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Loop groups and their actions on the corresponding completed affine Lie algebras

By

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Introduction

In his paper [2], Garland studied a certain group of loops in a connected, simply connected complex simple Lie group G, and its 1-dimensional central extensions, together with their Lie algebras. The class to which the loops belong, corresponds to the algebra L of formal Laurent series. For instance, the Lie algebra of the loop group is obtained as the coefficient extension of the Lie algebra $\mathfrak{g}$ of G by L.

To construct and analyze the groups, he utilized the well-known results about the structure of the original group G (see, for example, [8]), by regarding the loop groups as the group of L rational points of the Chevalley group scheme over $\mathbf{Z}$ corresponding to G.

In the present paper, we shall consider the group $\tilde{G}_k$ of C^k-loops in G and its action on the corresponding completed affine Lie algebra—a 1-dimensional central extension of the Lie algebra $\tilde{\mathfrak{g}}_k$ of $\tilde{G}_k$—and we follow Garland's method for our problems. We first prepare some basic results about general loop spaces, and then go towards our main purpose. Thus, this paper is divided into the respective two parts.

In the first part (§1 and §2), we analyze the structure of the space of C^k-loops in a finite-dimensional manifold M ($k = 0, 1, 2, \ldots, \infty$).

Assume that M is a vector space. Then the loop space, denoted by $M(L_k)$, is obtained by a coefficient extension, that is, $M(L_k)$ is regarded as the tensor product $M \otimes_{\mathbf{R}} L_{k,\mathbf{R}}$, where $L_k := C^k(S^1)$ and $L_{k,\mathbf{R}}$ is the real form of L_k consisting of all the real valued functions. So it is identified with the Banach (or Fréchet if $k = \infty$) space $(L_{k,\mathbf{R}})^{\dim M}$. Thus, $M(L_k)$ is a Banach (or Fréchet) manifold with the tangent space $(L_{k,\mathbf{R}})^{\dim M}$.

It is natural to expect that this fact is generalized to the case of a manifold in general. And, if M is a Lie group, we get an affirmative result as the first main result in this paper, that is, we can provide $M(L_k)$ with a canonical Lie group structure such that its tangent space is $(L_{k,\mathbf{R}})^{\dim M}$.

To show this, we first introduce certain classes of functions on an open

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subset of loop spaces $(L_k)^m$ with values in other loop spaces $(L_k)^{m'}$, in analogy with the usual differentiable functions on finite-dimensional spaces.

Locally, they are defined as limits of polynomials (and so differentiable in the sense of Fréchet) similarly as in the case of differentiable functions on finite-dimensional spaces, but the coefficients of the polynomials are taken from L_k. In other words, we extend the coefficients of the differentiable functions, from $\mathbf{C}$ to L_k.

We can prove that for any open subset V in $\mathbf{C}^m$, $V(L_k)$ is open in the topological vector space $(L_k)^m$, and that every smooth function f on V is extended to a function $\Psi(f)$ on $V(L_k)$ of the above class.

And then, we give a canonical topology to the loop space $M(L_k)$ for an arbitrary manifold M, with respect to which the connected components are given exactly by homotopy classes of the manifold. Further, if M is a Lie group, $M(L_k)$ becomes a topological group by pointwise products, with this topology.

Let M be any Lie group. Take any sufficiently small coordinate neighbourhood (V, ϕ) of 1 in M. Then, thanks to the above topology and the coefficient extension of functions, we get the pair $(V(L_k), \Psi(\phi))$ of the neighbourhood $V(L_k)$ of 1 in $M(L_k)$ and the homeomorphism $\Psi(\phi)$ of $V(L_k)$ onto an open subset in $(L_k)^{\dim M}$. This system satisfies the axioms for a coordinate neighbourhood of 1 in a (probably infinite-dimensional) Lie group. Thus, we arrive at the goal of the first part.

