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Mixed problems or Cauchy problems for semi-degenerate hyperbolic equations of 2-nd order with a parameter

By

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Introduction.

Let us consider linear hyperbolic operators of 2-nd order with real coefficients:

$$L=L(t, x;\partial_t, \partial_x)=\partial_t^2-2\sum_{j=1}^{n} a_{j0}(t, x)\partial_j\partial_t-\sum_{j,k=1}^{n} a_{jk}(t, x)\partial_j\partial_k$$

$$+b_0(t, x)\partial_t+\sum_{j=1}^{n} b_j(t, x)\partial_j+c(t, x)$$

in $I\times\Omega=[0, T]\times R_+^n=\{0\leqq t\leqq T,\ x_1>0,\ x'=(x_2, \cdots, x_n)\in R^{n-1}\}$, where $a_{jk}=a_{kj}$ and $\partial_j=\partial/\partial x_j$. It is well known that the mixed problem:

$$(\text{M.P})\qquad \begin{cases} Lu=f & \text{in } I\times\Omega, \\ u|_{x_1=0}=g_0 & \text{on } I\times\partial\Omega. \\ u|_{t=0}=u_0, \quad \partial_t u|_{t=0}=u_1 & \text{on } \Omega \end{cases}$$

is well posed, if

i) $\displaystyle\inf_{I\times\Omega\times S^{n-1}} \sum_{j,k=1}^{n} a_{jk}(t, x)\xi_j\xi_k>0,$

ii) $a_{jk}, b_j, c\in \mathcal{B}^\infty(I\times\Omega)$

are satisfied. How about the problem if i) and ii) are replaced by

i)′ $\displaystyle\inf_{I\times\Omega_\varepsilon\times S^{n-1}} \sum_{j,k=1}^{n} a_{jk}(t, x)\xi_j\xi_k>0$ (any $\varepsilon>0$),

ii)′ $a_{jk}, b_j, c\in \mathcal{B}^\infty(I\times\Omega_\varepsilon)$ (any $\varepsilon>0$),

where $\Omega_\varepsilon=\Omega\cap\{x_1>\varepsilon\}$? In this paper, assuming i)′ and ii)′, we consider two cases. One is a degenerate case, when i) is not satisfied, and the other is a singular case, when ii) is not satisfied. Their typical examples are as follows:

(I) $L=\partial_t^2-\rho\partial_1^2-\partial_2^2-(\mu+1)\partial_1,$

(II) $L=\partial_t^2-\partial_1^2-\partial_2^2-(\mu+1)\rho^{-1}\partial_1,$

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where μ is a real parameter, $\rho=\rho(x_1)\in\mathcal{B}^\infty(R_+)$, and $\rho=x_1$ near $x_1=0$. We consider the mixed problem (M.P) for $\mu<0$ and we consider the Cauchy problem:

$$\text{(C.P)}\qquad \begin{cases} Lu=f & \text{in } I\times\Omega\,, \\ u|_{t=0}=u_0\,, \quad \partial_t u|_{t=0}=u_1 & \text{on } \Omega \end{cases}$$

for $\mu>0$.

There are so many studies on semi-degenerate problems for parabolic or elliptic operators since W. Feller [1], but there are little about hyperbolic problems except for fully degenerate cases (e.g. [2], [3]). Nakaoka ([4]) considered

$$\partial_t^2 u=\rho(x)^\alpha\partial_x^2 u \qquad (0<\alpha<1)$$

in $\{t>0,\ x>0\}$ with initial data and with zero boundary data. By the change of variables

$$s=\beta t, \qquad y=x^\beta \qquad (\beta=2-\alpha),$$

it is transformed into

$$\partial_s^2 u=\rho(y)\partial_y^2 u+(\mu+1)\partial_y u \qquad \left(\mu=-\frac{1}{\beta}=-\frac{1}{2-\alpha}\right)$$

near $y=0$. Therefore the result of this paper is considered as a generalization of Nakaoka's. The simple idea in this paper is to reduce L to a Bessel type operator. The energy method is applicable to Bessel type operators. Examples in § 6 illustrate the structure of solutions relating to a parameter μ.

§ 1. Semi-degenerate problems and singular coefficient problems

Let us assume that L satisfies the following Ass. I$-\mu$ or Ass. II$-\mu$ in addition to i)′ and ii)′ in Introduction. Under Ass. I$-\mu$ or Ass. II$-\mu$, L is a Fuchsian on $\partial\Omega$ with characteristic roots $\{0,\ \mu\}$ (see [5]).

Assumption I-μ (degenerate case). $a_{ij},\ b_j,\ c\in\mathcal{B}^\infty\ (I\times\bar{\Omega})$, and

(I—A) $a_{1j}=\rho\tilde{a}_{1j}$, where $\tilde{a}_{1j}\in\mathcal{B}^\infty(I\times\bar{\Omega})$ $(j=0,\ 1,\ \cdots,\ n)$,

$$\inf_{I\times\partial\Omega}\tilde{a}_{11}>0\,, \qquad \inf_{I\times\partial\Omega\times S^{n-1}}\sum_{j,k=2}^{n} a_{jk}\xi_j\xi_k>0\,,$$

(I—B) $b_1=-\tilde{a}_{11}(\mu+1)+\rho\tilde{b}_1$, where μ is a real constant and $\tilde{b}_1\in\mathcal{B}^\infty(I\times\bar{\Omega})$.

Assumption II-μ (singular case). $a_{ij},\ b_j\ (j\neq1)\in\mathcal{B}^\infty(I\times\bar{\Omega})$, and

(II—A) $a_{1j}=\rho\tilde{a}_{1j}$, where $\tilde{a}_{1j}\in\mathcal{B}^\infty(I\times\bar{\Omega})$ $(j\neq1)$,

$$\inf_{I\times\partial\Omega} a_{11}>0\,, \qquad \inf_{I\times\partial\Omega\times S^{n-1}}\sum_{j,k=2}^{n} a_{jk}\xi_j\xi_k>0\,,$$

(II—B) $b_1=-a_{11}(\mu+1)\rho^{-1}+\tilde{b}_1$, where μ is a real constant and $\tilde{b}_1\in\mathcal{B}^\infty(I\times\bar{\Omega})$

(II—C) $c=\rho^{-1}\tilde{c},\ \tilde{c}\in\mathcal{B}^\infty(I\times\bar{\Omega})$,

We say that $(M.\ P)$ is *solvable* in H^h, if there exists a unique solution $u\in H^h(I\times\Omega)$ if

$$(f,\ g_0,\ u_0,\ u_1)\in H^l(I\times\Omega)\times H^l(I\times\partial\Omega)\times H^l(\Omega)\times H^l(\Omega)$$

with compatibility conditions of order l' for some l and l'. Compatibility conditions will be explained later. We say that (C.P) is *solvable* in H^h, if there exists a unique solution $u\in H^h(I\times\Omega)$ if

$$(f,\ u_0,\ u_1)\in H^l(I\times\Omega)\times H^l(\Omega)\times H^l(\Omega)$$

for some l.

Under Ass. I—μ, we have

$$\begin{aligned}L=&-\tilde{a}_{11}\{(\rho\partial_1)+\mu+1\}\partial_1\\&+\{-2(\tilde{a}_{10}\partial_t+\sum_j{}'\tilde{a}_{1j}\partial_j)+\tilde{b}_1\}(\rho\partial_1)\\&+\{\partial_t^2-2\sum_j{}'a_{j1}\partial_j\partial_t-\sum_{j,k}{}'a_{jk}\partial_j\partial_k+b_0\partial_t+\sum_j{}'b_j\partial_j+c\}\\=&-\tilde{a}_{11}\{\Phi(\rho\partial_1)\partial_1+\Psi_1(\partial_\tau)(\rho\partial_1)+\Psi_2(\partial_\tau)\}\\=&-\tilde{a}_{11}\{\Phi(\rho\partial_1)\partial_1+\Psi(\partial_\tau,\ \rho\partial_1)\}=-\tilde{a}_{11}L',\end{aligned}$$

where

$$\sum_j{}'=\sum_{j=2}^{n},\qquad \sum_{j,k}{}'=\sum_{j,k=2}^{n},\qquad \partial_\tau=(\partial_t,\ \partial_2,\ \cdots,\ \partial_n).$$

Under Ass. II—μ, we have

$$\begin{aligned}\rho L=&-a_{11}\{(\rho\partial_1)+\mu+1\}\partial_1\\&+\{-2\rho(\tilde{a}_{10}\partial_t+\sum_j{}'\tilde{a}_{1j}\partial_j)+\tilde{b}_1\}(\rho\partial_1)\\&+\{\rho[\partial_t^2-2\sum_j{}'a_{j0}\partial_j\partial_t-\sum_{j,k}{}'a_{jk}\partial_j\partial_k+b_0\partial_t+\sum_j{}'b_j\partial_j]+\tilde{c}\}\\=&-a_{11}\{\Phi(\rho\partial_1)\partial_1+[\Psi_1(\partial_\tau)(\rho\partial_1)+\Psi_2(\partial_\tau)]\}\\=&-a_{11}\{\Phi(\rho\partial_1)\partial_1+\Psi(\partial_\tau,\ \rho\partial_1)\}=-a_{11}L'.\end{aligned}$$

In both cases,

$$L'=\Phi(\rho\partial_1)\partial_1+\Psi(t,\ x\,;\,\partial_\tau,\ \rho\partial_1),$$

where $\Phi(\lambda)=\lambda+\mu+1$ and Ψ is a linear differential operator of 2-nd order with respect to $\{\partial_\tau,\ \rho\partial_1\}$ with $\mathcal{B}^\infty$ coefficients and

$$\Psi(t,\ x\,;\,\partial_\tau,\ \rho\partial_1)=\Psi_1(t,\ x\,;\,\partial_\tau)(\rho\partial_1)+\Psi_2(t,\ x\,;\,\partial_\tau).$$

To consider L' near $x_1=0$, we define $L_1\cong L_2$, if $L_1=L_2$ near $x_1=0$. Moreover, we define L_β by

$$\rho^\beta Lu=L_\beta(\rho^\beta u).$$

Lemma 1.1. *Let L satisfy Ass. $I-\mu$ (resp. Ass. $II-\mu$), then L_μ satisfies Ass. $I-(-\mu)$ (resp. Ass. $II-(-\mu)$).*

Proof. Let L satisfy Ass. I—μ, then

$$-\tilde{a}_{11}^{-1}L\cong\rho^{-1}(\rho\partial_1+\mu)(\rho\partial_1)+\{\Psi_1(\rho\partial_1)+\Psi_2\},$$

therefore

$$-\tilde{a}_{11}^{-1}L_{\mu}\cong\rho^{-1}(\rho\partial_1)(\rho\partial_1-\mu)+\{\Psi_1(\rho\partial_1-\mu)+\Psi_2\},$$

$$\cong\rho^{-1}(\rho\partial_1-\mu)(\rho\partial_1)+\{\Psi_1(\rho\partial_1)+(\Psi_2-\mu\Psi_1)\}.$$

Let L satisfy Ass. II$-\mu$, then

$$-a_{11}^{-1}L\cong\rho^{-2}((\rho\partial_1+\mu)(\rho\partial_1)+\rho^{-1}\{\Psi_1(\rho\partial_1)+\Psi_2\},$$

therefore

$$-a_{11}^{-1}L_{\mu}\cong\rho^{-2}(\rho\partial_1-\mu)(\rho\partial_1)+\rho^{-1}\{\Psi_1(\rho\partial_1)+(\Psi_2-\mu\Psi_1)\}, \quad \square$$

