J. Math. Kyoto Univ. (JMKYAZ)
32-1 (1992) 101-134

Twistor spaces of even dimensional Riemannian manifolds

By

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Introduction

The twistor space of a four-dimensional oriented Riemannian manifold X is a total space of a certain $\mathbf{P}^1$-bundle over X with an almost complex structure, and the integrability condition of the almost complex structure is equivalent to anti-self-duality of the underlying manifold X ([A.H.S]). Relations between certain field equations on the manifold X and cohomology groups of certain holomorphic line bundles over the twistor space are studied by Hitchin in [H].

The notion of twistor spaces of four-dimensional manifolds was generalized to higher dimensional oriented Riemannian manifolds by O'Brian and Rawnsley ([O.R]). Their definition of the twistor space $Z(X)$ of a $2n$-dimensional oriented Riemannian manifold X is $Z(X):=\mathrm{SO}(X)/\mathrm{U}(n)$, where $\mathrm{SO}(X)$ is the oriented orthonormal frame bundle of X with right $\mathrm{SO}(2n)$-action and $\mathrm{U}(n)$ is considered to be a subgroup of $\mathrm{SO}(2n)$. Murray ([M]) studied relations between certain field equations on the underlying manifold and cohomology groups of holomorphic line bundles over the twistor space, under the condition of the integrability of the almost complex structure of the twistor space.

In this paper, we shall give another definition of the twistor space $Z(X)$ of a $2n$-dimensional spin manifold X, and the hyperplane bundle H over it. More precisely, if we denote by $\Delta^+(X)$ a positive spin bundle over X, $Z(X)$ is defined as a submanifold of $\mathbf{P}(\Delta^+(X))$, and H is the pull-back of the hyperplane bundle over $\mathbf{P}(\Delta^+(X))$. Hence $Z(X)$ and H^2 are defined even if X has no spin structure, but H can be defined if and only if X is a spin manifold. This is a generalization of the original definition by using the twistor operator ([A.H.S]), and gives an immediate correspondence between solutions of the twistor equation and holomorphic sections of the hyperplane bundle (see Theorem 9.2 below). This is a generalization of a result in the four-dimensional case given by Hitchin in [H]. The correspondence will be proved without assuming the integrability of the almost complex structure of the twistor space.

To show the equivalence of two definitions of twistor spaces, we shall prove that there is an embedding of the twistor space in the sense of [O.R] to the projectivized spinor bundle $\mathbf{P}(\Delta^+(X))$ which is induced by the canonical

Received August 11, 1990

embedding of $SO(2n)/U(n)$ into $\mathbf{P}(\Delta^+)$ equivariant under the action of $SO(2n)$. The image is the twistor space in our sense and two definitions of almost complex structures coincide. Especially, the embedding is surjective if $n \leq 3$. Hence the twistor space of a six-dimensional Riemannian manifold can be defined to be a projectivized spinor bundle as in [W]. Another advantage of our definition is that the conformal invariance of the twistor space, proved in [O.R] and [M], can be reduced immediately from the conformal invariance of the twistor operator ([F]).

Using the geometric definition of twistor spaces in [O.R], we shall also show that twistor spaces enjoy similar properties as those in four-dimensional case described in [A.H.S]. Let X be a $2n$ dimensional oriented Riemannian manifold. $Z(X)$ and H^2 denote the twistor space and the square of the hyperplane bundle, respectively. The canonical bundle of the twistor space has a form:

$$K_{Z(X)} \simeq H^{-2n}$$

where the canonical bundle of an m-dimensional almost complex manifold is defined to be a complex line bundle consisting of $(m, 0)$-forms. This isomorphism is holomorphic, if the almost complex structure is integrable. The conformal structure of X can be recovered from the almost complex structure of $Z(X)$ (see Theorem 5.2 below). Furthermore, if the almost complex structure of $Z(X)$ is integrable and $n > 1$, we shall define a $2n$-dimensional holomorphic complex conformal manifold $X_{\mathbf{C}}$ as a family of certain submanifolds of $Z(X)$ (see Theorem 5.3). Hence we have a double fibration, which is used to define the Penrose transform,

$$\begin{tikzcd} & Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Z(X) & & X_{\mathbf{C}} \end{tikzcd}$$

where

$$Y := \{(z, x) \in Z(X) \times X_{\mathbf{C}} | z \in \text{the submanifold corresponding to } x\}$$

and p_1 (resp. p_2) is the projection to the first (resp. second) factor. Although $X_{\mathbf{C}}$ is defined in [M], the natural complex conformal structure of $X_{\mathbf{C}}$ is not mentioned there. The manifold X can be naturally considered as a submanifold of $X_{\mathbf{C}}$. Furthermore, there is an anti-holomorphic involution $\tilde{\tau}$ on $X_{\mathbf{C}}$, whose fixed locus is X and the conformal structure of X is recovered by restricting the complex conformal structure of $X_{\mathbf{C}}$ (see Theorem 5.2 and Theorem 7.3 below). Thus, even if we forget the fibration over X, from the twistor space $Z(X)$, we can recover information of the conformal manifold X to a certain extent.

In the following we shall study two examples, namely spheres and tori.

The twistor space of S^{2n} is $\mathrm{SO}(2n+2)/\mathrm{U}(n+1)$ with the holomorphic structure as a Hermitian symmetric space, and the complexification $S_{\mathbf{C}}^{2n}$ is a $2n$-dimensional non-singular complex hyperquadric Q_{2n}. The hyperquadric Q_{2n} is expressed as a homogeneous space: $\mathrm{SO}(2n+2)/\mathrm{U}(1)\times\mathrm{SO}(2n)$. Hence, in this case, the fibration has the form:

$$\begin{tikzcd} & \mathrm{SO}(2n+2)/\mathrm{U}(1)\times\mathrm{U}(n) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathrm{SO}(2n+2)/\mathrm{U}(n+1) & & \mathrm{SO}(2n+2)/\mathrm{U}(1)\times\mathrm{SO}(2n) \end{tikzcd}$$

where all spaces have natural holomorphic structures as Hermitian symmetric spaces and two projections are holomorphic mappings. The complex conformal structure of Q_{2n} is a natural one. This is a generalization of the so called the Penrose fibration in case $n=2$,

$$\begin{tikzcd} & F_{1,2;4} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbf{P}^3 & & G_{2,4} \end{tikzcd}$$

For general conformally flat manifold X, let $\tilde{X}$ be the universal covering space of X with conformally flat structure induced by X. There is a conformal map called the developing map:

$$\Phi\colon \tilde{X} \longrightarrow S^{2n}$$

which induces a group homomorphism:

$$\tilde{\Phi}\colon \pi_1(X) \longrightarrow \mathrm{SO}_0(1, 2n+1).$$

Here, $\mathrm{SO}_0(1, 2n+1)$ is considered to be the conformal transformation group of S^{2n}. Since twistor spaces are conformally invariant, the problem of studying twistor spaces of certain conformally flat manifolds is reduced to the study of discrete subgroups of $\mathrm{SO}_0(1, 2n+1)$.

Following the above method, for a lattice Γ of $\mathbf{R}^{2n}$ with $n>1$, we shall show that the complexification of $\mathbf{R}^{2n}/\Gamma$ is $(\mathbf{R}^{2n}\otimes\mathbf{C})/\Gamma$ with natural complex conformal structure.

Let us explain briefly the contents of this paper.

In § 1, we recall a general method to define a distribution by a first order differential operators. In § 2, to fix notation, we give an explicit description of the spinor group $\mathrm{SPIN}(2n)$ and the spin module $\Delta^{\pm}$ in terms of the Clifford algebra. In § 3, we define twistor spaces of even dimensional conformal manifolds and show the equivalence to the definition given in [O.R]. In §4, we study the canonical bundle of the twistor space. In §5, we define the complexification $X_{\mathbf{C}}$ of X when the almost complex structure of the twistor space $Z(X)$ is integrable, and define a real structure of $X_{\mathbf{C}}$ in case $\frac{1}{2}\dim X$ is even. In §6, we study the generalized Penrose fibration, by considering the complexification of even dimensional spheres. In §7, we define the real structure of $X_{\mathbf{C}}$ when $\frac{1}{2}\dim X$ is odd. In §8, for a lattice Γ of $\mathbf{R}^{2n}$, we study the complexification of $\mathbf{R}^{2n}/\Gamma$. In §9, we give an explicit correspondence between the solutions of the twistor equation over X and the holomorphic sections of H over the twistor space $Z(X)$.

I would like to thank Prof. K. Ueno and Dr. W. M. Oxbury for their encouragement and valuable suggestions.

Notation

$\mathbf{R}$: the real number field
$\mathbf{C}$: the complex number field with the imaginary unit $\sqrt{-1}$
$\mathbf{C}^{\times}$: the set of non-zero complex numbers
$\mathrm{so}(m)$: the Lie algebra of $\mathrm{SO}(m)$
$J := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ a natural complex structure of $\mathbf{R}^{2n}$
$\mathrm{U}(n) := \{A \in \mathrm{SO}(2n) | AJ = JA\}$
$u(n)$: the Lie algebra of $\mathrm{U}(n)$
$Z_n := \mathrm{SO}(2n)/\mathrm{U}(n)$ with the Hermitian symmetric structure
Q_{2n}: the $2n$-dimensional complex hyperquadric
Let E be a real vector bundle (or a real vector space)
$E \otimes \mathbf{C}$: the complexification of E
Let E be a complex vector bundle (or a complex vector space)
E^*: the dual of E
$E^{\times}$: the set of non-zero vectors of E
$\mathbf{P}(E)$: the set of one-dimensional subspaces of E
$[v]$: the image of $v \in E^{\times}$ by the projection map $E \to \mathbf{P}(E)$
$\mathcal{O}(1)$: the hyperplane bundle over $\mathbf{P}(E)$
$\mathcal{O}(-1)$: the dual bundle of $\mathcal{O}(1)$
$\Gamma(E)$: the set of sections of E
$s^{\vee}$: the function on E^* defined by a section $s \in \Gamma(E)$
$J_1(E)$: the bundle consisting of one-jets of sections of E
TM, T^*M: the real tangent and cotangent bundle of a smooth manifold M
$\mathrm{SPIN}(2n)$: the spinor group
Δ: the spin module
$\Delta^{\pm}$: the positive or negative spin module
$\{\theta_I, I \subset \{1, \ldots, n\}\}$: the orthonormal basis of Δ defined in §2
cl: $\mathbf{R}^{2n} \otimes \Delta \to \Delta$ the Clifford multiplication map
K_+: the cokernel of $cl^*: (\Delta^-)^* \to (\Delta^+)^* \otimes (\mathbf{R}^{2n})^*$
$V(D)$: the distribution defined by D

Let X be a oriented Riemannian manifold (or a spin manifold)

$\Delta^{\pm}(X)$: the positive or negative spin bundle

$K_{+}(X)$: the K_{+} bundle

$\bar{D}$: the twistor operator

$\mathrm{SO}(X)$: the oriented orthonormal frame bundle

$\mathrm{SPIN}(X)$: the spinor frame bundle

$Z(X)$: the twistor space

H: the hyperplane bundle over $Z(X)$

$X_{\mathbf{C}}$: the complexification of X

$\mathrm{SO}_0(1, m)$: the identity component of $\mathrm{SO}(1, m)$, or the conformal transformation group of S^{m-1}, for $m \geq 3$

§1. The distribution defined by a first order linear differential operator

In this section, we recall how to define the distribution by a first order linear differential operator. Let E be a complex vector bundle over a smooth manifold M. By the canonical pairing of E and its dual bundle E^*, the section s of E defines a complex function $s^\vee$ on the total space of E^*. Let $p\colon E^* \to M$ be the projection, and $J_1(E)$ denotes the one-jet bundle of E. We define a linear map

$$V\colon p^* J_1(E) \longrightarrow T^*(E^*) \otimes \mathbf{C}$$

by $V(p^* j_1(s)) = ds^\vee$ for all sections s of E, where $j_1(s) \in \Gamma(J_1(E))$ is the one-jet of s. Let F be another vector bundle over M. For a first order linear differential operator $D\colon \Gamma(E) \to \Gamma(F)$, we have a linear map

$$L(D)\colon J_1(E) \longrightarrow F.$$

Definition 1.1. For a differential operator $D\colon \Gamma(E) \to \Gamma(F)$, let R be the kernel of the linear map $L(D)$. A distribution $V(p^* R)$ on the total space of the dual vector bundle E^* of E is called the distribution of D, and denoted by $V(D)$.

Now consider the case in which a differential operator D has a form:

$$D\colon \Gamma(E) \xrightarrow{\nabla} \Gamma(E \otimes T^*M) \xrightarrow{\sigma} \Gamma(F)$$

where ∇ is a covariant derivative and $\sigma\colon E \otimes T^*M \to F$ is a linear map. In terms of jets, it is written as:

$$L(D)\colon J_1(E) \xrightarrow{L(\nabla)} E \otimes T^*M \xrightarrow{\sigma} F.$$

The linear map $L(\nabla)$ defines a splitting of the exact sequence

$$0 \longrightarrow E \otimes T^*M \underset{L(\nabla)}{\overset{\longrightarrow}{\longleftarrow}} J_1(E) \longrightarrow E \longrightarrow 0.$$

Hence there is a splitting

$$J_1(E) \simeq E \oplus (E \otimes T^*M),$$

where E is identified with the kernel of $L(\nabla)$, and by using the splitting, $R := \ker L(D)$ can be written as:

$$R \simeq E \oplus S, \tag{1.1}$$

where S denotes the kernel of σ.