In the second part (§§3–5), we concentrate on the group $\tilde{G}_k$ and the Lie algebra $\tilde{\mathfrak{g}}_k$. Thanks to results in the first part, $\tilde{G}_k$ is a Lie group with the Lie algebra $\tilde{\mathfrak{g}}_k$.

As is well-known, the Lie algebra $\tilde{\mathfrak{g}}$ of algebraic loops in $\mathfrak{g}$ has a universal central extension $\hat{\mathfrak{g}}$ called an affine Lie algebra, and the corresponding 2-cocycle $Z(\cdot, \cdot)$ was explicitly given in [2]. We extend the 2-cocycle to $\tilde{\mathfrak{g}}_k$ after [3], and get a central extension $\hat{\mathfrak{g}}_k$ of $\tilde{\mathfrak{g}}_k$.

$\hat{\mathfrak{g}}$ is one of the simplest infinite-dimensional Kac-Moody algebras (more precisely, the derived subalgebra of a Kac-Moody algebra, see [2] and [4]). And the corresponding Kac-Moody group $\hat{G}$ is a 1-dimensional central extension of the group $\tilde{G}$ of algebraic loops in G (cf. [2], [9], [5], and [6]).

Since the kernel of the adjoint action Ad of $\hat{G}$ on $\hat{\mathfrak{g}}$, is precisely the center C of $\hat{G}$, $\tilde{G} \simeq \hat{G}/C$ acts on $\hat{\mathfrak{g}}$ through Ad, and the set of invariants in $\hat{\mathfrak{g}}$ under this action is just the center of $\hat{\mathfrak{g}}$. Hence, the action on $\hat{\mathfrak{g}}$ induces an action of $\tilde{G}$ on $\tilde{\mathfrak{g}}$. The last action of $\tilde{G}$ coincides with the adjoint action.

The main purpose of the second part is to extend this fact to the pair of the infinite-dimensional Lie group $\tilde{G}_k$ and the Lie algebra $\hat{\mathfrak{g}}_k$, and describe the extended action explicitly. That is, we give explicitly an action of $\tilde{G}_k$ on $\hat{\mathfrak{g}}_k$ which induces the adjoint action Ad_0 of $\tilde{G}_k$ on $\tilde{\mathfrak{g}}$. Note that $\tilde{G}$ is a subgroup of $\tilde{G}_k$ and that $\hat{\mathfrak{g}}$ is a subalgebra of $\hat{\mathfrak{g}}_k$.

Moreover, we extend the action on $\hat{\mathfrak{g}}_k$ to that on a "Lie algebra" $\hat{\mathfrak{g}}_k^e$, called the extended affine Lie algebra after the terminology in [2]. $\hat{\mathfrak{g}}_k^e$ is a semi-direct sum of $\hat{\mathfrak{g}}_k$ and the "degree derivation" ∂ on $\hat{\mathfrak{g}}_k$. I remark that $\hat{\mathfrak{g}}_k^e$ is not really a Lie algebra. The bracket product on $\hat{\mathfrak{g}}_k^e$ is a bilinear map of $\hat{\mathfrak{g}}_k^e \times \hat{\mathfrak{g}}_k^e$ into $\hat{\mathfrak{g}}_{k-1}^e$, not into $\hat{\mathfrak{g}}_k^e$. Further, the "action" of $\tilde{G}_k$ on $\hat{\mathfrak{g}}_k^e$ is also not really a group action. Actually, we associate with each element in $\tilde{G}_k$ a linear map from $\hat{\mathfrak{g}}_k^e$ into $\hat{\mathfrak{g}}_{k-1}^e$. But we call $\hat{\mathfrak{g}}_k^e$ the extended affine Lie algebra, and say "$\tilde{G}_k$ acts on $\hat{\mathfrak{g}}_k^e$" for simplicity, because the bracket product on $\hat{\mathfrak{g}}_k^e$ and the above maps associated with each element in $\tilde{G}_k$, have the similar properties as the bracket product on the usual Lie algebra and the usual group action respectively, as shown in § 4.