Let u be a smooth solution of $L'u=f$. Let

$$g_j=\partial_1^j u\,|_{x_1=0}, \qquad f_j=\partial_1^j f\,|_{x_1=0},$$

then we have

$$\begin{aligned}L'u&=\Phi(\rho\partial_1)\partial_1 u+\Psi(\partial_\tau,\ \rho\partial_1)u\\ &\sim\Phi(\rho\partial_1)\{g_1+g_2\rho/1!+g_3\rho^2/2!+\cdots\}\\ &\quad+\Psi(\partial_\tau,\ \rho\partial_1)\{g_0+g_1\rho/1!+g_2\rho^2/2!+\cdots\}\\ &\sim\{\Phi(0)g_1+\Psi(\partial_\tau,\ 0)g_0\}+\{\Phi(1)g_2+\Psi(\partial_\tau,\ 1)g_1\}\rho/1!\\ &\quad+\{\Phi(2)g_3+\Psi(\partial_\tau,\ 2)g_2\}\rho^2/2!+\cdots,\end{aligned}$$

where $\sim$ means the asymptotic expansion at $x_1=0$. Since

$$\Psi(t,\ x;\ \partial_\tau,\ \rho\partial_1)\sim\sum_{k=0}^{\infty}\rho^k/k!\,\Psi^{(k)}(t,\ x';\ \partial_\tau,\ \rho\partial_1),$$

we have

$$(*)\qquad \Phi(j)g_{j+1}+\sum_{k=0}^{j}\binom{j}{k}\Psi^{(j-k)}(\partial_\tau,\ k)g_k=f_j \qquad (j=0,\ 1,\ 2,\ \cdots).$$

Conversely, let us define $\{g_1,\ \cdots,\ g_{l'-1}\}$ by $(*)$, making use of data $\{f_0,\ \cdots,\ f_{l'-2},\ g_0\}$, if $\mu\neq-1,\ -2,\ \cdots,\ -(l'-1)$. Then, we have

$$g_j\in H^{l-2j}(I\times\partial\Omega),$$

if $f\in H^l(I\times\Omega)$ and $g_0\in H^l(I\times\partial\Omega)$. Let us define

$$U(t,\ x)=\sum_{j=0}^{l'-1}(j!)^{-1}g_j(t,\ x')\tilde{\rho}(x_1)^j,$$

where $\tilde{\rho}\in\mathcal{B}^\infty(R_+)$, $\tilde{\rho}=x_1$ near $x_1=0$, and $\tilde{\rho}=0$ if $x_1>1$, then

$$U\in H^{l-2(l'-1)}\subset H^{l'+1}$$

if $l'\leqq(l+1)/3$.

In case when $\mu=-\nu$ $(\nu=1,\ 2,\ \cdots)$, let u be a solution of $L'u=f$ and

$$u=v+w\log\rho,$$

where v and w are smooth functions satisfying $\partial_1^j w\,|_{x_1=0}=0$ $(j=0,\ 1,\ \cdots,\ \nu-1)$. Let

$$g_j=\partial_1^j v|_{x_1=0}, \qquad h_j=\partial_1^j w|_{x_1=0} \qquad (h_0=h_1=\cdots=h_{\nu-1}=0),$$

then we have

$$\begin{aligned} L'v&=\Phi(\rho\partial_1)\partial_1 v+\Psi(\partial_\tau, \rho\partial_1)v \\ &\sim\{\Phi(0)g_1+\Psi(\partial_\tau, 0)g_0\}+\{\Phi(1)g_2+\Psi(\partial_\tau, 1)g_1\}\rho/1! \\ &\quad+\{\Phi(2)g_3+\Psi(\partial_\tau, 2)g_2\}\rho^2/2!+\cdots \end{aligned}$$

and

$$\begin{aligned} L'(w\log\rho)&\cong\Phi(\rho\partial_1)(w\rho^{-1}+\partial_1 w\log\rho)+\Psi(\partial_\tau, \rho\partial_1)(w\log\rho) \\ &\cong\{\Phi(\rho\partial_1)(w\rho^{-1})+\partial_1 w+\Psi_1(\partial_\tau)w\} \\ &\quad+\{\Phi(\rho\partial_1)\partial_1 w+\Psi(\partial_\tau, \rho\partial_1)w\}\log\rho \\ &=I_1+I_2, \end{aligned}$$

where

$$\begin{aligned} I_1&\sim h_\nu\rho^{\nu-1}/(\nu-1)! \\ &\quad+\{[\Phi(\nu)(\nu+1)^{-1}+1]h_{\nu+1}+\Psi_1(\partial_\tau)h_\nu\}\rho^\nu/\nu! \\ &\quad+\{[\Phi(\nu+1)(\nu+2)^{-1}+1]h_{\nu+2}+\Psi_1(\partial_\tau)h_{\nu+1}\}\rho^{\nu+1}/(\nu+1)!+\cdots, \\ (\log\rho)^{-1}I_2&\sim\{\Phi(\nu)h_{\nu+1}+\Psi(\partial_\tau, \nu)h_\nu\}\rho^\nu/\nu! \\ &\quad+\{\Phi(\nu+1)h_{\nu+2}+\Psi(\partial_\tau, \nu+1)h_{\nu+1}\}\rho^{\nu+1}/(\nu+1)!+\cdots. \end{aligned}$$

Hence we have

$$(**)\qquad \Phi(j)h_{j+1}+\sum_{k=\nu}^{j}\binom{j}{k}\Psi^{(j-k)}(\partial_\tau, k)h_k=0 \qquad (j=\nu, \nu+1, \nu+2, \cdots),$$

$$(***)\qquad \begin{cases} \Phi(j)g_{j+1}+\sum_{k=0}^{j}\binom{j}{k}\Psi^{(j-k)}(\partial_\tau, k)g_k=f_j \qquad (j=0, 1, 2, \cdots, \nu-2), \\ h_\nu+\sum_{k=0}^{\nu-1}\binom{\nu-1}{k}\Psi^{(\nu-1-k)}(\partial_\tau, k)g_k=f_{\nu-1} \\ \Phi(j)g_{j+1}+\sum_{k=0}^{j}\binom{j}{k}\Psi^{(j-k)}(\partial_\tau, k)g_k \\ \quad+[\Phi(j)(j+1)^{-1}+1]h_{j+1}+\sum_{k=\nu}^{j}\binom{j}{k}\Psi_1^{(j-k)}(\partial_\tau)h_k=f_j \quad (j=\nu, \nu+1, \cdots). \end{cases}$$

Conversely, let us define $\{g_1, \cdots, g_{l'-1}; h_\nu, h_{\nu+1}, \cdots, h_{l'-1}\}$ by (**) and (***), making use of data $\{f_0, \cdots, f_{l'-2}, g_0\}$, if $\mu=-\nu$ ($\nu=1, 2, \cdots$). Then we have

$$g_j, h_j\in H^{l-2j}(I\times\partial\Omega),$$

if $f\in H^l(I\times\Omega)$ and $g_0\in H^l(I\times\partial\Omega)$. Let us define

$$U=V+W\log\rho.$$

where

$$V(t, x)=\sum_{j=0}^{l'-1}(j!)^{-1}g_j(t, x')\tilde{\rho}(x_1)^j,$$

$$W(t, x)=\sum_{j=0}^{l'-1}(j!)^{-1}h_j(t, x')\tilde{\rho}(x_1)^j,$$

then

$$V, W\in H^{l'+1}\quad (l'\leqq(l+1)/3),\qquad U\in H^{\nu}\quad (\nu\leqq l').$$

Let us define

$$H_0^l=\{u\in H^l \mid \partial_1^j u|_{x_1=0}=0\qquad (j=0, 1, \cdots, l-1)\},$$

and say that data $\{f, g_0, u_0, u_1\}$ satisfy compatibility conditions of order l' if

$$\begin{cases} \tilde{u}_0=u_0-U|_{t=0}\in H_0^{l'-1}(\Omega), \\ \tilde{u}_1=u_1-\partial_t U|_{t=0}\in H_0^{l'-1}(\Omega). \end{cases}$$

Here we have

Lemma 1.2. *Let*

$$(f, g_0, u_0, u_1)\in H^l(I\times\Omega)\times H^l(I\times\partial\Omega)\times H^l(\Omega)\times H^l(\Omega)$$

satisfy compatibility conditions of order l' $(l'\leqq(l+1)/3)$, *then*

$$\tilde{f}=f-L'U\in H_0^{l'-1}(I\times\Omega),$$

$$\tilde{u}_j=u_j-\partial_t^j U\in H_0^{l'-1}(\Omega)\qquad (j=0, 1).$$

Let us say that (C.P) is solvable in H_0^h, if there exists a unique solution $u\in H_0^h(I\times\Omega)$ for any $(f, u_0, u_1)\in H_0^l(I\times\Omega)\times H_0^l(\Omega)\times H_0^l(\Omega)$ with some l. Here we have

Lemma 1.3. i) *In case when* $\mu\neq-1, -2, \cdots$, *if* (C.P) *is solvable in* H_0^h, *then* (M.P) *is solvable in* H^h.

ii) *In case when* $\mu=-\nu$ $(\nu=1, 2, \cdots)$ *if* (C.P) *is solvable in* H_0^h, *then* (M.P) *is solvable in* $H^{\min(h,\nu)}$.

Our aim is to establish the following theorems.

Theorem (C). *Let L satisfy Ass. I*$-\mu$ *or Ass. II*$-\mu$. *Let* $\mu>0$, *then* (C.P) *is solvable in* H^h *for any* h.

Theorem (M). *Let L satisfy Ass. I*$-\mu$ *or Ass. II*$-\mu$. *Let* $-h-1/2\leqq\mu<-h+1/2$, *then* (M.P) *is solvable in* H^h, *where* $h=2, 3, 4, \cdots$.

Theorem (M′). *Let L satisfy Ass. I*$-\mu$ *or Ass. II*$-\mu$. (M.P) *has a solution in* H^0, *if* $-1/2\leqq\mu<0$. (M.P) *has a solution in* H^1, *if* $-3/2\leqq\mu<-1/2$. *More precisely,* (M.P) *has a unique solution satisfying*

$$u\in H^2(I\times\Omega_1),$$

$$u\in\mathcal{B}^{|\mu|/2+\varepsilon}((0,1);H^2(I\times\partial\Omega)),$$

$$(x_1\partial_1)u\in\mathcal{B}^{|\mu|/2+\varepsilon}((0,1);H^1(I\times\partial\Omega)),$$

$$(x_1\partial_1)^2u\in\mathcal{B}^{|\mu|/2+\varepsilon}((0,1);H^0(I\times\partial\Omega))$$

for some $\varepsilon>0$.

§2. Hyperbolic operators of Bessel type

Let us define

$$\mathcal{B}_\rho^k(I\times\Omega)=\{u\,|\,\partial_\rho^\alpha u\in\mathcal{B}^0(I\times\Omega)\qquad(|\alpha|\leqq k)\},$$

$$H_\rho^k(I\times\Omega)=\{u\,|\,\rho^{-1/2}\partial_\rho^\alpha u\in H^0(I\times\Omega)\qquad(|\alpha|\leqq k)\},$$

where

$$\partial_\rho^\alpha=\partial_t^{\alpha_0}(\rho\partial_1)^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}.$$

Let us assume that L satisfies more general assumptions than Ass. I$-\mu$ or Ass. II$-\mu$. Let $\sigma=(\sigma_0,\sigma_1,\cdots,\sigma_n)$ satisfy

$$\sigma_1=1,\qquad 0<\sigma_0\leqq\sigma_j\qquad(j=1,\cdots,n),$$

and let us define $\rho_j=\rho^{\sigma_j}$, then we have

$$\rho_1=\rho,\qquad \rho_0\geqq\rho_j\qquad(j=1,\cdots,n).$$

Let us define

$$\tilde\partial_j=\rho_j\partial_j\qquad(j=0,1,\cdots,n),$$

where $\partial_0=\partial_t$.