On the other hand, the connection defines a splitting of the real cotangent bundle of E^*:

$$T^*(E^*) = T_V^*(E^*) \oplus T_H^*(E^*) \tag{1.2}$$

where the first component is the vertical cotangent bundle defined by the connection of E, and the second component is the horizontal cotangent bundle.

Let us study a relationship between two splitttings (1.1) and (1.2), by the map V.

First, we study $V(p^*E)$. The real vector bundle $T_V^*(E^*)$ has a natural complex structure, since E^* is a complex vector bundle. Hence the complexified bundle $T_V^*(E^*) \otimes \mathbf{C}$ has two components, $T_V^{*(0,1)}(E^*)$ and $T_V^{*(0,1)}(E^*)$, and the subbundle $T_V^{*(1,0)}(E^*)$ is isomorphic to p^*E. The restriction map $V|_{p^*E}$ is nothing but the inverse of this isomorphism. Hence we have

$$V(p^*E) = T_V^{*(1,0)}(E^*).$$

Next, we restrict V to $p^*(E \otimes T^*M)$. This is obtained by the canonical pairing of E and E^* and identifying $T^*M \otimes \mathbf{C}$ with the complexified horizontal cotangent bundle. Therefore, we have

$$V(p^*S) \subset T_H^*(E^*) \otimes \mathbf{C}.$$

Proposition 1.2. *For a differential operator $D = \sigma \circ \nabla$, where $\sigma: E \otimes T^*M \to F$ is a linear map and ∇ is a covariant derivative, there is a splitting of the distribution $V(D) = V(p^*E) \oplus V(p^*S)$ corresponding to the splitting* (1.2) *of the cotangent bundle, where S is the kernel of σ. Furthermore, we have*

$$V(p^*E) = T_V^{*(1,0)}(E^*).$$

Example 1.3. If $D = \nabla$, then $S = 0$ and $V(\nabla) = V(p^*E) = T_V^{*(1,0)}(E^*)$. Hence we have

$$T_V^*(E^*) \otimes \mathbf{C} = V(\nabla) \oplus \overline{V(\nabla)}.$$

This is a way to define the vertical cotangent bundle (hence also the horizontal tangent bundle) by a covariant derivative. We give here a basis of the bundle $V(\nabla)$. Let $(e_1, \dots, e_m)$ be a local frame of E, and $(e^1, \dots, e^m)$ be the dual frame. We write the covariant derivative in terms of this frame, $\nabla e_i = e_j \omega_i^j$. Let $(\tau_1, \dots, \tau_m)$ be the local coordinates of the fiber part of E^* corresponding to the frame $(e^1, \dots, e^m)$. Then $V(\nabla)$ is spanned by

$$d\tau_i - \tau_j \omega_i^j, \qquad i = 1, \dots, m.$$

§ 2. Spinor groups and spin modules

In this setion, to fix notation, we give a description of SPIN(2n), $\Delta^{\pm}$ and certain SPIN(2n)-equivariant maps in terms of the Clifford algebra CLIF(2n). (See [G] Chapter 3, for detail.)

Let E be a $2n$-dimensional real vector space with a positive definite inner product (,). The Clifford algebra CLIF(E) is an algebra generated by E subject to the relations

$$v * v + (v, v) = 0, \qquad \text{for } v \in E. \tag{2.1}$$

Since this relation is of even degree in the tensor algebra $\otimes V$ with respect to the canonical grading, we may regard CLIF(E) as a $\mathbf{Z}/2\mathbf{Z}$ graded algebra,

$$\mathrm{CLIF}_+(E) = \langle v_1 * \cdots * v_{2k} | v_1, \dots, v_{2k} \in E,\ k \geq 0 \rangle$$
$$\mathrm{CLIF}_-(E) = \langle v_1 * \cdots * v_{2k+1} | v_1, \dots, v_{2k+1} \in E,\ k \geq 0 \rangle.$$

Let $\Lambda(E)$ be the exterior algebra of E. The exterior multiplication by an element of E defines a mapping ext: $E \to \mathrm{END}(\Lambda(E))$. Let interior multiplication int(v) be the dual endomorphism of ext(v). Now define

$$c\colon E \longrightarrow \mathrm{END}(\Lambda(E))$$
$$v \longmapsto \mathrm{ext}(v) - \mathrm{int}(v).$$

Then we have $c(v)^2 + (v, v)\, id = 0$. Hence, by the universality of the Clifford algebra, there is a unique algebra homomorphism c' which is an extension of c

$$c'\colon \mathrm{CLIF}(E) \longrightarrow \mathrm{END}(\Lambda(E)).$$

Let $(e_1, \dots, e_{2n})$ be an orthonormal basis of E. For $I = \{i_1, i_2, \dots, i_k\}$, $1 \leq i_1 < i_2 < \cdots < i_k \leq 2n$, let e_I be $e_{i_1} * e_{i_2} * \cdots * e_{i_k}$. Then we have $c'(e_I) 1 = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$. Hence the map $w \mapsto c'(w) 1$ induces an isomorphism of vector spaces

$$\mathrm{CLIF}(E) \simeq \Lambda(E).$$

We define a positive definite inner product in CLIF(E) by pulling back that of $\Lambda(E)$. Then,

$$\{e_I | I \subset \{1, \dots, 2n\}\}$$

is an orthonormal basis of CLIF(E). The multiplication of e_1 to CLIF(E) from left or right induces a permutation of the basis, hence it is isometric. This means that the left or right multiplication of any unit vector v of E is isometric. In particular, for any unit vector v of E, the map $x \mapsto v * x * v$ is isometric, which preserves the subspace E and induces on it the reflection map with respect to the hyperplane with the normal vector v. We let SPIN(E) be the set of all

$w \in \mathrm{CLIF}(E)$ expressible as a product of an even number of unit vectors of E,

$$\mathrm{SPIN}(E) = \{v_1 * \cdots * v_{2k} \in \mathrm{CLIF}(E) | v_1, \ldots, v_{2k}\colon \text{unit vectors in } E\}.$$

This forms a group by the Clifford multiplication. An element w of $\mathrm{SPIN}(E)$ satisfies the equation ${}^t w * w = 1$, where ${}^t w$ denotes the transpose of w. For an element w of $\mathrm{SPIN}(E)$, the map

$$\begin{aligned} \rho(w)\colon \mathrm{CLIF}(E) &\longrightarrow \mathrm{CLIF}(E) \\ x &\longmapsto w * x * {}^t w \end{aligned}$$

preserves E and induces on it an isometric transformation preserving the orientation. Hence we get a homomorphism of groups

$$\begin{aligned} \pi\colon \mathrm{SPIN}(E) &\longrightarrow \mathrm{SO}(E) \\ w &\longmapsto \rho(w)|_E. \end{aligned}$$

The kernel is in the center of the $\mathrm{CLIF}(E)$, which consists of scalars, because E is even dimensional. We have an exact sequence of groups:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{SPIN}(E) \xrightarrow{\pi} \mathrm{SO}(E) \longrightarrow 1$$

The group $\mathrm{SPIN}(E)$ is the universal covering group of $\mathrm{SO}(E)$, if $n \geq 2$.

Next, we define a spin module. Put

$$\theta_\emptyset := (1 + \sqrt{-1}\, e_1 * e_{n+1}) * (1 + \sqrt{-1}\, e_2 * e_{n+2}) * \cdots * (1 + \sqrt{-1}\, e_n * e_{2n}),$$

and

$$\theta_I = e_I * \theta_\emptyset, \qquad I \subset \{1, \ldots, n\}.$$

We define a spin module $\Delta = \Delta(E)$ as a complex subspace of $\mathrm{CLIF}(E) \otimes \mathbf{C}$ spanned by θ_I, $I \subset \{1, \ldots, n\}$. We have

$$e_{n+i} * \theta_I = \begin{cases} \sqrt{-1}\, e_i * \theta_I, & \text{if } i \notin I \\ -\sqrt{-1}\, e_i * \theta_I, & \text{if } i \in I, \end{cases}$$

since we have

$$e_{n+i} * (1 + \sqrt{-1}\, e_j e_{n+j}) = \begin{cases} (1 + \sqrt{-1}\, e_j e_{n+j}) * e_{n+i}, & \text{if } i \neq j \\ \sqrt{-1}\, e_i * (1 + \sqrt{-1}\, e_j e_{n+j}), & \text{if } i = j \end{cases}$$

$$(1 + \sqrt{-1}\, e_j * e_{n+j}) * e_i = e_i * (1 + \sqrt{-1}\, e_j * e_{n+j}), \qquad \text{if } i \neq j.$$

Hence the spin module Δ is invariant under the left action of $\mathrm{CLIF}(E)$.

Since $\mathrm{SPIN}(E) \subset \mathrm{CLIF}_+(E)$, the positive spin module $\Delta^+ = \Delta^+(E)$ (resp. the negative spin module $\Delta^- = \Delta^-(E)$) defined as

$$\Delta^+(E) := \Delta(E) \cap (\mathrm{CLIF}_+(E) \otimes \mathbf{C})$$

$$(\text{resp. } \Delta^-(E) := \Delta(E) \cap (\mathrm{CLIF}_-(E) \otimes \mathbf{C}))$$

is invariant under the action of $\mathrm{SPIN}(E)$.

Now let us show that $\{\theta_I\}$ is a basis of Δ. For simplicity, we extend the notation of multi-indices. We define a multi-index to be a finite sequence of elements of $\{\pm 1, \dots, \pm n\}$. For a multi-index $I = (m_1, \dots, m_k)$, $-I$ and ${}^tI$ are the multi-indices defined as

$$-I := (-m_1, \dots, -m_k),$$
$${}^tI := (m_k, \dots, m_1).$$

For another multi-index $J = (m'_1, \dots, m'_l)$, we define the addition $I + J$ to be

$$I + J := (m_1, \dots, m_k, m'_1, \dots, m'_l).$$

For a multi-index $I = (\varepsilon i)$, where $\varepsilon = \pm 1$ and $1 \le i \le n$, we define

$$e_I := \varepsilon \cdot e_i,$$

and for a general multi-index J, we define e_J inductively, such that e_J's satisfy the following identities:

$$e_{J+J'} = e_J * e_{J'}, \qquad \text{for all } J \text{ and } J'.$$

With respect to other operations of multi-indices, we have

$$e_{-I} = (-1)^{|I|} e_I,$$
$$e_{{}^tI} = {}^te_I,$$

and

$$e_{I+J} = e_K \quad \text{implies} \quad e_J = e_{-{}^tI+K}.$$

Put $A := \{1, \dots, n\}$. For a multi-index I, there is a unique subset I' of A such that $e_I = \pm e_{I'}$. We define the length of I as the number of elements of I', and denote it by $|I|$.

$$|I| := \# I'.$$

We regard a subset of A as a multi-index by the natural order such that two definitions of e_I coincide. Define $\theta_I := e_I * \theta_\emptyset$ as before.

Proposition 2.1. (1) *There is a bilinear form* (,) *on* Δ *equivariant under the action of* $\mathrm{SPIN}(E)$.

(2) $\{\theta_I \mid I \subset A\}$ *is a basis of* Δ.

(3) *By the invariant bilinear form, there is an isomorphism between* Δ *and* Δ^*. *Let* $\{\theta^I \mid I \subset A\}$ *be the dual basis of* Δ^*. *The isomorphism is written as*:

$$\Delta \longrightarrow \Delta^*$$
$$\theta_I \longmapsto \theta^{-I+A},$$

where we use the same notation of multi-indices for the dual basis. Hence $(\Delta^{\pm})^* \simeq (\Delta^{\pm})$ *if* n *is even, and* $(\Delta^{\pm})^* \simeq (\Delta^{\mp})$ *if* n *is odd, as* $\mathrm{SPIN}(E)$*-modules.*

Proof. The bilinear map

$$\begin{aligned} \Delta \times \Delta &\longrightarrow \mathbf{C}\cdot 2^n\theta_A \\ (x, y) &\longmapsto {}^t x * y \end{aligned}$$

is well-defined, since we have

$$\begin{gathered} {}^t(1 + \sqrt{-1}\, e_i * e_{n+i}) * (1 + \sqrt{-1}\, e_i * e_{n+i}) = 0 \\ {}^t(1 + \sqrt{-1}\, e_i * e_{n+i}) * e_i * (1 + \sqrt{-1}\, e_i * e_{n+i}) = 2e_i * (1 + \sqrt{-1}\, e_i * e_{n+i}). \end{gathered}$$

We define a bilinear form $(\ ,\ )$ as the coefficients of $2^n\theta_A$. Since ${}^t w * w = 1$, $w \in \mathrm{SPIN}(E)$, we have

$${}^t(w * x) * (w * y) = {}^t x * ({}^t w * w) * y = {}^t x * y.$$

Hence it is invariant under the action of $\mathrm{SPIN}(2n)$. Now let us compute (θ_I, θ_J) for $I, J \subset A$.

$$\begin{aligned} {}^t\theta_I * \theta_J &= {}^t(e_I * \theta_\emptyset) * (e_J * \theta_\emptyset) \\ &= {}^t\theta_\emptyset * {}^t e_I * e_J * \theta_\emptyset \\ &= \begin{cases} 2^n \theta_{\iota_{I+J}}, & \text{if } I \amalg J = A \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.2}$$

Hence $\{\theta_I | I \subset A\}$ is linearly independent, that is, a basis of Δ.