As an application of the above results, we can calculate explicit forms of the normalizers and centralizers, in the loop group $\tilde{G}_k$, of the Cartan subalgebras of the affine Lie algebras $\hat{\mathfrak{g}}_k$ and $\hat{\mathfrak{g}}_k^e$. We see that, in both cases, the quotient groups of the normalizers by the centralizers are canonically isomorphic to the usual affine Weyl group.

This paper is organized as follows.

The contents of the first part, §§ 1–2, are general theories for the structure of spaces of loops in a manifold.

In § 1, we give the coefficient extension from $\mathbf{C}$ to L_k, for the differentiable functions.

In § 2, we define a topology for each space of loops in a manifold so that the connected components are given exactly by homotopy classes of the manifolds, and further, give a Lie group structure to the group of loops in a Lie group.

The second part, §§ 3–5, is devoted to study the actions of the loop group $\tilde{G}_k$ on the corresponding affine Lie algebras $\hat{\mathfrak{g}}_k$ and $\hat{\mathfrak{g}}_k^e$.

§ 3 is the preliminary section to §§ 4–5. We first prepare notations and well-known results about a connected, simply connected complex simple Lie group and its Lie algebra after [8]. And we define a completed version of the affine Lie algebra after the formulation of [3].

§ 4 is the main section of this article. We extend the adjoint action of the loop group $\tilde{G}_k$ from its Lie algebra $\tilde{\mathfrak{g}}_k$ to the completed affine Lie algebra $\hat{\mathfrak{g}}_k$, and further extend it to the "Lie algebra" $\hat{\mathfrak{g}}_k^e$, called an extended affine Lie algebra after [2], which is defined as a semidirect sum of $\hat{\mathfrak{g}}_k$ and the "degree derivation".

In § 5, we describe the explicit forms of the normalizers and centralizers of Cartan subalgebras of affine Lie algebras $\hat{\mathfrak{g}}_k$ and $\hat{\mathfrak{g}}_k^e$ with respect to the actions defined in § 4. And as the quotient groups of them, we get the affine Weyl group in both cases.

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§ 1. The coefficient extension from C to $L_k = C^k(S^1)$, for the differentiable functions

In this section, we introduce certain classes of functions on loop spaces with values in other loop spaces, in analogy with the usual differentiable functions on finite-dimensional spaces.

Locally, they are defined as limits of polynomials, similarly as in the case of differentiable functions on finite-dimensional spaces, but the coefficients of the polynomials are taken from the algebras of C^k-loops ($k = 1, 2, \ldots, \infty$) in $\mathbf{C}$, which will be denoted by L_k (for detail, see below).

1.1. The algebra of differentiable functions on the product space $(L_k)^n$ with values in L_k. Let $L_k := C^k(S^1)$, the algebra of C^k-functions on S^1. We define a norm $|\cdot|_k$ on L_k as

$$|a|_k := \sup_{\substack{r \in \mathbf{R} \\ j=0,\ldots,k}} |(\partial^j a)(e^{2\pi\sqrt{-1}r})| \qquad \text{for } a \in L_k ,$$

where ∂ is a differential operator on S^1, defined by

$$(\partial a)(e^{2\pi\sqrt{-1}r}) := \frac{1}{2\pi\sqrt{-1}} \lim_{t\to 0} \frac{1}{t} \{a(e^{2\pi\sqrt{-1}(r+t)}) - a(e^{2\pi\sqrt{-1}t})\} .$$

The space $(L_k, |\cdot|_k)$ is a Banach algebra.

We consider the polynomial ring $P_{k;n} := L_k[X_1, \ldots, X_n]$ with coefficients in L_k. Define derivations $D_j = \dfrac{\partial}{\partial X_j}$ on $P_{k;n}$ ($j = 1, 2, \ldots, n$) by

$$D_j X_i = \delta_{ij} \qquad \text{for } i = 1,\ 2,\ \ldots,\ n ,$$

$$D_j a = 0 \qquad \text{for } {}^{\forall}a \in L_k .$$

In this subsection, we investigate the properties of such L_k-valued functions on an open set in $(L_k)^n$ that can be uniformly approximated, together with their derivatives, on every bounded subset by polynomials in $P_{k;n}$.