Assumption III$-\mu-\sigma$. $a_{ij},b_j,c\in\mathcal{B}_\rho^\infty(I\times\Omega)$, and

$$L\cong\{\tilde\partial_1^2+\mu\tilde\partial_1-c\}+2\rho_0\sum_j{}^*\tilde a_{j1}\tilde\partial_j\tilde\partial_1+\sum_{j,k}{}^*a_{jk}\tilde\partial_j\tilde\partial_k+\rho_0\sum_j b_j\tilde\partial_j,$$

where $\sum_j{}^*=\sum_{j\neq1}$, $\sum_{j,k}{}^*=\sum_{j,k\neq1}$, where

(III-A) $$\sup_{I\times\partial\Omega}a_{00}<0,\qquad \inf_{I\times\partial\Omega\times S^{n-1}}\sum_{j,k=2}^n a_{jk}\xi_j\xi_k>0.$$

(III-D) $$\mu^2+4\inf_{I\times\partial\Omega}c>0.$$

Remark. If L satisfies Ass. I$-\mu$ with $\mu\neq0$, then $-\tilde a_{11}^{-1}\rho L$ satisfies Ass. III$-\mu-(1/2,1,1/2,\cdots,1/2)$. If L satisfies Ass. II$-\mu$ with $\mu\neq0$, then $-a_{11}^{-1}\rho^2L$ satisfies III$-\mu-(1,1,\cdots,1)$.

Let us define

$$\mathcal{H}_{\rho,\sigma}^{l+1}=\{u\,|\,\rho_j\partial_ju\in H_\rho^l\qquad(j=0,1,\cdots,n),\ u\in H_\rho^l\}\quad(l=0,1,\cdots),$$

$$\mathcal{H}_{\rho,\sigma}^0=\{u\,|\,\rho_0u\in H_\rho^0\},$$

then we have

$$H_\rho^{l+1}\subset\mathcal{H}_{\rho,\sigma}^{l+1}\subset H_\rho^l \qquad (l=0,1,\cdots).$$

Our aim in §2~§4 is to establish the following

Theorem 1. *Let L satisfy Ass. III$-\mu-\sigma$. Let*

$$\rho^{\mu/2}(\rho_0^{-1}f,\ u_0,\ u_1)\in H_\rho^l(I\times\Omega)\times\mathcal{H}_{\rho,\sigma}^{l+1}(\Omega)\times\mathcal{H}_{\rho,\sigma}^l(\Omega) \qquad (l\geqq 0),$$

then there exists a unique solution u of (C.P) *satisfying*

$$\rho^{\mu/2}u\in\mathcal{H}_{\rho,\sigma}^{l+1}(I\times\Omega).$$

Lemma 2.1. *Let Z_s be a variable transformation in Ω:*

$$Z_s:z_1=x_1^s,\qquad z'=x'\qquad \text{near } x_1=0,$$

and let L_* be the transformed operator of L. Assume that L satisfies Ass. III$-\mu-\sigma$, then L_* satisfies Ass. III$-\mu_*-\sigma_*$, where $\mu_*=\mu/s$ and $\sigma_*=(\sigma_0/s,\ 1,\ \sigma_2/s,\ \cdots,\ \sigma_n/s)$.

Proof. Let $\rho_*(z_1)=z_1$ near $z_1=0$, then

$$\rho\partial_1=s\rho_*\partial_{*1}$$

near $z_1=0$, where $\partial_{*1}=\partial_{z1}$. Hence we have

$$\begin{aligned}(\rho\partial_1)^2+\mu(\rho\partial_1)-c&=(s\rho_*\partial_{*1})^2+\mu(s\rho_*\partial_{*1})-c\\&=s^2\{(\rho_*\partial_{*1})^2+\mu s^{-1}(\rho_*\partial_{*1})-cs^{-2}\}\\&=s^2\{(\rho_*\partial_{*1})^2+\mu_*(\rho_*\partial_{*1})-c_*\}\end{aligned}$$

near $z_1=0$, where $\mu_*=\mu s^{-1}$, $c_*=cs^{-2}$, and

$$\mu_*^2+4\inf c_*=\mu^2s^{-2}+4\inf cs^{-2}>0. \qquad \square$$

Let us consider a transformation of dependent variables:

$$u\longrightarrow v=\rho^\beta u,$$

where β is a real number, then L is transformed to L_β i.e.

$$\rho^\beta Lu=L_\beta(\rho^\beta u).$$

Lemma 2.2. *Assume that L satisfies Ass. III$-\mu-\sigma$, then $L_{\mu/2}$ satisfies Ass. III$-0-\sigma$.*

Proof. Let L satisfy Ass. III$-\mu-\sigma$, then

$$L\cong\{\tilde\partial_1^2+\mu\tilde\partial_1-c\}+2\rho_0\mathcal{L}_1(\tilde\partial_\tau)\tilde\partial_1+\mathcal{L}_2(\tilde\partial_\tau)+\rho_0\sum b_j\tilde\partial_j,$$

wherc

$$\mathcal{L}_1=\sum_j{}^*a_{j1}\tilde\partial_j,\qquad \mathcal{L}_2=\sum_{j,k}{}^*a_{jk}\tilde\partial_j\tilde\partial_k.$$

Then we have

$$L_\beta \cong \{(\tilde{\partial}_1-\beta)^2+\mu(\tilde{\partial}_1-\beta)-c\}+2\rho_0\mathcal{L}_1(\tilde{\partial}_1-\beta)+\mathcal{L}_2$$
$$+\rho_0\{b_1(\tilde{\partial}_1-\beta)+\textstyle\sum^* b_j\tilde{\partial}_j\}$$
$$\cong \{\tilde{\partial}_1^2+(-2\beta+\mu)\tilde{\partial}_1-[-\beta^2+\mu\beta+c-\rho_0 b_1\beta]\}$$
$$+2\rho_0\mathcal{L}_1\tilde{\partial}_1+\mathcal{L}_2+\rho_0\{b_1\tilde{\partial}_1+(\textstyle\sum^* b_j\tilde{\partial}_j-2\beta\mathcal{L}_1)\},$$

therefore we have

$$L_{\mu/2}\cong \tilde{\partial}_1^2-c_*+2\rho_0\mathcal{L}_1\tilde{\partial}_1+\mathcal{L}_2+\rho_0\{b_1\tilde{\partial}_1+(\textstyle\sum^* b_j\tilde{\partial}_j-\mu\mathcal{L}_1)\},$$

where

$$c_*=(\mu/2)^2+c-\rho_0 b_1(\mu/2).$$

Since

$$c_*|_{x_1=0}=(\mu/2)^2+c|_{x_1=0},$$

$L_{\mu/2}$ satisfies Ass. III$-0-\sigma$. □

We say that L is a hyperbolic operator of *Bessel type*, if L satisfy Ass. III$-0-\sigma$.

§ 3. Energy estimates of (C. P) for hyperbolic operator of Bessel type

Let us define

$$(u, v)=(u, v)_{L^2(\Omega)}, \qquad (u, v)_\rho=(\rho^{-1/2}u, \rho^{-1/2}v)_{L^2(\Omega)},$$
$$\|u\|_{\rho,l}^2=\|u\|_{H_\rho^l(\Omega)}^2=\sum_{|\alpha|\le l}\|\partial_\rho^\alpha u\|_\rho^2,$$
$$|||u(t)|||_{\rho,l}^2=\sum_{j=0}^{l}\|\partial_t^j u(t)\|_{\rho,l-j}^2,$$

and

$$\|u\|_{H_\rho^l(I\times\Omega)}^2=\int_0^T |||u(t)|||_{\rho,l}^2 dt.$$

Moreover, let us define

$$\|u\|_{(\rho,l+1)}^2=\|u\|_{\mathcal{H}_{\rho\sigma}^{l+1}(\Omega)}^2=\sum_{j=1}^{n}\|\tilde{\partial}_j u\|_{\rho,l}^2+\|u\|_{\rho,l}^2,$$
$$|||u(t)|||_{(\rho,l+1)}^2=\sum_{j=0}^{n}|||\tilde{\partial}_j u(t)|||_{\rho,l}^2+|||u(t)|||_{\rho,l}^2,$$

and

$$\|u\|_{\mathcal{H}_{\rho,\sigma}^{l+1}(I\times\Omega)}^2=\int_0^T |||u(t)|||_{(\rho,l+1)}^2 dt.$$

Remark. If $u\in\mathcal{H}_{\rho,\sigma}^{l+1}$, then $(\rho\partial_1)^j|_{x_1=0}=0$ $(j=0, 1, \cdots, l)$.

Let us define

$$e^{-\gamma t}Lu=L^\wedge(e^{-\gamma t}u),$$

then

$$L^\wedge=L(t, x; \partial_t^\wedge, \partial_x)=L(t, x; \partial_t+\gamma, \partial_x).$$

Let us define

$$|||u(t)|||^2_{\rho,l+1,\gamma}=|||u(t)|||^2_{\rho,l+1}+\gamma^{2(l+1)}\|u(t)\|^2_\rho\,,$$

$$|||u(t)|||^2_{(\rho,l+1),\gamma}=|||u(t)|||^2_{(\rho,l+1)}+\gamma^{2(l+1)}\|\rho_0u(t)\|^2_\rho\,,$$

then we have

Lemma 3.1 (basic energy estimate). *Let L be of Bessel type, then there exist $\gamma_0(>0)$ and $C(>0)$ such that*

$$|||u(t)|||^2_{(\rho,1),\gamma}+\gamma\int_0^t|||u(t)|||^2_{(\rho,1),\gamma}dt$$

$$\leqq C\left\{|||u(0)|||^2_{(\rho,1),\gamma}+\gamma^{-1}\int_0^t\|\rho_0^{-1}L\hat{\ }u(t)\|^2_\rho dt\right\}\qquad(0<t<T)$$

for any $\gamma>\gamma_0$ and any $\{u\in\mathcal{H}^1_{\rho,\sigma}(I\times\Omega),\ \rho_0^{-1}L\hat{\ }u\in H^0_\rho(I\times\Omega)\}$.