(3) is immediate by (2.2).

By multiplying vectors of E from left to the spin module Δ, we get a $\mathrm{SPIN}(E)$-equivariant map:

$$\begin{aligned} cl\colon E \otimes \Delta &\longrightarrow \Delta \\ v \otimes w &\longmapsto v * w \end{aligned}$$

called the Clifford multiplication map. Since cl is surjective, the dual map cl^* is injective. By simple computation, we have an explicit description of this map.

Lemma 2.2. *The* $\mathrm{SPIN}(E)$*-equivariant map* cl^* *is written as*:

$$\begin{aligned} cl^*\colon \Delta^* &\longrightarrow \Delta^* \otimes E^* \\ \theta^I &\longmapsto \sum_{i \in I} \theta^{-i+I} \otimes \alpha^i + \sum_{i \notin I} \theta^{-i+I} \otimes \beta^i \end{aligned}$$

where $\alpha^i := e^i + \sqrt{-1}\, e^{n+i}$ *and* $\beta^i := e^i - \sqrt{-1}\, e^{n+i}$, *for* $i = 1, \ldots, n$.

Let K be the cokernel of cl^*. We have an exact sequence of equivariant maps:

$$0 \longrightarrow \Delta^* \xrightarrow{cl^*} \Delta^* \otimes E^* \xrightarrow{\kappa} K \longrightarrow 0. \tag{2.3}$$

Let $K_\pm$ be the image of $(\Delta^\pm)^* \otimes E$ by κ, which are SPIN(E)-submodules of K.

Lemma 2.3. (1) *The* SPIN(E)*-modules* $K_\pm$ *are irreducible.*
(2) $cl^*((\Delta^\mp)^*)$ *is a maximal proper* SPIN(E)*-submodule of* $(\Delta^\pm)^* \otimes E^*$.

Proof. By the representation theory of compact Lie groups, there is a splitting of (2.3) as SPIN(E)-submodules:

$$(\Delta^\pm)^* \otimes E^* = K_\pm \oplus (\Delta^\mp)^*.$$

Hence it suffices to show that $K_\pm$ are irreducible. The dimension of $K_\pm$ are equal to $(2n-1)2^{n-1}$. If $n = 1$, then it is equal to 1, hence the irreducibility is obvious. Since SPIN(E) is connected, it suffices to prove the irreducibility of their differential representation. Since the Lie algebra so(E) of SPIN(E) is semi-simple if $n \geq 2$, we can use the representation theory of semi-simple Lie algebras. We fix a Cartan decomposition of the Lie algebra so$(E) \otimes \mathbf{C}$. Let λ and λ' be the highest weight of $(E \otimes \mathbf{C})^*$ and $(\Delta^\pm)^*$, respectively. Then, there is an irreducible submodule of $(\Delta^\pm)^* \otimes E^*$ with the highest weight $\lambda + \lambda'$, whose dimension is $(2n-1)2^{n-1}$, which can be computed combinatorially by Weyl's dimensionality formula. Since this is greater than the dimension of $(\Delta^\mp)^*$, namely 2^{n-1}, this submodule must be $K_\pm$.

§3. Twistor spaces

Let X be an oriented Riemannian manifold of even dimension $2n$. For simplicity, we assume that X has a spin structure, and let SPIN(X) or SO(X) denote the spinor or oriented orthonormal frame bundle of X, respectively. The Levi-Civita connection on SO(X) induces a connection on SPIN(X).

By the positive (resp. the negative) spin representation Δ^+ (resp. Δ^-) of SPIN$(2n)$, we define the positive (resp. the negative) spin bundle of X and denote it by $\Delta^+(X)$ (resp. $\Delta^-(X)$),

$$\Delta^\pm(X) := \mathrm{SPIN}(X) \times_{\mathrm{SPIN}(2n)} \Delta^\pm.$$

The dual vector bundle is written as:

$$\Delta^\pm(X)^* = \mathrm{SPIN}(X) \times_{\mathrm{SPIN}(2n)} (\Delta^\pm)^*.$$

Similarly, we define

$$K_+(X) := \mathrm{SPIN}(X) \times_{\mathrm{SPIN}(2n)} K_+$$

where K_+ is a SPIN$(2n)$-module defined by the exact sequence:

$$0 \longrightarrow (\Delta^-)^* \xrightarrow{cl^*} (\Delta^+)^* \otimes (\mathbf{R}^{2n})^* \xrightarrow{\kappa} K_+ \longrightarrow 0. \tag{3.1}$$

The SPIN$(2n)$-equivariant map κ induces a linear map:

$$\kappa\colon \Delta^+(X)^*\otimes T^*X \longrightarrow K_+(X).$$

Definition 3.1. The twistor operator $\bar{D}$ is a first order linear differential operator defined as

$$\bar{D}\colon \Gamma(\Delta^+(X)^*) \xrightarrow{\nabla} \Gamma(\Delta^+(X)^*\otimes T^*X) \xrightarrow{\kappa} \Gamma(K_+(X)),$$

where ∇ is the covariant derivative induced by the Levi-Civita connection.

Now let us apply the theory developed in § 1 to the situation in which M is a Riemannian manifold X and D is the twistor operator $\bar{D}=\kappa\circ\nabla$. Then we have a distribution $V(\bar{D})$ on the total space of $\Delta^+(X)$. Since the kernel of κ is $(\Delta^-)^*$ by (3.1), by Proposition 1.2, the distribution has two components,

$$V(\bar{D}) = V(p^*\Delta^+(X)^*)\oplus V(p^*\Delta^-(X)^*)$$

where $p\colon \Delta^+(X)\to X$ is the projection. Furthermore, $V(p^*\Delta^+(X)^*)$ is a subbundle of $T_V^*(\Delta^+(X))\otimes \mathbf{C}$ consisting of $(1, 0)$ covectors with respect to the complex structure of the fibers. Hence, for each point z of $\Delta^+(X)$, $V(p^*\Delta^+(X)^*)_z$ is a maximal isotropic subspace of $T_V^*(\Delta^+(X))_z\otimes\mathbf{C}$.

Let us consider the subset $V(p^*\Delta^-(X)^*)$ of $T_H^*(\Delta^+(X))\otimes\mathbf{C}$. By trivializing the vector bundles locally, the composition of the mappings

$$p^*\Delta^-(X)^* \longrightarrow p^*(\Delta^+(X)^*\otimes T^*X) \longrightarrow T_H^*(\Delta^+(X))$$

is obtained by the composition of the mappings

$$l\colon \Delta^+\times(\Delta^-)^* \xrightarrow{1\times cl^*} \Delta^+\times((\Delta^+)^*\otimes(\mathbf{R}^{2n})^*) \longrightarrow (\mathbf{R}^{2n})^*\otimes\mathbf{C}$$

by Proposition 1.2, where the second map is induced by the canonical pairing of Δ^+ and $(\Delta^+)^*$.

Now we compute the rank of $V(p^*\Delta^-(X)^*)_w$, for $w\in\Delta^+(X)$. Let w be written as (A, z), where $A\in \mathrm{SPIN}(X)$ and $z\in\Delta^+$, then the rank of $V(p^*\Delta^-(X)^*)_w$ is equal to that of

$$\Xi(z) := \{l(z, \psi)\,|\,\psi\in(\Delta^-)^*\}.$$

Note that Ξ is equivariant under the action of $\mathrm{SPIN}(2n)$;

$$\Xi(A\cdot z) = A\cdot\Xi(z),\qquad \text{for all } A\in\mathrm{SPIN}(2n). \tag{3.2}$$

Since we have $\Xi(z)=\Xi(c\cdot z)$, for all $c\in\mathbf{C}^\times$, we define $\Xi[z]$ as $\Xi(z)$ where z is a representative of $[z]\in\mathbf{P}(\Delta^+)$.

Proposition 3.2. *For an element $[z]\in\mathbf{P}(\Delta^+)$, we have $\operatorname{rank}\Xi[z]\geq n$, and*

$$Z = Z_n := \{[z]\in\mathbf{P}(\Delta^+)\,|\,\operatorname{rank}\Xi[z]=n\}$$

is a non-singular projective variety of dimension $\frac{1}{2}n(n-1)$. There is an $\mathrm{SO}(2n)$-equivariant diffeomorphism between Z and $\mathrm{SO}(2n)/\mathrm{U}(n)$. Furthermore, for a point $[z]$ of Z, $\Xi[z]$ is a maximal isotropic subspace of $(\mathbf{R}^{2n})^*\otimes\mathbf{C}$.

Proof. We use the notation of Proposition 2.1. Let z be a point of Δ^+ written as $Z^I\theta_I$. Then, by Lemma 2.2, we have

$$\begin{aligned} l(z, \theta^I) &= \langle z, cl^*(\theta^I)\rangle \\ &= \sum_{i\in I} Z^{-i+I}\alpha^i + \sum_{i\notin I} Z^{-i+I}\beta^i. \end{aligned}$$

where $\langle\ ,\ \rangle$ denotes the pairing of Δ^+ and $(\Delta^+)^*$. In particular, since $l(\theta_\emptyset, \theta^i) = \alpha^i$, $\Xi[\theta_\emptyset] = \langle \alpha^i | i = 1,\dots,n\rangle$, which is regarded as the set of (1, 0) covectors with respect to a natural complex structure J of $\mathbf{R}^{2n}$ defined as

$$J := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Hence $[\theta_\emptyset]$ is in Z. On the other hand, the SPIN$(2n)$-action of Δ^+ induces an SO$(2n)$-action of Z. With respect to this action, by (3.2), the isotropic subgroup at $[\theta_\emptyset]$ is contained in

$$\mathrm{U}(n) = \{A\in \mathrm{SO}(2n) | A\cdot \Xi[\theta_\emptyset] = \Xi[\theta_\emptyset]\}.$$

Hence Z contains a manifold of real dimension at least $n(n-1)$.

Let z satisfy $Z^\emptyset \neq 0$, then, for $j = 1,\dots,n$, we have

$$l(z, \theta^j) = Z^\emptyset\alpha^j - \sum_{i\neq j} Z^{ij}\beta^i.$$

These covectors are linearly independent, and span a maximal isotropic subspace. Therefore, the rank of $\Xi[z]$ is not less than n. Since the rank of Ξ is invariant under the action of SPIN$(2n)$, this is true for all $[z]$ in $\mathbf{P}(\Delta^+)$.

Put $U_\emptyset := \{[z]\in\mathbf{P}(\Delta^+) | Z_\emptyset \neq 0\}$. We deduce the condition of $[z]\in U_\emptyset$ when the other covectors $l(z, \phi)$, for $\phi\in(\Delta^-)^*$, are in the subspace $\langle l(z, \theta^j) | j = 1,\dots,n\rangle$.

$$l(z, \theta^I) = c_j l(z, \theta^j) \Longleftrightarrow \sum_{i\in I} Z^{-i+I}\alpha^i + \sum_{i\notin I} Z^{-i+I}\beta^i = \sum_{j=1}^{n} c_j\Big(Z^\emptyset\alpha^j - \sum_{i\neq j} Z^{ij}\beta^i\Big)$$

Hence, by computing the coefficients of α^j, we have

$$\begin{cases} c_j = \dfrac{Z^{-j+I}}{Z^\emptyset}, & \text{if } j\in I \\ c_j = 0, & \text{if } j\notin I. \end{cases}$$

Substitute these for c_j's in the equation and compute the coefficients of β^i, for $i\notin I$. Then, we have

$$Z^{-i+I} = -\sum_{j\in I} \frac{Z^{-j+I}}{Z^\emptyset} Z^{ij} \tag{3.3}$$

Since $|-i+I| > |-j+I|$ for $j\in I$, the variables Z^I, $|I| > 2$ can be expressed in terms of $Z^\emptyset$, Z^{ij}, $1\leq i<j\leq n$, inductively. Therefore, the dimension of $Z\cap U_\emptyset$

is not greater than $n(n-1)/2$. Since $Z \cap U_\emptyset$ contains a real submanifold of dimension at least $n(n-1)$ (namely the open piece of $\mathrm{SO}(2n)\cdot[\theta_\emptyset]$), the dimension of $Z \cap U_\emptyset$ is just $n(n-1)/2$, and it is defined by (3.3). Hence $Z \cap U_\emptyset$ is non-singular and isomorphic to the affine space $\mathbf{C}^{n(n-1)/2}$. By multiplying $e_{-\iota J}$ to z from left, we conclude that Z is a non-singular variety defined by

$$Z^J Z^{J+i+I} + \sum_{j \in I} Z^{J+j+I} Z^{J+i+j} = 0, \qquad \text{for all } i, I, J, \text{ such that } i \notin I$$

Furthermore, since the isotropic subgroup at $[\theta_\emptyset]$ is $\mathrm{U}(n)$, the mapping

$$\begin{aligned} \mathrm{SO}(2n) &\longrightarrow \mathbf{P}(\Delta^+) \\ A &\longmapsto A\cdot[\theta_\emptyset] \end{aligned}$$

induces an $\mathrm{SO}(2n)$-equivariant embedding

$$\mathrm{SO}(2n)/\mathrm{U}(n) \longrightarrow Z.$$

The embedding is also surjective, since both sides are connected and compact.

