & \ddots & \\ & & & & & t_\ell^{-1} \end{pmatrix} \right\},$$

$$\mathbf{N} = \left\{ \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| a \in GL(\ell) \text{ is upper unipotent, } b \in M(\ell),\ {}^t b = b \right\},$$

$$\omega_0 = J.$$

Then χ is of the form

$$\chi(\mathrm{diag}[t_1,\dots,t_\ell,\ t_1^{-1},\dots,t_\ell^{-1}]) = \prod_{i=1}^{\ell} \chi_i(t_i), \qquad t_i \in k^\times,$$

with unramified quasi-characters χ_i of $k^\times$. We have $w_0\chi = \bar{\chi}^{-1}$ if and only if $\chi_i (1 \le i \le \ell)$ are real valued. Let P (resp. U) be the subgroup of $GL(\ell, k)$ consisting of all upper triangular (resp. upper unipotent) matrices. For $n = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in N$, we consider the decomposition $\omega_0 n \omega_0^{-1} = b_1 \omega_0 n_1$, $b_1 \in B$, $n_1 \in N$. We set

$$b_1 = \begin{pmatrix} x & 0 \\ 0 & {}^t x^{-1} \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \qquad n_1 = \begin{pmatrix} \alpha & 0 \\ 0 & {}^t \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix},$$

where $x \in P$, $\alpha \in U$ and $y, \beta \in M(\ell, k)$ are symmetric. Then we get

$$ab = {}^t x^{-1} \alpha. \tag{5.17}$$

For a matrix $C = (c_{ij}) \in M(\ell, k)$, let $M_i(C)$ denote the determinant of the minor

$$M_i(C) = \det(c_{rs};\ 1 \le r, s \le i)$$

for $1 \le i \le \ell$, and set $M_0(C) = 1$. The following Lemma can be verified immediately by induction on ℓ.

Lemma 5.12. *Let $C \in M(\ell, k)$ be given. Then the equation ${}^t pu = C$ with $p = (p_{ij}) \in P$, $u \in U$ can be solved if and only if $M_i(C) \neq 0$ for $1 \le i \le \ell$. If this is the case, the solution is unique and we have*

$$p_{ii} = M_i(C)/M_{i-1}(C), \qquad 1 \le i \le \ell.$$

By (5.17) and Lemma 5.12, the first ℓ diagonal components of b_1 are given by $M_{i-1}(ab)/M_i(ab)$, $1 \le i \le \ell$. We have

$$\delta_B(\mathrm{diag}[t_1,\dots,t_\ell,\ t_1^{-1},\dots,t_\ell^{-1}]) = (\prod_{i=1}^{\ell} |t_i|^{\ell+1-i})^2, \qquad t_i \in k^\times.$$

Therefore, by Lemma 5.10, the distribution $T_{w,\chi}$ is given by a function

$$T_{w,\chi}(n) = \prod_{i=1}^{\ell} \chi_i(M_i(ab)/M_{i-1}(ab)) \prod_{i=1}^{\ell} |M_i(ab)|^{-1} \tag{5.18}$$

for $n = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in N$, when χ is in the domain of absolute convergence, that is $1 > |\chi_\ell(\varpi)| > |\chi_{\ell-1}(\varpi)| > \cdots > |\chi_1(\varpi)|$. In the general case, $T_{w,\chi}$ is given by the analytic continuation of (5.18).

§6. Spherical functions

Let $\chi \in X$. Recall that π_χ^1 is the unique K-spherical constituent of $\pi(\chi)$. The spherical function associated with π_χ^1 is given by

$$\Gamma_\chi(g) = \int_K \varphi_{K,\chi}(kg)\, dk, \qquad g \in G, \tag{6.1}$$

where the Haar measure dk is normalized so that $\int_K dk = 1$. As is well known, π_χ^1 is unitarizable if and only if Γ_χ is of positive type (i.e., positive definite function). Let $\mathbf{P}$ denote the set of all $\chi \in X$ such that π_χ^1 is unitarizable. Since $\Gamma_\chi(g)$ is a continuous function of χ when g is fixed, it is obvious that $\mathbf{P}$ is closed in X. It is well known that $\mathbf{P}$ is bounded. Hence $\mathbf{P}$ is a compact W-stable subset of X.

Lemma 6.1. *Let $w \in W$, $\chi \in X_w$. Assume that $\chi(a_\alpha) \neq 1$ for every $\alpha \in \Psi_w^+$ and that π_χ^1 is unitarizable. Put $\bar{\varphi}_{K,\chi} = \varphi_{K,\chi} \bmod \mathrm{Ker}(T_w(\chi))$. If $PS(\chi)/\mathrm{Ker}(T_w(\chi))$ is generated by $\bar{\varphi}_{K,\chi}$, then*

$$(\varphi_1, \varphi_2) = c_w(\chi)^{-1} \int_{B\backslash G} (T_w(\chi)\varphi_1)(g)\, \overline{\varphi_2(g)}\, dg, \qquad \varphi_1, \varphi_2 \in PS(\chi)$$

is a positive definite hermitian form on $PS(\chi)/\mathrm{Ker}(T_w(\chi))$.

Proof. By the assumption, we have

$$T_w(\chi)\varphi_{K,\chi} = c_w(\chi)\varphi_{K,w\chi}, \qquad c_w(\chi) \neq 0$$

and every element of $PS(\chi)/\mathrm{Ker}(T_w(\chi))$ can be represented by a function of the form

$$\eta(g) = \sum_i \alpha_i \varphi_{K,\chi}(g y_i), \qquad \alpha_i \in \mathbf{C},\ y_i \in G.$$

We have

$$(\eta, \eta) = c_w(\chi)^{-1} \int_{B\backslash G} \sum_i \alpha_i c_w(\chi)\varphi_{K,w\chi}(gy_i) \sum_i \bar{\alpha}_j \overline{\varphi_{K,\chi}(gy_j)}\, dg$$

$$= \sum_{i,j}\left(\int_{B\backslash G} \varphi_{K,w\chi}(gy_i)\,\overline{\varphi_{K,\chi}(gy_j)}\, dg\right)\alpha_i\bar{\alpha}_j = \sum_{i,j}\left(\int_{B\backslash G} \varphi_{K,w\chi}(gy_j^{-1}y_i)\,\overline{\varphi_{K,\chi}(g)}\, dg\right)\alpha_i\bar{\alpha}_j$$

$$= \sum_{i,j}\left(\int_K \varphi_{K,w\chi}(ky_j^{-1}y_i)\,\overline{\varphi_{K,\chi}(k)}\, dk\right)\alpha_i\bar{\alpha}_j = \sum_{i,j} \Gamma_{w\chi}(y_j^{-1}y_i)\alpha_i\bar{\alpha}_j = \sum_{i,j} \Gamma_\chi(y_j^{-1}y_i)\alpha_i\bar{\alpha}_j \geq 0$$

since $\Gamma_{w\chi} = \Gamma_\chi$ and Γ_χ is positive definite. Hence the sesqui-linear form $(\ ,\ )$ must be hermitian. Then it is clear that $(\ ,\ )$ defines a non-degenerate hermitian form on $PS(\chi)/\mathrm{Ker}(T_w(\chi))$. Hence positive definiteness follows. This completes the proof.

Lemma 6.2. *Let* $\tilde{\mathbf{G}}$ *be the simply connected covering group of* $\mathbf{G}$ *and let the related notation be the same as in* §3. *Let* $\tilde{X}$ *be the set of all unramified quasi-characters of* $\tilde{T}$. *For* $\chi \in X$, *define* $\tilde{\chi} \in \tilde{X}$ *by* $\tilde{\chi}(t) = \chi(\psi(t))$, $t \in \tilde{T}$. *Then* $\pi^1_{\tilde{\chi}}$ *is unitarizable if and only if* π^1_χ *is unitarizable.*

Proof. We have

$$\Gamma_{\tilde{\chi}}(\tilde{g}) = \int_{\tilde{K}} \varphi_{\tilde{K},\tilde{\chi}}(\tilde{k}\tilde{g})\, d\tilde{k}, \qquad \tilde{g} \in \tilde{G}$$

where $\tilde{K}$ is the maximal compact subgroup of $\tilde{G}$ defined as in §3 and $\int_{\tilde{K}} d\tilde{k} = 1$. If $\tilde{g} = \tilde{b}\tilde{k}$, $\tilde{b} \in \tilde{B}$, $\tilde{k} \in \tilde{K}$, we have

$$\varphi_{\tilde{K},\tilde{\chi}}(\tilde{g}) = \delta_{\tilde{B}}^{1/2}(\tilde{b})\tilde{\chi}(\tilde{b}) = \delta_B^{1/2}(\psi(\tilde{b}))\,\chi(\psi(\tilde{b})) = \varphi_{K,\chi}(\psi(\tilde{g})).$$

Since $\mathrm{Ker}(\psi)$ is contained in $\tilde{T} \cap \tilde{K}$, we obtain

$$\Gamma_{\tilde{\chi}}(\tilde{g}) = \Gamma_\chi(\psi(\tilde{g})), \qquad \tilde{g} \in \tilde{G}. \tag{6.3}$$

Therefore $\Gamma_{\tilde{\chi}}$ is positive definite if Γ_χ is positive definite. Thus the unitarizability of π^1_χ implies that of $\pi^1_{\tilde{\chi}}$.

Conversely we assume that $\pi^1_{\tilde{\chi}}$ is unitarizable. Put $G' = \psi(\tilde{G})$. We note that G' is a normal subgroup of G. By (6.3) and by two-sided K-invariance of Γ_χ, we see that $\Gamma_\chi | G'K$ defines a positive definite function on $G'K$. This implies that $\pi^1_\chi | G'K$ is unitarizable. Since $[G: G'K] < \infty$, we can easily conclude that π^1_χ is unitarizable. This completes the proof.

§7. Deformations of representations

In the following sections, we shall determine all unitarizable $\pi(\chi)$ when $\mathbf{G}$ is of classical type and $\pi(\chi)$ is irreducible. Besides the result in §2, certain deformation arguments shall play important roles, which we shall prepare in this section.

Let $w \in W$, $w^2 = 1$. We set $X^i_w = X_w \cap X^i$ (cf. §3 and §4). Let $\chi \in X^i_w$ and assume that T_w is holomorphic at χ. In §5, we have shown that

$$(7.1)\qquad (\varphi_1, \varphi_2)_\chi = c_w(\chi)^{-1}\int_K (T_w(\chi)\varphi_1)(k)\,\overline{\varphi_2(k)}\,dk, \qquad \varphi_1, \varphi_2 \in PS(\chi)$$

gives an invariant non-degenerate hermitian form on $PS(\chi)$ and that $\pi(\chi)$ is unitarizable if and only if $(\ ,\ )_\chi$ is positive definite. We define a hermitian form H_χ on S by

$$(7.2)\qquad H_\chi(f_1, f_2) = (R(\chi)^{-1}f_1, R(\chi)^{-1}f_2), \qquad f_1, f_2 \in S.$$

We have

$$(7.3)\qquad H_\chi(f_1, f_2) = c_w(\chi)^{-1}\int_K (T'_w(\chi)\varphi_1)(k)\,\overline{\varphi_2(k)}\,dk, \qquad \varphi_1, \varphi_2 \in S,$$

where $T'_w(\chi) = R(\chi)T_w(\chi)R(\chi)^{-1} \in \mathrm{End}(S)$. For an open subgroup U of K, let H_χ^U be the restriction of H_χ to S^U. Then H_χ^U is a non-degenerate hermitian form on S^U; $\pi(\chi)$ is unitarizable if and only if H_χ^U is positive definite for every U.

The following Lemma 7.1 and Proposition 7.3 are well known as a general technique to construct complementary series (cf. Oršanskiĭ [15], p. 251). We include proofs for the sake of completeness.

Lemma 7.1. *Let $a < b$ be real numbers and let $p: [a, b] \to X_w^i$ be a continuous map. Set $\chi_t = p(t)$, $a \le t \le b$. Let U be an open compact subgroup of K. Then $H_{\chi_b}^U$ is positive definite if and only if $H_{\chi_a}^U$ is positive definite.*

Proof. For simplicity, set $H_t^U = H_{\chi_t}^U$, $a \le t \le b$. It suffices to prove "if" part. Assume H_a^U is positive definite and H_b^U is not positive definite. Since $c_w(\chi)^{-1}T_w(\chi)$ is holomorphic at χ as far as $\chi(a_\alpha) \ne q$ for every $\alpha \in \Psi_w^+$ (cf. (4.14) and Lemma 4.3), $c_w(\chi_t)^{-1}T'_w(\chi_t) \in \mathrm{End}(S)$ depends continuously on t by the assumption and (4.2). Hence when we fix a basis of the finite dimensional vector space S^U, H_t^U is represented by a hermitian matrix whose matrix coefficients depend continuously on t. It is clear that the set

$$P = \{t \in [a, b] \mid H_t^U \text{ is positive definite}\}$$

is open in $[a, b]$. Hence $[a, b] - P$ is a compact subset, which contains b, of $[a, b]$. Therefore there exists $a < t_0 \le b$ such that $[a, t_0) \subseteq P$, $t_0 \notin P$. Since $H_{t_0}^U = \lim_{t\to t_0} H_t^U$, we see that $H_{t_0}^U$ is positive semi-definite, but is not positive definite. Hence $H_{t_0}^U$ cannot be non-degenerate. This is a contradiction and completes the proof.

Lemma 7.2. *Let $p: [a, b] \to X_w^i$ be a continuous map. Set $\chi_t = p(t)$. Then $\pi(\chi_1)$ is unitarizable if and only if $\pi(\chi_0)$ is unitarizable.*

Proof. If the assertion is negative, there exists an open subgroup U of K such that one of $H_{\chi_0}^U$ and $H_{\chi_1}^U$ is positive definite but the other is not positive definite. This contradicts Lemma 7.1 and completes the proof.

Proposition 7.3. *Let $w \in W$, $w^2 = 1$ and $p: [a, b] \to X_w$ be a continuous* *map. Put* $\chi_t = p(t)$, $0 \le t \le 1$. *If the conditions*

(1) $\chi_0(a_\alpha) = 1$ *for every* $\alpha \in \Psi_w^+$,

(2) $\chi_t(a_\alpha) \ne 1$ *for every* $\alpha \in \Psi_w^+$, $0 < t \le 1$,

(3) $p(0, 1] \subseteq X_w^i$

are satisfied, then $\pi(\chi_t)$ *is unitarizable for* $0 < t \le 1$.

Proof. Let $w = \sigma_1 \cdots \sigma_{n-1}\sigma_n$ be a reduced expression of w. Let $0 < t \le 1$. By (4.15), we have

$$T_w(\chi_t) = T_{\sigma_1}(\sigma_2 \cdots \sigma_n \chi_t) \cdots T_{\sigma_{n-1}}(\sigma_n \chi_t)\, T_{\sigma_n}(\chi_t).$$

Set $\theta_i = (\sigma_n \sigma_{n-1} \cdots \sigma_{i+1})\alpha_i$, $1 \le i \le n-1$. We have

$$\Psi_w^+ = \{\theta_i (1 \le i \le n-1),\ \alpha_n\} \quad \text{and} \quad (\sigma_{i+1} \cdots \sigma_{n-1}\sigma_n \chi_t)(a_{\alpha_i}) = \chi_t(a_{\theta_i}).$$

Let U be an open subgroup of K and fix a basis of S^U. By the assumption (1) and Lemma 4.1, we have

$$\begin{aligned} c_{\alpha_i}(\sigma_{i+1} \cdots \sigma_{n-1}\sigma_n \chi_t)^{-1} T'_{\sigma_i}(\sigma_{i+1} \cdots \sigma_{n-1}\sigma_n \chi_t) | S^U &= 1 + o(1) \qquad (1 \le i \le n-1), \\ c_{\alpha_n}(\chi_t)^{-1} T'_{\sigma_n}(\chi_t) | S^U &= 1 + o(1) \end{aligned} \tag{7.4}$$

for $t \to +0$, where $1 \in \mathrm{End}(S)$ denotes the identity. Hence we get

$$c_w(\chi_t)^{-1} T'_w(\chi_t) | S^U = 1 + o(1), \qquad t \longrightarrow +0.$$

Then, by (7.3), there exists $0 < \varepsilon_U < 1$ such that $H^U_{\chi_t}$ is positive definite for $0 < t \le \varepsilon_U$. Applying Lemma 7.1 to the interval $[\varepsilon_U, 1]$, we see that $H^U_{\chi_t}$ is positive definite for $0 < t \le 1$. This completes the proof.

Many unitarizable $\pi(\chi)$, $\chi \in X_w^i$ can be constructed by means of this proposition. The following proposition gives a more elaborate study of deformations.

Proposition 7.4. *Let* $w \in W$, $w^2 = 1$ *and assume* w *is decomposed so that* $w = w_1 w_2$, $w_1^2 = 1$, $w_2^2 = 1$, $l(w) = l(w_1)\, l(w_2)$. *Let* $p\colon [a, b] \to X_w$ *and* $p_1\colon [a, b] \to X_{w_1}$ *be a continuous maps. For* $0 \le t \le 1$, *put* $\chi_t = p(t)$, $\chi_t^1 = p_1(t)$. *We assume that the following conditions are satisfied.*

(1) $\chi_0 = \chi_0^1$.

(2) $p(0, 1] \subseteq X_w^i$ *and* $p_1(0, 1] \subseteq X_{w_1}^i$.

(3) *For every* $\alpha \in \Psi_{w_1}^+$, $\chi_0(a_\alpha) \ne 1, q$.

(4) *For every* $\alpha \in \Psi_{w_2}^+$, $\chi_0(a_\alpha) = 1$.

If $\pi(\chi_{t_0}^1)$ *is unitarizable for some* $t_0 \in (0, 1]$, *then* $\pi(\chi_t)$ *is unitarizable for all* $0 < t \le 1$. *Conversely if* $\pi(\chi_{t_0})$ *is unitarizable for some* $t_0 \in (0, 1]$, *then* $\pi(\chi_t^1)$ *is unitarizable for all* $0 < t \le 1$.