Proof. It is sufficient to prove Lemma 3.1 for n satisfying $\mathrm{supp}[u]\subset\{x_1<\varepsilon\}$ ($\varepsilon(>0)$: small enough). Let us denote

$$L\cong a_{00}\rho_0^2\partial_t^2+2\rho_0\mathcal{A}_1\partial_t+\mathcal{A}_2+\rho_0\mathcal{B}\,,$$

where

$$\mathcal{A}_1=\rho_0a_{10}\tilde{\partial}_1-\sum_j{}'a_j\tilde{\partial}_j\,,$$

$$\mathcal{A}_2=-c+\tilde{\partial}_1^2+2\rho_0\sum_j{}'a_{j1}\tilde{\partial}_j\tilde{\partial}_1+\sum_{j,k}{}'a_{jk}\tilde{\partial}_j\tilde{\partial}_k\,,$$

$$\mathcal{B}=b_0\tilde{\partial}_t+\sum_{j=1}^n b_j\tilde{\partial}_j\,,$$

then we have

$$\begin{aligned}(L\hat{\ }u,\ \partial\hat{_t}u)_\rho&=(a_{00}\rho_0\partial\hat{_t}^2u,\ \rho_0\partial\hat{_t}u)_\rho+2(\mathcal{A}_1\partial\hat{_t}u,\ \rho_0\partial\hat{_t}u)_\rho\\&\quad+(\mathcal{A}_2u,\ \partial\hat{_t}u)_\rho+(\mathcal{B}u,\ \rho_0\partial\hat{_t}u)_\rho\\&=I_1+I_2+I_3+I_4=I\,.\end{aligned}$$

We have

$$-2\,\mathrm{Re}\,I_1=(\partial_t+2\gamma)(-a_{00}\rho_0\partial\hat{_t}u,\ \rho_0\partial\hat{_t}u)_\rho+R_1\,,$$

$$\begin{aligned}-2\,\mathrm{Re}\,I_3&=(\partial_t+2\gamma)\{(cu,\ u)_\beta+\|\tilde{\partial}_1u\|^2_\rho+2\sum_j{}'(\rho_0a_{1j}\tilde{\partial}_1u,\ \tilde{\partial}_ju)_\rho\\&\quad+\sum_{j,k}{}'(a_{jk}\tilde{\partial}_ku,\ \tilde{\partial}_ju)_\rho\}+R_3\,,\end{aligned}$$

where

$$|R_1|+|I_2|+|R_3|+|I_4|\leqq C(\|\rho_0\partial\hat{_t}u\|^2_\rho+\sum_{j=1}^n\|\tilde{\partial}_ju\|^2_\rho+\|u\|^2_\rho)\,.$$

Let

$$-2\,\mathrm{Re}\,I=(\partial_t+2\gamma)E(t)+R(t)\,,$$

where

$$\begin{aligned}E(t)&=(-a_{00}\rho_0\partial\hat{_t}u,\ \rho_0\partial\hat{_t}u)_\rho+(cu,\ u)_\rho\\&\quad+2\sum{}'(\rho_0a_{1j}\tilde{\partial}_1u,\ \tilde{\partial}_ja)_\rho+\sum{}'(a_{jk}\tilde{\partial}_ku,\ \tilde{\partial}_ju)_\rho\end{aligned}$$

then we have

$$c_1\{\|\rho_0\hat{\partial_t} u\|_\rho^2+\|u(t)\|_{(\rho,1)}^2\}\leqq E(t)\leqq c_2\{\|\rho_0\hat{\partial_t} u\|_\rho^2+\|u(t)\|_{(\rho,1)}^2\},\quad |R(t)|\leqq c_3E(t),$$

where $\{c_j\}$ are positive constants innependent of t, γ, u. On the other hand, since

$$|I|\leqq C\|\rho_0^{-1}L\hat{\ }u(t)\|_\rho E(t)^{1/2},$$

we have

$$(\partial_t+\gamma)E(t)\leqq C\gamma^{-1}\|\rho_0^{-1}L\hat{\ }u(t)\|_\rho^2 \qquad (\gamma>\gamma_0),$$

therefore

$$E(t)+\gamma\int_0^t E(t)\mathrm{d}t\leqq E(0)+C\gamma^{-1}\int_0^t\|\rho_0^{-1}L\hat{\ }u(t)\|_\rho^2\mathrm{d}t \qquad (\gamma>\gamma_0).$$

Remarking

$$\gamma\int_0^t\|\rho_0\hat{\partial_t} u\|_\rho^2\mathrm{d}t=\gamma\left\{\int_0^t(\|\rho_0\partial_t u\|_\rho^2+\gamma^2\|\rho_0 u\|_\rho^2)\mathrm{d}t+\gamma\|\rho_0 u(t)\|_\rho^2-\gamma\|\rho_0 u(0)\|_\rho^2\right\}$$

$$\leqq E(0)+C\gamma^{-1}\int_0^t\|\rho_0^{-1}L\hat{\ }u(t)\|_\rho^2\mathrm{d}t$$

and

$$\|\rho_0\hat{\partial_t} u\|_\rho^2\geqq\|\rho_0\partial_t u\|_\rho^2+\gamma^2\|\rho_0 u\|_\rho^2-2\gamma\|\rho_0\partial_t u\|_\rho\|\rho_0 u\|$$
$$\geqq\frac{1}{2}\|\rho_0\partial_t u\|_\rho^2-3\gamma^2\|\rho_0 u\|_\rho^2,$$

we have Lemma 3.1. □

Lemma 3.2. *Let L be of Bessel type, then*

$$\partial_\tau^\alpha L-L\partial_\tau^\alpha=P_\alpha(\partial_\tau)+\rho_0\sum_{j=1}^n Q_{\alpha j}(\partial_\tau)\tilde{\partial}_j,$$

where

$$P_\alpha(\partial_\tau)=\sum_{|\beta|\leqq|\alpha|-1}p_{\alpha\beta}(t,x)\partial_\tau^\beta,$$
$$Q_{\alpha j}(\partial_\tau)=\sum_{|\beta|\leqq|\alpha|}q_{\alpha j\beta}(t,x)\partial_\tau^\beta,$$

where $p_{\alpha\beta}$, $q_{\alpha j\beta}\in\mathscr{B}_\rho^\infty$.

Proof. It is proved by the mathematical induction about $\{|\alpha|=1, 2, \cdots\}$. Let us see the case when $|\alpha|=1$. Since

$$L\cong-c+\tilde{\partial}_1^2+2\rho_0\mathscr{L}_1(\tilde{\partial}_\tau)\tilde{\partial}_1+\mathscr{L}_2(\tilde{\partial}_\tau)+\rho_0\sum b_j\tilde{\partial}_j$$
$$=-c+\tilde{\partial}_1^2+2\rho_0\sum{}^*a_{j1}\tilde{\partial}_j\tilde{\partial}_1+\sum{}^*a_{jk}\tilde{\partial}_j\tilde{\partial}_k+\rho_0\sum b_j\tilde{\partial}_j\},$$

we have for $l\neq 1$

$$\begin{aligned}\partial_l L - L\partial_l &= -c^{(l)} + 2\rho_0\Sigma^* a_{j1}^{(l)}\tilde{\partial}_j\tilde{\partial}_1 + \Sigma^* a_{jk}^{(l)}\tilde{\partial}_j\tilde{\partial}_k + \rho_0\Sigma b_j^{(l)}\tilde{\partial}_j\} \\ &= -c^{(l)} + \rho_0\{2\Sigma^* a_{j1}^{(l)}\rho_j\partial_j\tilde{\partial}_1 + \Sigma^* a_{jk}^{(l)}(\rho_j/\rho_0)\partial_j\tilde{\partial}_k + \Sigma b_j^{(l)}\tilde{\partial}_j\} \\ &= -c^{(l)} + \rho_0\{[2\Sigma^* a_{j1}^{(l)}\rho_j\partial_j + b_1^{(l)}]\tilde{\partial}_1 + \sum_k{}^*[\sum_j{}^* a_{jk}^{(l)}(\rho_j/\rho_0)\partial_j + b_k^{(l)}]\tilde{\partial}_k\} \\ &= P_{(l)} + \rho_0\sum_j Q_{(l)j}(\partial_\tau)\tilde{\partial}_j\,,\end{aligned}$$

where $a_{jk}^{(l)} = \partial_l a_{jk} - a_{jk}\partial_l$, $\cdots$. Let us assume that it holds for $|\alpha| = N$:

$$\partial_\tau^\alpha L - L\partial_\tau^\alpha = P_\alpha + \rho_0\Sigma Q_{\alpha j}\tilde{\partial}_j\,.$$

Let $l \neq 1$, then

$$\begin{aligned}\partial_l\partial_\tau^\alpha L - L\partial_l\partial_\tau^\alpha &= \partial_l(\partial_\tau^\alpha L - L\partial_\tau^\alpha) + (\partial_l L - L\partial_l)\partial_\tau^\alpha \\ &= \partial_l(P_\alpha + \rho_0\Sigma Q_{\alpha j}\tilde{\partial}_j) + (P_{(l)} + \rho_0\Sigma Q_{(l)j}\tilde{\partial}_j)\partial_\tau^\alpha \\ &= \{\partial_l P_\alpha + P_{(l)}\partial_\tau^\alpha\} + \rho_0\Sigma\{\partial_l Q_{\alpha j} + Q_{(l)j}\partial_\tau^\alpha\}\tilde{\partial}_j\,. \qquad \square\end{aligned}$$

Lemma 3.3. *Let L be of Bessel type, then there exist γ_l (>0) and C_l (>0) such that for $|\alpha| \leqq l$*

$$\sum_{|\alpha|\leqq l} |||\partial_\tau^{\wedge\alpha} u(t)|||_{(\rho,1),\gamma}^2 + \gamma\int_0^t \sum_{|\alpha|\leqq l} |||\partial_\tau^{\wedge\alpha} u(t)|||_{(\rho,1),\gamma}^2 dt$$

$$\leqq C_l\Big\{\sum_{|\alpha|\leqq l} |||\partial_\tau^\wedge u(0)|||_{(\rho,1),\gamma}^2 + \gamma^{-1}\int_0^t \sum_{|\alpha|\leqq l} \|\rho_0^{-1}\partial_\tau^\wedge L^\wedge u(t)\|_\rho^2 dt\Big\} \quad (0<t<T)$$

for any $\gamma > \gamma_l$ and any $\{\partial_\tau^\alpha u \in \mathcal{H}_\rho^1,\ \rho_0^{-1}\partial_\tau^\alpha L^\wedge u \in H_\rho^0(I\times\Omega)\ (|\alpha| \leqq l)\}$.