Corollary 3.3. *The defining equations of Z are*

$$Z^J Z^{J+i+I} + \sum_{j \in I} Z^{J+j+I} Z^{J+i+j} = 0, \qquad \textit{for all } i, I, J, \textit{ such that } i \notin I$$

If we identify Z with $\mathrm{SO}(2n)/\mathrm{U}(n)$, Ξ has the following meaning. The homogeneous space $\mathrm{SO}(2n)/\mathrm{U}(n)$ can be considered to be the set of complex structures of $\mathbf{R}^{2n}$ preserving the metric and the orientation. Then, Ξ is nothing but a correspondence between complex structures of $\mathbf{R}^{2n}$ and (1, 0) subspaces of $(\mathbf{R}^{2n})^* \otimes \mathbf{C}$. On the other hand, a maximal isotropic subspace of $(\mathbf{R}^{2n})^* \otimes \mathbf{C}$ defines a complex structure of $(\mathbf{R}^{2n})^*$, hence of $\mathbf{R}^{2n}$, compatible with the metric, by restricting the projection $(\mathbf{R}^{2n})^* \otimes \mathbf{C} \to (\mathbf{R}^{2n})^*$ to the isotropic subspace. Hence the correspondence between the set of complex structures of $\mathbf{R}^{2n}$ compatible with the metric and the set of maximal isotropic subspaces of $\mathbf{R}^{2n} \otimes \mathbf{C}$ is bijective.

By considering orientations of corresponding complex structures, there are two kinds of maximal isotropic subspaces.

Definition 3.4. A maximal isotropic subspace of $(\mathbf{R}^{2n})^* \otimes \mathbf{C}$ is called an α-subspace if the corresponding complex structure of $\mathbf{R}^{2n}$ is compatible with the orientation. Otherwise, it is called a β-subspace.

Hence Ξ is a correspondence between Z_n and the set of α-subspaces of $(\mathbf{R}^{2n})^* \otimes \mathbf{C}$.

Example 3.5. Since $\dim \Delta^+ = 2^{n-1}$, Z_n is the projective space $\mathbf{P}(\Delta^+)$ if $n \leq 3$. If $n = 4$, $\dim Z_4 = 6$ and $\dim \mathbf{P}(\Delta^+) = 7$. Hence this is a 6-dimensional non-singular hyperquadric defined by an equation

$$Z^\emptyset Z^{1234} - Z^{12}Z^{34} + Z^{13}Z^{24} - Z^{14}Z^{23} = 0,$$

which is obtained by putting $J = \emptyset$, $i = 1$, and $I = \{2, 3, 4\}$.

n	$\dim Z_n$	Z_n	$\mathbf{P}(\Delta^+)$
1	0	$\mathbf{P}^0$	$\mathbf{P}^0$
2	1	$\mathbf{P}^1$	$\mathbf{P}^1$
3	3	$\mathbf{P}^3$	$\mathbf{P}^3$
4	6	Q_6	$\mathbf{P}^7$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{2}n(n-1)$	$SO(2n)/U(n)$	$\mathbf{P}^{2^{n-1}-1}$

By Proposition 3.2, the distribution $V(\bar{D})$ has the minimum rank on the submanifold:

$$W := \{z \in \Delta^+(X) \mid \operatorname{rank} V(p^*\Delta^-(X)^*)_z = n\}.$$

By pulling back $V(\bar{D})$ to W, we have a distribution which is an α-subspace at each point. Hence it defines an almost complex structure on W, by the correspondence in Definition 3.4. If we regard W as a fiber bundle over X, since the almost complex structure is compatible with the connection, we have a complex structure of the vertical and the horizontal cotangent bundle. The complex structure of the horizontal cotangent bundle is invariant under the $\mathbf{C}^\times$-action, and the complex structure of the vertical cotangent bundle is equal to the one induced by the complex structure of the fibers. Hence we have an almost complex structure of

$$Z(X) := \mathbf{P}(W) \subset \mathbf{P}(\Delta^+(X)),$$

by projecting the distribution on W to this space, which is a Z_n-bundle over X by Proposition 3.2. W is a $\mathbf{C}^\times$-bundle over $Z(X)$. We define H^* to be the associated line bundle over $Z(X)$. If we identify W with $(H^*)^\times := H^* \backslash \{\text{zero vectors}\}$, the almost complex structure of W can be extended to that of the total space of H^*. We have a commutative diagram:

$$\begin{array}{ccc} H^* & \longrightarrow & \mathcal{O}(-1) \\ \downarrow & & \downarrow \\ Z(X) & \longrightarrow & \mathbf{P}(\Delta^+(X)) \end{array} \tag{3.4}$$

where $\mathcal{O}(-1)$ is the tautological line bundle over $\mathbf{P}(\Delta^+(X))$. Since $\mathcal{O}(-1)$ has a connection induced by the connection and the Hermitian metric of $\Delta^+(X)$, H^* also has the induced connection by the diagram (3.4). We define H as the dual line bundle of H^* over $Z(X)$.

By definition, $Z(X)$ and H^2 does not depend on the spin structure of X, and they are well-defined even if X has no spin structure. But the line bundle H can be defined if and only if X has a spin structure.

Definition 3.6. Let X be a $2n$-dimensional oriented Riemannian manifold. The manifold $Z(X)$ with the almost complex structure is called the *twistor* space of X. If X has a spin structure, the line bundle H is called the hyperplane bundle of $Z(X)$.

The following theorem is an immediate consequence of the definition.

Theorem 3.7. *For $2n$-dimensional spin manifold X, we have a diagram*:

$$\begin{array}{ccc} H & \longrightarrow & \mathcal{O}(1) \\ \downarrow & & \downarrow \\ Z(X) & \longrightarrow & \mathbf{P}(\Delta^+(X)). \end{array}$$

For a point x of X, the fiber $Z(X)_x$ is an almost complex submanifold and the almost complex structure is equal to the one induced by the embedding $Z(X)_x \to \mathbf{P}(\Delta^+(X)_x)$.

Remarks. 1. By Proposition 3.2, $Z(X)$ is isomorphic to $\mathrm{SO}(X)/\mathrm{U}(n)$. This is the original definition of a twistor space of a Riemannian manifold. ([O.R])

2. Even if X is not orientable, we can still define its twistor space. Let $\tilde{X}$ be the double covering of X. We define $Z(X) := Z(\tilde{X})$ as the twistor space of X. Geometrically, this is nothing but $\mathrm{O}(X)/\mathrm{U}(n)$. In this case, H can be defined if and only if X has a *pin* structure.

3. The definition of $Z(X)$ only depends on an oriented conformal structure of X, since the twistor operator is conformally invariant ([F]). Hence a conformal map $f: X \to Y$ between conformal manifolds induces a *holomorphic* map $\tilde{f} : Z(X) \to Z(Y)$. ("*holomorphic*" means the map preserves the almost complex structures.)

4. If $n = 1$, $Z(X) = X$. This means that a conformal structure determines an almost complex structure uniquely, which is always integrable. If $n = 2$, the almost complex structure of $Z(X)$ is integrable if and only if X is anti-self-dual ([A.H.S]). On the other hand, if $n \geq 3$, the almost complex structure of $Z(X)$ is integrable if and only if X is conformally flat, hence the integrability of $Z(X)$ is independent of the choice of the orientation of X ([O.R]).

5. If $Z(X)$ is a complex manifold, H and H^* are holomorphic line bundles over $Z(X)$.

Now we study an important example of a twistor space.

Example 3.8. Let S^{2n} be a $2n$-dimensional sphere with the conformally flat metric. Then the twistor space $Z(S^{2n})$ is Z_{n+1}.

This is shown as follows. Since the orthonormal frame bundle of S^{2n} is identified with $\mathrm{SO}(2n+1)$, the twistor space $Z(S^{2n})$ is isomorphic to $\mathrm{SO}(2n+1)/$

U(n). The embedding SO$(2n+1) \to$ SO$(2n+2)$ induces an isomorphism

$$\mathrm{SO}(2n+1)/\mathrm{U}(n) \xrightarrow{\sim} \mathrm{SO}(2n+2)/\mathrm{U}(n+1)$$

Since the conformal transformation group on S^{2n} is $\mathrm{SO}_0(1, 2n+1)$, the identity component of SO$(1, 2n+1)$, the almost complex structure of $Z(S^{2n})$ is invariant under the action of $\mathrm{SO}_0(1, 2n+1)$. The action of $\mathrm{SO}_0(1, 2n+1)$ to $Z(S^{2n})$ can be complexified, and the almost complex structure is invariant under this PSO$(2n+2; \mathbf{C})$-action. Hence the almost complex structure of $Z(S^{2n})$ coincides with the one as a Hermitian symmetric space.

§4. Canonical bundles of twistor spaces

Let X be an oriented Riemannian manifold, and $Z(X)$ be the twistor space of X. In this section, we study the canonical bundle of $Z(X)$.

We begin with a useful lemma from the representation theory.

Lemma 4.1. *Any one-dimensional representation of* SU(n) *is trivial.*

It is convenient to use the geometric definition of $Z(X)$,

$$Z(X) = \mathrm{SO}(X)/\mathrm{U}(n) = \mathrm{SPIN}(X)/\mathrm{U}'(n)$$

where $\mathrm{U}'(n) := \pi^{-1}(\mathrm{U}(n))$ and $\pi\colon \mathrm{SPIN}(2n) \to \mathrm{SO}(2n)$ is the projection map. We give a geometric realization of the line bundle H^2, defined in [M].

Lemma 4.2. *If X is a spin manifold, the line bundle H^* over $Z(X)$ is isomorphic to* $\mathrm{SPIN}(X) \times_{\rho^*} \mathbf{C}$, *where ρ is a representation of $U'(n)$ satisfying* $\rho(\omega)^2 = \det(\pi(\omega))$ *for* $\omega \in \mathrm{U}'(n)$. *(Determinants of elements of* U(n) *are computed by regarding them as complex linear endomorphisms of* $(1, 0)$ *vectors.) Hence, for an oriented Riemannian manifold not necessarily spin, H^2 is isomorphic to* $\mathrm{SO}(X) \times_{\det} \mathbf{C}$.

Proof. As was shown in the proof of Proposition 3.2, $\mathrm{U}'(n)$ is the isotropic subgroup at $[\theta_\emptyset] \in \mathbf{P}(\Delta^+)$ with respect to the SPIN$(2n)$-action on $\mathbf{P}(\Delta^+)$. Hence $\mathrm{U}'(n)$ acts on the one-dimensional subspace $\mathbf{C}\cdot\theta_\emptyset$ of Δ^+. Put $\mathrm{SU}'(n) = \pi^{-1}(\mathrm{SU}(n))$. Then, by Lemma 4.1, the $\mathrm{SU}'(n)$-action is locally trivial. Since $-1 \in \mathrm{SU}'(n)$ is not in the isotropic subgroup at $\theta_\emptyset$, $\mathrm{SU}'(n)$ is not connected. Hence the isotropic subgroup contains the identity component of $\mathrm{SU}'(n)$, which is isomorphic to SU(n). Thus, if we denote it by SU(n), the $\mathrm{U}'(n)$-action factors through.

$$\mathrm{U}'(n) \longrightarrow \mathrm{U}'(n)/\mathrm{SU}(n) \longrightarrow \mathrm{GL}(\mathbf{C}\cdot\theta_\emptyset) \simeq \mathbf{C}^\times.$$

Since $\overline{\pi|_{\mathrm{U}'(n)}}\colon \mathrm{U}'(n)/\mathrm{SU}(n) \to \mathrm{U}(n)/\mathrm{SU}(n)$ is a double covering, we have a representation $\rho\colon \mathrm{U}'(2n) \to \mathbf{C}^\times$ satisfying the relation: $\rho(\omega)^2 = \det(\pi(\omega))$. By definition of ρ, $\mathrm{U}'(n)$ acts as ρ^m on $\mathbf{C}\cdot\theta_\emptyset$ for some integer m. Hence it suffices to show that $m = -1$, by computing the action of an element

$$\omega(\alpha) := \cos\alpha + e_1 * e_{n+1} \sin\alpha = e_1 * (-e_1 \cos\alpha + e_{n+1} \sin\alpha) \in \mathrm{U}'(n).$$

Put $c(\alpha) = \cos\alpha + \sqrt{-1}\sin\alpha$. Then $\rho(\omega(\alpha))$ is equal to $c(\alpha)$, because det $\pi(\omega(\alpha)) = c(\alpha)^2$. On the other hand, we have $\omega(\alpha)\theta_\emptyset = c(\alpha)^{-1}\theta_\emptyset$. Hence we complete the proof.