Proof. By (3), we see that $T_{w_1}(\chi)$ is holomorphic at $\chi = \chi_0$. From $w_1 w_2 \chi_0 = \bar{\chi}_0^{-1}$, $w_1 \chi_0 = \bar{\chi}_0^{-1}$, we get $w_2 \chi_0 = \chi_0$. Hence $T_{w_1}(w_2 \chi)$ is holomorphic at $\chi = \chi_0$. Also by (3), we have

$$\lim_{t\to+0} c_{w_1}(w_2\chi_t)^{-1} = \lim_{t\to+0} c_{w_1}(\chi_t^1)^{-1} \neq \infty.$$

Therefore we get

$$\lim_{t\to+0} c_{w_1}(w_2\chi_t)^{-1} T'_{w_1}(w_2\chi_t)f = \lim_{t\to+0} c_{w_1}(\chi_t^1)^{-1} T'_{w_1}(\chi_t^1)f$$

for every $f\in S$. By (4), using the same argument as in the proof of Proposition 7.3, we obtain

(7.5) $$c_{w_2}(\chi_t)^{-1} T'_{w_2}(\chi_t)|S^U = 1 + o(1), \qquad t \longrightarrow +0$$

for any open subgroup U of K. Since

$$T'_w(\chi_t) = T'_{w_1}(w_2\chi_t)\, T'_{w_2}(\chi_t) \qquad \text{for } t>0,$$

we get

$$\lim_{t\to+0} c_w(\chi_t)^{-1} T'_w(\chi_t)f = \lim_{t\to+0} c_{w_1}(\chi_t^1)^{-1} T'_{w_1}(\chi_t^1)f$$

for every $f\in S$, by (7.4) and (7.5). Therefore we obtain, by (7.3),

(7.6) $$\lim_{t\to+0} H_{\chi_t}(f_1, f_2) = \lim_{t\to+0} H_{\chi_t^1}(f_1, f_2), \qquad f_1, f_2\in S.$$

Now assume that $\pi(\chi_{t_0}^1)$ is unitarizable for some $t_0\in(0, 1]$. Then by Lemma 7.2, $\pi(\chi_t^1)$ is unitarizable for all $t\in(0, 1]$. Let U be any open subgroup of K. By (7.6), we see that there exists $0<\varepsilon_U<1$ such that $H_{\chi_t}^U$ is positive definite for $0<t\leq\varepsilon_U$. By Lemma 7.1, $H_{\chi_t}^U$ is then positive definite for all $0<t\leq 1$. Threfore $\pi(\chi_t)$ is unitarizable for $0<t\leq 1$. The converse assertion can be proved similarly. This completes the proof.

Remark 7.5. The assumption (4) can be replaced by the condition (7.5) for all open subgroup U of K.

Theorem 7.6. *Assume* **G** *is of adjoint type. Let* $\chi\in X$ *and assume* $\pi(\chi)$ *is irreducible and unitarizable. Then* χ *belongs to the closure of* $\mathbf{P}\cap X^i\cap X^r$ *in* X.

Proof. By the assumption of irreducibility, we have $\pi(\chi)\cong\pi_\chi^1$ and $\pi(w\chi)\cong\pi(\chi)$ for every $w\in W$. We have $w\chi=\bar{\chi}^{-1}$ for some $w\in W$. Hence, by Lemma 3.3, we get $w_1\chi=\bar{\chi}^{-1}$ with $w_1\in W$, $w_1^2=1$. By Lemma 3.5, we may assume that $\chi\in X_{w_J}$ for some $J\subseteq\Delta$ and that w_J acts on J by multiplication by -1, replacing χ by $w'\chi$, $w'\in W$.

Assume $\chi(a_\alpha)=1$ for some $\alpha\in\Sigma_J^+$. Let $w_J=\sigma_1\sigma_2\cdots\sigma_n$ be a reduced expression of w_J with $\sigma_i=\sigma_{\alpha_i}$, $\alpha_i\in J$, $1\leq i\leq n$. By Lemma 4.4, there exists i such that $w'=(\sigma_1\cdots\sigma_{i-1})(\sigma_{i+1}\cdots\sigma_n)$, $w'\chi=\bar{\chi}^{-1}$. We have $w_Jw'=(\sigma_n\cdots\sigma_{i+1})\sigma_i(\sigma_{i+1}\cdots\sigma_n)$. Hence w_Jw' is of order 2 and we get $w'^2=1$. Therefore w' is conjugate in W_J to $w_{J'}$ for some $J'\subsetneq J$ and $w_{J'}=-1$ on J'. Repeating this procedure, we may assume $\chi \in X_{w_J}$ and $\chi(a_\alpha) \neq 1$ for every $\alpha \in \Sigma_J^+$. By Lemma 3.4 and its proof, there exists a continuous map $p\colon [0, 1] \to X_{w_J}$ such that $p(0) = \chi$, $p(0, 1] \subseteq X_{w_J}^i \cap X^r$. Now the assertion follows from Lemma 7.2.

§8. The unitarizability of $\pi(\chi)$ when $\pi(\chi)$ is irreducible

In this section, we shall explain basic principles of determining the unitarizability of $\pi(\chi)$ for $\chi \in X^i$.

Theorem 8.1. *Assume that* $\mathbf{G}$ *is simply connected. Let* $J \subseteq \Delta$, $\chi \in X_{w_J}^i$ *and assume* $|\chi(a_\alpha)| < 1$ *for every* $\alpha \in \Sigma_J^+$. *Define* G_J, T_J *and* X_J *as in Lemma* 5.8. *Put* $\eta = \chi | T_J \in X_J$. *Then* $\pi(\chi)$ *is unitarizable if and only if the representation* $\pi(\eta)$ *of* G_J *is unitarizable.*

Proof. By Lemma 5.4, $\pi(\chi)$ is unitarizable if and only if $cT_{w_J,\chi}$ is of positive type for some $c \in \mathbf{C}^\times$. By Lemma 5.8 and Theorem 2.3, $cT_{w_J,\chi}$ is of positive type if and only if $cT_{w_J,\eta}^J$ is of positive type. Again by Lemma 5.4, this is equivalent to the unitarizability of $\pi(\eta)$. This completes the proof.

Now we consider the determination of the unitarizability of $\pi(\chi)$ for $\chi \in X^i$. By Lemmas 3.2, 6.2 and the irreducibility criterion, we may assume that $\mathbf{G}$ is of adjoint type. By Theorem 7.6, it suffices to consider the case where $\chi \in X^i \cap X^r$. Replacing χ by $w\chi$ with $w \in W$, we may assume that $\chi \in X_{w_J}^i$ for some $J \subseteq \Delta$ and that w_J acts on J by -1 (cf. Lemma 3.5). Replacing χ by $w\chi$ with $w \in W_J$, we may further assume $|\chi(a_\alpha)| \leq 1$ for every $\alpha \in \Sigma_J^+$. By Lemmas 3.4 and 7.2, it suffices to consider the case $|\chi(a_\alpha)| < 1$ for every $\alpha \in \Sigma_J^+$. Let $\psi\colon \tilde{\mathbf{G}} \to \mathbf{G}$ be the simply connected covering map as in §3. Put $\tilde{\chi} = \psi \circ \chi$. Then we may assume that $\pi(\tilde{\chi})$ is irreducible and $\tilde{\chi}$ is regular by Lemma 7.2. Now we apply Theorem 8.1. If $J \subsetneq \Delta$, we can reduce the unitarizability of $\pi(\tilde{\chi})$, which is equivalent to that of $\pi(\chi)$, to the untarizability of $\pi(\eta)$ of a lower rank group. Therefore it suffices to consider the case where $J = \Delta$, $\chi \in X_{w_0}^i \cap X^r$, w_0 acts on Δ by -1 and $\mathbf{G}$ is of adjoint type.

We may set χ in the form

$$\chi(\beta(\varpi)) = q^{\langle x, \beta \rangle} = \exp(\pi\sqrt{-1}\langle z, \beta \rangle) q^{\langle y, \beta \rangle}, \qquad \beta \in P(\check{\Sigma}),$$

where $x = y + \sqrt{-1}\dfrac{\pi}{\log q} z$, $y, z \in V$. The condition $w_0\chi = \bar{\chi}^{-1}$ is equivalent to

$$w_0 y = -y, \qquad w_0 z - z \in 2Q(\Sigma).$$

Since $w_0 = -1$, we have $z \in Q(\Sigma)$. Thus we may set $x = y + \sqrt{-1}\dfrac{\pi}{\log q} z$ with $y \in V$, $z \in Q(\Sigma)$ and we have

$$\chi(\beta(\varpi)) = (-1)^{\langle z, \beta \rangle} q^{\langle y, \beta \rangle} \qquad \beta \in P(\check{\Sigma}). \tag{8.1}$$

We fix $z \in Q(\Sigma)$. In V, we consider the family of hyperplanes $\mathfrak{H}_z$ defined by

(8.2) $$\langle v, \check{\alpha}\rangle = \pm 1.$$

for every $\check{\alpha}\in\check{\Sigma}$ such that $\langle z, \check{\alpha}\rangle \equiv 0 \bmod 2$. Let D_z be a connected component of $V-\mathfrak{H}_z$. For $v\in D_z$, define $\chi(v)\in X^i_{w_0}$ by

(8.3) $$\chi(v)(\beta(\varpi)) = (-1)^{\langle z,\beta\rangle} q^{\langle v,\beta\rangle} \qquad \beta\in P(\check{\Sigma}).$$

By Lemma 7.2, either all of $\pi(\chi(v))$ is unitarizable or non-unitarizable for $v\in D_z$, i.e., the unitarizability on D_z remains the same.

Lemma 8.2. *Let* **G** *be of rank* 1, $\Sigma^+ = \{\alpha\}$ *and* $w = \sigma_\alpha$. *Let* $\chi\in X^i_w$. *Then* $\pi(\chi)$ *is unitarizable if and only if* $q^{-1} < \chi(a_\alpha) < q$.

Proof. This is well known. We shall give a short proof. It suffices to consider the case when **G** is of adjoint type. By $\chi\in X^i_w$, we find $\chi(a_\alpha)\in\mathbf{R}$. If $\chi(a_\alpha) < q^{-1}$ or $\chi(a_\alpha) > q$, we find that D is not bounded. If $q^{-1} < \chi(a_\alpha) < q$, we can immediately conclude the unitarizability of $\pi(\chi)$ by Proposition 7.3.

Suppose that there exist $\check{\alpha}_0\in\check{\Sigma}$ and $v_0\in D_z$ such that $\langle z, \check{\alpha}_0\rangle \equiv 0 \bmod 2$, $\langle v_0, \check{\alpha}_0\rangle = 0$. Put $\chi_0 = \chi(v_0)$. We shall show that the unitarizability of $\pi(\chi_0)$ can be reduced to that for a lower rank group. Take $w\in W$ so that $w^{-1}\alpha_0\in\Delta$. Replacing χ_0 by $w\chi_0$, we may assume $\alpha_0\in\Delta$. We can find a reduced expression of w_0 such that $w_0 = \sigma_1\cdots\sigma_n$, $\sigma_n = \sigma_{\alpha_0}$. Set $w_1 = \sigma_1\cdots\sigma_{n-1}$. Then $w_1^2 = 1$ since $w_0 = -1$. We have $\chi_0\in X^i_{w_1}$ and w_1 is conjugate to w_J for some $J\subsetneqq\Delta$. Then as we have shown above, the unitarizability of $\pi(\chi_0)$ can be reduced to that for a lower rank group.

Remark 8.3. Assume that $J = \{\alpha\}\cup J_1$ where all roots of J_1 and α are orthogonal. Then, by Lemma 8.2, we see that $\pi(\chi_0)$ is unitarizable if and only if $q^{-1} < \chi(a_\alpha) < q$ and $\pi(\eta_0)$, $\eta_0 = \chi_0 | T_{J_1}\in X_{J_1}$ is unitarizable, in the notation of Theorem 8.1.

Therefore it suffices to consider only those D_z which satisfy

(8.4) $\langle v, \check{\alpha}\rangle \neq 0$ for every $v\in D_z$ and for every $\check{\alpha}\in\check{\Sigma}$ such that $\langle z, \check{\alpha}\rangle \equiv 0 \bmod 2$.

In the following sections, we shall show that if D_z satisfies (8.4), then all the points of D_z represent non-unitarizable representations. This shall complete the determination of the unitarizability for $\pi(\chi)$ which are irreducible.

We shall prove two more Lemmas which are useful in later considerations.

Lemma 8.4. *Let* $D = D_z$ *be as above and let* $\alpha\in\Sigma$. *There exists* $v_0\in D$ *such that* $\langle v_0, \check{\alpha}\rangle = 0$ *if and only if* $\sigma_\alpha D\cap D \neq \emptyset$.

Proof. It $v_0\in D$ satisfies $\langle v_0, \check{\alpha}\rangle = 0$, we have $\sigma_\alpha v_0 = v_0$ by the formula of reflexions. Hence $v_0\in\sigma_\alpha D\cap D$. Conversely assume $v_1\in\sigma_\alpha D\cap D$. Then we have

(8.5) $$\langle v_1, \check{\alpha}\rangle = \langle \sigma_\alpha v_1, \sigma_\alpha\check{\alpha}\rangle = \langle \sigma_\alpha v_1, -\check{\alpha}\rangle = -\langle \sigma_\alpha v_1, \check{\alpha}\rangle.$$

We may assume $\langle v_1, \check{\alpha}\rangle \neq 0$. Then (8.5) shows that the real valued continuous function $v \to \langle v, \check{\alpha} \rangle$ on D changes the sign. Hence there exists $v_0 \in D$ such that $\langle v_0, \check{\alpha} \rangle = 0$. This completes the proof.

Lemma 8.5. *Let $\chi \in X$. We assume that all composition factors of $PS(\chi)$ are hermitian. Furthermore we assume the following conditions* (1) $\sim$ (3).

(1) *There exists a non-degenerate invariant hermitian form on $PS(\chi)$.*

(2) $|W_\chi| = 2$.

(3) *$PS(\chi)$ has two irreducible G-submodules.*

Then there exists irreducible G-submodules V_1 and V_2 such that $PS(\chi) = V_1 \oplus V_2$.

Proof. Let $(\, , \,)$ be the non-degenerate invariant hermitian form on $PS(\chi)$. Let V be an irreducible G-submodule of $PS(\chi)$. We shall show that $(\, , \,)|V$ can be assumed to be non-degenerate. Assume, on the contrary, that $(\, , \,)$ is degenerate. Since V is irreducible, $(\, , \,)|V$ must be the zero form. Let $V^\perp$ be the annihilator of V in $PS(\chi)$. We have $V^\perp \supseteq V$. Since there exists a non-degenerate G-invariant sesqui-linear form

$$V \times PS(\chi)/V^\perp \longrightarrow \mathbf{C},$$

we get $PS(\chi)/V^\perp \cong \bar{\tilde{V}}$ as G-modules, and we have $\bar{\tilde{V}} \cong V$ by the assumption. Let (W_i) be the set of all irreducible constituents of $PS(\chi)$. We may set $W_1 = V$, $W_2 \cong V$. By the exactness of the Jacquet functor, we obtain

$$\bigoplus_{w \in W} \mathbf{C}_{\delta_B^{1/2} w\chi} \cong \bigoplus_i \text{ the semi-simplification of } (W_i)_N$$

as T-modules. Since $\mathbf{C}_{\delta_B^{1/2}\chi}$ occurs in V_N (cf. the proof of Lemma 4.6), $\mathbf{C}_{\delta_B^{1/2}\chi}$ does not occur in $(W_i)_N$, $i \geq 3$, by the assumption (2). Therefore W_i is not isomorphic to V for $i \geq 3$ and any irreducible submodule of $PS(\chi)$ must be isomorphic to V. Let $V' \neq V$ be an irreducible submodule. We obviously have $V' \cap V^\perp = V' \cap V = \{0\}$. Therefore $PS(\chi) = V^\perp \oplus V'$ and we get a non-degenerate sesqui-linear form on $V \times V'$ by the restriction of $(\, , \,)$. This implies that $(\, , \,)|V \oplus V'$ is non-degenerate. Hence $V \oplus V'$ has the orthogonal complement W such that $PS(\chi) = V \oplus V' \oplus W$. (The existence of the orthogonal complement can be justified easily by considering $(\, , \,)$ on the spaces of U-fixed vectors for open subgroups U of K.) Therefore by (2) and Lemma 4.6, we must have $W = \{0\}$; hence the assertion follows.

Thus we may assume that $(\, , \,)|V$ is non-degenerate for every irreducible G-submodule V. Let V_1 be an irreducible G-submodule of $PS(\chi)$. Let V_2 be an irreducible G-submodule of the orthogonal complement of V_1. Then $(\, , \,)|V_1 \oplus V_2$ is non-degenerate. If $PS(\chi) \supsetneq V_1 \oplus V_2$, we obtain an irreducible G-submodule V_3 in the orthogonal complement of $V_1 \oplus V_2$, which contradicts (2) and Lemma 4.6. This completes the proof.

§ 9. The case of type C_ℓ

For the root system, we use the notation given in Bourbaki [7]. The simple roots are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3, \cdots, \alpha_{\ell-1} = \varepsilon_{\ell-1} - \varepsilon_\ell,\ \alpha_\ell = 2\varepsilon_\ell.$$

Put $\sigma_i = \sigma_{\alpha_i}$, $1 \le i \le \ell$ as before. Then we have

$$w_0 = (\sigma_1\sigma_2\cdots\sigma_{\ell-1}\sigma_\ell\sigma_{\ell-1}\cdots\sigma_2\sigma_1)\cdots(\sigma_{\ell-1}\sigma_\ell\sigma_{\ell-1})\sigma_\ell, \quad w_0\varepsilon_i = -\varepsilon_i,\ 1 \le i \le \ell.$$

We have

$$P(\Sigma) = \oplus_{i=1}^{\ell} \mathbf{Z}\varepsilon_i, \qquad Q(\Sigma) = \{\sum_{i=1}^{\ell} a_i\varepsilon_i | a_i \in \mathbf{Z},\ \sum_{i=1}^{\ell} a_i \equiv 0 \bmod 2\}.$$

We identify $\{\varepsilon_i\}$ with its dual basis with respect to $\langle\ ,\ \rangle$. Then we have

$$2\check{\varepsilon}_i = \varepsilon_i\ (1 \le i \le \ell), \qquad \varepsilon_i \check{\pm} \varepsilon_j = \varepsilon_i \pm \varepsilon_j\ (1 \le i < j \le \ell),$$

$$P(\check{\Sigma}) = \oplus_{i=1}^{\ell} \mathbf{Z}\varepsilon_i + \mathbf{Z}\left(\frac{1}{2}\sum_{i=1}^{\ell} \varepsilon_i\right), \qquad Q(\check{\Sigma}) = \oplus_{i=1}^{\ell} \mathbf{Z}\varepsilon_i.$$

We assume that $\mathbf{G}$ is of adjoint type. For $z = \sum_{i=1}^{\ell} c_i\varepsilon_i \in Q(\Sigma)$ and $v = \sum_{i=1}^{\ell} a_i\varepsilon_i \in V$, $a_i \in \mathbf{R}$, we define $\chi(v) \in X$ by

$$\chi(v)(\beta(\varpi)) = (-1)^{\langle z,\beta\rangle} q^{\langle v,\beta\rangle}, \qquad \beta \in P(\check{\Sigma}). \tag{9.1}$$

We have

$$\begin{aligned} &\chi(v)((2\check{\varepsilon}_i)(\varpi)) = (-1)^{c_i} q^{a_i}, && 1 \le i \le \ell, \\ &\chi(v)((\varepsilon_i \check{-} \varepsilon_j)(\varpi)) = (-1)^{c_i - c_j} q^{a_i - a_j}, && 1 \le i < j \le \ell, \\ &\chi(v)((\varepsilon_i \check{+} \varepsilon_j)(\varpi)) = (-1)^{c_i + c_j} q^{a_i + a_j}, && 1 \le i < j \le \ell. \end{aligned} \tag{9.2}$$

First we consider the case $z = 0$. The family of hyperplanes $\mathfrak{H}$ in V considered in §8 are

$$a_i = \pm 1\ (1 \le i \le \ell), \qquad a_i \pm a_j = \pm 1\ (1 \le i < j \le \ell).$$

We consider a connected component D of $V - \mathfrak{H}$. Replacing D by wD, $w \in W$, we may assume that D contains a point $v = \sum_{i=1}^{\ell} a_i\varepsilon_i$ such that $\langle v, \check{\alpha}_i\rangle > 0$ for every $1 \le i \le \ell$. This condition is equivalent to $a_1 > a_2 > \cdots > a_\ell > 0$. Take i so that $a_i > 1$, $a_{i+1} < 1$. Then D contains a point $v = \sum_{i=1}^{\ell} a_i\varepsilon_i$ such that

$$a_1 > a_2 > \cdots > a_i > 1 > a_{i+1} > \cdots > a_\ell > 0. \tag{9.3}$$

We may assume that D is bounded. Let $1 \le j \le i$ and suppose $a_j - a_{j+1} > 1$. Then we have

$$a_u - a_s > 1, \quad 1 \le u \le j, \quad j < s \le \ell.$$

Hence we see that

$$\sum_{u=1}^{j} (a_u + t)\varepsilon_i + \sum_{u=j+1}^{\ell} a_u\varepsilon_j \in D$$

for every $t > 0$. Thus D is not bounded. We must have

$$(9.4) \qquad a_j - a_{j+1} < 1, \qquad 1 \leq j \leq i.$$

For $1 \leq j \leq \ell - 1$, we define $p(j)$ as the greatest integer p such that $a_j - a_p < 1$, $j < p \leq \ell$. Obviously we have

$$2 \leq p(1) \leq p(2) \leq \cdots \leq p(i) \leq p(i+1) = \cdots = p(\ell - 1) = \ell.$$

For $2 \leq j \leq \ell$, we define $q(j)$ as the least integer q such that $a_q - a_j < 1$, $1 \leq q < j$. We have

$$1 = q(2) \leq q(3) \leq \cdots \leq q(\ell) \leq i + 1.$$

The function q can be determined by the function p in the obvious manner. For $i + 1 \leq j \leq \ell$, let $r(j)$ be the greatest integer such that $a_j + a_r > 1$, $j \neq r$, $1 \leq r \leq \ell$. If such $r(j)$ does not exist for some j, we have $i = 0$, $a_1 + a_\ell < 1$. Then we find $\sigma_\alpha D = D$ for $\alpha = 2\varepsilon_\ell$. Therefore in this case, we can apply the results in §8. We may assume that $r(j)$ exists for $i + 1 \leq j \leq \ell$. We have

$$\ell \geq r(i+1) \geq \cdots \geq r(\ell) \geq i.$$

The functions p and r completely determine the shape of the domain D.