Proof. From Lemma 3.2, we have for $|\alpha| \leqq l$

$$\begin{aligned}L^\wedge\partial_\tau^{\wedge\alpha} u &= \partial_\tau^{\wedge\alpha} L^\wedge u - P_\alpha(\partial_\tau^\wedge)u - \rho_0\sum_{j=1}^n Q_{\alpha j}(\partial_\tau^\wedge)\tilde{\partial}_j u \\ &= F_1 + F_2 + F_3 = F\,,\end{aligned}$$

where

$$\int_0^t \|\rho_0^{-1}F_1(t)\|_\rho^2 dt \leqq C\int_0^t \|\rho_0^{-1}\partial_\tau^{\wedge\alpha} L^\wedge u\|_\rho^2 dt\,,$$

$$\int_0^t \|\rho_0^{-1}F_3(t)\|_\rho^2 dt \leqq C\int_0^t \sum_{|\beta|\leqq l} |||\partial_\tau^{\wedge\beta} u(t)|||_{(\rho,1),\gamma}^2 dt\,.$$

Since

$$\begin{aligned}\int_0^t (F_2, \partial_t^\wedge\partial_\tau^{\wedge\alpha} u)_\rho dt &= -\int_0^t (P_\alpha(\partial_\tau^\wedge)u, \partial_t^\wedge\partial_\tau^{\wedge\alpha} u)_\rho dt \\ &= \{-(P_\alpha(\partial_\tau^\wedge)u(t), \partial_\tau^{\wedge\alpha} u(t))_\rho + (P_\alpha(\partial_\tau^\wedge)u(0), \partial_\tau^{\wedge\alpha} u(0))_\rho\} \\ &\quad -2\gamma\int_0^t (P_\alpha(\partial_\tau^\wedge)u, \partial_\tau^{\wedge\alpha} u)_\rho dt + \int_0^t (\partial_t^\wedge P_\alpha(\partial_\tau^\wedge)u, \partial_\tau^{\wedge\alpha} u)_\rho dt\,,\end{aligned}$$

we have

$$\Big|\int_0^t (F_2, \partial_t^\wedge\partial_\tau^{\wedge\alpha} u)_\rho dt\Big| \leqq C\gamma^{-1}\{E_{l+1}(t) + E_{l+1}(0)\} + C\int_0^t E_{l+1}(t)dt\,,$$

where

$$E_{l+1}(t)=\sum_{|\alpha|\leq l}|||\partial_{\hat{\tau}}^{\alpha}u)t)|||_{(\rho,1),\gamma}^2 .$$

Here we have

$$\left|\int_0^t(F,\ \partial_{\hat{t}}\partial_{\hat{\tau}}^{\alpha}u)_\rho dt\right|\leqq C\left\{\int_0^t\|\rho_0^{-1}\partial_{\hat{\tau}}L^{\wedge}u(t)\|_\rho^2 dt\right\}^{1/2}\left\{\int_0^t E_{l+1}(t)dt\right\}^{1/2}$$

$$+C\gamma^{-1}\{E_{l+1}(t)+E_{l+1}(0)]+C\int_0^t E_{l+1}(t)dt .$$

Considering

$$\sum_{|\alpha|\leq l}2\,\mathrm{Re}\,(L^{\wedge}(\partial_{\hat{\tau}}^{\alpha}u),\ \partial_{\hat{t}}(\partial_{\hat{\tau}}^{\alpha}u))_\rho$$

as in the proof of Lemma 3.1, we have

$$E_{l+1}(t)+\gamma\int_0^t E_{l+1}(t)dt\leqq C_l\left\{E_{l+1}(0)+\gamma^{-1}\int_0^t\sum_{|\alpha|\leq l}\|\rho_0^{-1}\partial_{\hat{\tau}}^{\alpha}L^{\wedge}u(t)\|_\rho^2 dt\right\}$$

for large γ. □

Lemma 3.4. *Let L be of Bessel type, then*

$$\tilde{\partial}_1^{k+2}=M_k(\partial_\rho)L+P_k(\partial_\tau)+\sum_{j=0}^n Q_{kj}(\partial_\tau)\tilde{\partial}_j \qquad (k=0,\ 1,\ \cdots).$$

where

$$M_k=\sum_{|\beta|\leq k}m_{k\beta}(t,\ x)\partial_\rho^\beta ,$$

$$P_k=\sum_{|\beta|\leq k}p_{k\beta}(t,\ x)\partial_\tau^\beta .$$

$$Q_{kj}=\sum_{|\beta|\leq k+1}q_{kj\beta}(t,\ x)\partial_\tau^\beta ,$$

where $m_{k\beta}$, $p_{k\beta}$, $q_{kj\beta}\in\mathcal{B}_\rho^\infty$.

Proof. Since

$$L\cong-c+\tilde{\partial}_1^2+2\rho_0\mathcal{L}_1(\tilde{\partial}_\tau)\tilde{\partial}_1+\mathcal{L}_2(\tilde{\partial}_\tau)+\rho_0\sum b_j\tilde{\partial}_j ,$$

we have

$$\begin{aligned}\tilde{\partial}_1^2&\cong L+c-\{2\rho_0\mathcal{L}_1(\tilde{\partial}_\tau)\tilde{\partial}_1+\mathcal{L}_2(\tilde{\partial}_\tau)+\rho_0\sum b_j\tilde{\partial}_j\},\\&=L+c+\{-2\rho_0{\sum}^* a_{j1}\rho_j\partial_j-\rho_0 b_1\}\tilde{\partial}_1+\sum_j{}^*\{\sum_k{}^* a_{jk}\rho_k\partial_k-\rho_0 b_j\}\tilde{\partial}_j\\&=L+c+\sum Q_{0j}(\partial_\tau)\tilde{\partial}_j .\end{aligned}$$

Let us assume that

$$\tilde{\partial}_1^{N+2}=M_N(\partial_\rho)L+P_N(\partial_\tau)+\sum_{j=0}^n Q_{Nj}(\partial_\tau)\tilde{\partial}_j ,$$

then

$$\begin{aligned}\tilde{\partial}_1^{N+3}=&\tilde{\partial}_1 M_N(\partial_\rho)L+P_N(\partial_\tau)\tilde{\partial}_1+P_N'(\partial_\tau)\\&+Q_{N1}(\partial_\tau)\tilde{\partial}_1^2+{\sum}^* Q_{Nj}(\partial_\tau)\tilde{\partial}_j\tilde{\partial}_1+\sum Q_{Nj}'(\partial_\tau)\tilde{\partial}_j\end{aligned}$$

$$=\tilde{\partial}_1 M_N(\partial_\rho)L+P_N(\partial_\tau)\tilde{\partial}_1+P'_N(\partial_\tau)$$

$$+Q_{N1}(\partial_\tau)\{L+c+\sum_j Q_{0j}(\partial_\tau)\tilde{\partial}_j\}+\Sigma^* Q_{Nj}(\partial_\tau)\tilde{\partial}_j\tilde{\partial}_1+\sum Q'_{Nj}(\partial_\tau)\tilde{\partial}_j$$

$$=\{\tilde{\partial}_1 M_N(\partial_\rho)+Q_{N1}(\partial_\tau)\}L+\{P'_N(\partial_\tau)+Q_{N1}(\partial_\tau)c\}$$

$$+\{P_N(\partial_\tau)+Q_{N1}(\partial_\tau)Q_{01}(\partial_\tau)+\Sigma^* Q_{Nj}(\partial_\tau)\tilde{\partial}_j+Q'_{N1}(\partial_\tau)\}\tilde{\partial}_1$$

$$+\Sigma^*\{Q_{N1}(\partial_\tau)Q_{0j}(\partial_\tau)+Q'_{Nj}(\partial_\tau)\}\tilde{\partial}_j,$$

where $P'_N=\tilde{\partial}_1 P_N-P_N\tilde{\partial}_1, \cdots$. □

From Lemma 3.3 and Lemma 3.4, we have

Proposition 3.5. *Let L be of Bessel type, then there exist γ_l (>0) and C_l (>0) such that*

$$|||u(t)|||^2_{(\rho,l+1),\gamma}+\gamma\int_0^t |||u(t)|||^2_{(\rho,l+1),\gamma}dt$$

$$\leqq C_l\left\{|||u(0)|||^2_{(\rho,l+1),\gamma}+\gamma^{-1}\int_0^t |||\rho_0^{-1}L^\wedge u(t)|||^2_{\rho,l,\gamma}dt\right\} \qquad (0<t<T)$$

for any $\gamma>\gamma_l$ and any $\{u\in\mathcal{H}^{l+1}_\rho,\ \rho_0^{-1}L^\wedge u\in H^l_\rho(I\times\Omega)\}$.

§4. Existence theorem of (C.P) for hyperbolic operators of Bessel type

To obtain existence theorem of (C.P) for L, we construct approximate solutions for approximate problems. Let

$$L_{(\varepsilon)}=L(t,\ x_1+\varepsilon,\ x';\ \partial_t,\ \partial_x), \qquad \rho_\varepsilon=\rho(x_1+\varepsilon), \cdots \quad (\varepsilon>0),$$

then we have

Lemma 4.1. *Let L be of Bessel type, then there exists γ_l (>0) and C_l (>0) such that*

$$|||u(t)|||^2_{(\rho_\varepsilon,l+1),\gamma}+\gamma\int_0^t |||u(t)|||^2_{(\rho_\varepsilon,l+1),\gamma}dt$$

$$\leqq C_l\left\{|||u(0)|||^2_{(\rho_\varepsilon,l+1),\gamma}+\gamma^{-1}\int_0^t |||\rho_{0\varepsilon}^{-1}L^\wedge_{(\varepsilon)}u(t)|||^2_{\rho_\varepsilon,l,\gamma}dt\right\}$$

$(0<t<T)$ *for any $\gamma>\gamma_l$, any $\{u\in\mathcal{H}^{l+1}_{\rho_\varepsilon},\ \rho_{0\varepsilon}^{-1}L^\wedge u\in H^l_{\rho_\varepsilon}(I\times\Omega),\ u|_{x_1=0}=0\}$* and any $0<\varepsilon<1$.

Proof. Lemma 4.1 with $l=0$ is proved in the same way as in Lemma 3.1. Lemma 4.1 with $l\geqq 1$ is proved in the same way as in Proposition 3.5, remarking that Lemma 3.2 and Lemma 3.4 are valid also when L, ρ, $p_{\alpha\beta}, \cdots$ are replaced by $L_{(\varepsilon)}$, ρ_ε, $p_{\alpha\beta\varepsilon}, \cdots$, where

$$p_{\alpha\beta\varepsilon}(t,\ x)=p_{\alpha\beta}(t,\ x_1+\varepsilon,\ x'), \cdots. \qquad \square$$

Remarking that $\rho\leqq\rho_\varepsilon$ and

$$(\rho\partial_1)^j=\sum_{k=0}^{j} c_{jk}\rho^k\partial_1^k \qquad (c_{jk}\in\mathscr{B}^\infty),$$

we have

Lemma 4.2. *Let* $s\geqq 1/2$, *then there exists* $C(>0)$ *such that*

$$\|\rho^s u\|_{\rho,l}\leqq C\|\rho_\varepsilon^s u\|_{\rho_\varepsilon,l}$$

for any $u\in H^l$ *and any* $0<\varepsilon<1$.

Proposition 4.3. *Let* L *be of Bessel type. Let*

$$\rho_0^{-1}f\in H_\rho^l(I\times\Omega), \qquad u_1\in\mathscr{H}_{\rho,\sigma}^l(\Omega), \qquad u_0\in\mathscr{H}_{\rho,\sigma}^{l+1}(\Omega),$$

then there exists a unique solution $u\in\mathscr{H}_{\rho,\sigma}^{l+1}(I\times\Omega)$ *of* (C. P).