Example 4.3. We give here the hyperplane bundle over the twistor space of a $2n$-dimensional sphere. As was mentioned in Example 3.8, the twistor space $Z(S^{2n})$ is $\mathrm{SPIN}(2n+1)/\mathrm{U}'(n)$. Hence, by the above lemma, we have

$$H = \mathrm{SPIN}(2n+1) \times_\rho \mathbf{C}$$

where ρ is the representation of $\mathrm{U}'(n)$. If we extend the group $\mathrm{SPIN}(2n+1)$ to $\mathrm{SPIN}(2n+2)$ as in Example 3.8,

$$H = \mathrm{SPIN}(2n+2) \times_\rho \mathbf{C}.$$

Here, $\mathrm{SPIN}(2n+2)$ is considered to be a $\mathrm{U}'(n+1)$-bundle over $Z(S^{2n})$, and ρ is considered to be a representation of $\mathrm{U}'(n+1)$ in this case. This line bundle is nothing but the pull-back of the hyperplane bundle over the projectivized positive spin module of $\mathbf{R}^{2n+2}$. $\mathbf{P}(\Delta^+(\mathbf{R}^{2n+2}))$ by the canonical embedding defined in Proposition 3.2.

Now we can express the canonical bundle $K_{Z(X)}$ in terms of the hyperplane bundle H. For an almost complex manifold M of dimension m, the canonical bundle K_M is the complex line bundle $\Lambda^{m,0} M$.

Theorem 4.4. *Let X be an oriented Riemannian manifold of dimension $2n$. Then the canonical bundle of $Z(X)$ is isomorphic to H^{-2n}.*

Proof. By the splitting of the cotangent bundle of $Z(X)$, the $(1, 0)$ cotangent bundle has two components, namely $\Lambda_H^{1,0} Z(X)$ and $\Lambda_V^{1,0} Z(X)$, of rank n and $n(n-1)/2$, respectively. Hence the canonical bundle can be written as:

$$K_{Z(X)} \simeq \Lambda_H^{n,0} Z(X) \otimes \Lambda_V^{n(n-1)/2,0} Z(X).$$

Therefore, it suffices to prove the following lemma.

Lemma 4.5. (1) *$\Lambda_H^{n,0} Z(X)$ is isomorphic to H^{-2}.*
(2) *$\Lambda_V^{n(n-1)/2,0} Z(X)$ is isomorphic to $H^{-2(n-1)}$.*

Proof. The both statements are proved in a similar way. First define a form on $\mathrm{SO}(X)$, which spans a subbundle invariant under the right $\mathrm{U}(n)$-action. By Lemma 4.1, the $\mathrm{U}(n)$-action only depends on the multiplication by m-th power of determinants of elements of $\mathrm{U}(n)$, for some integer m. By Lemma 4.2, this means that the form determines a section of $H^{-2m} \otimes \Lambda(Z(X))$. Hence we need to consider the following.

- There are forms on $\mathrm{SO}(X)$ which determine non-zero sections of $H^{-2m} \otimes$

$\Lambda_H^{n,0} Z(X)$ in (1) and $H^{-2m} \otimes \Lambda_V^{n(n-1)/2,0} Z(X)$ in (2), for some integers m, respectively.

- The integer m is -1 in (1), and $-n+1$ in (2), respectively.

(1) For a point x of X, the fiber $\mathrm{SO}(X)_x$ is the set of orthonormal frames of $T_x X$. Hence, for each point of $\mathrm{SO}(X)_x$, we define an n-form ω as

$$\omega := (e^1 + \sqrt{-1}\, e^{n+1}) \wedge (e^2 + \sqrt{-1}\, e^{n+2}) \wedge \cdots \wedge (e^n + \sqrt{-1}\, e^{2n})$$

where, at each point, $(e^1, \ldots, e^{2n})$ is the dual frame of the corresponding frame. This is a basis of $\Lambda^{n,0}(T_x^* X \otimes \mathbf{C})$ when we take a complex structure J in $T_x^* X$ with respect to the frame $(e^1, \ldots, e^{2n})$ defined as

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \tag{4.1}$$

Hence $\mathbf{C} \cdot \omega \subset \Lambda_H^n \mathrm{SO}(X)$ is a $\mathrm{U}(n)$ invariant subbundle. By a simple computation, the integer m is -1 in this case. Hence ω determines a non-zero global section of $H^2 \otimes \Lambda_H^{n,0} Z(X)$ over $Z(X)$.

(2) We define a connection on $\mathrm{SO}(2n)$ as a $\mathrm{U}(n)$ principal bundle over $\mathrm{SO}(2n)/\mathrm{U}(n)$. Let v be a subspace of $\mathrm{so}(2n)$ defined as

$$v = \{\alpha \in \mathrm{so}(2n) \mid J\alpha J^{-1} = -\alpha\}.$$

where J is the complex structure of $\mathbf{R}^{2n}$ defined as (4.1). Since we have

$$u(n) = \{\alpha \in \mathrm{so}(2n) \mid J\alpha J^{-1} = \alpha\},$$

we have a decomposition of $\mathrm{so}(2n)$

$$\mathrm{so}(2n) = v \oplus \mathrm{u}(n)$$

If we regard $\mathrm{so}(2n)$ as a set of left invariant vector fields, v determines a horizontal tangent bundle over $\mathrm{SO}(2n)$ as a principal $\mathrm{U}(n)$-bundle over $\mathrm{SO}(2n)/\mathrm{U}(n)$, since v is invariant under the adjoint action of an element of $\mathrm{u}(n)$. The almost complex structure of $\mathrm{SO}(2n)/\mathrm{U}(n)$ is induced by a complex structure of the vector space v which is invariant under the adjoint action of $\mathrm{u}(n)$. By taking a basis of $(v^* \otimes \mathbf{C})^{1,0}$, and taking the exterior multiplication of its elements, we define a horizontal $n(n-1)/2$-form on $\mathrm{SO}(2n)$ as a fiber bundle over $\mathrm{SO}(2n)/\mathrm{U}(n)$. Since this form is invariant under the left action of $\mathrm{SO}(2n)$, by defining the vertical form (as a fiber bundle over X) locally and patched them together, we have a global $n(n-1)/2$ form on $\mathrm{SO}(X)$ which spans a subbundle invariant under the right $\mathrm{U}(n)$-action. Hence if we shall show $m = -n+1$, it gives a non-zero global section of $H^{2(n-1)} \otimes \Lambda_V^{n(n-1)/2,0} Z(X)$ and we complete the proof.

The value of m is computed by the induction with respect to n. By Example 3.8, for a point x of X, the fiber $Z(X)_x$ can be considered to be the twistor space of a $(2n-2)$-dimensional sphere S^{2n-2}. And by Example 4.3, the restriction of H to $Z(X)_x$ is nothing but the hyperplane bundle as the twistor space $Z(S^{2n-2})$. Hence m is $-n+1$.

Remark. If X is conformally flat, then the theorem is also true in the holomorphic category.

§5. Complexification of some conformal manifolds

The twistor space $Z(X)$ of a $2n$-dimensional oriented Riemannian manifold X is considered to be a fiber bundle over X with fiber $Z \simeq \mathrm{SO}(2n)/\mathrm{U}(n)$. First, we study the fiber $Z(X)_x$ for each point x of X. From now on, we assume that n is grater than one. There is a splitting of the $(1, 0)$ tangent bundle of $Z(X)$ corresponding to the splitting of the tangent bundle as a fiber bundle over X.

$$T^{(1,0)}Z(X) \simeq T_V^{(1,0)}Z(X) \oplus T_H^{(1,0)}Z(X)$$

Furthermore, by the definition of the almost complex structure, $T_V^{(1,0)}Z(X)$ is the set of complex tangent vectors of holomorphic directions of $Z(X)_x$ considered as a complex manifold by Theorem 3.7, and

$$T_H^{(1,0)}Z(X) \simeq \mathrm{SO}(X) \times_{\rho_1} \mathbf{C}^n$$

where ρ_1 is a natural representation of $\mathrm{U}(n)$ to $\mathbf{C}^n$. Hence the normal bundle N_x of $Z(X)_x$ is isomorphic to $\mathrm{SO}(X)_x \times_{\rho_1} \mathbf{C}^n$. Hence N_x is a homogeneous vector bundle over $Z(X)_x$. Since the argument is independent of the choice of $x \in X$, we omit the suffix for a while and denote $Z(X)_x$, N_x and T_xX by Z, N and $\mathbf{R}^{2n}$, respectively. For each vector v of $\mathbf{R}^{2n} \otimes \mathbf{C}$, we define a section $s(v)$ of N over Z by

$$\begin{aligned} s(v)\colon Z &\longrightarrow N \\ \bar{g} &\longmapsto (g, \pi^{(1,0)}(g^{-1} \cdot v)) \end{aligned}$$

where $\pi^{(1,0)}\colon \mathbf{R}^{2n} \otimes \mathbf{C} \to (\mathbf{R}^{2n} \otimes \mathbf{C})^{(1,0)}$ is the projection to the $(1, 0)$ subspace with respect to the complex structure of $\mathbf{R}^{2n}$ defining the embedding $\mathrm{U}(n) \subset \mathrm{SO}(2n)$, and g is one of the inverse image of $\bar{g}$ by the projection $\mathrm{SO}(2n) \to Z$. This correspondence is well-defined, since $\pi^{(1,0)}$ is $\mathrm{U}(n)$-equivariant. This is the way to construct all holomorphic sections of N over Z by the theorem of Borel-Weil. Moreover, if we apply the theorem of Bott-Borel-Weil-Kostant ([B]), we have the following lemma.

Lemma 5.1. *Let N be the holomorphic vector bundle over Z as above. Then the cohomology groups are*

$$H^i(Z, \mathcal{O}(N)) = \begin{cases} \mathbf{R}^{2n} \otimes \mathbf{C}, & \text{if } i = 0 \\ 0, & \text{if } i > 0. \end{cases}$$

A explicit description of holomorphic sections is obtained by the map s defined above.

Now, let us consider the fiber $Z(X)_x$ over $x \in X$. By the definition of s, the set of holomorphic sections of the normal bundle N_x which vanish at some points corresponds precisely to the set of all null vectors of $T_xX \otimes \mathbf{C}$ with the canonical complex conformal structure induced by the Riemannian metric of X. Hence we have proved the next theorem.

Theorem 5.2. *The conformal structure of X can be recovered from the almost complex structure of $Z(X)$.*

If the almost complex structure of $Z(X)$ is integrable, we can go further by deformation theory of complex manifolds ([K]). By Lemma 5.1, the set

$$X' := \{W \subset Z(X) \mid W \simeq Z \text{ with the normal bundle } N_W \text{ isomorphic to } N\}$$

inherits a $2n$-dimensional holomorphic structure, and at each point W, there is an isomorphism:

$$T_W^{(1,0)} X' \simeq H^0(W, \mathcal{O}(N_W)). \tag{5.1}$$

The fibers $Z(X)_x$, for $x \in X$, are points of X'. We define the complexification $X_{\mathbf{C}}$ of X as the components of X' containing fibers $Z(X)_x$ for all $x \in X$. The points of $X \subset X_{\mathbf{C}}$ are called real points of $X_{\mathbf{C}}$. As mentioned above, the set of holomorphic tangent vectors corresponding, by (5.1), to the set of sections which vanish at some points determines a holomorphic complex conformal structure on $X_{\mathbf{C}}$, which is an extension of the conformal structure of X. Furthermore, if n is even, the symplectic structure of $(\Delta^+)^*$ (i.e. the SPIN$(2n)$-equivariant isomorphism $\iota\colon \Delta^+ \to (\Delta^+)^*$ defined in Proposition 2.1) induces a fixed-point free anti-holomorphic involution on Z. If we regard Z as a set of compatible complex structure of $\mathbf{R}^{2n}$, it is nothing but a map sending a complex structure J' to $-J'$. Since it is SPIN$(2n)$-equivariant, it extends to an involution on $Z(X)$

$$\tau\colon Z(X) \longrightarrow Z(X),$$

which is anti-holomorphic and preserves the fibration over X. Hence τ induces an anti-holomorphic involution $\tilde{\tau}$ on $X_{\mathbf{C}}$, and X is the fixed locus of $\tilde{\tau}$.

Theorem 5.3. *Let X be a $2n$-dimensional oriented conformal manifold with $n > 1$. Assume that the almost complex structure of the twistor space $Z(X)$ is integrable. Then, there is a $2n$-dimensional complex manifold $X_{\mathbf{C}}$ with a complex conformal structure. The manifold X is considered to be a real submanifold of $X_{\mathbf{C}}$ and the conformal structure of X is a restriction of that of $X_{\mathbf{C}}$. Furthermore, if n is even, there is an anti-holomorphic involution τ on $Z(X)$ and $\tilde{\tau}$ on $X_{\mathbf{C}}$, respectively, such that X is the fixed-point set of $\tilde{\tau}$.*

§6. Even dimensional spheres and generalized Penrose fibrations

For twistor spaces of conformal manifolds of dimension greater than or equal to six, the integrability of the almost complex structure is equivalent to the conformally flatness of the underlying manifolds. For a conformally flat manifold X of dimension m grater than two, there is a conformal map called a developing map

$$\Phi : \tilde{X} \longrightarrow S^m$$

where $\tilde{X}$ is the universal covering space of X with a conformal structure induced from X. Furthermore, Φ is unique up to conformal transformations of S^m. Hence, for a conformal transformation a of $\tilde{X}$, there is a unique conformal transformation b on S^m such that the following diagram commutes:

$$\begin{tikzcd} \tilde{X} \arrow[r, "\Phi"] \arrow[d, "a"'] & S^m \arrow[d, "b"] \\ \tilde{X} \arrow[r, "\Phi"] & S^m \end{tikzcd}$$

Hence we have a group homomorphism:

$$\tilde{\Phi} : \pi_1(X) \longrightarrow \mathrm{SO}_0(1, m+1).$$

where $\mathrm{SO}_0(1, m+1)$ is the conformal transformation group of S^m. These facts are easily deduced from a theorem of Liouville (see, for example, [D.F.N] Theorem 15.2).