For a bounded closed subset B in $(L_k)^n$, and $j = 0,\ 1,\ 2,\ \ldots,$ put

$$|f|_{k;B,j} := \sup_{m,b} |(D^m f)(b)|_k \qquad \text{for } f \in P_{k;n} ,$$

where sup is taken over all $m = (m_1, \ldots, m_n) \in (\mathbf{Z}_{\geqq 0})^n$ satisfying

$$|m| := m_1 + \cdots + m_n \leqq j ,$$

and all $b = (b_1, \ldots, b_n) \in B$, and D^m is a short form of $D_1^{m_1} D_2^{m_2} \ldots D_n^{m_n}$. Let $C^{k;j}(B)$ be the completion of the normed space $(P_{k;n}, |\cdot|_{k;B,j})$.

By definition, for any $m \in (\mathbf{Z}_{\geqq 0})^n$ such that $|m| \leqq j$, and any $b \in B$, the L_k-linear map $P_{k;n} \ni f \to (D^m f)(b) \in L_k$ is continuously extended to $C^{k;j}(B)$. It is easy to prove the following lemma.

Lemma 1.1. *Let $f \in C^{k;j}(B)$. We assume that $f(b) = 0$ for any $b \in B$. Then, for any $b_0 \in \mathrm{Int}\,(B)$, the interior of B, and any $m \in (\mathbf{Z}_{\geqq 0})^n$ such that $|m| \leqq j$, it holds that*

$$(D^m f)(b_0) = 0 .$$

Let U be an open subset of $(L_k)^n$, and $C^{k;j}(U)$ the space of L_k-valued functions f on U which satisfies

(*) for any $u \in U$, there exists a bounded closed neighbourhood B of u in $(L_k)^n$ and $g \in C^{k;j}(B)$ such that

$$f(b) = g(b) \qquad \text{for } {}^\forall b \in B\,.$$

Thanks to Lemma 1.1, we can define $D^m f(u)$ as $D^m g(u)$ for $m \in (\mathbf{Z}_{\geqq 0})^n$, $|m| \leqq j$. We consider $C^{k;j}(U)$ as the inductive limit of the spaces $C^{k;j}(B)$, where B are bounded subsets of U closed in $(L_k)^n$.

The following lemmas are clear from definition.

Lemma 1.2. *$f \to D^m f$ is a continuous linear map from $C^{k;j}(U)$ into $C^{k;j-|m|}(U)$.*

Lemma 1.3. *$D_i(fg) = (D_i f)g + f(D_i g)$.*

Lemma 1.4. *$D^m D^{m'} f = D^{m+m'} f$.*

The functions in $C^{k;j}(U)$ are j-times differentiable in the sense of Fréchet, that is, we have the following lemma.

Lemma 1.5. i) *Every $f \in C^{k;0}(U)$ is continuous, and further uniformly continuous on any bounded subset of U.*

ii) *Every $f \in C^{k;1}(U)$ is C^1-map, that is, it holds that*

(1) *for ${}^\forall u \in U$, there exists an operator $df_u \in L((L_k)^n, L_k)$ such that*

$$\lim_{v \to 0} |v|_k^{-1}(f(u+v) - f(u) - df_u(v)) = 0\,,$$

(2) *the map $df\colon U \ni u \to df_u \in L((L_k)^n, L_k)$ is continuous.*

Here, we consider $(L_k)^n$ as a Banach space with the norm

$$|u|_k := \max_{1 \leqq i \leqq n} |u_i|_k \qquad \text{for } u = (u_1, \dots, u_n) \in (L_k)^n\,,$$

and denote by $L((L_k)^n, L_k)$ the Banach space of all continuous linear maps from $(L_k)^n$ into L_k.