Proof. Let

$$\rho_0^{-1}f\in H_\rho^l(I\times\Omega), \qquad u_1\in\mathscr{H}_{\rho,\sigma}^l(\Omega), \qquad u_0\in\mathscr{H}_{\rho,\sigma}^{l+1}(\Omega),$$

and set

$$f_\varepsilon=f(t,\ x_1+\varepsilon,\ x'), \qquad u_{j\varepsilon}=u_j(x_1+\varepsilon,\ x') \qquad (j=0,\ 1),$$

then there exists $u_\varepsilon\in H^{l+1}$ satisfying

$$\begin{cases} L_{(\varepsilon)}u_\varepsilon=f_\varepsilon\,, \\ u_\varepsilon|_{x_1=0}=0\,, \\ \partial_t^j u_\varepsilon|_{t=0}=u_{j\varepsilon} \qquad (j=0,\ 1). \end{cases}$$

Moreover, from Lemma 4.1,

$$\|u_\varepsilon\|_{\mathscr{H}_{\rho_\varepsilon,\sigma}^{l+1}(I\times\Omega)}\leqq C\{\|\rho_{0\varepsilon}^{-1}f_\varepsilon\|_{H_{\rho_\varepsilon}^l(I\times\Omega)}+\|u_{1\varepsilon}\|_{\mathscr{H}_{\rho_\varepsilon,\sigma}^l(\Omega)}+\|u_{0\varepsilon}\|_{\mathscr{H}_{\rho_\varepsilon,\sigma}^{l+1}(\Omega)}\}.$$

On the other hand, we have

$$\|\rho_{0\varepsilon}^{-1}f_\varepsilon\|_{H_{\rho_\varepsilon}^l(I\times\Omega)}=\|\rho_0^{-1}f\|_{H_\rho^l(I\times\Omega_\varepsilon)}\leqq\|\rho_0^{-1}f\|_{H_\rho^l(I\times\Omega)}$$

and so on. Let

$$s=\max(\sigma_0,\ \sigma_1,\ \cdots,\ \sigma_n) \qquad (\geqq 1),$$

then, remarking Lemma. 4.2, we have

$$\begin{aligned} c\{\sum_j \|\partial_j(\rho^s u_\varepsilon)\|_{H_\rho^l(I\times\Omega)}&+\|\rho^s u_\varepsilon\|_{H_\rho^l(I\times\Omega)}+\|\rho^s u_\varepsilon\|_{H_\rho^l(I\times\Omega)}\} \\ &\leqq\sum\|u_\varepsilon^{\sigma_j}\partial_j u_\varepsilon)\|_{H_{\rho_\varepsilon}^l(I\times\Omega)}+\|u_\varepsilon\|_{H_{\rho_\varepsilon}^l(I\times\Omega)} \\ &\leqq C\{\|\rho_0^{-1}f\|_{H_\rho^l(I\times\Omega)}+\|u_1\|_{\mathscr{H}_{\rho,\sigma}^l(\Omega)}+\|u_0\|_{\mathscr{H}_{\rho\,\sigma}^{l+1}(\Omega)}\}\equiv CK\,. \end{aligned}$$

Hence, there exists $\{u_{\varepsilon_k}\}$ such that

$$v_k=\rho^s u_{\varepsilon_k}\longrightarrow v \qquad \text{in } H_\rho^l,$$

where $u=\rho^{-s}v$ satisfies (C. P) and

$$\|\partial v\|_{H^l_\rho}+\|v\|_{H^l_\rho}\leqq K.$$

Moreover, let $\delta>0$ and $|\alpha|\leqq l$, then we have

$$\|\rho_\delta^{-s+\sigma j}\partial_\tau^\alpha\partial_j v\|+\|\rho_\delta^{-s}\partial_\tau^\alpha v\|_\rho$$
$$\leqq\varliminf_k\{\|\rho_\delta^{-s+\sigma j}\partial_\tau^\alpha\partial_j v_k\|_\rho+\|\rho_\delta^{-s}\partial_\tau^\alpha v_k\|_\rho\}$$
$$\leqq\varliminf_k\{\|\rho_{\varepsilon_k}^{-s+\sigma j}\partial_\tau^\alpha\partial_j v_k\|_\rho+\|\rho_{\varepsilon_k}^{-s}\partial_\tau^\alpha v_k\|_\rho\}\leqq CK,$$

because

$$\|\rho_{\varepsilon_k}^{-s+\sigma j}\partial_\tau^\alpha\partial_j v_k\|_\rho+\|\rho_{\varepsilon_k}^{-s}\partial_\tau^\alpha v_k\|_\rho$$
$$\leqq C_1\{\|\rho_{\varepsilon_k}^{-s+\sigma j}\rho^{s-1/2}\partial_\tau^\alpha\partial_j u_{\varepsilon_k}\|+\|\rho_{\varepsilon_k}^{-s}\rho^{s-1/2}\partial_\tau^\alpha u_{\varepsilon_k}\|\}$$
$$\leqq C_2\{\|\rho_{\varepsilon_k}^{\sigma j-1/2}\partial_\tau^\alpha\partial_j u_{\varepsilon_k}\|+\|\rho_{\varepsilon_k}^{-1/2}\partial_\tau^\alpha u_{\varepsilon_k}\|\}$$
$$=C_2\{\|\rho_{\varepsilon_k}^{\sigma j}\partial_\tau^\alpha\partial_j u_{\varepsilon_k}\|_{\rho_{\varepsilon_k}}+\|\partial_\tau^\alpha u_\varepsilon\|_{\rho_{\varepsilon_k}}\}\leqq C_3 K.$$

Since

$$|\rho_\delta^{-s+\sigma j}\partial_\tau^\alpha\partial_j v|\nearrow|\rho^{-s+\sigma j}\partial_\tau^\alpha\partial_j v|\qquad\text{as }\delta\searrow 0,$$
$$|\rho_\delta^{-1}\partial_\tau^\alpha v|\nearrow|\rho^{-1}\partial_\tau^\alpha v|\qquad\text{as }\delta\searrow 0,$$

we have

$$\rho^{-s+\sigma j}\partial_\tau^\alpha\partial_j v,\qquad \rho^{-s}\partial_\tau^\alpha v\in H^0_\rho,$$
$$\|\rho^{-s+\sigma j}\partial_\tau^\alpha\partial_j v\|_\rho+\|\rho^{-s}\partial_\tau^\alpha v\|_\rho\leqq K.$$

Here we have $u=\rho^{-s}v\in\mathcal{H}^{l+1}_{\rho,\sigma}$. □

Owing to Lemma 2.2, Theorem 1 follows from Proposition 4.3 as its corollary.

§5. Problems (C.P) or (M.P) for L under Ass. I$-\mu$ or Ass. II$-\mu$

Lemma 5.1.

$$u\in H^l_0\Longleftrightarrow\rho^{-l+1/2}u\in H^l_\rho.$$

Proof. $\Rightarrow$) Let $u\in H^l_0$ and let $|\alpha|\leqq l$. Since

$$\partial^\alpha u(x)=\{(l-|\alpha|-1)!\}^{-1}x_1^{l-|\alpha|}\int_0^1(1-\theta)^{l-|\alpha|-1}(\partial_1^{l-|\alpha|}\partial^\alpha u)(x_1\theta,x')d\theta,$$

we have

$$\|x_1^{-l+|\alpha|}\partial^\alpha u\|\leqq\int_0^1\|(\partial_1^{l-|\alpha|}\partial^\alpha u)(x_1\theta,x')\|d\theta,$$

where $\|\ \|=\|\ \|_{L^2((0,1)\times R^{n-1})}$. Hence we have

$$\rho^{-l+|\alpha|+1/2}\partial^\alpha u(x)\in H^0_\rho.$$

$\Leftarrow$) Let $\rho^{-l+1/2}u\in H^l_\rho$. Since it is easy to see that $u\in H^l$, let us show that $\partial_1^k u|_{x_1=0}=0$ $(k\leqq l-1)$. We remark that $(\rho\partial_1)^k v|_{x_1=0}$ $(k\leqq l-1)$ if $v\in H^l_\rho$. Therefore it holds that $\partial_1^k v|_{x_1=0}$ if $\rho^{-k}v\in H^l_\rho$ $(k\leqq l-1)$, because

$$\partial_1^k v=\rho^{-1}(\rho\partial_1)\cdots\rho^{-1}(\rho\partial_1)\cong(\rho\partial_1+1)(\rho\partial_1+2)\cdots(\rho\partial_1+k)(\rho^{-k}v),$$

Since $\rho^{-l+1/2}u\in H_\rho^l$, it holds that

$$\partial_1^k u|_{x_1=0}\qquad (k\leqq l-1). \qquad \square$$

Let us assume that L satisfy Ass. I$-\mu$ or Ass. II$-\mu$ in §5, then

$$L'\cong(\rho\partial_1+\mu+1)\partial_1+\Psi(t,\ x;\partial_\tau,\ \rho\partial_1)$$

is a Fuchsian, therefore regularity theorem holds [5]. Here we see it in our situation.

Lemma 5.2. *Let* $\beta<\mu$. *Assume that*

$$\rho^{\beta+\varepsilon}u\in H_\rho^l,\qquad \rho^{\beta+\varepsilon}f\in H_\rho^{l-2}\quad (\textit{any}\ \varepsilon>0),$$

and $L'u=f$, *then*

$$\rho^{\beta+\varepsilon}\partial_1 u\in H_\rho^{l-2}\qquad (\textit{any}\ \varepsilon>0).$$

Proof. Let $\chi(x_1)\in\mathcal{B}^\infty(R_+)$ such that $\chi(x_1)=1$ near $x_1=0$ and $\chi(x_1)=0$ for $x_1\geqq 1$. Multiplying both sides of $L'u=f$ by χ, we have

$$(x_1\partial_1+\mu+1)\partial_1 v=g,$$

where $v=\chi u$ and

$$x_1^{\beta+\varepsilon}v\in H_\rho^l,\qquad x_1^{\beta+\varepsilon}g\in H_\rho^{l-2}.$$

Multiplying both sides by x_1^μ, we have

$$\partial_1(x_1^{\mu+1}\partial_1 v)=x_1^\mu g.$$

Since

$$x_1^{\mu+1}\partial_1 v=x_1^{\mu-\beta-\varepsilon}(x_1^{\beta+1+\varepsilon}\partial_1 v)=x_1^{\mu-\beta-\varepsilon}w,\qquad w\in H_\rho^{l-1},$$

we have from Lemma 5.1

$$x_1^{\mu+1}\partial_1 v|_{x_1=0}=0,$$

taking ε small enough to satisfy $\mu-\beta-\varepsilon>0$. Heace we have

$$x_1^{\mu+1}\partial_1 v=\int_0^{x_1}x_1^\mu g\mathrm{d}x_1$$

$$=\int_0^{x_1}(x_1^{\mu+1-\beta-\varepsilon})(x_1^{\beta+\varepsilon}g)x_1^{-1}\mathrm{d}x_1,$$

therefore

$$|x_1^{\mu+1}\partial_1 v|\leqq Cx_1^{\mu+1-\beta-\varepsilon}\Big(\int|x_1^{\beta+\varepsilon}g|^2x_1^{-1}\mathrm{d}x_1)^{1/2},$$

that is,

$$|x_1^{\beta+\varepsilon'}\partial_1 v|\leqq Cx_1^{\varepsilon'-\varepsilon}\Big(\int|x_1^{\beta+\varepsilon}g|^2x_1^{-1}\mathrm{d}x_1\Big)^{1/2}.$$

Taking $0<\varepsilon<\varepsilon'$ for any $\varepsilon'>0$, we have

$$x_1^{\beta+\varepsilon'}\partial_1 v\in H_\rho^0\qquad (\text{any}\ \varepsilon'>0).$$

In the same way, we have

$$x_1^{\beta+\varepsilon'}\partial_1 v \in H_\rho^{l-2} \qquad (\text{any } \varepsilon'>0). \qquad \square$$

Lemma 5.3. *Let* $0<\beta<\mu$ *and let* $l \geqq [\beta]+1$. *Assume that*

$$\rho^\beta u \in H_\rho^{2l}(I\times\Omega), \qquad f\in H^{2l}(I\times\Omega),$$

and $L'u=f$, *then*

$$u \in H^{l-[\beta]-1}(I\times\Omega).$$

Proof. Multiplying both sides of $L'u=f$ by χ, defined in Lemma 5.2, we have

$$(x_1\partial_1+\mu+1)\partial_1 v+\Psi(t, x; \partial_\tau, x_1\partial_1)v=g,$$

where $v=\chi u$,

$$x_1^\beta v\in H_\rho^{2l}, \qquad g\in H^{2l-1}.$$

We have only to prove $v\in H^{l-[\beta]-1}(I\times\Omega)$.