Lemma 9.1. *Suppose that one of the following conditions are satisfied.*
(1) *For some* $1 \leq j \leq i - 1$, $p(j) = p(j+1)$ *and* $q(j) = q(j+1)$. (*If* $j = 1$, *we understand* $q(1) = q(2)$.)
(2) *For some* $i + 1 \leq j \leq \ell - 1$, $q(j) = q(j+1)$, *and* $r(j) = r(j+1)$ *or* $r(j) = j + 1$, $r(j+1) = j$.
Then there exists $\check{\alpha} \in \check{\Sigma}$ *such that* $\langle v_0, \check{\alpha} \rangle = 0$ *for some* $v_0 \in D$.

Proof. Suppose that (2) is satisfied. Among the inequalities defining the domain D, we consider those involving the variables a_j and a_{j+1}. Put $q = q(j)$. We have

$$(9.5) \qquad 1 > a_t - a_j > -1,\ 1 > a_t - a_{j+1} > -1,\ t \geq q,\ t \neq j,\ j+1,$$

$$(9.6) \qquad a_t - a_j > 1,\ a_t - a_{j+1} > 1,\ 1 \leq t < q,$$

$$(9.7) \qquad 1 > a_j - a_{j+1} > -1.$$

If $r(j) = r(j+1)$, we have, putting $r = r(j)$,

$$(9.8) \qquad a_t + a_j > 1,\ a_t + a_{j+1} > 1,\ t \leq r,$$

$$(9.9) \qquad -1 < a_t + a_j < 1,\ -1 < a_t + a_{j+1} < 1,\ t > r,\ t \neq j,\ j+1,$$

$$(9.10) \qquad a_j + a_{j+1} > 1 \text{ if } r > j+1,\ -1 < a_j + a_{j+1} < 1 \text{ if } r < j.$$

If $r(j) = j + 1$, $r(j+1) = j$, we have

$$(9.8') \qquad a_t + a_j > 1,\ a_t + a_{j+1} > 1,\ t < j,$$

$$(9.9') \qquad -1 < a_t + a_j < 1,\ -1 < a_t + a_{j+1} < 1,\ t > j+1,$$

(9.10′) $$a_j + a_{j+1} > 1.$$

By (9.5) ~ (9.7), and (9.8) ~ (9.10) or (9.8′) ~ (9.10′) according as the cases, we see that $\sigma_\alpha D = D$ for $\alpha = \varepsilon_j - \varepsilon_{j+1}$, i.e., D is invariant under the permutation of variables a_j, a_{j+1}. Our assertion follows from Lemma 8.4.

If (1) is satisfied, we find $\sigma_\alpha D = D$ for $\alpha = \varepsilon_j - \varepsilon_{j+1}$ similarly as above. This completes the proof.

Our main objective in this section is to prove the following Theorem.

Theorem 9.2. *If the points of D represent unitarizable representations, we have*

$$-1 < a_i < 1, \ 1 \le i \le \ell.$$

First we state a consequence of this Theorem.

Lemma 9.3. *If all the points of D satisfy*

$$-1 < a_i < 1, \ 1 \le i \le \ell,$$

then $\sigma_\alpha D = D$ for some $\alpha \in \Sigma^+$.

Proof. We may assume that D contains a point $v = \sum_{i=1}^{\ell} a_i \varepsilon_i$ such that

$$1 > a_1 > a_2 > \cdots > a_\ell > 0.$$

We have $q(i) = 1$, $2 \le i \le \ell$. Assume that there does not exist $\alpha \in \Sigma^+$ such that $\sigma_\alpha D = D$. By Lemma 9.1, (2), we must have

(9.11) $$r(1) > r(2) > \cdots > r(\ell - 1) > r(\ell).$$

Since $\ell \ge r(1)$, $r(\ell) \ge 1$, we immediately obtain

$$r(i) = \ell + 1 - i, \qquad 1 \le i \le \ell$$

by (9.11). If ℓ is odd, we have $r\left(\dfrac{\ell+1}{2}\right) = \dfrac{\ell+1}{2}$, which contradicts the definition of r. If ℓ is even, put $\ell = 2n$. We have $r(n) = n + 1$, $r(n+1) = n$. Applying Lemma 9.1, (2), we get a contradiction. This completes the proof.

Therefore as explained in §8, it suffices to prove Theorem 9.2 to determine the unitarizability. The following Lemma shall play a crucial role in the proof.

Lemma 9.4. *Let $\ell \ge 4$. Suppose that $\overline{wD}$, the closure of wD for some $w \in W$, contains a point v_0 of the form*

$$v_0 = (a-1)(\varepsilon_1 - \varepsilon_2) - a\varepsilon_3 + (a-2)\varepsilon_4 + \sum_{j=5}^{\ell} b_j \varepsilon_j$$

which satisfies the following conditions.

(1) $b_j \pm b_t \ne 0, \pm 1$, $5 \le j < t \le \ell$, $b_j \ne 0, \pm 1$, $5 \le j \le \ell$.

(2) $a \neq -1, 0, 1/2, 1, 3/2, 2, 3.$

(3) $b_j \neq \pm(a+1),\ \pm a,\ \pm(a-1),\ \pm(a-2),\ \pm(a-3),\ 5 \leq j \leq \ell.$

Then all the points of D represent non-unitarizable representations.

Proof. Assume that a point of D represents a unitarizable representation. Then all the points of wD represent unitarizable representations as shown in §8. By virtue of a result of Tadič [26], Theorem 2.7, we see that all the composition factors of $PS(\chi(v))$ for $v \in \overline{wD}$ are unitarizable, in particular hermitian.

Define $w_2 \in W$ by

$$w_2 \varepsilon_j = \varepsilon_j,\ 1 \leq j \leq 2, \qquad w_2 \varepsilon_j = -\varepsilon_j,\ 3 \leq j \leq \ell.$$

Put $\chi_0 = \chi(v_0)$. By the assumptions (1) ~ (3) and by Lemma 4.3, we see that $T_{\sigma_1 w_2}$ is holomorphic at χ_0 and $\mathrm{Ker}(T_{\sigma_1 w_2}(\chi_0)) = \{0\}$. We have $\sigma_1 w_2 \chi_0 = \bar{\chi}_0^{-1}$. Hence

$$(\varphi_1, \varphi_2) = c_{\sigma_2 w_2}(\chi_0)^{-1} \int_{B\backslash G} (T_{\sigma_2 w_2}(\chi_0)\varphi_1)(g)\, \overline{\varphi_2(g)}\, dg,\ \varphi_1, \varphi_2 \in PS(\chi_0)$$

defines a non-degenerate invariant hermitian form on $PS(\chi_0)$. We have $|W_{\chi_0}| = 2$ by the assumptions (1) ~ (3). Let V be the G-submodule of $PS(\chi_0)$ generated by φ_{K,χ_0}. By (4.18) and (4.14), we see that

$$T_{\sigma_2\sigma_1 w_2}(\chi_0)\varphi_{K,\chi_0} = c\varphi_{K,\chi_0},\ c \neq 0, \qquad PS(\chi_0) \supsetneq \mathrm{Ker}(T_{\sigma_2\sigma_1 w_2}(\chi_0)) \supsetneq \{0\}.$$

Let

$$\psi(g) = \sum_i c_i \varphi_{K,\chi_0}(gg_i) \in V, \qquad c_i \in \mathbf{C},\ g, g_i \in G.$$

Then we have

$$(T_{\sigma_2\sigma_1 w_2}(\chi_0)\psi)(g) = c\sum_i c_i \varphi_{K,\chi_0}(gg_i), \qquad g \in G.$$

By this formula, we find $V \cap \mathrm{Ker}(T_{\sigma_2\sigma_1 w_2}(\chi_0)) = \{0\}$. Hence $PS(\chi_0)$ has two irreducible G-submodules. Therefore, by Proposition 8.5, $PS(\chi_0)$ must be a direct sum of two irreducible G-submodules. Since

$$PS(\chi_0) \supsetneq \mathrm{Ker}(T_{\sigma_2}(\chi_0)) \supsetneq \{0\},$$

$\mathrm{Ker}(T_{\sigma_2}(\chi_0))$ must be irreducible. On the otherhand, it is easy to see that

$$\begin{aligned} &\mathrm{Ker}(T_{\sigma_1\sigma_2\sigma_3}(\chi_0)) \supsetneq \mathrm{Ker}(T_{\sigma_2}(\chi_0)), \\ &\mathrm{Ker}(T_{\sigma_1\sigma_2\sigma_3}(\chi_0)) \cap \mathrm{Ker}(T_{\sigma_2}(\chi_0)) \ni \varphi_{K,\chi_0}. \end{aligned} \tag{9.12}$$

(A simple way to see (9.12) is to use Casselman's formula [9], 3.4.) This is a contradiction and completes the proof.

Proof of Theorem 9.2. We may assume $i \geq 1$ in (9.3). It suffices to show that a point of D represents non-unitarizable representation. We assume $\ell \geq 2$ and set the hypothesis of induction, i.e., we assume that the theorem holds for groups of type C whose ranks $\leq \ell - 1$. Throughout the proof, we let $\tilde{V} = \bigoplus_{i=1}^{\ell+2} \mathbf{R}\varepsilon_i$ be the vector space attached to the adjoint group of type $C_{\ell+2}$ similarly as above. In the proof, we shall consider a domain $\tilde{D}$ which contains a point $v \in \tilde{V}$. By this term, we shall always understand that $v \notin \tilde{\mathfrak{H}}$ and that $\tilde{D} \ni v$ is the connected component of $\tilde{V} - \tilde{\mathfrak{H}}$, where $\tilde{\mathfrak{H}}$ is the family of hyperplanes in $\tilde{V}$ similary defined as above. We shall prove the theorem by contradiction. Thus we assume that all the points of D reresent unitarizable representations. The proof is rather involved. We divide it into several cases.

CASE (A) We assume $i = \ell - 1$.

Thus we assume that D contains a point $v = \sum_{j=1}^{\ell} a_j \varepsilon_j$ such that

$$a_1 > a_2 > \cdots > a_{\ell-1} > 1 > a_\ell > 0.$$

First we note the following fact. Suppose that $a_{j+1}, \ldots, a_\ell$ are chosen so that

$$a_{j+1} > \cdots > a_{\ell-1} > 1 > a_\ell > 0,\ a_t - a_{p(t)} < 1,\ a_t - a_{p(t)+1} > 1,\ j+1 \leq t \leq \ell - 1.$$

Then choose a_j so that

$$1 + a_{p(j)+1} < a_j < 1 + a_{p(j)}, \qquad a_j > a_{j+1}.$$

This choice is always possible since $a_{j+1} < 1 + a_{p(j+1)} \leq 1 + a_{p(j)}$. Repeating this procedure, we can construct a point in D.

Let $q = q(\ell)$. We have

$$a_q - a_\ell < 1, \qquad a_{q-1} - a_\ell > 1. \tag{9.13}$$

We can choose a_ℓ so that

$$0 < a_\ell < 1/2. \tag{9.14}$$

We have $p(q) = \ell$, $p(q-1) \leq \ell - 1$. By the remark above, a_q can attain values less than $1 + a_\ell$ and a_{q-1} can attain values less than $1 + a_{p(q-1)}$. Hence we find easily that we can choose $a_{q-1} > a_q$ so that

$$a_{q-1} + a_q > 3. \tag{9.15}$$

By (9.13), (9.14) and $a_q > a_{q+1}$, we have

$$a_q < 3/2, \qquad a_q + a_{q+1} < 3. \tag{9.16}$$

In $\tilde{V}$, we consider a domain $\tilde{D}$ which contains a point $\sum_{j=1}^{\ell} a_j \varepsilon_j + b_1 \varepsilon_{\ell+1} + b_2 \varepsilon_{\ell+2}$ such that $\sum_{j=1}^{\ell} a_j \varepsilon_j \in D$, $a_1 > \cdots > a_{\ell-1} > 1 > a_\ell > 0$, $1 > b_1$, $b_2 > 0$,

$$a_q - b_j < 1,\ a_{q-1} - b_j > 1,\ a_\ell + b_j < 1,\ j = 1, 2, \tag{9.17}$$

$$b_1 + b_2 < 1. \tag{9.18}$$

The conditions on b_j are

$$a_q - 1 < b_j < a_{q-1} - 1, \qquad b_j < 1 - a_\ell$$

which are satisfied by 1/2. Hence we see that there exists b_1 and b_2 which satisfy (9.17) and (9.18), i.e., the domain $\tilde{D}$ is non-empty.

We have $\sigma_\alpha \tilde{D} = \tilde{D}$ for $\alpha = \varepsilon_{\ell+1} - \varepsilon_{\ell+2}$. By Remark 8.3 and (9.18), we see that all the points of $\tilde{D}$ represent unitarizable representations. Assume $\sum_{j=1}^{\ell} a_j \varepsilon_j + b_1 \varepsilon_{\ell+1} + b_2 \varepsilon_{\ell+2} \in \tilde{D}$, the conditions (9.13) $\sim$ (9.18) being retained. We set

$$a'_\ell = a_q - 1, \quad b'_1 = a_q - 1, \quad b'_2 = 2 - a_q.$$

Then we have $a'_\ell = b'_1 < b'_2$ by (9.16). We have

$$a_q - a'_\ell = 1, \qquad a_{q-1} - b'_2 = a_{q-1} + a_q - 2 > 1$$

by (9.15). We also have

$$a'_\ell + b'_1 = 2a_q - 2 < 1, \qquad a'_\ell + b'_2 = b'_1 + b'_2 = 1$$

by (9.16). Therefore we see that

$$\sum_{j=1}^{\ell-1} a_j \varepsilon_j + (a_q - 1)\varepsilon_\ell + (a_q - 1)\varepsilon_{\ell+1} + (2 - a_q)\varepsilon_{\ell+2} \in \bar{\tilde{D}}.$$

Hence, for some $w \in \tilde{W}$, we have

$$(a_q - 1)(\varepsilon_1 - \varepsilon_2) - a_q \varepsilon_3 + (a_q - 2)\varepsilon_4 + \sum_{j=1}^{q-1} a_j \varepsilon_{j+4} + \sum_{j=q+1}^{\ell-1} a_j \varepsilon_{j+3} \in \overline{w\tilde{D}}.$$

Here $\tilde{W}$ is the Weyl group of type $C_{\ell+2}$. Since we could have chosen a_j, $1 \leq j \leq \ell - 1$, $j \neq q$ in "generic position"[2], we can apply Lemma 9.4. We have obtained a contradiction.

We therefore may assume $\ell - i \geq 2$, $i \geq 1$. We shall consider three cases.

CASE (B) We assume $a_{i+1} + a_{i+2} > 1$, $a_{\ell-1} + a_\ell < 1$.

The assumption implies $\ell - i \geq 3$. We can find s, $i + 2 \leq s \leq \ell - 1$ so that

$$a_{s-1} + a_s > 1, \qquad a_s + a_{s+1} < 1. \tag{9.19}$$

First we note the following fact. We choose $a_s > a_{s+1} > \cdots > a_\ell > 0$ so that $a_s + a_{s+1} < 1$. We choose a_j for $s - 1 \geq i + 1$ in the following way. Suppose that $a_{j+1}, \ldots, a_\ell$ are chosen. Then choose a_j so that

$$1 - a_{r(j)} < a_j < 1 - a_{r(j)+1}, \qquad a_{j+1} < a_j.$$

Since we have $r(j) \geq s$ for $s - 1 \geq j \geq i + 1$, we see easily that this is always possible. Then $a_{i+1}, \ldots, a_\ell$ satisfy the required properties to belong D concerning the additions among them. Let $1 \leq j \leq i$ and suppose that $a_{j+1}, \ldots, a_\ell$ are chosen. We can choose a_j so that

[2] This remark shall apply to the succeeding arguments as well. We shall not repeat it.

$$1 + a_{p(j)+1} < a_j < 1 + a_{p(j)}, \qquad a_j > a_{j+1}.$$

By successive application of this procedure, we can construct a point in D.

Let $q = q(s)$. We have

$$a_q - a_s < 1, \qquad a_{q-1} - a_s > 1. \tag{9.20}$$

First we assume

SUBCASE (I) $a_q - a_{s+1} > 1$.

By the remark above and the assumption, we can choose a_q, a_{q+1} and a_s so that

$$a_s > 1/2, \quad a_q > 3/2, \quad a_q + a_{q+1} < 3. \tag{9.21}$$

In $\tilde{V}$, we consider a domain $\tilde{D}$ which contains a point $\sum_{j=1}^{\ell} a_j\varepsilon_j + b_1\varepsilon_{\ell+1} + b_2\varepsilon_{\ell+2}$ such that

$$\sum_{j=1}^{\ell} a_j\varepsilon_j \in D, \quad a_1 > \cdots > a_i > 1 > a_{i+1} > \cdots > a_\ell > 0, \quad 1 > b_1,\ b_2 > 0,$$

$$a_q - b_j > 1,\ a_{q+1} - b_j < 1,\ a_s + b_j > 1,\ a_{s+1} + b_j < 1,\ j = 1, 2, \tag{9.22}$$

$$b_1 + b_2 < 1. \tag{9.23}$$

The conditions on b_j are

$$a_q - 1 > b_j > a_{q+1} - 1, \qquad 1 - a_{s+1} > b_j > 1 - a_s,$$

which are satisfied by $1/2$. Hence we see that $\tilde{D} \neq \emptyset$.