Proof. i) By definition of $C^{k;j}(U)$, it is enough to prove the assertion for $f \in P_{k;n}$, and this is clear because L_k is a Banach algebra.

ii) We can define df_u by

$$df_u(v) := \textstyle\sum_{i=1}^n D_i f(u) v_i \qquad \text{for } v = (v_1, \dots, v_n) \in (L_k)^n\,.$$

Indeed, firstly, by Lemma 1.2 and the above i), $u \to df_u$ is continuous.

Secondly, for any $u \in U$, there exists $\varepsilon > 0$ such that

$$u + C_\varepsilon \subset U\,,$$

where C_ε is a closed ball in $(L_k)^n$ with diameter ε and center 0. Since $u + C_\varepsilon$ is a bounded closed neighbourhood of u, by definition of $C^{k;j}(U)$, there exists a sequence $\{p_l\}_{l \geqq 1}$ of polynomials in $P_{k;n}$ such that

$$\lim_{l \to \infty} |f - p_l|_{k;u+C_\varepsilon,1} = 0\,.$$

For each p_l, by an easy calculation, we see that

$$p_l(u+v) = p_l(u) + \sum_{i=1}^{n} \int_0^1 (D_i p_l)(u+tv)v_i dt\,, \qquad {}^\forall v = (v_1, \ldots, v_n) \in C_\varepsilon\,.$$

As $l \to \infty$, we have

$$f(u+v) = f(u) + \sum_{i=1}^{n} \int_0^1 (D_i f)(u+tv)v_i dt\,, \qquad {}^\forall v = (v_1, \ldots, v_n) \in C_\varepsilon\,.$$

Therefore, it holds that

$$\begin{aligned}&\frac{1}{|v|_k}(f(u+v) - f(u) - df_u(v)) \\ &= \sum_{i=1}^{n} \int_0^1 (D_i f(u+tv) - D_i f(u))\frac{1}{|v|_k} v_i dt\,, \qquad {}^\forall v \in C_\varepsilon \backslash \{0\}\,.\end{aligned}$$

Since each $D_i f$ is continuous by i), we obtain the equality in (1). Q.E.D.

1.2. An extension of usual differentiable functions on C^n to those on $(L_k)^n$. For a subset S of $\mathbf{C}^n$ and a closed subset P of S^1, put

$$S(P) := \{(a_1, \ldots, a_n) \in (L_0)^n;\, (a_1(t), \ldots, a_n(t)) \in S,\, {}^\forall t \in P\}\,,$$

$$S(P)_k := S(P) \cap (L_k)^n\,, \qquad S(L_k) := S(S^1)_k\,.$$

Then, $S(P)$ inherits topological properties of S as follows.

Lemma 1.6. *If S is closed or open in $\mathbf{C}^n$, then so is $S(P)$ in $(L_0)^n$.*

Proof. When S is closed, the assertion is clear because $(L_0)^n \ni (a_1, \ldots, a_n) \to (a_1(t), \ldots, a_n(t)) \in \mathbf{C}^n$ is continuous for any $t \in S^1$.

Let S be open, and $a = (a_1, \ldots, a_n)$ be an arbitrary element in $S(P)$. Since a is continuous as a map from S^1 into $\mathbf{C}^n$, $a(P)$ ($\subset S$) is compact.

Hence, the distance, say d, between $a(P)$ and $\mathbf{C}^n \backslash S$ is positive, and the open ball with diameter d and center a in $(L_0)^n$ is contained in $S(P)$. Q.E.D.

Let V be an open subset of $\mathbf{C}^n$. We may regard $V = V(L_k) \cap \mathbf{C}^n$. So, for any $f \in C^{k;j}(V(L_k))$, we can define an L_k-valued function $\Phi(f)$ on V by

$$\Phi(f) := f|_V\,.$$

By Lemma 1.5 ii), if $f(V) \subset \mathbf{C}$ ($\subset L_k$), then $\Phi(f) \in C^j(V)$.