In the following, we consider the following problems.

- Determine the complexification of a $2n$-dimensional sphere S^{2n}. It is a $2n$-dimensional non-singuar complex hyperquadric Q_{2n}.
- Determine the action of $\mathrm{SO}_0(1, 2n+1)$ to the twistor space $Z(S^{2n})$ and Q_{2n}.

First, we recall the case $n = 2$, in which we have the Penrose fibration:

$$\begin{tikzcd} & F_{1,2;4} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbf{P}^3 & & G_{2,4} \end{tikzcd}$$

where $\mathbf{P}^3$ is the twistor space of S^4 and $G_{2,4}$ is a Grassmannian manifold identified with a four-dimensional complex hyperquadric Q_4 by the Plücker embedding. By definition, $G_{2,4}$ parametrizes all the lines in $\mathbf{P}^3$. The normal bundle of a line $\mathbf{P}^1$ in $\mathbf{P}^3$ is isomorphic to $H \oplus H$, where H is the hyperplane bundle of $\mathbf{P}^1$. Hence the complexification of S^4 is $G_{2,4} \simeq Q_4$. The flag manifold $F_{1,2;4}$ is considered as a submanifold of $\mathbf{P}^3 \times G_{2,4}$ defined by

$$F_{1,2;4} \simeq \{(z, x) \in \mathbf{P}^3 \times G_{2,4} | z \in \text{the line corresponding to } x.\}.$$

To generalize the situation to higher dimensional cases, we neglect the holomorphy for a while and write them as quotient spaces of SO(6) instead of SU(4),

$$F_{1,2;4} \simeq \mathrm{SO}(6)/\mathrm{U}(1) \times \mathrm{U}(2),$$

$$\mathbf{P}^3 \simeq \mathrm{SO}(6)/\mathrm{U}(3),$$

$$Q_4 \simeq \mathrm{SO}(6)/\mathrm{U}(1) \times \mathrm{SO}(4).$$

Then the maps p_1 and p_2 of the Penrose fibration are given by the canonical projections:

$$p_1\colon \mathrm{SO}(6)/\mathrm{U}(1) \times \mathrm{U}(2) \longrightarrow \mathrm{SO}(6)/\mathrm{U}(3)$$
$$p_2\colon \mathrm{SO}(6)/\mathrm{U}(1) \times \mathrm{U}(2) \longrightarrow \mathrm{SO}(6)/\mathrm{U}(1) \times \mathrm{SO}(4).$$

Furthermore, we have $\mathrm{U}(1) \times \mathrm{U}(2) = \mathrm{U}(3) \cap (\mathrm{U}(1) \times \mathrm{SO}(4))$.

In another point of view, $\mathbf{P}^3$ determines a family of submanifolds of Q_4 isomorphic to $\mathbf{P}^2$, since we have

$$\mathbf{P}^2 \simeq \mathrm{U}(3)/\mathrm{U}(1) \times \mathrm{U}(2)$$

Above consideration can be generalized in higher dimensional cases, by considering the fibration:

$$\begin{tikzcd} & Y_n \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Z_{n+1} & & Q_{2n} \end{tikzcd} \tag{6.1}$$

$$Y_n := \mathrm{SO}(2n+2)/\mathrm{U}(1) \times \mathrm{U}(n),$$
$$Z_{n+1} := \mathrm{SO}(2n+2)/\mathrm{U}(n+1),$$
$$Q_{2n} := \mathrm{SO}(2n+2)/\mathrm{U}(1) \times \mathrm{SO}(2n).$$

Then, we have also $\mathrm{U}(1) \times \mathrm{U}(n) = \mathrm{U}(n+1) \cap (\mathrm{U}(1) \times \mathrm{SO}(2n))$. Note that Z_{n+1} is the twistor space of S^{2n} and Q_{2n} is identified with a $2n$-dimensional non-singular complex hyperquadric. The identification is obtained as follows.

Let $(e_1, \ldots, e_{2n+2})$ be the orthonormal basis of $\mathbf{R}^{2n+2}$. For simplicity, we take a complex structure of $\mathbf{R}^{2n+2}$ defined by

$$\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -I_n \\ & & I_n & 0 \end{pmatrix}. \tag{6.2}$$

Let us consider a map

$$\begin{aligned} \mathrm{SO}(2n+2) &\longrightarrow \mathbf{P}(\mathbf{R}^{2n+2} \otimes \mathbf{C}) \\ A &\longmapsto A \cdot [e_1 - \sqrt{-1}\, e_2], \end{aligned} \tag{6.3}$$

where, for $v \in \mathbf{R}^{2n+2} \otimes \mathbf{C}$, $[v]$ denotes the image of the projection map to the projective space. Then, by the definition of the complex structure of $\mathbf{R}^{2n}$, it induces an injection

$$\mathrm{SO}(2n+2)/\mathrm{U}(1) \times \mathrm{SO}(2n) \longrightarrow \mathbf{P}(\mathbf{R}^{2n+2} \otimes \mathbf{C}), \tag{6.4}$$

and the image is the set of null vectors with respect to the bilinear form on $\mathbf{R}^{2n+2} \otimes \mathbf{C}$, hence it is isomorphic to a $2n$-dimensional non-singular complex hyperquadric Q_{2n}.

By the generalized Penrose fibration, Q_{2n} parametrizes submanifolds of Z_{n+1} isomorphic to $Z_n \simeq \mathrm{U}(1) \times \mathrm{SO}(2n)/\mathrm{U}(1) \times \mathrm{U}(n)$. Since the fibration are $\mathrm{SO}(2n+2)$-equivariant, the submanifolds of Z_{n+1} corresponding to a point of Q_{2n} is analytically isomorphic to Z_n with the normal bundle isomorphic to N. Since the dimension of Q_{2n} is $2n$, the complexification of S^{2n} is identified with Q_{2n} by the fibration (6.1).

Theorem 6.1. *The complexification of the* $2n$*-dimensional sphere with a conformally flat structure is a* $2n$*-dimensional non-singular complex hyperquadric* Q_{2n}. *Put*

$$Y_n := \{(z, x) \in Z_{n+1} \times Q_{2n} | z \in \text{ the submanifold corresponding to } x\}.$$

Then, the projections give a generalization of the Penrose fibration (6.1).

Remark. In case of $n = 2$, Y_2, Z_2 and Q_4 are all flag manifolds. (We use the notation 'a flag manifold' to be a set of flags in $\mathbf{C}^m$.) The reason is that SPIN(6) is isomorphic to SU(4) by a positive spin representation.

Since twistor spaces are conformally invariant, a conformal transformation on S^{2n} induces a holomorphic transformation on Z_{n+1}. Hence we have a group homomorphism

$$\mathrm{SO}_0(1, 2n+1) \longrightarrow \mathrm{Aut}(Z_{n+1}).$$

Since the automorphism group of Z_{n+1} is $\mathrm{PSO}(2n+2; \mathbf{C})$, the above homomorphism is nothing but the complexification map of the real Lie group $\mathrm{SO}_0(1, 2n+1)$.

On the other hand, Z_{n+1} parametrizes submanifolds of Q_{2n} isomorphic to

$$\mathbf{P}^n \simeq \mathrm{U}(n+1)/\mathrm{U}(1) \times \mathrm{U}(n),$$

which are identified with projectivized α-subspaces in $\mathbf{P}(\mathbf{R}^{2n+2} \otimes \mathbf{C})$ by the injection (6.4) induced by (6.3). This gives a geometric description of the correspondence Ξ in §3.

Now, we describe the real structure of the generalized Penrose fibration, when n is even. We regard $\mathrm{SO}(2n+1)$ and $\mathrm{SO}(2n)$ as the subgroups of $\mathrm{SO}(2n+2)$ by

$$\mathrm{SO}(2n+1) = \{A \in \mathrm{SO}(2n+2) | Ae_1 = e_1\},$$
$$\mathrm{SO}(2n) = \{A \in \mathrm{SO}(2n+1) | Ae_2 = e_2\}.$$

$\mathbf{R}^{2n+1}$ and $\mathbf{R}^{2n}$ are regarded as the subspaces of $\mathbf{R}^{2n+2}$ spanned by $e_2, \dots, e_{2n+2}$ and $e_3, \dots, e_{2n+2}$, respectively. They are defined to be a natural representation spaces of $\mathrm{SO}(2n+1)$ and $\mathrm{SO}(2n)$, respectively, by restricting the $\mathrm{SO}(2n+2)$-

action. Furthermore, the $2n$-dimensional sphere S^{2n} is considered to be the unit sphere of $\mathbf{R}^{2n+1}$, which is identified with $\mathrm{SO}(2n+1)/\mathrm{SO}(2n)$ by the map:

$$\begin{aligned} \mathrm{SO}(2n+1) &\longrightarrow S^{2n} \\ A &\longmapsto Ae_2 \end{aligned} \tag{6.5}$$

By Example 3.8, the twistor space of S^{2n} is a Hermitian symmetric space:

$$\mathrm{SO}(2n+1)/\mathrm{U}(n) \simeq \mathrm{SO}(2n+2)/\mathrm{U}(n+1)$$

where $\mathrm{U}(n+1)$ is considered to be a subgroup of $\mathrm{SO}(2n+2)$ defined by the complex structure (6.2) of $\mathbf{R}^{2n+2}$, and $\mathrm{U}(n)$ is considered to be the intersection of $\mathrm{U}(n+1)$ and $\mathrm{SO}(2n)$. The projection p of the twistor space to the sphere as a fiber bundle is written as:

$$p\colon \mathrm{SO}(2n+1)/\mathrm{U}(n) \longrightarrow \mathrm{SO}(2n+1)/\mathrm{SO}(2n).$$

Now, under the above notation, we describe the embedding

$$i\colon S^{2n} \longrightarrow Q_{2n} \subset \mathbf{P}(\mathbf{R}^{2n+2} \otimes \mathbf{C}).$$

The submanifold of $\mathrm{SO}(2n+2)/\mathrm{U}(n+1)$ corresponding to the point $\bar{I}_{2n+2} \in \mathrm{SO}(2n+2)/\mathrm{U}(1) \times \mathrm{SO}(2n)$ is $\mathrm{SO}(2n)/\mathrm{U}(n)$. By (6.5), this is the fiber over

$$\mathrm{SO}(2n+1)/\mathrm{SO}(2n) \ni \bar{I}_{2n+1} = e_2 \in S^{2n}.$$

Hence $\bar{I}_{2n+2}$ is a real point corresponding to $[e_1 - \sqrt{-1}\, e_2] \in \mathbf{P}(\mathbf{R}^{2n+2} \otimes \mathbf{C})$, by (6.3). Since i is equivariant under the conformal transformation group, especially the isometry group $\mathrm{SO}(2n+1)$, we have

$$\begin{aligned} i\colon S^{2n} &\longrightarrow \mathbf{P}(\mathbf{R}^{2n+2} \otimes \mathbf{C}) \\ x &\longmapsto [e_1 - \sqrt{-1}\, x] \end{aligned} \tag{6.6}$$

where, as (6.5), S^{2n} is now considered as a unit sphere of $\mathbf{R}^{2n+1}$ spanned by $e_2, \ldots, e_{2n+2}$. Hence, if n is even, the anti-holomorphic involution $\tilde{\tau}$ on Q_{2n} defined in Theorem 5.3 is the one induced by the anti-linear transformation σ of $\mathbf{R}^{2n+2} \otimes \mathbf{C}$ defined by:

$$\sigma(e_j) = \begin{cases} e_1, & j = 1; \\ -e_j, & j > 1. \end{cases}$$

This is deduced from the following proposition.

Proposition 6.2. *A holomorphic transformation of Q_{2n} fixing the real points is identity.*

Proof. This is because any holomorphic transformation of Q_{2n} is induced by a linear transformation of $\mathbf{R}^{2n} \otimes \mathbf{C}$ preserving the bilinear form, and the set of real points $i(S^{2n})$ spans the whole space $\mathbf{R}^{2n} \otimes \mathbf{C}$.

Note that σ also defines an anti-holomorphic involution on Q_{2n} even if n is odd. In all cases, the set of fixed points coincides with the set of real points $i(S^{2n})$, and this characterizes the anti-holomorphic involution on Q_{2n}.

For a point x of Q_{2n}, Z_n denotes the corresponding submanifold $p_1(p_2^{-1}(x))$ of $Z(S^{2n})$ by the fibration (6.1). There is a useful criterion when two such submanifolds intersect.

Proposition 6.3. *Let x and y be two points of Q_{2n}. The two submanifold Z_x and Z_y intersect if and only if the projective line pathing both x and y lies in Q_{2n}.*

Proof. We identify Q_{2n} with the set of projectivized null vectors (i.e. one-dimensional isotropic subspaces) in $\mathbf{P}(\mathbf{R}^{2n+2} \otimes \mathbf{C})$ as above. Then, for two points x and y in Q_{2n}, the line joining x and y lies in Q_{2n} if and only if the subspace spanned by x and y is isotropic.