By Remark 8.3 and (9.23), we see that all the points of $\tilde{D}$ represent unitarizable representations. Take $\sum_{j=1}^{\ell} a_j\varepsilon_j + b_1\varepsilon_{\ell+1} + b_2\varepsilon_{\ell+2} \in \tilde{D}$, the conditions (9.20) $\sim$ (9.23) being retained. Set

$$a_s' = a_q - 1, \quad b_1' = a_q - 1, \quad b_2' = 2 - a_q.$$

We have $a_s' = b_1' > b_2'$ by (9.21). We have

$$a_q - a_s' = 1, \quad a_q - b_2' > 1, \quad a_{q+1} - b_2' = a_q + a_{q+1} - 2 < 1$$

by (9.23). We also have

$$a_{s-1} + b_2' = 2 - (a_q - a_{s-1}) > 1, \qquad a_{s+1} + a_s' = (a_q - a_s) + (a_s + a_{s+1}) - 1 < 1.$$

Therefore we see that

$$\sum_{j=1, j\neq s}^{\ell} a_j\varepsilon_j + (a_q - 1)\varepsilon_s + (a_q - 1)\varepsilon_{\ell+1} + (2 - a_q)\varepsilon_{\ell+2} \in \bar{\tilde{D}}.$$

In the same way as in Case (A), we obtain a contradiction.

Next we assume

SUBCASE (II) $a_q - a_{s+1} < 1$.

By (9.20) and the assumption, we have $q(s) = q(s+1) = q$. By Lemma 9.1,

(2) and the induction hypothesis, we may assume $r(s+1) < r(s)$. By (9.19), we have $r(s) = s - 1$. Thus we have $a_{s+1} + a_{s-1} < 1$. Then it follows $r(s-1) = s$. Again by Lemma 9.1, (2) and the induction hypothesis, we may assume $q(s-1) < q(s)$. Thus we have

$$(9.24) \qquad a_{s-1} + a_{s+1} < 1, \qquad a_{q-1} - a_{s-1} < 1.$$

By $a_{q-1} + a_{s+1} = (a_{q-1} - a_{s-1}) + (a_{s-1} + a_{s+1})$, we get

$$(9.25) \qquad a_{q-1} + a_{s+1} < 2.$$

Now we can choose a_{q-1}, a_q and a_s so that

$$(9.26) \qquad a_s > 1/2, \quad a_{q-1} > 3/2, \quad a_{q-1} + a_q < 3.$$

In $\tilde{V}$, we consider a domain $\tilde{D}$ which contains a point $\sum_{j=1}^{\ell} a_j\varepsilon_j + b_1\varepsilon_{\ell+1} + b_2\varepsilon_{\ell+2}$ such that

$$\sum_{j=1}^{\ell} a_j\varepsilon_j \in D, \quad a_1 > \cdots > a_i > 1 > a_{i+1} > \cdots > a_\ell > 0, \quad 1 > b_1,\ b_2 > 0,$$

$$(9.27) \qquad a_q - b_j < 1,\ a_{q-1} - b_j > 1,\ a_s + b_j > 1,\ a_{s+1} + b_j < 1,\ j = 1, 2,$$

$$(9.28) \qquad b_1 + b_2 < 1.$$

The conditions on b_j are

$$a_{q-1} - 1 > b_j > a_q - 1, \qquad 1 - a_{s+1} > b_j > 1 - a_s,$$

which are satisfied by $1/2$. Hence have $\tilde{D} \neq \emptyset$.

We see that all the points of $\tilde{D}$ represent unitarizable representations. Take $\sum_{j=1}^{\ell} a_j\varepsilon_j + b_1\varepsilon_{\ell+1} + b_2\varepsilon_{\ell+2} \in \tilde{D}$, the conditions (9.24) ~ (9.28) being retained. Set

$$a_s' = a_{q-1} - 1, \quad b_1' = a_{q-1} - 1, \quad b_2' = 2 - a_{q-1}.$$

We have $a_s' = b_1' > b_2'$ by (9.26). We have

$$a_{q-1} - a_s' = 1, \qquad a_q - b_2' = a_{q-1} + a_q - 2 < 1$$

by (9.26). We also have $a_{s-1} + b_2' = 2 - (a_{q-1} - a_{s-1}) > 1$ by (9.24), $a_{s+1} + a_s' = (a_{q-1} + a_{s+1}) - 1 < 1$ by (9.25). Therefore we see that

$$\sum_{j=1, j\neq s} a_j\varepsilon_j + (a_{q-1} - 1)\varepsilon_s + (a_{q-1} - 1)\varepsilon_{\ell+1} + (2 - a_{q-1})\varepsilon_{\ell+2} \in \bar{\tilde{D}}.$$

In the same way as in Case (A), we obtain a contradiction. This finishes the proof in Case (B).

Case (C) We assume $a_{i+1} + a_{i+2} < 1$.

In this case, we have

$$(9.29) \qquad r(j) = i, \qquad i + 1 \leq j \leq \ell.$$

Let $q = q(i+1)$. We have

$$a_q - a_{i+1} < 1, \qquad a_{q-1} - a_{i+1} > 1. \tag{9.30}$$

By (9.29), Lemma 9.1, (2) and the induction hypothesis, we may assume $q(i+2) < q(i+1)$. Hence we get

$$a_q - a_{i+2} > 1. \tag{9.31}$$

By $a_q + a_{i+2} = (a_q - a_{i+1}) + (a_{i+1} + a_{i+2})$, we have

$$a_q + a_{i+2} < 2. \tag{9.32}$$

By (9.31), we see that we can choose a_q, a_{q+1} and a_{i+1} so that

$$a_{i+1} > 1/2, \quad a_q > 3/2, \quad a_q + a_{q+1} < 3. \tag{9.33}$$

In $\tilde{V}$, we consider a domain $\tilde{D}$ which contains a point $\sum_{j=1}^{\ell} a_j\varepsilon_j + b_1\varepsilon_{\ell+1} + b_2\varepsilon_{\ell+2}$ such that

$$a_q - b_j > 1, \quad a_{q+1} - b_j < 1, \quad a_{i+1} + b_j > 1, \quad a_{i+2} + b_j < 1, \ j = 1, 2, \tag{9.34}$$

$$1 > b_1, \ b_2 > 0, \qquad b_1 + b_2 < 1. \tag{9.35}$$

The conditions on b_j are

$$a_q - 1 > b_j > a_{q+1} - 1, \qquad 1 - a_{i+2} > b_j > 1 - a_{i+1},$$

which are satisfied by $1/2$. Hence we have $\tilde{D} \neq \emptyset$. We see that all the points of $\tilde{D}$ represent unitarizable representations. Take $\sum_{j=1}^{\ell} a_j\varepsilon_j + b_1\varepsilon_{\ell+1} + b_2\varepsilon_{\ell+2} \in \tilde{D}$, the conditions (9.30) $\sim$ (9.35) being retained. Set

$$a'_{i+1} = a_q - 1, \quad b'_1 = a_q - 1, \quad b'_2 = 2 - a_q.$$

Then we have $a'_{i+1} = b'_1 > b'_2$. We get

$$a_q - a'_{i+1} = 1, \qquad a_{q+1} - b'_2 = a_q + a_{q+1} - 2 < 1$$

by (9.33). We also have

$$a'_{i+1} + b'_1 > 1, \qquad a'_{i+1} + a_{i+2} = a_q + a_{i+2} - 1 < 1$$

by (9.32). Therefore we have

$$\sum_{j=1, j\neq i+1}^{\ell} a_j\varepsilon_j + (a_q - 1)\varepsilon_{i+1} + (a_q - 1)\varepsilon_{\ell+1} + (2 - a_q)\varepsilon_{\ell+2} \in \bar{\tilde{D}}.$$

In the same way as in Case (A), we obtain a contradiction.

It remains to consider

CASE (D) We assume $a_{\ell-1} + a_\ell > 1$.

In this case, we have

$$r(j) = \ell, \quad i + 1 \leq j \leq \ell - 1, \quad r(\ell) = \ell - 1. \tag{9.36}$$

Let $q = q(\ell)$. We have

$$a_q - a_\ell < 1, \qquad a_{q-1} - a_\ell > 1. \tag{9.37}$$

By (9.36), Lemma 9.1, (2) and the induction hypothesis, we may assume $q(\ell - 1) < q(\ell)$. Then we have

$$a_{q-1} - a_{\ell-1} < 1. \tag{9.38}$$

We can choose a_{q-1}, a_q and a_ℓ so that

$$a_\ell > 1/2, \quad a_{q-1} > 3/2, \quad a_{q-1} + a_q < 3. \tag{9.39}$$

In $\tilde{V}$, we consider a domain $\tilde{D}$ which contains a point $\sum_{j=1}^{\ell} a_j\varepsilon_j + b_1\varepsilon_{\ell+1} + b_2\varepsilon_{\ell+2}$ such that

$$a_q - b_j < 1, \quad a_{q-1} - b_j > 1, \quad a_\ell + b_j > 1, \quad j = 1, 2, \tag{9.40}$$

$$1 > b_1,\ b_2 > 0, \qquad b_1 + b_2 < 1. \tag{9.41}$$

The conditions on b_j are

$$a_q - 1 < b_j < a_{q-1} - 1, \qquad 1 - a_\ell < b_j,$$

which are satisfied by $1/2$. Hence we have $\tilde{D} \neq \emptyset$. We see that all the points of $\tilde{D}$ represent unitarizable representations. Take $\sum_{j=1}^{\ell} a_j\varepsilon_j + b_1\varepsilon_{\ell+1} + b_2\varepsilon_{\ell+2} \in \tilde{D}$, the conditions (9.37) $\sim$ (9.41) being retained. Set

$$a'_\ell = a_{q-1} - 1, \quad b'_1 = a_{q-1} - 1, \quad b'_2 = 2 - a_{q-1}.$$

We have $a'_\ell = b'_1 > b'_2$. We get $a_q - b'_2 = a_{q-1} + a_q - 2 < 1$ by (9.39). We also have $a_{\ell-1} + b'_2 = 2 - (a_{q-1} - a_{\ell-1}) > 1$ by (9.38). Therefore we obtain

$$\sum_{j=1}^{\ell} a_j\varepsilon_j + (a_{q-1} - 1)\varepsilon_\ell + (a_{q-1} - 1)\varepsilon_{\ell+1} + (2 - a_{q-1})\varepsilon_{\ell+2} \in \bar{\tilde{D}}.$$

In the same way as in Case (A), we obtain a contradiction. This completes the proof of Theorem 9.2.

Now we consider the general case where $z = \sum_{i=1}^{\ell} c_i\varepsilon_i \in Q(\Sigma)$ is not necessarily 0. Replacing $\chi(v)$ by $w(\chi(v))$, we may assume

$$c_i = 0,\ 1 \le i \le n, \qquad c_i = 1,\ n+1 \le i \le \ell.$$

We see that $\ell - n$ is even. The family of hyperplanes in V considered in §8 are

$$a_i = \pm 1 \ (1 \le i \le n), \quad a_i \pm a_j = \pm 1 \ (1 \le i < j \le n), \quad a_i \pm a_j = \pm 1 \ (n+1 \le i < j \le \ell).$$

This shows that we can treat the variables a_i $(1 \le i \le n)$ and a_j $(n+1 \le j \le \ell)$ separately. We can normalize $a_{n+1}, \dots, a_\ell$ so that

$$a_{n+1} > a_{n+2} > \cdots > a_{\ell-1} > a_\ell > 0.$$

Then we obtain

(9.42) $$a_{n+1} - a_{\ell} < 1$$

by the same proof as Theorem 9.2. Then we see that $\sigma_{\alpha} D = D$ for some $\alpha \in \Sigma^+$ by the same proof as Lemma 9.3. This completes the determination of the unitarizability for groups of type C_{ℓ}.

§10. The case of type B_{ℓ}

The simple roots are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3, \cdots, \alpha_{\ell-1} = \varepsilon_{\ell-1} - \varepsilon_{\ell},\ \alpha_{\ell} = \varepsilon_{\ell},$$

$$P(\Sigma) = \oplus_{i=1}^{\ell} \mathbf{Z}\varepsilon_i + \mathbf{Z}\left(\frac{1}{2}\sum_{i=1}^{\ell} \varepsilon_i\right), \qquad Q(\Sigma) = \oplus_{i=1}^{\ell} \mathbf{Z}\varepsilon_i.$$

We identify $\{\varepsilon_i\}$ with its dual basis with respect to $\langle\ ,\ \rangle$. Then we have

$$\check{\varepsilon}_i = 2\varepsilon_i (1 \le i \le \ell),\ \varepsilon_i \check{\pm} \varepsilon_j = \varepsilon_i \pm \varepsilon_j\ (1 \le i < j \le \ell),$$

$$P(\check{\Sigma}) = \oplus_{i=1}^{\ell} \mathbf{Z}\varepsilon_i, \qquad Q(\check{\Sigma}) = \{\sum_{i=1}^{\ell} a_i \varepsilon_i | a_i \in \mathbf{Z},\ \sum_{i=1}^{\ell} a_i \equiv 0 \bmod 2\}.$$

We have

$$w_0 \varepsilon_i = -\varepsilon_i,\ 1 \le i \le \ell,\ w_0 = (\sigma_1 \sigma_2 \cdots \sigma_{\ell-1} \sigma_{\ell} \sigma_{\ell-1} \cdots \sigma_2 \sigma_1) \cdots (\sigma_{\ell-1} \sigma_{\ell} \sigma_{\ell-1}) \sigma_{\ell}.$$

We assume that $\mathbf{G}$ is of adjoint type. For $z = \sum_{i=1}^{\ell} c_i \varepsilon_i \in Q(\Sigma)$ and $v = \sum_{i=1}^{\ell} a_i \varepsilon_i \in V$, $a_i \in \mathbf{R}$, we define $\chi(v) \in X$ by

(10.1) $$\chi(v)(\beta(\varpi)) = (-1)^{\langle z, \beta \rangle} q^{\langle v, \beta \rangle}, \qquad \beta \in P(\check{\Sigma}).$$

We have

(10.2)
$$\begin{aligned}
&\chi(v)((\check{\varepsilon}_i)(\varpi)) = q^{2a_i}, && 1 \le i \le \ell,\\
&\chi(v)((\varepsilon_i \check{-} \varepsilon_j)(\varpi)) = (-1)^{c_i - c_j} q^{a_i - a_j}, && 1 \le i < j \le \ell,\\
&\chi(v)((\varepsilon_i \check{+} \varepsilon_j)(\varpi)) = (-1)^{c_i + c_j} q^{a_i + a_j}, && 1 \le i < j \le \ell.
\end{aligned}$$

First we consider the case $z = 0$. The family of hyperplanes $\mathfrak{H}$ in V considered in §8 are

$$a_i = \pm 1/2\ (1 \le i \le \ell), \qquad a_i \pm a_j = \pm 1\ (1 \le i < j \le \ell).$$

Let D be a connected component of $V - \mathfrak{H}$. Replacing D by wD, $w \in W$, we may assume that D contains a point $v = \sum_{i=1}^{\ell} a_i \varepsilon_i$ such that $\langle v, \check{\alpha}_i \rangle > 0$ for every $1 \le i \le \ell$. This condition is equivalent to $a_1 > a_2 > \cdots > a_{\ell} > 0$. Take i so that $a_i > 1/2$, $a_{i+1} < 1/2$. Then D contains a point $v = \sum_{i=1}^{\ell} a_i \varepsilon_i$ such that

(10.3) $$a_1 > a_2 > \cdots > a_i > 1/2 > a_{i+1} > \cdots > a_{\ell} > 0.$$

In a similar manner as in the case of type C_{ℓ}, we see that the boundedness of D implies

(10.4) $$a_j - a_{j+1} < 1, \qquad 1 \leq j \leq \ell - 1.$$

For $1 \leq j \leq \ell - 1$, we define $p(j)$ as the greatest integer p such that $a_j - a_p < 1$, $j < p \leq \ell$. For $2 \leq j \leq \ell$, we define $q(j)$ as the least integer q such that $a_q - a_j < 1$, $1 \leq q < j$.

Lemma 10.1. *If $a_1 + a_\ell < 1$, we have $\sigma_\alpha D = D$ for $\alpha = \varepsilon_\ell$.*

Proof. We have $a_t \pm a_\ell < 1$, $1 \leq t \leq \ell - 1$ from the assumption. Hence the assertion is obvious.

We assume

(10.5) $$a_1 + a_\ell > 1.$$

For $i + 1 \leq j \leq \ell$, we define $r(j)$ as the greatest integer r such that $a_j + a_r > 1$, $j \neq r$, $1 \leq r \leq \ell$.

Lemma 10.2. *Suppose that one of the following conditions are satisfied.*

(1) *For some $1 \leq j \leq i - 1$, $p(j) = p(j+1)$ and $q(j) = q(j+1)$. (If $j = 1$, we understand $q(1) = q(2)$.)*
(2) *For some $i + 1 \leq j \leq \ell - 1$, $q(j) = q(j+1)$, and $r(j) = r(j+1)$ or $r(j) = j + 1$, $r(j+1) = j$.*

Then there exists $\check{\alpha} \in \check{\Sigma}$ such that $\langle v_0, \check{\alpha} \rangle = 0$ for some $v_0 \in D$.

The proof is identical to that of Lemma 9.1.

Lemma 10.3. *Let $\ell \geq 4$. Suppose that $\overline{wD}$, the closure of wD for some $w \in W$, contains a point v_0 of the form*

$$v_0 = (a-1)(\varepsilon_1 - \varepsilon_2) - a\varepsilon_3 + (a-2)\varepsilon_4 + \sum_{j=5}^{\ell} b_j \varepsilon_j$$

which satisfies the following conditions.

(1) $b_j \pm b_t \neq 0, \pm 1$, $5 \leq j < t \leq \ell$, $b_j \neq 0, \pm 1/2$ $5 \leq j \leq \ell$.
(2) $a \neq -1/2, 0, 1/2, 1, 3/2, 2, 5/2$.
(3) $b_j \neq \pm(a+1), \pm a, \pm(a-1), \pm(a-2), \pm(a-3)$, $5 \leq j \leq \ell$.

then all the points of D represent non-unitarizable representations.

Again the proof is identical to that of Lemma 9.4.

Theorem 10.4. *The points of D represent unitarizable representations if and only if*

$$-1/2 < a_i < 1/2, \qquad 1 \leq i \leq \ell.$$

Proof. Assume that this condition holds. Let $v_1 \in D$ and put $\chi_1 = \chi(v_1)$. Since 0, the origin of V, belongs to $\bar{D}$, we can easily find a continuous map $p\colon [0, 1] \to X_{w_0}$ so that the conditions of Proposition 7.3 are satisfied. Hence $\pi(\chi(v_1))$ is unitarizable.