Conversely, for any $g \in C^j(V)$, define an L_k-valued function $\Psi(g)$ on $V(L_k)$ by

$$\Psi(g)(a_1, \ldots, a_n)(t) := g(a_1(t), \ldots, a_n(t)) \qquad \text{for } (a_1, \ldots, a_n) \in V(L_k),\ t \in S^1\,.$$

Clearly, $(\Psi \circ \Phi)(f) = f$, if $f(V) \subset \mathbf{C}$. Moreover, we have

Lemma 1.7. *Ψ maps $C^{k+j}(V)$ into $C^{k;j}(V(L_k))$, and it holds that*

$$\Phi \circ \Psi = \mathrm{Id}_{C^{k+j}(V)}\,.$$

Proof. Take a sequence $\{C_i\}_{i\geqq 1}$ of bounded closed subsets of $\mathbf{C}^n$ such that

$$C_i \subset \operatorname{Int}(C_{i+1}), \qquad \bigcup_{i=1}^{\infty} C_i = V.$$

For each i, put

$$B_i := \{a \in C_i(L_k);\ |a|_k \leqq i\}.$$

By Lemma 1.6, the subsets

$$V_i := \{a \in (\operatorname{Int}(C_i))(L_k);\ |a|_k < i\}$$

of B_i are open, and so they are contained in $\operatorname{Int}(B_i)$. On the other hand, since $a(S^1)$ is a compact subset of V for each $a \in V(L_k)$, $a(S^1)$ is contained in some $\operatorname{Int}(C_i)$, and so the V_i's cover $V(L_k)$. Thus, we get the equality

$$V(L_k) = \bigcup_{i=1}^{\infty} \operatorname{Int}(B_i).$$

Let g be an arbitrary element of $C^{j+k}(V)$, and let $\{g_l\}_{l\geqq 1}$ be a sequence of polynomials with coefficients in $\mathbf{C}$ which satisfies

$$\lim_{l\to\infty} \sup_{m,x} |D^m(g-g_l)(x)| = 0,$$

where sup is taken over $m \in (\mathbf{Z}_{\geqq 0})^n$, $|m| \leqq j+k$, and $x \in C_i$.

For l, $l' \geqq 1$, and $m \in (\mathbf{Z}_{\geqq 0})^n$, $|m| \leqq j$, we have

$$\begin{aligned}
\sup_{a\in B_i} |D^m(g_l-g_{l'})(a)|_k &\leqq \sup_{a\in B_i} \sup_{\substack{t\in S^1\\ 0\leqq p\leqq k}} |\partial^p D^m(g_l-g_{l'})(a)(t)| \\
&= \sup_{a\in B_i} \sup_{t,p} |\partial^p D^m(g_l-g_{l'})(a(t))| \\
&\leqq \Big(\sup_{a\in B_i} |a|_1\Big)^k \Big(\sup_{\substack{|m'|\leqq j+k\\ x\in C_i}} |D^{m'}(g_l-g_{l'})(x)|\Big) \\
&\leqq i^k \Big(\sup_{m',x} |D^{m'}(g_l-g_{l'})(x)|\Big) \to 0 \qquad (l, l' \to \infty).
\end{aligned}$$

Here, we identify $\mathbf{C}[X_1, \ldots, X_n]$ with the subalgebra of elements in $L_k[X_1, \ldots, X_n]$ with coefficients in $\mathbf{C}$ $(\subset L_k)$.

Hence, $\{g_l\}_{l\geqq 1}$ is convergent in $C^{k;j}(B_i)$, and its limit $g_\infty = \lim_{l\to\infty} g_l \in C^{k;j}(B_i)$ satisfies

$$g_\infty(b) = \Psi(g)(b), \qquad {}^\forall b \in B_i.$$

This implies that $\Psi(g) \in C^{k;j}(V(L_k))$, because interiors of B_i's cover $V(L_k)$. The equality of the lemma is clear from definition. Q.E.D.