The twistor space $Z(S^{2n})$ is identified with $\mathrm{SO}(2n+2)/\mathrm{U}(n+1)$, which parametrizes the set of complex structures of $\mathbf{R}^{2n+2}$ compatible with the metric and the orientation, or equivalently the set of α-subspaces of $\mathbf{R}^{2n+2} \otimes \mathbf{C}$. L_z demotes the α-subspace corresponding to $z \in Z(S^{2n})$. Then, for $z \in Z(S^{2n})$ and $x \in Q_{2n}$,

$$z \in Z_x \Longleftrightarrow x \subset L_z, \tag{6.7}$$

where x is considered to be a one-dimensional subspace of $\mathbf{R}^{2n+2} \otimes \mathbf{C}$. Hence, for a point x of Q_{2n}, Z_x is the set of α-subspaces containing x.

First, if $Z_x \cap Z_y \neq \emptyset$, its elements are identified with α-subspaces containing both x and y. Hence, as mentioned above, x and y can be joined by a line in Q_{2n}.

On the other hand, if x and y can be joined by a line in Q_{2n}, the subspace spanned by x and y is a 2-dimensional isotropic subspace of $\mathbf{R}^{2n+2} \otimes \mathbf{C}$. Since we assume $n > 1$, there is an α-subspace of $\mathbf{R}^{2n+2} \otimes \mathbf{C}$ containing x and y. Hence there is a point z of $Z(S^{2n})$ such that L_z contains both x and y. This means that z is contained in both Z_x and Z_y by (6.7).

Remark. If x and y satisfy the above condition, the intersection $Z_x \cap Z_y$ is always analytically isomorphic to $\mathrm{SO}(2n-2)/\mathrm{U}(n-1)$, which is identified with the set of α-subspaces of $\mathbf{R}^{2n+2} \otimes \mathbf{C}$ containing both x and y.

§7. Definition of real structures when n is odd

In this section, we want to use the anti-holomorphic involution $\tilde{\tau}$ on Q_{2n} defined in the previous section to define an anti-holomorphic involution on the complexification of $2n$-dimensional conformally flat manifolds such that the set of fixed points are the set of real points, in case n is odd and $n > 1$.

As in the previous section, Z_x denotes the submanifold of $Z(S^{2n})$ corresponding to a point x of Q_{2n}. Let $p\colon Z(S^{2n}) \to S^{2n}$ be the projection. The subset $p(Z_x)$ is either a point, namely x, if x is a real point, or otherwise a submanifold isomorphic to S^{2n-2} which is cut out by two hyperplanes.

Proposition 7.1. *If $x \in Q_{2n}$ is not a real point, i.e. $\tilde{\tau}(x) \neq x$, the subset $p(Z_x) \subset S^{2n}$ is isomorphic to S^{2n-2} and obtained by cutting out by two hyperplanes of $\mathbf{R}^{2n+1}$. Conversely, for any such submanifold, there are precisely two points of Q_{2n} corresponding to the submanifold, and they are conjugate to each other by $\tilde{\tau}$.*

Proof. Let x be a point of Q_{2n}. Then, for a point $y \in S^{2n}$, $y \in p(Z_x)$ if and only if the submanifolds Z_x and Z_y intersect. Hence, by Proposition 6.3,

$$p(Z_x) = \{y \in S^{2n} \subset Q_{2n} | x \text{ and } y \text{ are joined by a line in } Q_{2n}\}$$

We compute the right-hand condition explicitly. For a point y of S^{2n}, the corresponding projectivized null vector is $[e_1 - \sqrt{-1}\, y]$ by (6.6), where S^{2n} is considered as the unit sphere of $\mathbf{R}^{2n+1} \subset \mathbf{R}^{2n+1} \otimes \mathbf{C}$ with the basis $(e_2, \ldots, e_{2n+2})$. Let $\hat{x}$ be a representative of x written as:

$$\hat{x} = x^1 e_1 + \dot{x},$$

where $x^1 \geq 0$ and $\dot{x} \in \mathbf{R}^{2n+1} \otimes \mathbf{C}$. The condition for y is written as:

$$\begin{aligned} (\hat{x}, e_1 - \sqrt{-1}\, y) = 0 &\Longleftrightarrow x^1 - \sqrt{-1}\,(\dot{x}, y) = 0 \\ &\Longleftrightarrow x^1 + (\Im \dot{x}, y) = 0,\ (\Re \dot{x}, y) = 0 \end{aligned} \tag{7.1}$$

where $\Re \dot{x}$ and $\Im \dot{x}$ denote the real and imaginary part of the vector $\dot{x}$, respectively. Since $\hat{x}$ is a null vector,

$$(\hat{x}, \hat{x}) = (x^1)^2 + |\Re \dot{x}|^2 - |\Im \dot{x}|^2 + 2\sqrt{-1}\,(\Re \dot{x}, \Im \dot{x}) = 0,$$

hence, we have

$$(\Re \dot{x}, \Im \dot{x}) = 0,$$
$$\Im \dot{x} \neq 0.$$

Furthermore, if $\Re \dot{x} = 0$, then x is a real point and $p(Z_x)$ is a point. If $\Re \dot{x} \neq 0$, $p(Z_x)$ is the $(2n-2)$-dimensional sphere cut out by two hyperplanes (7.1). Note that they are independent because $\Re \dot{x}$ and $\Im \dot{x}$ are perpendicular.

On the other hand, let us given a $(2n-2)$-dimensional sphere of S^{2n} cut out by two hyperplanes defined by

$$h + (v, y) = 0$$
$$(w, y) = 0$$

where h is a real number with $0 \leq h < 1$ and v and w are unit vectors of $\mathbf{R}^{2n+1}$ such that $(v, w) = 0$. Then, it is written as $p(Z_x)$ for a non-real point x of Q_{2n} of the form:

$$x = [he_1 + \sqrt{1 - h^2}\, w + \sqrt{-1}\, v].$$

If h is non-zero, h and v are unique and w is unique up to sign. If we change the sigh of w, we have the conjugate point by $\tilde{\tau}$. If h is zero, v and w are unique up to O(2). Changing by the action of an elements of SO(2) gives the same point of Q_{2n}, and by the action of an elements of O(2)\SO(2) gives the conjugate point by $\tilde{\tau}$.

Now we can give a definition of a real structure for the complexification of a conformally flat manifold. Let X be a $2n$-dimensional conformally flat manifold and $Z(X)$ and $X_{\mathbf{C}}$ be the twistor space and the complexification of X, respectively, where we assume $n > 1$. Let $p: Z(X) \to X$ be the projection.

Definition 7.2. For a point x of $X_{\mathbf{C}} \backslash X$, there is a unique point $x' \neq x$ such that $p(Z_{x'}) = p(Z_x)$. For a point x of X, we put $x' = x$. Then, we define a map $\tilde{\tau}$ by

$$\begin{aligned} \tilde{\tau}: X_{\mathbf{C}} &\longrightarrow X_{\mathbf{C}} \\ x &\longmapsto x' \end{aligned}$$

We call this map $\tilde{\tau}$ the real structure of $X_{\mathbf{C}}$.

Theorem 7.3. *Let X, $X_{\mathbf{C}}$ and $\tilde{\tau}$ be as above. Then $\tilde{\tau}$ is an anti-holomorphic involution on $X_{\mathbf{C}}$, and $\tilde{\tau}(x) = x$ if and only if x is a real point, that is, $x \in X$.*

Proof. This is immediate by considering the developing map $\Phi: \tilde{X} \to S^{2n}$, where $\tilde{X}$ is the universal covering space with the conformally flat structure induced from X.

§8. Complexification of tori

In this section, we show that the complexification of $\mathbf{R}^{2n}/\Gamma$ for a lattice Γ is $(\mathbf{R}^{2n} \otimes \mathbf{C})/\Gamma$. The conformal structure on it is a complexification of the (conformally) flat structure of $\mathbf{R}^{2n}/\Gamma$.

First, we prove the complexification $\mathbf{R}^{2n}_{\mathbf{C}}$ of $\mathbf{R}^{2n}$ with flat metric is $\mathbf{R}^{2n} \otimes \mathbf{C}$, and the complex conformal structure is the complex linear extension of the metric of $\mathbf{R}^{2n}$. Since $\mathbf{R}^{2n}$ can be conformally embedded to S^{2n}, $\mathbf{R}^{2n}_{\mathbf{C}}$ is obtained as a submanifold of Q_{2n}. As above, consider Q_{2n} as a submanifold of $\mathbf{P}(\mathbf{R}^{2n+2} \otimes \mathbf{C})$, and S^{2n} as the unit sphere of $\mathbf{R}^{2n+1}$, which is spanned by $e_2, \dots, e_{2n+2}$ in $\mathbf{R}^{2n+2}$. The embedding i of S^{2n} to Q_{2n} as the set of real points is written as:

$$\begin{aligned} S^{2n} &\longrightarrow \mathbf{P}(\mathbf{R}^{2n+2} \otimes \mathbf{C}) \\ y &\longmapsto [e_1 - \sqrt{-1}\, y]. \end{aligned} \tag{8.1}$$

The image $i(S^{2n})$ is the set of fixed-points of the anti-holomorphic involution $\tilde{\tau}$ on Q_{2n} induced by the anti-linear transformation σ of $\mathbf{R}^{2n+2} \otimes \mathbf{C}$:

$$\sigma(Y^j e_j) = \bar{Y}^1 e_1 - \sum_{j=2}^{2n+2} \bar{Y}^j e_j. \tag{8.2}$$

We take a conformal embedding of $\mathbf{R}^{2n}$ to S^{2n} such that e_2 is a point at infinity. Let y be a point of Q_{2n} with a representative $Y^i e_i$, Z_y be the corresponding submanifold of $Z(S^{2n})$ and $p: Z(S^{2n}) \to S^{2n}$ be the projection map. Then y is a point of $\mathbf{R}^{2n}_{\mathbf{C}}$ if and only if $p(Z_y)$ does not contain e_2, by Proposition 6.3 and (8.1), which is equivalent to the condition:

$$Y^1 - \sqrt{-1}\, Y^2 \neq 0. \tag{8.3}$$

On the other hand, if we change the homogeneous coordinates,

$$\begin{aligned} X^1 &= Y^1 - \sqrt{-1}\, Y^2, \\ X^2 &= Y^1 + \sqrt{-1}\, Y^2, \\ X^j &= -\sqrt{-1}\, Y^j, \qquad j = 3, \ldots, 2n+2, \end{aligned}$$

the quadratic equation defining Q_{2n} becomes

$$X^1 X^2 - \sum_{j=3}^{2n+2} (X^j)^2 = 0, \tag{8.4}$$

and by (8.2) the anti-linear map σ of $\mathbf{R}^{2n+2} \otimes \mathbf{C}$ is written as:

$$\sigma(X^1, \ldots, X^{2n+2}) = (\bar{X}^1, \ldots, \bar{X}^{2n+2}). \tag{8.5}$$

Furthermore, by (8.3), the open subset $\mathbf{R}^{2n}_{\mathbf{C}}$ of Q_{2n} is defined by

$$X^1 \neq 0.$$

Hence, by (8.4) we take

$$x^j := X^j / X^1, \qquad j = 3, \ldots, 2n+2$$

as coordinates of $\mathbf{R}^{2n}_{\mathbf{C}}$. The point $(x^3, \ldots, x^{2n+2})$ is a real point if and only if all coordinates are real numbers by (8.5). Furthermore, this system of coordinates is a standard one of $\mathbf{R}^{2n}$. Hence we have an identification between $\mathbf{R}^{2n}_{\mathbf{C}}$ and $\mathbf{R}^{2n} \otimes \mathbf{C}$.

By conformal invariance of twistor spaces, a translation of $\mathbf{R}^{2n}$ induces a holomorphic transformation of the twistor space $Z(\mathbf{R}^{2n})$, which also induces a holomorphic transformation of $\mathbf{R}^{2n}_{\mathbf{C}} = \mathbf{R}^{2n} \otimes \mathbf{C}$. This is nothing but a translation of $\mathbf{R}^{2n} \otimes \mathbf{C}$, which can be easily proved by using Proposition 6.2.

Theorem 8.1. *Let Γ be a lattice of $\mathbf{R}^{2n}$. Then the complexification of $\mathbf{R}^{2n}/\Gamma$ is $(\mathbf{R}^{2n} \otimes \mathbf{C})/\Gamma$.*

Proof. It is enough to verify that a submanifold Z_x does not intersect with Z_{Tx} when x is a point of $\mathbf{R}^{2n}_{\mathbf{C}}$ and T is a non-trivial translation of $\mathbf{R}^{2n}$. Let x be the point with coordinates $(x^3, \ldots, x^{2n+2})$ and T be the translation by a vector $(y^3, \ldots, y^{2n+2})$. The point Tx is written as $(x^3 + y^3, \ldots, x^{2n+2} + y^{2n+2})$. By (8.4), the two points x and Tx are written in the homogeneous coordinates $(X^1, \ldots, X^{2n+2})$ as:

$$x = (1, (x, x), x),$$
$$Tx = (1, (x + y, x + y), x + y),$$

and the bilinear product of these two vectors is:

$$\frac{1}{2}(x + y, x + y) + \frac{1}{2}(x, x) - (x, x + y) = \frac{1}{2}|y|^2 \neq 0.$$

Hence, by Proposition 6.3, Z_x and Z_{Tx} are disjoint.

§9. Solutions of the twistor equation

In this section, we study relationship between *holomorphic* sections of the hyperplane bundle H over $Z(X)$ and solutions of the twistor equations on X.