We shall prove "only if" part. It suffices to show that a point of D represents a non-unitarizable representation assuming $i \geq 1$ in (10.3). We assume $\ell \geq 2$ and set the hypothesis of induction on ℓ. Throughout the proof, we let $\tilde{V} = \bigoplus_{i=1}^{\ell+2} \mathbf{R}\varepsilon_i$ be the vector space attached to the adjoint group of type $B_{\ell+2}$. In the proof, we consider a domain $\tilde{D}$ which contains a point $v \in \tilde{V}$. By this term, we shall always understand that $v \notin \tilde{\mathfrak{H}}$ and that $\tilde{D} \ni v$ is the connected component of $\tilde{V} - \tilde{\mathfrak{H}}$, where $\tilde{\mathfrak{H}}$ is the family of hyperplanes in $\tilde{V}$ similarly defined as above. We shall prove the theorem by contradiction. Thus we assume that all the points of D represent unitarizable representations.

CASE (A) We assume $a_1 - a_\ell < 1$.

Since $a_t - a_u < 1$, $1 \leq t < u \leq \ell$, we obtain

$$p(t) = \ell, \quad 1 \leq t \leq \ell - 1, \quad q(t) = 1, \quad 2 \leq t \leq \ell. \tag{10.6}$$

By Lemma 10.2 and the induction hypothesis, we may assume

$$\ell \geq r(1) > r(2) > \cdots > r(i) \geq r(i+1) > \cdots > r(\ell) \geq 1. \tag{10.7}$$

SUBCASE (I) We assume $r(i) > r(i+1)$.

By (10.7), we obtain

$$r(j) = \ell + 1 - j, \qquad 1 \leq j \leq \ell.$$

If ℓ is odd, we get $r\left(\frac{\ell+1}{2}\right) = \frac{\ell+1}{2}$, which contradicts the definition of r. We assume ℓ is even and put $\ell = 2n$. Then we have $r(n) = n+1$, $r(n+1) = n$. By Lemma 10.2 and induction hypothesis, we have $n = i$. Assume $n = 1$. Then we have $\bar{D} \ni v_1$, $v_1 = \frac{3}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2$. We see that $\chi(v_1) = \delta_B^{-1/2}$. Since $PS(\delta_B^{-1/2})$ contains a non-unitarizable constituent (cf. Borel-Wallach [6], XI), this is a contradiction. We assume $n \geq 2$. We can choose a_1 and a_2 so that $a_1 > 1$, $a_2 < 1$. In $\tilde{V}$, we consider a domain $\tilde{D}$ which contains a point $\sum_{i=1}^{\ell} a_i\varepsilon_i + b_1\varepsilon_{\ell+1} + b_2\varepsilon_{\ell+2}$ such that

$$a_1 - b_t > 1,\ a_2 - b_t < 1,\ a_1 + b_t > 1,\ a_2 + b_t < 1,\ t = 1, 2 \tag{10.8}$$

$$1/2 > a_{2n} > b_1, b_2 > 0. \tag{10.9}$$

Then we find $\tilde{D} \neq \emptyset$. We have $\sigma_\alpha \tilde{D} = \tilde{D}$ for $\alpha = \varepsilon_{\ell+1} - \varepsilon_{\ell+2}$. By Remark 8.3 and (10.9), we see that all the points of $\tilde{D}$ represent unitarizable representations.[3] Set

$$a_1' = 1 + a_{2n}, \quad a_2' = 1 - a_{2n}, \quad b_1' = a_{2n}$$

and choose b_2' so that $a_{2n} > b_2' > 0$. We have $a_2' > a_3$ since $a_3 < 1 - a_{2n}$, and $a_2' + a_{2n-1} > 1$. Hence we have

$$(1 + a_{2n})\varepsilon_1 + (1 - a_{2n})\varepsilon_2 + \sum_{t=3}^{\ell} a_t\varepsilon_t + a_{2n}\varepsilon_{\ell+1} + b_2'\varepsilon_{\ell+2} \in \bar{\tilde{D}}.$$

Note that we could have chosen a_t, $3 \le t \le \ell$ and b'_2 in "generic position"[3]. Hence we obtain a contradiction by Lemma 10.3.

In the following subcases, we shall assume $r(i) = r(i+1)$. This implies $\ell \ge 3$.

SUBCASE (II) We assume $r(j) - r(j+1) \le 1$ for all $1 \le j \le \ell - 1$.

We have either $r(1) = \ell$, $r(\ell) = 2$ or $r(1) = \ell - 1$, $r(\ell) = 1$. First assume $r(1) = \ell$, $r(\ell) = 2$. By (10.7), we have

$$r(j) = \begin{cases} \ell + 1 - j, & 1 \le j \le i, \\ \ell + 2 - j, & i + 1 \le j \le \ell. \end{cases}$$

If ℓ is odd, put $\ell = 2n - 1$, $n \ge 2$. If $n \le i$, we get $r(n) = n$, a contradiction. If $n > 1$, we get $r(n) = n + 1$, $r(n+1) = n$. By Lemma 10.2 and the induction hypothesis, we get a contradiction. If ℓ is even, put $\ell = 2n$, $n \ge 2$. If $n \ge i$, we get $r(n+1) = n + 1$, a contradiction. If $n < i$, we get $r(n) = n + 1$, $r(n+1) = n$. By Lemma 10.2 and the induction hypothesis, we get a contradiction.

Next assume $r(1) = \ell - 1$, $r(\ell) = 1$. By (10.7), we have

$$r(j) = \begin{cases} \ell - j, & 1 \le j \le i, \\ \ell + 1 - j, & i < j \le \ell. \end{cases}$$

If ℓ is odd, put $\ell = 2n - 1$, $n \ge 2$. If $n > i$, we get $r(n) = n$, a contradiction. If $n \le i$, we have $r(n-1) = n$, $r(n) = n - 1$, a contradiction. If ℓ is even, put $\ell = 2n$, $n \ge 2$. If $n \le i$, we get $r(n) = n$, a contradiction. If $n > i$, we get $r(n) = n + 1$, $r(n+1) = n$, a contradiction.

SUBCASE (III) We assume $r(j) - r(j+1) = 2$ for some j, $1 \le j \le \ell - 1$.

By (10.7), we have

$$r(1) = \ell, \qquad r(\ell) = 1. \tag{10.10}$$

Put $r_0 = r(j+1)$. By definition, we have

$$a_{j+1} + a_{r_0} > 1, \quad a_{j+1} + a_{r_0+1} < 1, \quad a_j + a_{r_0+2} > 1.$$

Hence we have $r(r_0 + 1) = r(r_0 + 2) = j$. By (10.7), we have $i = r_0 + 1$. Thus we get

$$j = r(i) = r(i+1), \qquad j \ne i,\ i + 1. \tag{10.11}$$

First assume $j > i + 1$. By (10.7), we have

$$r(t) = \begin{cases} \ell + 1 - t, & 1 \le t \le i \\ \ell + 2 - t, & i + 1 \le t \le j \\ \ell + 1 - t, & j < t \le \ell - 1. \end{cases}$$

By (10.11), we have

$$i + j = \ell + 1. \tag{10.12}$$

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[3] These remarks shall apply to the succeeding arguments as well. We shall not repeat them.

Assume ℓ is even and put $\ell = 2n$. Since $2n+1 = i+j > 2i+1$ by (10.12), we get $n \geq i+1$. If $n+1 \leq j$, we have $r(n+1) = n+1$, a contradiction. We may assume $n \geq j$. If $n = j$. we get $i = j+1$ by (10.12) which contradicts the assumption $j > i+1$. We may assume $n \geq j+1$. Then we have $r(n) = n+1$, $r(n+1) = n$. We get a contradiction by Lemma 10.2 and induction hypothesis. Assume ℓ is odd and put $\ell = 2n-1$, $n \geq 2$. By (10.12), we get $n \geq i+1$. If $n+1 \leq j$, we have $r(n) = n+1$, $r(n+1) = n$, a contradiction. We may assume $n \geq j$. If $n = j$, we get $i = j$ by (10.12), a contradiction. Hence $n \geq j+1$ and we get $r(n) = n$, a contradiction.

Next we assume $j < i$. By (10.7), we have

$$r(t) = \begin{cases} \ell + 1 - t, & 1 \leq t \leq j \\ \ell - t, & j+1 \leq t \leq i \\ \ell + 1 - t, & i+1 \leq t \leq \ell - 1. \end{cases}$$

By (10.11), we have

$$i + j = \ell. \tag{10.13}$$

Assume ℓ is even and put $\ell = 2n$. By (10.13), we have $n \geq j+1$. If $n \leq i$, we get $r(n) = n$, a contradiction. If $n \geq i+1$, we have $r(n) = n+1$, $r(n+1) = n$. By Lemma 10.2 and induction hypothesis, we get a contradiction. Assume ℓ is odd and put $\ell = 2n-1$. By (10.13), we have $n \geq j+1$. If $n \geq i+1$, we get $r(n) = n$, a contradiction. Hence we may assume $j+1 \leq n \leq i$. If $n \geq j+2$, we have $r(n-1) = n$, $r(n) = n-1$. By Lemma 10.2 and induction hypothesis, we get a contradiction. Therefore we may assume $n = j+1$. By (10.13), we obtain

$$\ell = 2n-1, \quad i = n, \quad j = n-1, \quad n \geq 2.$$

We note that

$$a_1 + a_{2n-1} > 1, \ a_2 + a_{2n-1} < 1, \quad \text{and} \quad a_2 + a_{2n-2} > 1 \ \text{if } n \geq 3.$$

We can choose a_1 and a_2 so that $a_1 > 1$, $a_2 < 1$. In $\tilde{V}$, we consider a domain $\tilde{D}$ which contains a point $\sum_{i=1}^{\ell} a_i\varepsilon_i + b_1\varepsilon_{\ell+1} + b_2\varepsilon_{\ell+2}$ such that

$$a_1 - b_t > 1, \quad a_2 - b_t < 1, \quad a_1 + b_t > 1, \quad a_2 + b_t < 1, \quad t = 1, 2, \tag{10.14}$$

$$1/2 > a_{2n-1} > b_1, b_2 > 0. \tag{10.15}$$

Then we find $\tilde{D} \neq \emptyset$. Set

$$a_1' = 1 + a_{2n-1}, \quad a_2' = 1 - a_{2n-1}, \quad b_1' = a_{2n-1}$$

and choose b_2' so that $a_{2n-1} > b_2' > 0$. We have $a_2' > a_3$ since $a_3 < 1 - a_{2n-1}$, and $a_2' + a_{2n-2} > 1$ if $n \geq 3$. Hence we obtain

$$(1 + a_{2n-1})\varepsilon_1 + (1 - a_{2n-1})\varepsilon_2 + \sum_{t=3}^{\ell} a_t\varepsilon_t + a_{2n-1}\varepsilon_{\ell+1} + b_2'\varepsilon_{\ell+2} \in \bar{\tilde{D}},$$

which contradicts Lemma 10.3. This finishes the proof in Case (A).

In the following cases, we shall assume

(10.16) $$a_1 - a_\ell > 1.$$

Let $1 \leq j \leq i$. If $a_j - a_\ell > 1$, we have $a_j + a_\ell > 1$. Assume that $a_j + a_\ell < 1$ holds for every j, $1 \leq j \leq i$ such that $a_j - a_\ell < 1$. Then we find easily that $\sigma_\alpha D = D$ for $\alpha = \varepsilon_\ell$. Therefore we may assume

(10.17) There exists j, $1 \leq j \leq i$ such that $a_j - a_\ell < 1$, $a_j + a_\ell > 1$.

We divide the series (10.3) in the form

(10.18) $$a_1 > a_2 > \cdots > a_j > 1 + a_\ell > a_{j+1} > \cdots > a_i > 1/2 > a_{i+1} > \cdots > a_\ell > 0.$$

By (10.16) and (10.17), we have

(10.19) $$j \geq 1, \quad i - j \geq 1, \quad \ell - i \geq 1.$$

(If $\ell = i$, D is not bounded.) By (10.17), we get

(10.20) $$a_{j+1} + a_\ell > 1.$$

CASE (B) We assume $i - j \geq 2$.

SUBCASE (I) We assume $a_{j+2} + a_\ell > 1$.

We note the following fact. First choose $a_{i+1}, \ldots, a_\ell$ so that $1/2 > a_{i+1} > \cdots > a_\ell > 0$. Then choose a_t, $j + 3 \leq t \leq i$ so that

$$a_t + a_{r(t)} > 1, \quad a_t + a_{r(t)+1} < 1, \quad a_t > a_{t+1}.$$

Then we choose a_{j+1}, a_{j+2} so that

(10.21) $$1 + a_\ell > a_{j+1} > a_{j+2} > 1 - a_\ell, \qquad a_{j+2} > a_{j+3}.$$

When $a_{t+1}, \ldots, a_\ell$ are chosen for $t \leq j$, we can choose a_t so that

$$1 + a_{p(t)+1} < a_t < 1 + a_{p(t)}, \qquad a_t > a_{t+1}.$$

By successive applications of these choices, we can obtain a point in D. Since (10.21) is the only constraints on a_{j+1}, a_{j+2}, we can choose a_{j+1} and a_{j+2} so that

(10.22) $$a_{j+1} + a_{j+2} = 2, \qquad a_{j+1} > a_{j+2}.$$

Then we have $1 < a_{j+1} < 3/2$. In $\tilde{V}$, we consider a domain $\tilde{D}$ which contains a point $\sum_{i=1}^{\ell} a_i \varepsilon_i + b_1 \varepsilon_{\ell+1} + b_2 \varepsilon_{\ell+2}$ such that

(10.23) $$a_{j+1} + b_t > 1,\ a_{j+2} + b_t < 1,\ a_{j+1} - b_t > 1,\ a_{j+2} - b_t < 1,\ t = 1, 2,$$

(10.24) $$1/2 > a_\ell > b_1,\ b_2 > 0.$$

The above conditions reduce to $a_{j+1} - 1 > b_t > 0$, $t = 1, 2$ and we find $\tilde{D} \neq \emptyset$. We set

$$b_1' = a_{j+1} - 1, \qquad b_2' = a_{j+1} - 1.$$

Then we see that

$$\sum_{t=1,t\neq j+1}^{\ell} a_t\varepsilon_t + (2-a_{j+1})\varepsilon_{j+2} + (a_{j+1}-1)\varepsilon_{\ell+1} + (a_{j+1}-1)\varepsilon_{\ell+2} \in \bar{\tilde{D}},$$

which contradicts Lemma 10.3.

In the following subcases, we shall assume

$$a_{j+2} + a_\ell < 1. \tag{10.25}$$

SUBCASE (II) We assume $\ell - i \geq 2$.

First we assume

$$a_{j+2} + a_{\ell-1} > 1. \tag{10.26}$$

We can choose a_{j+1} and a_{j+2} so that $a_{j+1} > 1$, $a_{j+2} < 1$. In $\tilde{V}$, we consider a domain $\tilde{D}$ which contains a point $\sum_{i=1}^{\ell} a_i\varepsilon_i + b_1\varepsilon_{\ell+1} + b_2\varepsilon_{\ell+2}$ such that

$$a_{j+1} + b_t > 1,\ a_{j+2} + b_t < 1,\ a_{j+1} - b_t > 1,\ a_{j+2} - b_t < 1,\ t = 1, 2, \tag{10.27}$$

$$1/2 > a_\ell > b_1, b_2 > 0. \tag{10.28}$$

We find $\tilde{D} \neq \emptyset$. Then put

$$a'_{j+1} = 1 + a_\ell, \quad a'_{j+2} = 1 - a_\ell, \quad b'_1 = a_\ell,$$

and choose $0 < b'_2 < a_\ell$. We find $a'_{j+2} > a_u$ for $u > j+2$ since $a_u + a_\ell < 1$ by (10.25). Since we can choose $a_1, \dots, a_j$ by the procedure described above, we find

$$\sum_{t=1,t\neq j+1,j+2}^{\ell} a_t\varepsilon_t + (1+a_\ell)\varepsilon_{j+1} + (1-a_\ell)\varepsilon_{j+2} + a_\ell\varepsilon_{\ell+1} + b'_2\varepsilon_{\ell+2} \in \bar{\tilde{D}},$$

which contradicts Lemma 10.3.

Next we assume

$$a_{j+2} + a_{\ell-1} < 1. \tag{10.29}$$

We have $r(\ell-1) = r(\ell) = j+1$ by (10.20), (10.25) and (10.29). By definition, we have $q(\ell) = j+1$. If $a_j - a_{\ell-1} > 1$, we get $q(\ell-1) = j+1$, which leads to a contradiction by Lemma 10.2 and induction hypothesis. Therefore we may assume

$$a_j - a_{\ell-1} < 1. \tag{10.30}$$

We can choose a_{j+1} so that $a_{j+1} < 1$. In $\tilde{V}$, we consider a domain $\tilde{D}$ which contains a point $\sum_{i=1}^{\ell} a_i\varepsilon_i + b_1\varepsilon_{\ell+1} + b_2\varepsilon_{\ell+2}$ such that

$$a_j - b_t > 1,\ a_{j+1} - b_t < 1,\ a_{j+1} + b_t < 1,\ t = 1, 2, \tag{10.31}$$

$$a_\ell > b_1, b_2 > 0. \tag{10.32}$$

We find $\tilde{D} \neq \emptyset$. Set

$$a'_j = 1 + a_\ell, \quad a'_{j+1} = 1 - a_\ell, \quad b'_1 = a_\ell$$

and choose $a_\ell > b'_2 > 0$. By (10.25), we have $a'_{j+1} > a_{j+2}$ and $a'_j - a_{\ell-1} < 1$ holds (cf. (10.30)). Since we can choose $a_1, \ldots, a_{j-1}$ by the procedure described above, we have

$$\sum_{t=1, t\neq j, j+1}^{\ell} a_t\varepsilon_t + (1 + a_\ell)\varepsilon_j + (1 - a_\ell)\varepsilon_{j+1} + a_\ell\varepsilon_{\ell+1} + b'_2\varepsilon_{\ell+2} \in \bar{\tilde{D}},$$

which contradicts Lemma 10.3.

SUBCASE (III) We assume $\ell - i = 1$.

By (10.25), we have

$$a_u < 1 - a_\ell, \qquad j + 2 \le u \le \ell - 1. \tag{10.33}$$

We can choose a_{j+1} so that $a_{j+1} > 1$. In $\tilde{V}$, we consider a domain $\tilde{D}$ which contains a point $\sum_{i=1}^{\ell} a_i\varepsilon_i + b_1\varepsilon_{\ell+1} + b_2\varepsilon_{\ell+2}$ such that

$$a_{j+1} - b_t > 1, \; a_{j+2} - b_t < 1, \; a_{j+1} + b_t > 1, \; a_{j+2} + b_t < 1, \tag{10.34}$$

We find $\tilde{D} \neq \emptyset$. By the specialization

$$a'_{j+1} = 1 + a_\ell, \; a'_{j+2} = 1 - a_\ell, \; b'_1 = a_\ell, \; a_\ell > b'_2 > 0,$$

we obtain a contradiction as before.

CASE (C) We assume $i = j + 1$.

In this case, (10.18) takes the form

$$a_1 > \cdots > a_j > 1 + a_\ell > a_{j+1} > 1/2 > a_{j+2} > \cdots > a_\ell > 0. \tag{10.35}$$

SUBCASE (I) We assume $\ell - j \ge 3$.