i) From Lemma 5.2, we have

$$x_1^{\beta+\varepsilon}\partial_1 v\in H_\beta^{2(l-1)} \qquad (\text{any } \varepsilon>0).$$

Since

$$|\partial_\tau^\alpha v| = \left|\int_1^{x_1}\partial_1\partial_\tau^\alpha v \mathrm{d}x_1\right|$$

$$\leqq C x_1^{-\beta+1}\Big(\int_0^1 x_1^{2\beta}|\partial_1\partial_\tau^\alpha v|^2 x_1^{-1}\mathrm{d}x_1\Big)^{1/2}$$

$(|\alpha|\leqq 2(l-1))$ if $\beta>1$, we have

$$x_1^{\beta-1+\varepsilon}\partial_\tau^\alpha v\in H_\rho^0 \qquad (\text{any } \varepsilon>0),$$

therefore, from Lemma 5.2,

$$x_1^{\beta-1-\varepsilon}v\in H_\rho^{2(l-1)}, \qquad x_1^{\beta-1+\varepsilon}\partial_1 v\in H_\rho^{2(l-2)}.$$

In this way, step by step, we have

$$x_1^{\beta-[\beta]+\varepsilon}v\in H_\rho^{2(l-[\beta])},$$

$$x_1^{\beta-[\beta]+\varepsilon}\partial_1 v\in H_\rho^{2(l-[\beta]-1)}.$$

ii) We have

$$|\partial_\tau^\alpha v| = \left|\int_1^{x_1}\partial_1\partial_\tau^\alpha v \mathrm{d}x_1\right|$$

$$\leqq C\Big(\int_0^1 x_1^{2(\beta-[\beta]+\varepsilon)}|\partial_1\partial_\tau^\alpha v|^2 x_1^{-1}\mathrm{d}x_1\Big)^{1/2}$$

$(|\alpha|\leqq 2(l-[\beta]-1))$, taking ε small enough to satisfy $-2(\beta-[\beta]+\varepsilon)>-2$, therefore

$$x_1^\varepsilon\partial_\tau^\alpha v\in H_\rho^0 \qquad (\text{any } \varepsilon>0).$$

Hence we have

$$x_1^\varepsilon v\in H_\rho^{2(l-[\beta]-1)}, \qquad x_1^{\beta-[\beta]+\varepsilon}\partial_1 v\in H_\rho^{2(l-[\beta]-2)}.$$

iii) Multiplying both sides of $L'v=g$ by ∂_1, we have

$$(x_1\partial_1+\mu+2)\partial_1^2 v$$
$$=-\Psi(t,\ x\ ;\ \partial_\tau,\ x_1,\ \partial_1+1)\partial_1 v-\Psi_{x_1}(t,\ x\ ;\ \partial_\tau,\ x_1\partial_1)v+\partial_1 g=g_2\ ,$$

where

$$x_1^\varepsilon g_2\in H_\rho^{2(l-[\beta]-3)}\ ,$$

hence we have

$$x_1^\varepsilon\partial_1^2 v\in H_\rho^{2(l-[\beta]-3)}\ .$$

In the same way, we have

$$x_1^\varepsilon\partial_1^k v\in H_\rho^{2(l-[\beta]-1-k)}\qquad (k\leqq l-[\beta]-1),$$

therefore

$$x_1^\varepsilon\partial_\tau^\alpha\partial_1^k v\in H_\rho^0\qquad (|\alpha|+k\leqq l-[\beta]-1),$$

therefore $v\in H^{l-[\beta]-1}$. □

Set $\beta=\mu/2$ in Lemma 5.3, then we have

Theorem 2. *Let L satisfy Ass. $I-\mu$ or Ass. $II-\mu$ with $\mu>0$. Let $l\geqq[\mu/2]+1$ and $L'u=f$, where*

$$\rho^{\mu/2}u\in H_\rho^{2l}(I\times\Omega),\qquad f\in H^{2l}(I\times\Omega),$$

then $u\in H^{l-[\mu/2]-1}(I\times\Omega)$.

Owing to Theorem 2, Theorem (C) follows from Theorem 1.

Theorem 3. *Let L satisfy Ass. $I-\mu$ or Ass. $II-\mu$ with $\mu<0$. For any N, there exists a solution of* (C.P) *satisfying $\rho^\mu u\in H^N$, if*

$$f\in H_0^l(I\times\Omega),\qquad u_0\in H_0^l(\Omega),\qquad u_1\in H_0^l(\Omega)$$

for some l.

Proof. Owing to Lemma 1.1, it follows from Theorem (C) that there exists a solution $v\in H^N$ of the problem:

$$\begin{cases} L_\mu v=\rho^\mu f\ , \\ v|_{t=0}=\rho^\mu u_0\ ,\qquad \partial_t v|_{t=0}=\rho^\mu u_1\ , \end{cases}$$

because

$$\rho^\mu(f,\ u_0,\ u_1)\in H^{l'}$$

from Lemma 5.1. Set $u=\rho^{-\mu}v$, then u satisfies

$$\begin{cases} Lu=f\ , \\ u|_{t=0}=u_0\ ,\qquad \partial_t u|_{t=0}=u_1\ . \end{cases}\qquad \square$$

Remember Lemma 1.2, then Theorem (M) and Theorem (M)′ follow from Theorem 3.

§6. Examples

Let us consider examples in one-dimensional x-space, whose solutions can be constructed exactly. First, let us consider

$$(\mathrm{P})\qquad \begin{cases} u_{tt}=xu_{xx}+1/2u_x & (t>0,\ x>0), \\ u|_{t=0}=u_0, \quad u_t|_{t=0}=u_1 & (x>0), \end{cases}$$

where u_0, $u_1\in C^N(R_+)$, and u_0, $u_1=O(x^N)$ as $x\to+0$ (N: large).

Set

$$\phi_\pm(x)=u_1(x)\pm\sqrt{x}\,u_0'(x),$$

then we have

$$\phi_\pm\in C^{N-1}(R_+),\qquad \phi_\pm^{(k)}=O(x^{N-k-1/2})\qquad \text{as } x\to 0\ (k\leqq N-1).$$

Moreover, set

$$\Phi_\pm(t,x)=\begin{cases} \{\phi_+((t/2+\sqrt{x})^2)\pm\phi_+((t/2-\sqrt{x})^2)\}/2 & \text{if } t/2-\sqrt{x}>0, \\ \{\phi_+((t/2+\sqrt{x})^2)+\phi_-((t/2-\sqrt{x})^2)\}/2 & \text{if } t/2-\sqrt{x}\leqq 0, \end{cases}$$

and $\Phi_-(t,x)=\sqrt{x}\Phi_0(t,x)$.

Lemma 6.1.

$$\Phi_0,\ \Phi_+\in C^M(\overline{R_+\times R_+})\qquad (M\leqq(N-2)/3).$$

Proof. It is easy to see

$$\Phi_\pm\in C^{N-1}(\bar{R}_+\times R_+).$$

In the following, we shall see the regularity near $x=0$.

i) Regularity of Φ_+. Let $t/2-\sqrt{x}>0$, then

$$\begin{aligned}
\Phi_+(t,x)&=\{\phi_+((t/2+\sqrt{x})^2)+\phi_+((t/2-\sqrt{x})^2)\}/2 \\
&=\{\phi_+(t^2/4+x)+\phi_+'(t^2/4+x)(t\sqrt{x})+\cdots \\
&\quad\cdots+\phi_+^{(2M-1)}(t^2/4+x)(t\sqrt{x})^{2M-1}/(2M-1)! \\
&\quad+(t\sqrt{x})^{2M}/(2M-1)!\int_0^1(1-\theta)^{2M-1}\phi_+^{(2M)}(t^2/4+x+t\sqrt{x}\theta)d\theta\}/2 \\
&\quad+\{\phi_+(t^2/4+x)+\phi_+'(t^2/4+x)(-t\sqrt{x})+\cdots \\
&\quad\cdots+\phi_+^{(2M-1)}(t^2/4+x)(-t\sqrt{x})^{2M-1}/(2M-1)! \\
&\quad+(-t\sqrt{x})^{2M}/(2M-1)!\int_0^1(1-\theta)^{2M-1}\phi_+^{(2M)}(t^2/4+x-t\sqrt{x}\theta)\theta)\}/2 \\
&=\phi_+(t^2/4+x)+\phi_+''(t^2/4+x)t^2x/2!+\cdots \\
&\quad\cdots+\phi_+^{(2M-2)}(t^2/4+x)t^{2M-2}x^{M-1}/(2M-2)! \\
&\quad+t^{2M}x^M/\{2(2M-1)!\}\int_0^1(1-\theta)^{2M-1}\{\phi_+^{(2M)}(t^2/4+x+t\sqrt{x}\theta) \\
&\qquad\qquad+\phi_+^{(2M)}(t^2/4+x-t\sqrt{x}\theta)\}d\theta\,.
\end{aligned}$$

Here we can see that Φ_+ is differentiable up to order M at $(t, 0)$ $(t>0)$ and

$$\partial_t^j\partial_x^k\Phi(t, x) \longrightarrow 0 \quad \text{as } (t, x)\to(0, 0) \text{ in } \{t/2-\sqrt{x}>0\} \quad (j+k\leqq M),$$

if $3M\leqq N-1$.

Let $t/2-\sqrt{x}\leqq 0$ and $M\leqq N-1$, then we have

$$\sum_{j+k\leqq M} |\partial_t^j\partial_x^k\phi_\pm((t/2\pm\sqrt{x})^2)|$$
$$\leqq C\sum_{h\leqq M} |\phi_\pm^{(h)}((t/2\pm\sqrt{x})^2)|\, x^{-M+h}\leqq C x^{N-1/2-M},$$

therefore

$$\partial_t^j\partial_x^k\Phi_+(t, x) \longrightarrow 0 \quad \text{as } (t, x)\to(0, 0) \text{ in } \{t/2-\sqrt{x}\leqq 0\} \quad (j+k\leqq N-1).$$

ii) Regularity of Φ_0. Let $t/2-\sqrt{x}>0$, then

$$\begin{aligned}\Phi_-(t, x)&=\{\phi_+((t/2+\sqrt{x})^2)-\phi_+((t/2-\sqrt{x})^2)\}/2\\
&=\{\phi_+(t^2/4+x)+\phi_+'(t^2/4+x)(t\sqrt{x})+\cdots\\
&\qquad\cdots+\phi_+^{(2M)}(t^2/4+x)(t\sqrt{x})^{2M}/(2M)!\\
&\qquad+(t\sqrt{x})^{2M+1}/(2M)!\int_0^1(1-\theta)^{2M}\phi_+^{(2M+1)}(t^2/4+x+t\sqrt{x}\,\theta)d\theta\}/2\\
&\qquad-\{\phi_+(t^2/4+x)+\phi_+'(t^2/4+x)(-t\sqrt{x})+\cdots\\
&\qquad\cdots+\phi_+^{(2M)}(t^2/4+x)(-t\sqrt{x})^{2M}/(2M)!\\
&\qquad+(-t\sqrt{x})^{2M+1}/(2M)!\int_0^1(1-\theta)^{2M}\phi_+^{(2M+1)}(t^2/4+x-t\sqrt{x}\,\theta)d\theta\}/2\\
&=t\sqrt{x}\,\{\phi_+'(t^2/4+x)+\phi_+'''(t^2/4+x)t^2x/3!+\cdots\\
&\qquad\cdots+\phi_+^{(2M-1)}(t^2/4+x)t^{2M-2}x^{M-1}/(2M-1)!\\
&\qquad+t^{2M}x^M/\{2(2M)!\}\int_0^1(1-\theta)^{2M}[\phi_+^{(2M+1)}(t^2/4+x+t\sqrt{x}\,\theta)\\
&\qquad\qquad-\phi_+^{(2M+1)}(t^2/4+x-t\sqrt{x}\,\theta)]d\theta\}\\
&=\sqrt{x}\,\Phi_0(t, x).\end{aligned}$$

Here we can see that Φ_0 is differentiable up to order M at $(t, 0)$ $(t>0)$ and

$$\partial_t^j\partial_x^k\Phi(t, x) \longrightarrow 0 \quad \text{as } (t, x)\to(0, 0) \text{ in } \{t/2-\sqrt{x}>0\} \quad (j+k\leqq M),$$

if $3M\leqq N-2$.