Corollary 1.7.1 (of the proof). *For any $j \geqq 1$, $g \in C^{k+j}(V)$, and $i = 1, \ldots, n$, there holds that*

$$D_i \Psi(g) = \Psi(D_i(g)).$$

For an open subset U of $(L_k)^n$, and $m \in \mathbf{Z}_{>0}$, put

$$C^{k;j}(U; (L_k)^m) := C^{k;j}(U) \times \cdots \times C^{k;j}(U) \qquad (m \text{ times direct product}).$$

We regard each element of $C^{k;j}(U; (L_k)^m)$ as a map from U into $(L_k)^m$. For a subset S of $(L_k)^m$, we define a subspace $C^{k;j}(U; S)$ by

$$C^{k;j}(U; S) := \{f \in C^{k;j}(U; (L_k)^m); f(U) \subset S\}.$$

The following lemma is clear from Lemma 1.5.

Lemma 1.8. *Every element in $C^{k;1}(U; (L_k)^m)$ is C^1-map from U into $(L_k)^m$.*

For every element $g = (g_1, \ldots, g_n)$ in $C^{k+j}(V; \mathbf{C}^m)$ (= the space of C^{k+j}-maps from an open set V in $\mathbf{C}^n$ into $\mathbf{C}^m$), we put

$$\Psi(g) := (\Psi(g_1), \ldots, \Psi(g_n)).$$

And we also define $\Phi(f)$ for $f \in C^{k;j}(V(L_k); (L_k)^m)$, similarly. Clearly, the same fact as Lemma 1.7 holds for this new Ψ as follows.

Lemma 1.9. *For any $g \in C^{j+k}(V; \mathbf{C}^m)$, $\Psi(g)$ belongs to $C^{k;j}(V(L_k); (L_k)^m)$, and we have*

$$\Phi \circ \Psi = \mathrm{Id}_{C^{j+k}(V; \mathbf{C}^m)}.$$

Let $L_{k,\mathbf{R}}$ be the real form of L_k consisting of the real valued functions. Obviously, all the above results are true for $L_{k,\mathbf{R}}$, replacing $\mathbf{C}$ with $\mathbf{R}$.

§ 2. Structure of the space of loops in a manifold

In this section, we provide the space of C^k-loops in an arbitrary finite-dimensional C^k-manifold with a canonical topology, and, in case where the manifold is a Lie group, we give to the loop group a Banach Lie group structure.

In subsection 2.1, we shall give a canonical topology to spaces of loops in manifolds, with respect to which the connected components of the loop spaces are given exactly by homotopy classes of the manifolds.

In subsection 2.2, making use of the classes of functions in § 1 and the topology in 2.1, we give Lie group structure to an arbitrary loop group canonically in such a manner that their tangent spaces are isomorphic to a product of some copies of L_k.

2.1. Topology of loop spaces. Let $k = 0, 1, 2, \ldots$, and M be an n-dimensional C^k-manifold ($n < \infty$). We denote by $M(L_k)$ the space of C^k-loops in M:

$$M(L_k) = \{f: S^1 \to M;\ f \text{ is of class } C^k\}.$$

For another finite-dimensional manifold M', and a C^k-map F from M into M', we define a map $\Psi(F)$ from $M(L_k)$ into $M'(L_k)$ by

$$\Psi(F)(f)(s) := F(f(s)) \qquad \text{for } f \in M(L_k),\ s \in S^1 .$$

Of course, this definition is an extension of that of Ψ in 1.1.

Let $\mathscr{V}_k = \mathscr{V}_k(M)$ be the totality of the double sequence $(V_j^{(i)})_{1 \leq i \leq l, 0 \leq j \leq k}$ (l is a positive integer) with $V_0^{(i)}$ an open subset of M and $V_1^{(i)}, \ldots, V_k^{(i)}$ open subsets of $\mathbf{C}^n$ for each i. And let $\mathscr{P}$ be the totality of the sequence $(P_1, \ldots, P_l)$ of closed subsets of S^1 such that the union $\bigcup_i P_i$ is equal to the whole set S^1.