By (10.20), we get $r(\ell) = r(\ell - 1) = j + 1$. By definition, we have $q(\ell) = j + 1$. If $a_j - a_{\ell-1} > 1$, we have $q(\ell - 1) = j + 1$ and we get a contradiction by Lemma 10.2 and induction hypothesis. Hence we may assume

$$a_j - a_{\ell-1} < 1. \tag{10.36}$$

We can choose a_{j+1} so that $a_{j+1} < 1$. In $\tilde{V}$, we consider a domain $\tilde{D}$ which contains a point $\sum_{i=1}^{\ell} a_i\varepsilon_i + b_1\varepsilon_{\ell+1} + b_2\varepsilon_{\ell+2}$ such that

$$a_j - b_t > 1, \; a_{j+1} - b_t < 1, \; a_j + b_t > 1, \; a_{j+1} + b_t < 1, \; t = 1, 2, \tag{10.37}$$

$$1/2 > a_\ell > b_1, \; b_2 > 0. \tag{10.38}$$

We find $\tilde{D} \neq \emptyset$. Set

$$a'_j = 1 + a_\ell, \quad a'_{j+1} = 1 - a_\ell, \quad b'_1 = a_\ell$$

and choose $a_\ell > b'_2 > 0$. By (10.25), we have $a'_{j+1} > a_{j+2}$. We also have $a'_j - a_{\ell-1} < 1$ (cf. (10.36)). Since we can choose $a_1, \ldots, a_{j-1}$ by the procedure described above, we have

$$\sum_{t=1,t\neq j,j+1}^{\ell} a_t\varepsilon_t + (1+a_\ell)\varepsilon_j + (1-a_\ell)\varepsilon_{j+1} + a_\ell\varepsilon_{\ell+1} + b_2'\varepsilon_{\ell+2} \in \bar{\tilde{D}},$$

which contradicts Lemma 10.3.

SUBCASE (II) We assume $\ell - j = 2$.

In this subcase, (10.35) takes the form

$$a_1 > \cdots > a_{\ell-2} > 1 + a_\ell > a_{\ell-1} > 1/2 > a_\ell > 0. \tag{10.39}$$

We have

$$a_{\ell-1} + a_\ell > 1, \quad a_{\ell-2} - a_\ell > 1, \quad a_{\ell-2} - a_{\ell-1} < 1. \tag{10.40}$$

We can choose $a_{\ell-1}$ so that $a_{\ell-1} < 1$. In $\tilde{V}$, we consider a domain $\tilde{D}$ which contains a point $\sum_{i=1}^{\ell} a_i\varepsilon_i + b_1\varepsilon_{\ell+1} + b_2\varepsilon_{\ell+2}$ such that

$$a_{\ell-2} - b_t > 1,\ a_{\ell-1} - b_t < 1,\ a_{\ell-1} + b_t < 1,\ t = 1, 2, \tag{10.41}$$

$$1/2 > a_\ell > b_1,\ b_2 > 0. \tag{10.42}$$

We find $\tilde{D} \neq \emptyset$. By setting

$$a'_{\ell-2} = 1 + a_\ell, \quad a'_{\ell-1} = 1 - a_\ell, \quad b'_1 = a_\ell,$$

and choosing $a_\ell > b'_2 > 0$, we can find a desired point.

This completes the proof of Theorem 10.4.

Now we consider the general case where $z = \sum_{i=1}^{\ell} c_i\varepsilon_i \in Q(\Sigma)$ is not necessarily 0. Replacing $\chi(v)$ by $w(\chi(v))$, we may asume

$$c_i = 0,\ 1 \leq i \leq n, \qquad c_i = 1,\ n+1 \leq i \leq \ell.$$

The family of hyperplanes in V considered in §8 are

$$a_i = \pm 1/2\ (1 \leq i \leq n), \qquad a_i \pm a_j = \pm 1\ (1 \leq i < j \leq n),$$
$$a_i = \pm 1/2\ (n+1 \leq i \leq \ell), \qquad a_i \pm a_j = \pm 1\ (n+1 \leq i < j \leq \ell).$$

This shows that we can treat the variables a_i $(1 \leq i \leq n)$ and a_j $(n+1 \leq j \leq \ell)$ separately. Assume that the non-unitarizability of the domain for $\ell = 4$, $n = 2$:[4]

$$\begin{aligned} &-1/2 < a_1, a_3 < 1/2 < a_2, a_4, \quad -1 < a_2 - a_1 < 1, \quad -1 < a_4 - a_3 < 1, \\ &1 < a_1 + a_2, \ 1 < a_3 + a_4. \end{aligned} \tag{10.43}$$

Then all the arguments in the proof of Theorem 10.4 apply to this case and we obtain

$$-1/2 < a_i < 1/2, \qquad 1 \leq i \leq \ell \tag{10.44}$$

as the necessary and sufficient condition for the unitarizability.

[4] In Case (A), Subcase (I) in the proof of Theorem 10.4, we have used the existence of a non-unitarizable constituent of $PS(\delta_B^{-1/2})$. It becomes necessary to consider this case when Case (A), Subcase (I) occurs simultaneously for separated variables $a_i(1 \leq i \leq n)$ and $a_j(n+1 \leq j \leq \ell)$.

It remains to show that the domain D given by (10.43) represents non-unitarizable representations. Changing D to wD, $w \in W$, we may assume $c_1 = c_2 = 1$, $c_3 = c_4 = 0$. We take a point

$$v_0 = -\frac{1}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2 + a_3\varepsilon_3 + a_4\varepsilon_4 \in \bar{D}$$

so that

$$a_3 > 1/2 > a_4 > 0, \quad a_3 - a_4 < 1, \quad a_3 + a_4 > 1.$$

Put $\chi_0 = \chi(v_0)$. We have $|W_{\chi_0}| = 2$. Define $w_2 \in W$ by

$$w_2\varepsilon_i = \varepsilon_i, \ i = 1, 2, \qquad w_2\varepsilon_i = -\varepsilon_i, \ i = 3, 4.$$

Then we have $\sigma_1 w_2 \chi_0 = \bar{\chi}_0^{-1}$. Set

$$(\varphi_1, \varphi_2) = \int_{B\backslash G} (T_{\sigma_1 w_2}(\chi_0)\varphi_1)(g)\,\overline{\varphi_2(g)}\,dg, \qquad \varphi_1, \varphi_2 \in PS(\chi_0).$$

Then $(\ ,\ )$ defines a non-degenerate invariant hermitian form on $PS(\chi_0)/\operatorname{Ker}(T_{\sigma_1 w_2}(\chi_0))$. Let V be the G-submodule of $PS(\chi_0)$ generated by φ_{K,χ_0}. We have $T_{\sigma_1 w_2}(\chi_0)\varphi_{K,\chi_0} = c\varphi_{K,\chi_0}$ with $c \neq 0$. Set $V_0 = \operatorname{Ker}(T_{\sigma_1 w_2}(\chi_0))$. Then we obtain

$$V \cap V_0 = \{0\} \tag{10.45}$$

in the same way as in the proof of Lemma 9.4. Hence, by restricting $(\ ,\ )$ to V, we obtain a non-degenerate invariant hermitian form on V. By (10.45), $0 \subsetneqq V_0 \subsetneqq PS(\chi_0)$ and by Lemma 4.6, V must be irreducible. Let $v_1 = \sigma_2 w_2(v_0) = \frac{1}{2}\varepsilon_1 - \frac{1}{2}\varepsilon_2 - a_3\varepsilon_3 - a_4\varepsilon_4$ and put $\chi_1 = \chi(v_1)$. Set

$$(\varphi_1, \varphi_2)' = \int_{B\backslash G} (T_{\sigma_1 w_2}(\chi_0)\varphi_1)(g)\,\overline{\varphi_2(g)}\,dg, \qquad \varphi_1, \varphi_2 \in PS(\chi_1).$$

Then $(\ ,\ )'$ defines a non-degenerate invariant hermitian form on $PS(\chi_1)/\operatorname{Ker}(T_{\sigma_1 w_2}(\chi_1))$. Put $V_1 = \operatorname{Ker}(T_{\sigma_1 w_2}(\chi_1))$. Since $\varphi_{K,\chi_0} \in V_1$, all constituents of $PS(\chi_1)/V_1$ are not spherical. If the G-module $PS(\chi_1)/V_1$ is irreducible, we see that the distribution $cT_{\sigma_1 w_2, \chi_1}$ is of positive type for $c = \pm 1$ since it is unitarizable. By Lemma 5.4, this implies that the domain

$$0 > -a_4 > -1/2 > -a_3, \quad a_3 - a_4 < 1, \quad a_3 + a_4 > 1$$

represents unitarizable representations. We have already shown that this is not the case. Therefore we may assume that $PS(\chi_1)/V_1$ is not irreducible. Let W be an irreducible G-submodule of $PS(\chi_1)/V_1$. If $(\ ,\ )'|W$ is degenerate (i.e., zero form), we find $PS(\chi_1)/V_1$ contains a constituent W' isomorphic to W as in the proof of Lemma 8.5. By $T_{\sigma_1 w_2}(\chi_1)$, W is isomorphically mapped to a G-submodule $\bar{W}$ of $PS(\chi_0)$ and W' to a constituent of $PS(\chi_0)$. Then $\mathbf{C}_{\delta_B^{1/2}\chi_0}$ occurs in $(\bar{W})_N$. This leads to a contradiction as in Lemma 8.5. If $(\ ,\ )'|W$ is non-degenerate, there exists an irreducible G-submodule of $PS(\chi_1)/V_1$ such that $W \cap W_1 = \{0\}$. By $T_{\sigma_1 w_2}(\chi_1)$, W and W_1 are mapped isomorphically to G-submodules $\bar{W}$ and $\bar{W}_1$ of $PS(\chi_0)$. This contradicts Lemma 4.6 since V is a spherical irreducible G-submodule of $PS(\chi_0)$. This proves the desired non-unitarizability and we complete the determination of the unitarizability in case of type B_ℓ.

§11. The case of type D_ℓ

The simple roots are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3, \cdots, \alpha_{\ell-1} = \varepsilon_{\ell-1} - \varepsilon_\ell,\ \alpha_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell.$$

$$P(\Sigma) = \bigoplus_{i=1}^{\ell} \mathbf{Z}\varepsilon_i + \mathbf{Z}\left(\frac{1}{2}\sum_{i=1}^{\ell} \varepsilon_i\right),\ Q(\Sigma) = \{\sum_{i=1}^{\ell} a_i\varepsilon_i | a_i \in \mathbf{Z},\ \sum_{i=1}^{\ell} a_i \equiv 0 \bmod 2\}.$$

We identify $\{\varepsilon_i\}$ with its dual basis with respect to $\langle\ ,\ \rangle$. Then we have

$$\varepsilon_i \check{\pm} \varepsilon_j = \varepsilon_i \pm \varepsilon_j,\ 1 \le i < j \le \ell,\ P(\check{\Sigma}) = P(\Sigma),\ Q(\check{\Sigma}) = Q(\Sigma).$$

We assume ℓ is even, $\ell \ge 4$. (Since we have reduced to the case $w_0 = -1$ on Δ in §8, we lose no generality by this assumption.) Put $\ell = 2n$, $n \ge 2$. We have

$$w_0\varepsilon_i = -\varepsilon_i, \qquad 1 \le i \le \ell,$$
$$w_0 = (\sigma_1\sigma_2\cdots\sigma_{\ell-2}\sigma_{\ell-1}\sigma_\ell\sigma_{\ell-2}\cdots\sigma_2\sigma_1)\cdots(\sigma_{\ell-2}\sigma_{\ell-1}\sigma_\ell\sigma_{\ell-2})(\sigma_{\ell-1}\sigma_\ell).$$

We assume that $\mathbf{G}$ is of adjoint type. For $z = \sum_{i=1}^{\ell} c_i\varepsilon_i \in Q(\Sigma)$ and $v = \sum_{i=1}^{\ell} a_i\varepsilon_i \in V$, $a_i \in \mathbf{R}$, we define $\chi(v) \in X$ by

$$\chi(v)(\beta(\varpi)) = (-1)^{\langle z,\beta\rangle} q^{\langle v,\beta\rangle}, \qquad \beta \in P(\check{\Sigma}). \tag{11.1}$$

We have

$$\begin{aligned} &\chi(v)((\varepsilon_i \check{-} \varepsilon_j)(\varpi)) = (-1)^{c_i - c_j} q^{a_i - a_j}, \qquad 1 \le i < j \le \ell, \\ &\chi(v)((\varepsilon_i \check{+} \varepsilon_j)(\varpi)) = (-1)^{c_i + c_j} q^{a_i + a_j}, \qquad 1 \le i < j \le \ell. \end{aligned} \tag{11.2}$$

First we consider the case $z = 0$. The family of hyperplanes $\mathfrak{H}$ in V considered in §8 are

$$a_i \pm a_j = \pm 1 \qquad (1 \le i < j \le \ell). \tag{11.3}$$

We consider a connected component D of $V - \mathfrak{H}$. Replacing D by wD, $w \in W$, we may assume that D contains a point $v = \sum_{i=1}^{\ell} a_i\varepsilon_i$ such that $\langle v, \check{\alpha}_i\rangle > 0$ for every $1 \le i \le \ell$. This condition is equivalent to

$$a_1 > a_2 > \cdots > a_{\ell-1} > a_\ell, \qquad a_{\ell-1} + a_\ell > 0,$$

which is equivalent to

$$a_1 > a_2 > \cdots > a_{\ell-1} > |a_\ell| \ge 0. \tag{11.4}$$

Since D is open, we may assume $a_\ell \neq 0$. We may and shall assume that D is bounded. Let $1 \leq j \leq \ell - 2$ and suppose $a_j - a_{j+1} > 1$. Then we have

$$a_j \pm a_u > 1,\ j < u \leq \ell, \qquad a_i + a_u > 1,\ 1 \leq i < u \leq \ell.$$

Hence we see that

$$\sum_{u=1}^{j} (a_u + t)\varepsilon_i + \sum_{u=j+1}^{\ell} a_u \varepsilon_u \in D$$

for every $t > 0$. Thus D is not bounded. Therefore we have

$$a_j - a_{j+1} < 1, \qquad 1 \leq j \leq \ell - 2. \tag{11.5}$$

For $1 \leq j \leq \ell - 2$, we define $p(j)$ as the greatest integer p such that $a_j - a_p < 1$, $j < p \leq \ell$. If $a_{\ell-1} - a_\ell < 1$, we set $p(\ell - 1) = \ell$. If $a_{\ell-1} - a_\ell > 1$ and $a_{\ell-1} + a_\ell > 1$, we see that D is not bounded in the same way as above. Therefore we have

$$a_{\ell-1} + a_\ell < 1 \qquad \text{if } a_{\ell-1} - a_\ell > 1. \tag{11.6}$$

For $2 \leq j \leq \ell - 1$, we define $q(j)$ as the least integer such that $a_q - a_j < 1$, $1 \leq q < j$.

Lemma 11.1. *Assume* $a_1 + |a_\ell| < 1$. *Then we have* $\sigma_\alpha D = D$ *for* $\alpha = \varepsilon_{\ell-1} - \varepsilon_\ell$.

Proof. From the assumption, we see easily that

$$a_i \pm a_{\ell-1} < 1, \quad a_i \pm a_\ell < 1, \quad 1 \leq i \leq \ell - 2.$$

This shows that $\sigma_\alpha D = D$, $\alpha = \varepsilon_{\ell-1} - \varepsilon_\ell$; hence the assertion follows.

Assume

$$a_1 + |a_\ell| > 1. \tag{11.7}$$

For $1 \leq j \leq \ell - 1$, we can define $r(j)$ as the greatest integer r such that $a_j + a_r > 1$, $1 \leq r \leq \ell$, $r \neq j$.

Lemma 11.2. *Assume that there exists* j, $1 \leq j \leq \ell - 2$ *such that* $p(j) = p(j+1)$, $q(j) = q(j+1)$. *(If* $j = 1$, *we understand* $q(1) = q(2)$.*) Furthermore assume* (11.7) *and* $r(j) = r(j+1)$ *or* $r(j) = j + 1$, $r(j+1) = j$. *Then we have* $\sigma_\alpha D = D$ *for* $\alpha = \varepsilon_j - \varepsilon_{j+1}$.

The proof is identical to that of Lemma 9.1. We shall prove the following Theorem.

Theorem 11.3. *We normalize* D *so that* D *contains a point of the form* (11.4). *If the points of* D *represent unitarizable representations, we have*

$$a_1 - |a_\ell| < 1.$$

First we state a consequence of this Theorem.

Lemma 11.4. *If D contains a point which satisfies* (11.4) *and* $a_1 - |a_\ell| < 1$, *then* $\sigma_\alpha D = D$ *for some* $\alpha \in \Sigma^+$.

Proof. By Lemma 11.1, we may assume that (11.7) is satisfied. Hence $r(j)$ is defined for $1 \leq j \leq \ell - 1$. Assume $\sigma_\alpha D \neq D$ for every $\alpha \in \Sigma^+$. First we consider the case $a_\ell > 0$. We have

$$p(1) = p(2) = \cdots = p(\ell - 1) = \ell, \qquad q(2) = q(3) = \cdots = q(\ell - 1) = 1.$$

By Lemma 11.2, we must have

$$r(1) > r(2) > \cdots > r(\ell - 1). \tag{11.8}$$

By (11.7), we have $r(1) = \ell$. If $r(\ell - 1) = 1$, we have

$$a_1 + a_{\ell-1} > 1,\ a_2 + a_{\ell-1} < 1,\ a_1 + a_\ell > 1,\ a_2 + a_\ell < 1,\ a_1 - a_\ell < 1,\ a_1 - a_{\ell-1} < 1.$$

Hence we have $\sigma_\alpha D = D$ for $\alpha = \varepsilon_{\ell-1} - \varepsilon_\ell$. Thus we may assume $r(\ell - 1) \geq 2$. Then, by (11.8), we get

$$r(i) = \ell + 1 - i, \qquad 1 \leq i \leq \ell - 1.$$

Therefore $r(n) = n + 1$, $r(n + 1) = r(n)$. By Lemma 11.2, this is a contradiction.

Next we assume $a_\ell < 0$. If $a_1 - a_\ell < 1$, the same argument applies. We may assume

$$a_1 + a_\ell < 1, \qquad a_1 - a_\ell > 1. \tag{11.9}$$

Assume $a_j - a_\ell > 1$, $1 \leq j \leq \ell - 1$. Then we get

$$p(1) = p(2) = \cdots = p(\ell - 2) = \ell - 1,\ q(2) = q(3) = \cdots q(\ell - 1) = 1.$$

Hence, by Lemma 11.2, we must have

$$r(1) > r(2) > \cdots > r(\ell - 2).$$

If $r(\ell - 2) = r(\ell - 1)$, we find easily that $\sigma_\alpha D = D$, $\alpha = \varepsilon_{\ell-2} - \varepsilon_{\ell-1}$. Therefore we may assume (11.8). By (11.9), we have $r(1) \leq \ell - 1$. Hence we must have

$$r(i) = \ell - i, \qquad 1 \leq i \leq \ell - 1.$$

Then we get $r(n) = n$, which is a contradiction. Thus it turned out that we may assume

$$a_i - a_\ell > 1, \qquad a_{i+1} - a_\ell < 1 \tag{11.10}$$

for some $1 \leq i \leq \ell - 2$. Then we have

$$p(1) = p(2) = \cdots = p(i) = \ell - 1, \qquad p(i + 1) = \cdots = p(\ell - 1) = \ell,$$
$$q(2) = q(3) = \cdots = q(\ell - 1) = 1.$$

By Lemma 11.2, we must have

(11.11) $$\ell - 1 \geq r(1) > r(2) > \cdots > r(i) \geq r(i+1) > \cdots > r(\ell - 1) \geq 1.$$

If $r(i) > r(i+1)$, we get $r(j) = \ell - j$, $1 \leq j \leq \ell - 1$. Hence $r(n) = n$, a contradiction. We may assume

(11.12) $$r(i) = r(i+1).$$

By (11.10), we get $a_i + a_{\ell - 1} > 1$. Hence, by (11.9), we have $r(i) = \ell - 1$. By (11.11), we must have $i = 1$. Thus we obtain

(11.13) $$\ell - 1 = r(1) = r(2) > \cdots > r(\ell - 1) \geq 1.$$

We shall consider two cases.