Let $t/2-\sqrt{x}\leqq 0$ and $M\leqq N-1$, then we have

$$\sum_{j+k\leqq M} |\partial_t^j\partial_x^k\phi_\pm((t/2\pm\sqrt{x})^2)\sqrt{x}\,|$$
$$\leqq C\sum |\phi_\pm^{(h}((t/2\pm\sqrt{x})^2)|\, x^{-M+h-1/2}\leqq C x^{N-1-M},$$

therefore

$$\partial_t^j\partial_x^k\Phi_0(t, x) \longrightarrow 0 \quad \text{as } (t, x)\to(0, 0) \text{ in } \{t/2-\sqrt{x}\leqq 0\} \quad (j+k\leqq N-2). \qquad \square$$

Now, let us define

$$u_\pm(t, x)=u_0(x)+\int_0^t \Phi_\pm(t, x)dt,$$

then we have

Lemma 6.2. i) *u_- is a solution of* (P) *satisfying*

$$(*)\qquad \begin{cases} u\in C^2(\bar{R}_+\times R_+), \\ u, u_t\in C^0(\overline{R_+\times R_+}), \\ u|_{x=0}=0. \end{cases}$$

Conversely, let u be a solution of (P) *satisgying* (*),*then* $u=u_-$.

ii) *u_+ is a solution of* (P) *satisfying*

$$(**)\qquad u\in C^2(\overline{R_+\times R_+}).$$

Conversely, let u be a solution of (P) *satısfyinp* (**), *then* $u=u_+$.

Proof. Let u satisfy

$$u_{tt}=xu_{xx}+1/2\, u_x,$$

that is,

$$(\partial_t-\sqrt{x}\partial_x)(\partial_t+\sqrt{x}\partial_x)u=0,$$

then we have

$$(\partial_t\pm\sqrt{x}\partial_x)u=\text{const.}\quad \text{on } t\pm2\sqrt{x}=\text{const.}.$$

Since

$$(\partial_t\pm\sqrt{x}\partial_x)u|_{t=0}=\phi_\pm,$$

we have

$$(\partial_t\pm\sqrt{x}\partial_x)u=\phi_\pm(\xi)\quad \text{on } t\pm2\sqrt{x}=\pm2\sqrt{\xi}.$$

Here we remark that

$$(\partial_t+\sqrt{x}\partial_x)u|_{(t,x)=(2\sqrt{\xi},0)}=\phi_+(\xi).$$

In case of (i), since

$$\partial_t u \longrightarrow 0\quad \text{as } (t, x)\to(2\sqrt{\xi}, 0),$$

we have

$$\sqrt{x}\partial_x u \longrightarrow \phi_+(\xi)\quad \text{as } (t, x)\to(2\sqrt{\xi}, 0).$$

On the other hand, since

$$(\partial_t-\sqrt{x}\partial_x)u=\text{const.}\quad \text{on } t-2\sqrt{x}=2\sqrt{\xi},$$

we have

$$(\partial_t-\sqrt{x}\partial_x)u=-\phi_+(\xi)\quad \text{on } t-2\sqrt{x}=2\sqrt{\xi}.$$

Here we have

$$(\partial_t\pm\sqrt{x}\partial_x)u=\pm\phi_+((t/2\pm\sqrt{x})^2)\quad (t>2\sqrt{x}).$$

therefore we have

$$\partial_t u(t, x)=\{\phi_+((t/2+\sqrt{x})^2)-\phi_+((t/2-\sqrt{x})^2)\}/2,$$

$$\sqrt{x}\partial_x u(t, x)=\{\phi_+((t/2+\sqrt{x})^2)+\phi_+((t/2-\sqrt{x})^2)\}/2$$

for $t>2\sqrt{x}$.

In case of (ii), since

$$\sqrt{x}\partial_x u \longrightarrow 0 \quad \text{as } (t, x)\to(2\sqrt{\xi}, 0),$$

we have

$$\partial_t u \longrightarrow \phi_+(\xi) \quad \text{as } (t, x)\to(2\sqrt{\xi}, 0).$$

On the other hand, since

$$(\partial_t-\sqrt{x}\partial_x)u=\text{const.} \quad \text{on } t-2\sqrt{x}=2\sqrt{\xi},$$

we have

$$(\partial_t-\sqrt{x}\partial_x)u=\phi_+(\xi) \quad \text{on } t-2\sqrt{x}=2\sqrt{\xi}.$$

Here we have

$$(\partial_t\pm\sqrt{x}\partial_x)u=\pm\phi_+((t/2\pm\sqrt{x})^2) \quad (t>2\sqrt{x}),$$

therefore we have

$$\partial_t u(t, x)=\{\phi_+((t/2+\sqrt{x})^2)+\phi_+((t/2-\sqrt{x})^2)\}/2,$$

$$\sqrt{x}\partial_x u(t, x)=\{\phi_+((t/2+\sqrt{x})^2)-\phi_+((t/2-\sqrt{x})^2)\}/2$$

for $t>2\sqrt{x}$.

In both cases, we have

$$(\partial_t\pm\sqrt{x}\partial_x)u=\pm\phi_\pm((t/2\pm\sqrt{x})^2) \quad (t\leqq 2\sqrt{x}),$$

therefore we have

$$\partial_t u(t, x)=\{\phi_+((t/2+\sqrt{x})^2)+\phi_-((t/2-\sqrt{x})^2)\}/2,$$

$$\sqrt{x}\partial_x u(t, x)=\{\phi_+((t/2+\sqrt{x})^2)-\phi_-((t/2-\sqrt{x})^2)\}/2$$

for $t\leqq 2\sqrt{x}$. Here we have

$$u=u_\pm=u_0(x)+\int_0^t \Phi_\pm(t, x)\mathrm{d}t. \qquad \square$$

Since $\Phi_\pm$ is defined by initial data $\{u_0, u_1\}$, we also use the notations:

$$\Phi_\pm(t, x)=\Phi_\pm(t, x; u_0, u_1).$$

Next, we consider

$$(\mathrm{P})_{\mu=k-1/2} \qquad \begin{cases} u_{tt}=xu_{xx}+(k+1/2)u_x & (t>0, x>0), \\ u|_{t=0}=u_0, \quad u_t|_{t=0}=u_0 & (x>0), \end{cases}$$

where $u_0, u_1\in C^N(R_+)$, $u_0, u_1=O(x^N)$ as $x\to+0$ (N: large), and k is a positive integer. Let us define

$$\partial_x^{-1}u=\int_0^x u(x)\mathrm{d}x, \qquad \partial_x^{-k}u=(\partial_x^{-1})^k u,$$

and

$$U_{k-1/2}^\pm=u_0(x)+\int_0^t \partial_x^k\Phi_\pm(t, x; \partial_x^{-k}u_0, \partial_x^{-k}u_1)\mathrm{d}t,$$

then we have

Proposition 6.3. *$U^{\pm}_{k-1/2}$ are solutions of* $(\mathrm{P})_{\mu=k-1/2}$, *and*

$$U^{+}_{k-1/2}\in C^2(\overline{R_+\times R_+}),$$

$$U^{-}_{k-1/2}\in C^2(\bar{R}_+\times R_+), \qquad U^{-}_{k-1/2}=O(x^{1/2-k}) \quad \textit{as } x\to 0.$$

Proof. Let us consider

$$(\mathrm{P})_{\mu=-1/2} \qquad \begin{cases} v_{tt}=xv_{xx}+1/2v_x & (t>0,\ x>0), \\ v|_{t=0}=v_0, \quad v_t|_{t=0}=v_1 & (x>0), \end{cases}$$

where $v_0=\partial_x^{-k}u_0$, $v_1=\partial_x^{-k}u_1$, then there exist solutions:

$$v=v_{\pm}=v_0(x)+\int_0^t \Phi_{\pm}(t,\ x\,;\ v_0,\ v_1)\mathrm{d}t\,.$$

It is easy to see that $\partial_x^k v_{\pm}$ satisfies $(\mathrm{P})_{\mu=k-1/2}$. □

Finally, we consider

$$(\mathrm{P})_{\mu=-k+1/2} \qquad \begin{cases} u_{tt}=xu_{xx}+(-k+3/2)u_x & (t>0,\ x>0), \\ u|_{t=0}=u_0, \quad u_t|_{t=0}=u_1 & (x>0), \end{cases}$$

where $u_0,\ u_1\in C^N(R_+)$, $u_0,\ u_1=O(x^N)$ as $x\to+0$ (N: large), and k is a positive integer. Let us define

$$U^{\pm}_{-k+1/2}=u_0(x)+\int_0^t x^{k-1/2}\partial_x^k\Phi_{\pm}(t,\ x\,;\ \partial_x^{-k}(x^{-k+1/2}u_0),\ \partial_x^{-k}(x^{-k+1/2}u_1))\mathrm{d}t\ ,$$

then we have

Proposition 6.4. *$U^{\pm}_{-k+1/2}$ are solutions of* $(\mathrm{P})_{\mu=-k+1/2}$, *and*

$$U^{+}_{-k+1/2}\in C^2(\bar{R}_+\times R_+), \qquad U^{+}_{-k+1/2}=O(x^{k-1/2}) \quad \textit{as } x\to 0.$$

$$U^{-}_{-k+1/2}\in C^2(\overline{R_+\times R_+}).$$

Proof. Let us consider

$$(\mathrm{P})_{\mu=k-1/2} \qquad \begin{cases} w_{tt}=xw_{xx}+(k+1/2)w_x & (t>0,\ x>0), \\ w|_{t=0}=w_0, \quad w_t|_{t=0}=w_1 & (x>0), \end{cases}$$

where

$$w_0=x^{-k+1/2}u_0, \qquad w_1=x^{-k+1/2}u_1,$$

then there exist solutions of $(\mathrm{P})_{\mu=k-1/2}$:

$$w=w_{\pm}=w_0(x)+\int_0^t \partial_x^k\Phi_{\pm}(t,\ x\,;\ \partial_x^{-k}w_0,\ \partial_x^{-k}w_1)\mathrm{d}t\,.$$

It is easy to see that $x^{k-1/2}w_{\pm}$ satisfies $(\mathrm{P})_{\mu=-k+1/2}$. Moreover, since

$$u_-=u_0(x)+x^{k-1/2}\partial_x^k\left\{\sqrt{x}\int_0^t \Phi_0(t,\ x)\mathrm{d}t\right\}$$

$$=u_0(x)+x^{k-1/2}\sum_{j=0}^{k} c_{kj}x^{1/2-j}\int_0^t \partial_x^{k-j}\Phi_0(t,\ x)\mathrm{d}t\ ,$$

u_- is smooth up to the boundary. □

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