Let E be a complex vector bundle over a manifold M and s be a section of E. As in §1, we have a function $s^\vee$ on the total space of E^* (the dual bundle of E) by the canonical pairing.

Conversely, let $f: E^* \to \mathbf{C}$ be a complex function such that the restriction map $f|_{E_x^*}: E_x^* \to \mathbf{C}$ is complex linear for all $x \in M$. Then, for each point x of M, there is a point $s(x) \in E_x$ such that

$$f|_{E_x^*}(\varphi) = \langle \varphi, s(x) \rangle, \qquad \text{for all } \varphi \in E_x^*$$

that is, $s^\vee = f$, where $\langle\ ,\ \rangle$ denotes the canonical pairing of E^* and E.

Lemma 9.1. *There is a one to one correspondence between the set of sections of E and the set of complex functions of E^* which are linear on E_x^* for all $x \in M$. When M is an almost complex manifold and E has a connection, s is a holomorphic section if and only if $s^\vee$ is a holomorphic function. (The word "holomorphic" is explained in the proof.)*

Proof. We need to prove the last part. First we define an almost complex structure of E^* by a connection of E. A connection of E induces a connection of E^*, which defines a splitting of the exact sequence

$$0 \longrightarrow T_V(E^*) \longrightarrow T(E^*) \longrightarrow p^* T(M) \longrightarrow 0$$

where $p: E^* \to M$ is the projection.

$$T(E^*) = T_V(E^*) \oplus T_H(E^*)$$
$$T_H(E^*) \simeq p^* T(M)$$

The vertical part $T_V(E^*)$ has a complex structure, since E^* is a complex vector bundle. On the other hand, the horizontal part $T_H(E^*)$ has a complex structure induced by the complex structure of $T(M)$ (M is assumed to be an almost complex manifold). Hence $T(E^*)$ has a complex structure and E^* is an almost complex manifold. Note that the almost complex structure of the hyperplane bundle H over the twistor space $Z(X)$ is same as the one induced from the almost complex structure of $Z(X)$ and the connection of H.

Now we define a *holomorphic* section of E as a solution of the $\bar{\partial}$ operator

$$\bar{\partial}\colon \Gamma(E) \xrightarrow{\nabla} \Gamma(E \otimes \Lambda^1 M) \xrightarrow{\pi^{(0,1)}} \Gamma(E \otimes \Lambda^{0,1} M)$$

where $\pi^{(0,1)}\colon \Lambda^1 M \to \Lambda^{0,1} M$ is the projection. A *holomorphic* function on an almost complex manifold is simply a *holomorphic* section of a trivial line bundle with a trivial connection.

We use the notation in Example 1.3. If we take a section s of E and write it locally $s = e_i s^i$, then the function $s^\vee$ on E^* is written as $s^\vee = \tau_i s^i$. Now we give a condition when the function $s^\vee$ is *holomorphic*.

$$\begin{aligned}
\bar{\partial} s^\vee &= \bar{\partial}(\tau_i s^i) \\
&= \tau_i \bar{\partial} s^i + \pi^{(0,1)}(d\tau_i - \tau_j \omega_i^j + \tau_j \omega_i^j) s^i \\
&= \tau_i \bar{\partial} s^i + \tau_j \pi^{(0,1)}(\omega_i^j) s^i \qquad \text{(by Example 1.3)} \\
&= \tau_i (\bar{\partial} s^i + \pi^{(0,1)}(\omega_j^i) s^j)
\end{aligned}$$

Hence $s^\vee$ is *holomorphic* if and only if

$$\bar{\partial} s^i + \pi^{(0,1)}(\omega_j^i) s^j = 0, \qquad i = 1, \ldots, m.$$

This is precisely the condition $\bar{\partial} s = 0$.

Let X be a $2n$-dimensional spin manifold. Then the twistor operator is the following operator defined in Definition 3.1.

$$\bar{D}\colon \Gamma(\Delta^+(X)^*) \xrightarrow{\nabla} \Gamma(\Delta^+(X)^* \otimes T^*X) \xrightarrow{\kappa} \Gamma(K_+(X)).$$

Let $Z(X)$ and H denote the twistor space and the hyperplane bundle. We define the notion of *holomorphic* section of H as in the proof of Lemma 9.1.

Theorem 9.2. *There is a one to one correspondence between the solutions of the twistor equation and the holomorphic sections of H over $Z(X)$.*

Proof. First, let $s \in \Gamma(\Delta^+(X)^*)$ be a solution of the twistor equation: $\bar{D}s = 0$. Let R be the kernel of $L(\bar{D})$, where $L(\bar{D})$ denotes the linear map defining $\bar{D}$. Then, we have

$$ds^\vee|_z = V(p^* j_1(s))_z \in V(p^* R)_z = V(\bar{D})_z \qquad \text{for all } z \in \Delta^+(X)$$

by the definition of the map V. Hence, by the definition of the almost complex structure of the total space of $(H^*)^\times$, we have on $(H^*)^\times$

$$\bar{\partial} s^\vee = 0.$$

Thus, $s^\vee$ is a *holomorphic* function on H^*. It is easy to show that the function $s^\vee|_{H^*}$ is linear on H_z^*, for each $z \in Z(X)$. Hence, by Lemma 9.1, we obtain a *holomorphic* section f of H over $Z(X)$ such that

$$f^{\vee} = s^{\vee}|_{H^*}.$$

Conversely, let f be a *holomorphic* section of H over $Z(X)$. By Theorem 3.7, if we restrict H to the fiber $Z(X)_x$, $x \in X$, this is a line bundle obtained by pulling back the hyperplane bundle of $\mathbf{P}(\Delta^+(X)_x)$ by the embedding,

$$i\colon Z(X)_x \longrightarrow \mathbf{P}(\Delta^+(X)_x).$$

Thus we obtain holomorphic sections by the pull-back:

$$i^*\colon \Gamma(\mathbf{P}(\Delta^+(X)_x), \mathcal{O}(1)) \longrightarrow \Gamma(Z(X)_x, H_x)$$

By the theorem of Borel-Weil, we have the following lemma.

Lemma 9.3. *The pull-back map i^* is an isomorphism. Hence the holomorphic sections of H_x over $Z(X)_x$ are parametrized by the space of the linear forms of $\Delta^+(X)_x$, that is, $\Delta^+(X)^*_x$.*

Therefore, for each point x of X, there is a unique point $s(x) \in \Delta^+(X)^*_x$ such that

$$f|_{Z(X)_x} = i^* s(x),$$

or

$$f^{\vee}|_{H^*_x} = s(x)^{\vee}|_{H^*_x}.$$

Thus, we obtain a section s of $\Delta^+(X)^*$ over X. Since, by Lemma 9.1, $s^{\vee}|_{H^*}$ is a *holomorphic* function, for $z \in (H^*)^{\times}$, we have

$$ds^{\vee}|_z \in V(\bar{D})_z. \tag{9.1}$$

Now, let us describe this condition by using the splitting of the distribution $V(\bar{D})$. If we take a local frame $(e_1, \dots, e_m)$ of $\Delta^+(X)^*$, where $m = \dim(\Delta^+)^* = 2^{n-1}$, s can be expressed locally as $s = e_i s^i$. Let $(\tau_1, \dots, \tau_m)$ be a system of coordinates of the fiber direction of $\Delta^+(X)$, then $s^{\vee} = \tau_i s^i$ and,

$$ds^{\vee} = (d\tau_i - \tau_j \omega^j_i) s^i + \tau_i (ds^i + \omega^i_j s^j).$$

where ω^i_j is the connection form of $\Delta^+(X)^*$. By Example 1.3, for a point z of $\Delta^+(X)^{\times}$, $V(p^*\Delta^+(X)^*)_z$ can be expressed explicitly as

$$V(p^*\Delta^+(X)^*)_z = \langle d\tau_i - \tau_j \omega^j_i|_z \,|\, i = 1, \dots, m\rangle.$$

Hence (9.1) is equivalent to

$$\tau_i(ds^i + \omega^i_j s^j)|_z \in V(p^*\Delta^-(X)^*)_z$$

or

$$\langle z, (\nabla s)|_{p'(z)}\rangle \in \langle z, cl^*(\Delta^-(X)^*)_{p'(z)}\rangle, \tag{9.2}$$

where $p'\colon H^* \to X$ is the projection and $\langle\ ,\ \rangle$ denotes the canonical pairing of

$\Delta^+(X)$ and $\Delta^+(X)^*$.

Hence, to prove that the section s constructed from the holomorphic section f of H is a solution of the twistor equation, we need to show that s satisfies the condition (9.2), for all $z \in \Delta^+(X)^\times$. Since s satisfies (9.2), for a point $z \in (H^*)^\times$, the next lemma completes the proof of the theorem.

Lemma 9.4. *The submodule $cl^*((\Delta^-)^*)$ of $(\Delta^+)^* \otimes (\mathbf{R}^{2n})^*$ is equal to the following subspace.*

$$\{\alpha \in (\Delta^+)^* \otimes (\mathbf{R}^{2n}) | \langle z, \alpha \rangle \in \langle z, cl^*((\Delta^-)^*) \rangle, \quad \textit{for all } z \in \Delta^+ \textit{ such that } [z] \in Z\}$$

where $\langle\ ,\ \rangle$ denotes the canonical pairing of Δ^+ and $(\Delta^+)^*$, and Z is the submanifold defined in Proposition 3.2.

Proof. The above subspace is invariant under the action of SPIN$(2n)$ containing $cl^*((\Delta^-)^*)$. By letting α be $\theta^{\emptyset} \otimes e^1$ and z be $\theta_\emptyset$, one can see easily that it is a proper subspace of $(\Delta^+)^* \otimes (\mathbf{R}^{2n})^*$. Hence, by Lemma 2.3, we complete the proof.

Example 9.5. (1) By Example 4.3, the hyperplane bundle of the twistor space $Z(S^{2n}) = Z_{n+1}$ of the sphere S^{2n} is isomorphic to the pull-back of the hyperplane bundle over the projectivized positive spin module of $\mathbf{R}^{2n+2}$. Hence, by Lemma 9.3, the solution space of the twistor equation on S^{2n} is 2^n-dimensional.

(2) By the conformal invariance of the twistor equation, we have solutions on $\mathbf{R}^{2n}$ by the conformal embedding $\mathbf{R}^{2n} \subset S^{2n}$. If $n > 1$, since the codimension of $Z(S^{2n}) \backslash Z(\mathbf{R}^{2n})$ is greater than 2, there are no other solutions. Hence the dimension of the solution space is also 2^n. If $n = 1$, a solution of the twistor equation is identified with a holomorphic function on $\mathbf{C}$. Hence there are infinite dimensional solutions.

(3) Let Γ be a non-trivial lattice of $\mathbf{R}^{2n}$. Since $\mathbf{R}^{2n}/\Gamma$ is flat, the spin bundle $\Delta^+(\mathbf{R}^{2n}/\Gamma)$ is trivial. Solutions of the twistor equation on $\Delta^+(\mathbf{R}^{2n}/\Gamma)$ is identified with solutions on $\mathbf{R}^{2n}$ invariant under the action of Γ. If $n > 1$, it is easy to show that there are no non-constant solutions invariant under a non-trivial translation. Hence the solution space is 2^{n-1}-dimensional. If $n = 1$ and Γ is generated by $c \in \mathbf{C}^\times$, the solutions are holomorphic functions with a period c. Hence the solution space is infinite dimensional. If the rank of Γ is two, the solution space is one-dimensional consisting of constant functions, since $\mathbf{R}^{2n}/\Gamma$ is compact.

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References

[A] M. F. Atiyah, Geometry of Yang-Mills Fields, Fermi Lectures Notes, Scuola Normale Sup. di Pisa, Pisa, 1979.

[A.H.S] M. F. Atiyah, N. J. Hitchin and I. M. Singer, Self-duality in four-dimensional Riemannian geometry, Proc. R. Soc. Lond. A., **362** (1978), 425–461.

[B] R. Bott, Homogeneous vector bundles, Ann. of Math., **66** (1957), 203–248.

[D.F.N] B. A. Dubrovin, A. T. Fomenko, S. P. Novikov, Modern Geometry-Methods and Applications Vol. 1. G. T. M. **93**, Springer-Verlag, 1984.

[F] H. D. Fegan, Conformally invariant first order differential operators, Q. J. Math., **27** 1976, 371–378.

[G] P. B. Gilkey, Invariance Theory, The Heat Equation, And the Atiyah-Singer Index Theorem. Publish or Perish, Inc., 1984.

[H] N. J. Hitchin, Linear field equations on self-dual spaces, Proc. Roy. Soc. Lond. A, **370** 1980, 173–191.

[K] K. Kodaira, A theorem of completeness of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds, Annals. of Math., **75** 1962, 146–162.

[M] M. K. Murray, A Penrose transform for the twistor space of an even dimensional conformally flat Riemannian manifold, Ann. Global Anal. Geom., **4**–3 (1986), 71–88.

[O.R] N. R. O'Brian, J. H. Rawnsley, Twistor spaces, Ann. Global Anal. Geom., **8**–1 (1985), 29–58.

[W] P. M. Wong, Twistor spaces over 6-dimensional Riemannian manifolds, Illinois J. Math., **31**–2 (1987), 274–311.