(I) The case where $r(j) - r(j+1) \leq 1$ for every $1 \leq j \leq \ell - 2$.

Then, by (11.13), we have

$$r(j) = \ell + 1 - j, \qquad 2 \leq j \leq \ell - 1.$$

We get $r(n) = n + 1$, $r(n+1) = n$. By Lemma 11.2, we obtain a contradiction.

(II) The case where there exists j such that $1 \leq j \leq \ell - 2$, $r(j) - r(j+1) = 2$.

Put $r_0 = r(j+1)$. We have $r(j) = r_0 + 2 \leq \ell - 1$. By definition, we have

$$a_{j+1} + a_{r_0} > 1, \quad a_{j+1} + a_{r_0+1} < 1, \quad a_j + a_{r_0+2} > 1.$$

Hence we have $r(r_0 + 1) = r(r_0 + 2) = j$. By (11.13), we have $r_0 + 1 = 1$, $j = \ell - 1$. This is a contradiction and completes the proof.

Lemma 11.5. *Let $\ell \geq 4$. Suppose that $\overline{wD}$, the closure of wD for some $w \in W$, contains a point v_0 of the form*

$$v_0 = (a-1)(\varepsilon_1 - \varepsilon_2) - a\varepsilon_3 + (a-2)\varepsilon_4 + \sum_{j=5}^{\ell} b_j \varepsilon_j$$

which satisfies the following conditions.

(1) $b_j \pm b_t \neq 0, \ \pm 1, \ 5 \leq j < t \leq \ell.$

(2) $a \neq 1/2, \ 1, \ 3/2.$

(3) $b_j \neq \pm(a+1), \ \pm a, \ \pm(a-1), \ \pm(a-2), \ \pm(a-3), \ 5 \leq j \leq \ell.$

Then all the points of D represent non-unitarizable representations.

Since the proof is almost same as that of Lemma 9.4, it is omitted.

Proof of Theorem 11.3. We set

(11.14) $$a_1 > a_2 > \cdots > a_i > 1 + |a_\ell| > a_{i+1} > \cdots > a_{\ell - 1} > |a_\ell| > 0.$$

It suffices to show that a point of D represents non-unitarizable representation assuming $i \geq 1$ in (11.14). We shall prove the theorem by contradiction. Thus we assume that all the points of D represent unitarizable representations. We also make the hypothesis of induction on ℓ for groups of type D_ℓ. Throughout the proof, we let $\tilde{V} = \bigoplus_{i=1}^{\ell+2} \mathbf{R}\varepsilon_i$ be the vector space attached to the adjoint groups of type $D_{\ell+2}$. In the proof, we consider a domain $\tilde{D}$ which contains a point $v \in \tilde{V}$. By this term, we shall always understand that $v \notin \tilde{\mathfrak{H}}$ and that $\tilde{D} \ni v$ is the connected component of $\tilde{V} - \tilde{\mathfrak{H}}$, where $\tilde{\mathfrak{H}}$ is the family of hyperplanes in $\tilde{V}$ similary defined as above.

We have

$$(11.15) \qquad a_t + a_u > 1, \quad 1 \le t \le i, \quad t \le u \le \ell.$$

If $i = \ell - 1$, we see that $\sum_{j=1}^{\ell-1} (a_j + v)\varepsilon_j + a_\ell \varepsilon_\ell \in D$ for every $v > 0$. Hence D is not bounded. We may assume

$$(11.16) \qquad \ell - i \ge 2.$$

First we assume $a_\ell > 0$. Since this case is pararell to the case of type C_ℓ, we shall briefly describe the proof. We have

$$(11.17) \qquad a_t - a_u < 1, \qquad i + 1 \le t < u \le \ell.$$

CASE (A) We assume $a_{i+1} + a_{i+2} > 1$, $a_{\ell-1} + a_\ell < 1$.

In this case, $\ell - i \ge 3$. Choose s, $i + 2 \le s \le \ell - 1$, so that

$$(11.18) \qquad a_{s-1} + a_s > 1, \qquad a_s + a_{s+1} < 1.$$

Let $q = q(s)$. We get

$$(11.19) \qquad a_q - a_s < 1, \qquad a_{q-1} - a_s > 1.$$

SUBCASE (I) We assume $a_q - a_{s+1} > 1$.

We have $q \le i$. By the assumption of the subcase, we can choose a_s, a_q and a_{q+1} so that

$$(11.20) \qquad a_s > 1/2, \quad a_q > 3/2, \quad a_q + a_{q+1} < 3.$$

In $\tilde{V}$, we consider the domain $\tilde{D}$ which contains a point $\sum_{i=1}^{\ell} a_j\varepsilon_j + b_1\varepsilon_{\ell+1} + b_2\varepsilon_{\ell+2}$ such that

$$(11.21) \qquad a_q - b_j > 1,\ a_{q+1} - b_j < 1,\ a_s + b_j > 1,\ a_{s+1} + b_j < 1,\ j = 1, 2,$$

$$(11.22) \qquad 1 + a_\ell > b_1, b_2 > a_\ell, \qquad b_1 + b_2 < 1.$$

Then $\tilde{D} \ne \emptyset$. We have $\sigma_\alpha \tilde{D} = \tilde{D}$ for $\alpha = \varepsilon_{\ell+1} - \varepsilon_{\ell+2}$. By Remark 8.3 and (11.22), we see that all the points of $\tilde{D}$ represent unitarizable representations.[5] On the other hand, we have

$$\sum_{j=1, j\ne s}^{\ell} a_j\varepsilon_j + (a_q - 1)\varepsilon_s + (a_q - 1)\varepsilon_{\ell+1} + (2 - a_q)\varepsilon_{\ell+2} \in \overline{\tilde{D}}.$$

Hence

$$(a_q - 1)(\varepsilon_1 - \varepsilon_2) - a_q\varepsilon_3 + (a_q - 2)\varepsilon_4 + \sum_{j=1}^{q-1} a_j\varepsilon_{j+4} + \sum_{j=q+1}^{s-1} a_j\varepsilon_{j+3} + \sum_{j=s+1}^{\ell} a_j\varepsilon_{j+2} \in \overline{w\tilde{D}},$$

[5] These remarks shall apply to the succeeding argument as well. We shall not repeat them.

for some $w \in \tilde{W}$, where $\tilde{W}$ is the Weyl group for $D_{\ell+2}$. Note that we could have chosen a_t, $1 \leq t \leq \ell$, $t \neq s, q$ in "generic position"[5]. This contradicts Lemma 11.5.

SUBCASE (II) We assume $a_q - a_{s+1} < 1$.

By (11.18) and the assumption of the subcase, we find $q = q(s) = q(s+1)$. Since $a_\ell > 0$, we may assume $r(s) < r(s+1)$. Hence we get $a_{s+1} + a_{s-1} < 1$, which implies $r(s-1) = s$. Since $r(s) = s-1$, we may assume $q(s-1) < q(s)$. Summing up, we have

$$a_{s-1} + a_{s+1} < 1, \qquad a_{q-1} - a_{s-1} < 1. \tag{11.23}$$

Then we get $a_{q-1} + a_{s+1} < 2$. Choose a_s, a_{q-1} and a_q so that

$$a_s > 1/2, \quad a_{q-1} > 3/2, \quad a_{q-1} + a_q < 3. \tag{11.24}$$

In $\tilde{V}$, we consider the domain $\tilde{D}$ which contains a point $\sum_{i=1}^{\ell} a_j\varepsilon_j + b_1\varepsilon_{\ell+1} + b_2\varepsilon_{\ell+2}$ such that

$$a_q - b_j < 1,\ a_{q-1} - b_j > 1,\ a_s + b_j > 1,\ a_{s+1} + b_j < 1,\ j = 1, 2, \tag{11.25}$$

$$1 + a_\ell > b_1,\ b_2 > a_\ell,\ b_1 + b_2 < 1. \tag{11.26}$$

Then $\tilde{D} \neq \emptyset$ and all the points of $\tilde{D}$ represent unitarizable representations. On the other hand, we have

$$\sum_{j=1, j\neq s}^{\ell} a_j\varepsilon_j + (a_{q-1} - 1)\varepsilon_s + (a_{q-1} - 1)\varepsilon_{\ell+1} + (2 - a_{q-1})\varepsilon_{\ell+2} \in \bar{\tilde{D}},$$

which contradicts Lemma 11.5.

We omit the cases (B) $a_{i+1} + a_{i+2} < 1$, (C) $a_{\ell-1} + a_\ell > 1$, since they can be dealt with in completely pararell way as in the case of type C_ℓ.

Next we assume $a_\ell < 0$. Though this case can be dealt with in a similar but more complicated way compared with the case $a_\ell > 0$, we prefer the following argument. Let $\psi: \tilde{\mathbf{G}} \to \mathbf{G}$ be the simply connected covering map as in §3. Take any $v = \sum_{i=1}^{\ell} a_i\varepsilon_i \in D$ and put $\chi = \chi(v)$, $\tilde{\chi} = \chi \circ \psi$. Corresponding to the automorphism which interchanges $\alpha_{\ell-1}$ and α_ℓ of the Dynkin diagram of Σ, there exists an automorphism $\sigma: \tilde{G} \to \tilde{G}$ (cf. Steinberg [21], Theorem 29). We have

$$\tilde{B}^\sigma = \tilde{B},\ \tilde{T}^\sigma = \tilde{T},\ \tilde{K}^\sigma = \tilde{K}. \tag{11.27}$$

$$\begin{aligned} &(\check{\alpha}_i(t))^\sigma = \check{\alpha}_i(t), && 1 \leq i \leq \ell - 2, \\ &(\check{\alpha}_{\ell-1}(t))^\sigma = \check{\alpha}_\ell(t), && (\check{\alpha}_\ell(t))^\sigma = \check{\alpha}_{\ell-1}(t), \qquad t \in k^\times. \end{aligned} \tag{11.28}$$

(To see (11.28), use relations in [21], p. 30.) For an admissible representation π of $\tilde{G}$, set $\pi^\sigma(g) = \pi(g^\sigma)$, $g \in \tilde{G}$. Then π^σ is also an admissible representation; π is unitarizable if and only if π^σ is. For $\varphi \in PS(\chi)$, define φ^σ by $\varphi^\sigma(g) = \varphi(g^\sigma)$, $g \in \tilde{G}$. By (11.27), we have $\varphi^\sigma \in PS(\chi^\sigma)$ where $\tilde{\chi}^\sigma(t) = \tilde{\chi}(t^\sigma)$, $t \in \tilde{T}$. Hence we obtain $\pi(\tilde{\chi})^\sigma = \pi(\tilde{\chi}^\sigma)$. By (11.28), we obtain

$$\tilde{\chi}^{\sigma} = \psi \circ \chi(v^{\sigma}), \qquad v^{\sigma} = \sum_{i=1}^{\ell-1} a_i \varepsilon_i - a_{\ell} \varepsilon_{\ell}.$$

Since the unitarizability of $\pi(\chi(v))$, $\pi(\tilde{\chi})$, $\pi(\tilde{\chi}^{\sigma})$ and $\pi(\chi(v^{\sigma}))$ are equivalent, we are reduced to the case $a_{\ell} > 0$. Hence $a_1 - |a_{\ell}| < 1$ follows from the previous case. This completes the proof of Theorem 11.3.

Now we consider the general case where $z = \sum_{i=1}^{\ell} c_i \varepsilon_i \in Q(\Sigma)$ is not necessarily 0. We may assume

$$c_i = 0,\ 1 \le i \le n, \qquad c_i = 1,\ n+1 \le i \le \ell.$$

We see that $\ell - n$ is even. The family of hyperplanes in V considered in §8 are

$$a_i \pm a_j = \pm 1 \ (1 \le i < j \le n), \qquad a_i \pm a_j = \pm 1 \ (n+1 \le i < j \le \ell).$$

Therefore we can treat the variables a_i $(1 \le i \le n)$ and a_j $(n+1 \le j \le \ell)$ separately. We can normalize $a_{n+1}, \ldots, a_{\ell}$ so that

$$a_{n+1} > a_{n+2} > \cdots > a_{\ell-1} > |a_{\ell}| \ge 0.$$

Then we obtain $a_{n+1} - |a_{\ell}| < 1$ by the same proof as Theorem 11.3 and we obtain $\sigma_{\alpha} D = D$ for some $\alpha \in \Sigma^{+}$. This completes the determination of the unitrizability for groups of type D_{ℓ}.

§12. Reduction of the unitarizability of π_{χ}^{1} to the case of real quasi-characters

In this section, we shall consider the unitarizability problem of π_{χ}^{1}, the spherical consitituent of $PS(\chi)$. We shall show that the problem can be reduced to the case when χ is real valued, if $\mathbf{G}$ is a simply connected group of classical type. We begin with the following Lemma which is an elaboration on Lemma 5.4.

Lemma 12.1. *Let the notation and the assumptions be the same as in Lemma 5.4. Assume that* $\mathbf{G}$ *is of adjoint type and that* $cT_{w,\chi}$ *if of positive type. Let* $\{\beta_j\}$ *be the dual basis of* $\{\alpha_j\} = \Delta$ *with respect to* $\langle\ ,\ \rangle$ *(i.e., the fundamental weights of* $P(\check{\Sigma})$*). Set* $m = \max_{\alpha \in \Sigma^{+}, 1 \le j \le \ell} \langle \alpha, \beta_j \rangle$. *Then if* $|\chi(a_{\alpha})| < q^{1/m}$ *for all* $\alpha \in \Psi_{w}^{+}$, $(\ ,\ )$ *is positive semi-definite.*

Proof. We use the same notation as in the proof of Lemma 5.4. It suffices to prove (5.14), i.e.,

$$\lim_{i \to \infty} (\varphi - \iota_{\chi}(\Phi_i),\ \iota_{\chi}(\Phi_i)) = 0. \tag{12.1}$$

Since $\| \iota_{\chi}(\Phi_i) \|_{L^{\infty}(K)} \le \| \varphi \|_{L^{\infty}(K)}$, (12.1) follows from

$$\lim_{i \to \infty} \| T_w(\chi)(\varphi - \iota_{\chi}(\Phi_i)) \|_{L^1(K)} = 0. \tag{12.2}$$

Take any $f \in PS(\chi)$ such that $f|K$ is right invariant under K_L, where $L \in \mathbf{N}$. Note that $K_L \lhd K$. Let $\alpha \in \Delta$ and assume $|\chi(a_\alpha)| < q^{1/m}$. Choose $\mu_\alpha > 1$ so that $|\chi(a_\alpha)| < \mu_\alpha < q^{1/m}$. Then by (4.14), we obtain

$$\| T_{\sigma_\alpha}(\chi) f \|_{L^1(K)} \leq c_1 \mu_\alpha^L \| f \|_{L^1(K)}$$

with a constant $c_1 > 0$ which does not depend on f and L. Therefore we obtain

$$\| T_w(\chi) f \|_{L^1(K)} \leq c_2 \mu^{|\Psi_w^+| L} \| f \|_{L^1(K)} \tag{12.3}$$

with constants $c_2 > 0$ and $1 < \mu < q^{1/m}$ which do not depend on f and L. Let L_i be a strictly increasing sequence of positive integers and set $t_i = \prod_{j=1}^{\ell} \beta_j(\varpi^{-L_i}) \in T$. Since

$$t_i x_\alpha(u) t_i^{-1} = x_\alpha\Big(\prod_{j=1}^{\ell} \varpi^{-L_i \langle \alpha, \beta_j \rangle} u\Big), \qquad \alpha \in \Sigma,\ u \in k, \tag{12.4}$$

we have $N = \bigcup_{i=1}^{\infty} N_i$ for $N_i = t_i U_1^+ t_i^{-1}$; it suffices to prove (12.2) for this choice of N_i. Let M be a positive integer such that φ is right K_M-invariant. Obviously if

$$B\omega_0 N_i K_{M_1} \cap K = (B\omega_0 N_i \cap K) K_{M_1} \qquad \text{for } M_1 \geq M, \tag{12.5}$$

then $\iota_\chi(\Phi_i)|K$ is right K_{M_1} invariant. By (12.4), we have $K \cap t_i K_1 t_i^{-1} \supseteq K_{mL_i+1}$. By Lemma 3.1, we see that (12.5) is satisfied for $M_1 = mL_i + 1$ when i is sufficiently large. Set $U_i = K - (B\omega_0 N_i \cap K)$. By (12.3), if i is sufficiently large, we obtain

$$\| T_w(\chi)(\varphi - \iota_\chi(\Phi_i)) \|_{L^1(K)} \leq c_3 \mu^{m|\Psi_w^+| L_i} \cdot \mathrm{vol}(U_i), \tag{12.6}$$

with a constant $c_3 > 0$ which does not depend on i, since $\| \varphi - \iota_\chi(\Phi_i)) \|_{L^1(K)} \leq \| \varphi \|_{L^\infty(K)} \cdot \mathrm{vol}(U_i)$. To evaluate $\mathrm{vol}(U_i)$, define $\varphi_i \in PS(\delta_B^{1/2})$ by

$$(R(\chi)\varphi_i)(k) = \begin{cases} 1, & k \in U_i, \\ 0, & k \notin U_i, \end{cases}$$

for $k \in K$. Then we have

$$\varphi_i(\omega_0 n) = \begin{cases} 0 & \text{if } n \in N_i, \\ \varphi_{K,\delta_B^{1/2}}(\omega_0 n) & \text{if } n \in N - N_i \end{cases}$$

By Lemma 5.2, we get

$$\begin{aligned} \mathrm{vol}(U_i) &= \int_K \varphi_i(k)\, dk = \int_{B\backslash G} \varphi_i(g)\, dg = \int_N \varphi_i(\omega_0 n)\, dn \\ &= \int_{N - N_i} \varphi_{K,\delta_B^{1/2}}(\omega_0 n)\, dn = \delta_B(t_i)^{-1} \int_{N - U_1^+} \varphi_{K,\delta_B^{1/2}}(\omega_0 n)\, dn. \end{aligned}$$

By (3.6), we have

$$\delta_B(t_i)^{-1} = q^{-L_i n}, \qquad n = \sum_{\alpha \in \Sigma^+} \sum_{j=1}^{\ell} \langle \alpha, \beta \rangle.$$

For $\alpha\in\Sigma^+$, $\sum_{j=1}^{\ell}\langle\alpha,\beta_j\rangle\geq 1$. Hence we have $n\geq|\Sigma^+|$. We obtain

$$\mathrm{vol}(U_i)\leq c_4q^{-L_in},\qquad n\geq|\Sigma^+|, \tag{12.7}$$

with a constant $c_4>0$ which does not depend on i. Then (12.2) follows from (12.6) and (12.7) since $\mu^m<q$. This completes the proof.

Let $\mathbf{G}$ be the adjoint group and let $\psi\colon\tilde{\mathbf{G}}\to\mathbf{G}$ be the simply connected covering map as in §3. For $v\in V\otimes_{\mathbf{R}}\mathbf{C}$, we define $\chi(v)\in X$ by

$$\chi(v)(\beta(\varpi))=q^{\langle v,\beta\rangle},\qquad \beta\in P(\check{\Sigma}). \tag{12.8}$$

We have $\chi(v)=\chi(v')$ for $v,v'\in V\otimes_{\mathbf{R}}\mathbf{C}$ if and only if $v-v'\in\dfrac{2\pi\sqrt{-1}}{\log q}Q(\Sigma)$. Set $\chi=\chi(v)$ and assume π_χ^1 is unitarizable. Then we have $w\chi=\bar{\chi}^{-1}$ for some $w\in W$ which is equivalent to

$$w(v)+\bar{v}\in\frac{2\pi\sqrt{-1}}{\log q}Q(\Sigma). \tag{12.9}$$

If we replace χ to $w_1\chi$, $w_1\in W$, we have $(w_1ww_1^{-1})(w_1\chi)=\overline{(w_1\chi)}^{-1}$. By Lemmas 3.3 and 3.5, we may assume that $w_J\chi=\bar{\chi}^{-1}$ for some $J\subseteq\Delta$ and that w_J acts on J by -1.

Hereafter we shall assume that $\mathbf{G}$ is of type B, C or D. First we consider the case of type C_ℓ. We use the notation in §9. Set $v=\sum_{i=1}^{\ell}a_i\varepsilon_i$, $a_i\in\mathbf{C}$. We may assume

$$J=\{\alpha_m,\alpha_{m+2},\cdots,\alpha_{m+2u-2},\alpha_n,\alpha_{n+1},\cdots,\alpha_{\ell-1},\alpha_\ell\}, \tag{12.10}$$

where $n=m+2u$. Then we get

$$\begin{aligned}&w_J\varepsilon_i=\varepsilon_i,\qquad 1\leq i\leq m-1,\\ &w_J\varepsilon_{m+2v-2}=\varepsilon_{m+2v-1},\ w_J\varepsilon_{m+2v-1}=\varepsilon_{m+2v-2},\qquad 1\leq v\leq u,\\ &w_J\varepsilon_i=-\varepsilon_i,\qquad n+1\leq i\leq\ell.\end{aligned}$$

The condition (12.9) reduces to

$$\begin{aligned}&\sum_{v=1}^{u}\{(a_{m+2v-2}+\bar{a}_{m+2v-1})\varepsilon_{m+2v-1}+(a_{m+2v-1}+\bar{a}_{m+2v-2})\varepsilon_{m+2v-2}\}\\ &+\sum_{i=1}^{m-1}(a_i+\bar{a}_i)\varepsilon_i+\sum_{i=n+1}^{\ell}(\bar{a}_i-a_i)\varepsilon_i\in\frac{2\pi\sqrt{-1}}{\log q}Q(\Sigma)\end{aligned} \tag{12.11}$$

Set $a_i=b_i+\sqrt{-1}c_i$, $b_i,c_i\in\mathbf{R}$, $1\leq i\leq\ell$ and $w(v)+\bar{v}=\dfrac{2\pi\sqrt{-1}}{\log q}\sum_{i=1}^{\ell}d_i\varepsilon_i$. By (12.11), we obtain

$$a_i\in\sqrt{-1}\,\mathbf{R},\ d_i=0,\qquad 1\leq i\leq m. \tag{12.12}$$

$$(12.13)\quad \begin{aligned} &b_{m+2v-2} + b_{m+2v-1} = 0, \\ &c_{m+2v-2} + c_{m+2v-1} = -\frac{2\pi}{\log q} d_{m+2v-2} = \frac{2\pi}{\log q} d_{m+2v-1}, \qquad 1 \le v \le u, \end{aligned}$$

$$(12.14)\quad c_i = -\frac{\pi}{\log q} d_i, \qquad n+1 \le i \le \ell.$$

Hence we have

$$(12.15)\quad \sum_{i=n+1}^{\ell} d_i \equiv 0 \mod 2.$$

We have $\psi \circ \chi(v) = \psi \circ \chi(v + v')$ for $v' \in \frac{2\pi\sqrt{-1}}{\log q} P(\Sigma)$. By Lemma 6.2, we can assume $c_{m+2v-2} = c_{m+2v-1}$, $1 \le v \le u$ without losing any generality. Replacing $\chi(v)$ to $w\chi(v)$ for some $w \in W$, the series q^{a_i}, $1 \le i \le \ell$ takes the form

$$(12.16)\quad (\theta_1, \cdots, \theta_s, \mu_1, \bar{\mu}_1^{-1}, \cdots, \mu_m, \bar{\mu}_m^{-1}, \eta_1, \cdots, \eta_n), \qquad \ell = s + 2m + n.$$

Here $\theta_i \notin \mathbf{R}^\times$, $|\theta_i| = 1$, $\mu_i \notin \mathbf{R}^\times$, $|\mu_i| \neq 1$, $\eta_i \in \mathbf{R}^\times$ for every i. (For simplicity, we have used the letters m and n which may be different from the previous ones.) By (12.15), we observe:

(12.17)
The number of η_i such that $\eta_i < 0$ is even if $\eta_i \neq -1$ for every $1 \le i \le n$.

Set

$$v = (v_1, \cdots, v_p) = (\theta_1, \cdots, \theta_s, \mu_1, \bar{\mu}_1^{-1}, \cdots, \mu_m, \bar{\mu}_m^{-1}), \qquad p = s + 2m,$$

and let W' be the Weyl group of type C_p. Replacing v by wv, $w \in W'$, we can assume that v contains a series ("segment")

$$\text{(A)}\quad (q^{-t}\mu, q^{-(t-1)}\mu, \cdots, \mu), \qquad \mu \notin \mathbf{R},\ t \in \mathbf{Z},\ t \ge 0$$

which is maximal in the sense that no v_i or v_i^{-1} are equal to $q^{-(t+1)}\mu$ or $q\mu$. Consider the "conjugate" segment

$$\text{(B)}\quad (\bar{\mu}^{-1}, q\bar{\mu}^{-1}, \cdots, q^t\bar{\mu}^{-1}).$$

By the maximality of (A), it is clear that if one element of (A) is of absolute value 1, then (B) coincides with (A). Suppose that (A) and (B) contain a common element. Then we have

$$q^{-a}\mu = q^b\bar{\mu}^{-1} \quad \text{for some} \quad 0 \le a, b \le t.$$

Hence (B) equals

$$(q^{-(a+b)}\mu, q^{1-(a+b)}\mu, \cdots, q^{t-(a+b)}\mu).$$

If $a + b \neq t$, it contradicts the maximality of (A). If $a + b = t$, (A) coincides with

(B). Therefore either (A) coincides with (B), or (A) and (B) have no common elements. In the latter case, we may assume that (B) is contained in v.

We note the following properties of the maximal segment. Suppose that $(q^{-a}\mu)v_i^{-1} = q$ for some $0 \leq a \leq t$. Then $v_i = q^{-a-1}\mu$. If $a = t$, this contradicts the maximality of (A). If $a < t$, v_i appears in (A). In a similar manner, we see: Assume (A) = (B). If $xv_i^{-1} = q$ or $xv_i = q$ with $x \in$(A), then v_i or v_i^{-1} appears in (A). Assume (A) $\neq$ (B). If $xv_i^{-1} = q$ or $xv_i = q$ with $x \in$(A)$\cup$(B), then v_i or v_i^{-1} appears in (A)$\cup$(B).

Since $\mu \notin \mathbf{R}^\times$, we have $q^{-a}\mu \cdot q^{-b}\mu \neq q$ for $0 \leq a$, $b \leq t$. Assume (A) $\neq$ (B). Replacing (A) to (B) if necessary, we may assume $\mu\bar{\mu} \leq q^t$. Let $0 \leq a$, $b \leq t$. If $q^{-a}\mu \cdot q^{-b}\bar{\mu}^{-1} = q$, we have $\mu \in \mathbf{R}^\times$, a contradiction. If $q^{-a}\mu/(q^{-b}\bar{\mu}^{-1}) = q$, we have $\mu\bar{\mu} = q^u$ with $1 \leq u \leq t$. Then we see that (A) and (B) have a common element, a contradiction.

Now we take off (A) or (A)$\cup$(B) from v according as cases and apply the same procedure. By successive application of this procedure, we can bring v to the following form.

$$v = (I_1, I_2, \cdots, I_N),$$

(i) $$I_j = (q^{-t_j}\mu_j, \cdots, q^{-t_j}\mu_j, q^{-(t_j-1)}\mu_j, \cdots, q^{-(t_j-1)}\mu_j, \cdots, \mu_j, \cdots, \mu_j),$$

or

(ii) $$I_j = (q^{-t_j}\mu_j, \cdots, q^{-t_j}\mu_j, q^{-(t_j-1)}\mu_j, \cdots, q^{-(t_j-1)}\mu_j, \cdots, \mu_j, \cdots, \mu_j, \bar{\mu}_j^{-1}, \cdots, \bar{\mu}_j^{-1}, q\bar{\mu}_j^{-1}, \cdots, q\bar{\mu}_j^{-1}, \cdots, q^{t_j}\bar{\mu}_j^{-1}, \cdots, q^{t_j}\bar{\mu}_j^{-1}).$$

Here $(q^{-t_j}\mu_j, q^{-(t_j-1)}\mu_j, \cdots, \mu_j)$ is a maximal segment as in (A) such that $\mu_j\bar{\mu}_j \leq q^{t_j}$ and I_j takes the form (i) or (ii) according as (A) = (B) or (A) $\neq$ (B); $q^{-a}\mu_j$ and $q^b\bar{\mu}_j^{-1}$ may appear with certain multiplicities. All elements of the form $(q^{-a}\mu_j)^{\pm 1}$, $0 \leq a \leq t_j$ in case (i), $(q^{-a}\mu_j)^{\pm 1}$, $(q^b\bar{\mu}_j^{-1})^{\pm 1}$, $0 \leq a$, $b \leq t_j$ in case (ii) which appear in v are included in I_j. Furthermore if we make a suitable permutation on elements of I_j, I_j can be written as a sum of segments of the form of (A) or (A)$\cup$(B).

Now replacing (q^{a_i}) by $w(q^{a_i})$ with $w \in W$, we can bring it to the form

(12.18) $$(I_1, I_2, \cdots, I_N, \eta_1, \cdots, \eta_n), \qquad 1 \geq |\eta_n| \geq |\eta_{n-1}| \geq \cdots \geq |\eta_1|.$$

Then by this construction, we obviously have

(12.19) $$q^{a_i} \neq q, \ 1 \leq i \leq \ell, \qquad q^{a_i \pm a_j} \neq q, \ 1 \leq i < j \leq \ell.$$

Let $J_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_{\ell-n-1}\}$, $J_2 = \{\alpha_{\ell-n+1}, \cdots, \alpha_\ell\}$. Then W_{J_1} acts as the permutation group $S_{\ell-n}$ on $\varepsilon_1, \cdots, \varepsilon_{\ell-n}$. We have

(12.20) $$w_1 w_2(q^{a_i}) = \overline{(q^{a_i})}^{-1}$$

with some $w_1 \in W_{J_1}$, $w_2 \in W_{J_2}$. By the obsevation (12.17), we see easily that (12.20) can be lifted to the relation

(12.21) $$w_1' w_2' \chi(v) = \overline{\chi(v)}^{-1}$$

with $w_1' \in W_{J_1}$, $w_2' \in W_{J_2}$, replacing v by $v + v'$, $v' \in \frac{2\pi\sqrt{-1}}{\log q} P(\Sigma)$ if necessary. Then, replacing $\chi(v)$ by $w\chi(v)$, $w \in W_{J_1}$, we can assume that $\chi(v)$ satisfies the following conditions.

(12.22) $\quad w_1 w_2 \chi(v) = \overline{\chi(v)}^{-1} \quad$ for some $w_1 \in W_{J_1}$, $w_2 \in W_{J_2}$.

(12.23) $\quad \chi(v)(a_\alpha) \neq q \quad$ for every $\alpha \in \Sigma^+$.

(12.24) $\quad |\chi(v)(a_\alpha)| \leq 1 \quad$ for every $\alpha \in \Psi^+_{w_1 w_2}$.

(12.25) $\quad \chi(v)(a_\alpha) \neq 1 \quad$ for every $\alpha \in \Psi^+_{w_1 w_2}$.

Set $\chi = \chi(v)$, $\tilde{\chi} = \chi \circ \psi$. By (12.23), Lemma 3.2 and Kato's criterion (4.3), (4.4), $PS(\chi)$ is generated by $\varphi_{K,\chi}$ as G-module. By Lemmas 6.1, 5.4 and 12.1, π^1_χ is unitarizable if and only if $cT_{w_1 w_2, \chi}$ is of positive type for some $c \in \mathbf{C}^\times$. By Lemma 5.7, this is equivalent to that $cT_{w_1 w_2, \tilde{\chi}}$ is of positive type. By Lemma 5.8 and Theorem 2.3, this is the case if and only if $c_1 T_{w_1, \tilde{\chi}_{J_1}}$ and $c_2 T_{w_2, \tilde{\chi}_{J_2}}$ are positive type with some $c_1, c_2 \in \mathbf{C}^\times$. Here $\tilde{\chi}_{J_i}$, $i = 1, 2$ is defined as in Lemma 5.8, i.e., $\tilde{\chi}_{J_i} = \tilde{\chi} | T_{J_i}$. By the construction above, we can assume $\tilde{\chi}_{J_i} = \chi_{J_i} \circ \psi_i$ with $\chi_{J_i} \in X_{w_i}$ for the adjoint group, $i = 1, 2$, where ψ_i: $\tilde{\mathbf{G}}_{J_i} \to \mathbf{G}_{J_i}$ is the simply connected covering map. By Lemmas 5.7, 5.4 and 12.1, this is the case if and only if both of $\pi^1_{\chi_1}$ and $\pi^1_{\chi_2}$ are unitarizable, $\chi_i = \chi_{J_i}$, $i = 1, 2$. Summing up, π^1_χ is unitarizable if and only if $\pi^1_{\chi_1}$ and $\pi^1_{\chi_2}$ are unitarizable.

Now $\mathbf{G}_{J_1}$ (resp. $\mathbf{G}_{J_2}$) is the adjoint group of type A (resp. type C). The unitarizabilty of $\pi^1_{\chi_1}$ is solved in Tadič [26]. Therefore we have reduced the unitarizability problem of π^1_χ to that of $\pi^1_{\chi_2}$. We see that $\tilde{\chi}_2 = \chi_2 \circ \psi_2$ is real valued. (χ_2 itself may not be real valued because $\frac{1}{2}\sum_{i=1}^{\ell} \varepsilon_i \in P(\check{\Sigma})$. That is, if the number of η_i such that $\eta_i < 0$ is odd, χ_2 is not real valued.) This is what we claimed at the beginning of this section.

We assume that $\mathbf{G}$ is of type B_ℓ. We can start with $\chi(v) \in X_{w_J}$ with J of the form (12.10) assuming $\mathbf{G}$ is of adjoint type. All arguments in the previous case of type C_ℓ can be applied. It is not necessary to take care of the signature condition (12.17). We can reduce the unitarizability of π^1_χ for the simply connected group to the case when χ is real valued.

We assume that $\mathbf{G}$ is the adjoint group of type D_ℓ. We may assume $\chi(v) \in X_{w_J}$ for some $J \subseteq \Delta$ and that w_J acts on J by -1. We can assume that J takes either one of the following forms.

(I) $$J = \{\alpha_m, \alpha_{m+2}, \cdots, \alpha_{m+2u-2}, \alpha_n, \alpha_{n+1}, \cdots, \alpha_{\ell-1}, \alpha_\ell\},$$
$$n = m + 2u, \ \ell - n + 1 \geq 4 \text{ is even.}$$

(II) $\quad J$ consists of isolated summits in the Dynkin diagram of Σ.

Set $v = \sum_{i=1}^{\ell} a_i \varepsilon_i$, $a_i \in \mathbf{C}$ in the notation of §11. First we consider case (I). We may assume that the series q^{a_i}, $1 \leq i \leq \ell$ takes the form (12.16). Instead of

(12.17), we have

(12.26) Let a, b and c be the numbers of η_i such that $\eta_i = 1$, $\eta_i = -1$ and $\eta_i < 0$ respectively. Then $n - (a + b)$ and $c - b$ are even.

By this observation, we find easily that $\chi(v)$ satisfies (12.22) $\sim$ (12.25) after replacing $\chi(v) \to w\chi(v)$, $v \to v + v'$, $v' \in \frac{2\pi\sqrt{-1}}{\log q} P(\Sigma)$. In this way, we can obtain the same conclusion as in the case of type C_ℓ.

Now let us consider the case (II). If $\alpha_{\ell-1}$, $\alpha_\ell \in J$, then we have $\alpha_{\ell-2} \notin J$ by the assumption. Since $\sigma_{\ell-1}\sigma_\ell \varepsilon_i = -\varepsilon_i$ for $i = \ell - 1, \ell$, this case is completely similar to Case (I). Assume $\{\alpha_{\ell-1}, \alpha_\ell\} \nsubseteq J$. If $\alpha_\ell \notin J$, we observe the following fact. The series q^{a_i}, $1 \leq i \leq \ell$ takes the form (12.16). Then (12.26) also holds in this case. If there exists an η_i such that $\eta_i \neq \pm 1$, the same argument as above can be applied. If all η_i satisfies $\eta_i = \pm 1$, we can bring $w(q^{a_i})$ to the form

$$(\eta_1, \cdots, \eta_n, I_1, I_2, \cdots, I_N) \tag{12.18'}$$

instead of (12.18). Put

$$J_1 = \{\alpha_{n+1}, \alpha_{n+2}, \cdots, \alpha_{\ell-1}\}.$$

Then (12.22) $\sim$ (12.25) holds with $J_2 = \emptyset$. Therefore we obtain the desired reduction. Assume $\alpha_\ell \in J$, $\alpha_{\ell-1} \notin J$. Let σ be the automorphism of $\tilde{G}$ induced by the graph automorphism of Σ, $\alpha_{\ell-1} \to \alpha_\ell$, $\alpha_\ell \to \alpha_{\ell-1}$. We use the same notation as in §11. Then $(\pi_\chi^1)^\sigma$ occurs in $PS(\tilde{\chi}^\sigma)$ and is the unique spherical constituent of $PS(\tilde{\chi}^\sigma)$ since $\tilde{K}^\sigma = \tilde{K}$. $\pi_{\tilde{\chi}}^1$ is unitarizable if and only if $\pi_{\tilde{\chi}^\sigma}^1$ is unitarizable. Then we have $\tilde{\chi}^\sigma = \psi \circ \chi'$, $\chi' \in X$ such that $\chi' \in X_{w_{J'}}$, $J' \ni \alpha_{\ell-1}$. Thus we are reduced to the case $\alpha_{\ell-1} \in J$. This completes the reduction to the case of real quasi-characters.

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