For $\Delta = (P_1, \ldots, P_l) \in \mathscr{P}$, and $\mathbf{V} = (V_j^{(i)})_{i,j} \in \mathscr{V}_k$, put

$$U(\Delta; \mathbf{V}) := \bigcap_i \{f \in M(L_k);\ d^j f(P_i) \subset V_j^{(i)}\} ,$$

where $d^0 f(x) = f(x)$, and $d^j f(x) = d(d^{(j-1)} f)_x$ for $j > 0$.

For the family of these subsets $U(\Delta; \mathbf{V})$, we have

Lemma 2.1. i) *There exists a topology of $M(L_k)$ for which an open basis is given by the family $\{U(\Delta; \mathbf{V})\}_{\Delta \in \mathscr{P}, \mathbf{V} \in \mathscr{V}_k}$.*

ii) *If $F: M \to M'$ is a map of class C^k, then $\Psi(F): M(L_k) \to M'(L_k)$ is continuous with respect to the topology in* i).

Proof. i) is clear from definition of the family.

Let n and n' be dimensions of M and M' respectively. For each $j = 1, \ldots, k$, we may regard $d^j F$ as a C^{k-j}-map from M into the space $L_j(\mathbf{R}^n, \mathbf{R}^{n'})$ of $\mathbf{R}^{n'}$-valued j-linear forms on the product space of j-copies of $\mathbf{R}^n$. And for any $s \in S^1$ and $f \in M(L_k)$, derivatives of $\Psi(F)(f) = F \circ f$ are given by

$$d^j (F \circ f)_s = \textstyle\sum_{l=0}^{j} \sum d^1 F_{f(s)}(d^{p_1} f_s, \ldots, d^{p_l} f_s),$$

where the second sum is taken over the positive integers $p_1, \ldots, p_l$ such that $p_1 + \cdots + p_l = j$. Since the both maps $L_l(\mathbf{R}^n; \mathbf{R}^{n'}) \times (\mathbf{R}^n)^l \ni (b, a_1, \ldots, a_l) \to b(a_1, \ldots, a_l) \in \mathbf{R}^{n'}$, and $S^1 \ni s \to (d^j F_{f(s)}, d^{p_1} f_s, \ldots, d^{p_l} f_s) \in L_1(\mathbf{R}^n; \mathbf{R}^{n'}) \times (\mathbf{R}^n)^l$ are continuous, the assertion ii) holds. Q.E.D.

In the following, we always consider the topology of this lemma on $M(L_k)$.

If M is smooth manifold, then each $M(L_{k+1})$ is continuously and densely imbedded into $M(L_k)$. So let $M(L_\infty)$ be the inverse limit of $\{M(L_k)\}_{k \geq 0}$.

Remark 2.2. i) Of course, $M(L_\infty)$ coincides with the totality of smooth loops in M as sets.

ii) If we take $\mathbf{R}^n$ as M, then $M(L_k) = (L_{k,\mathbf{R}})^n$, and the above topology is the same as that given by the norm $|\cdot|_k$.

iii) For $k = 0$, the above topology is nothing but the compact-open topology.

With respect to this topology, the homotopy classes of M give the decomposition of $M(L_k)$ into connected components as follows.

For each $f_0 \in M(L_k)$, we define a subset $M(f_0)$ of $M(L_k)$ by

$$M(f_0) := \{f \in M(L_k);\ f \overset{h}{\sim} f_0,\ \text{that is, } f \text{ is homotopic to } f_0\}.$$

Lemma 2.3. *$M(f_0)$ is closed in $M(L_k)$.*

Proof. Let f_1 be an element of the closure of $M(f_0)$. Since $f_1(S^1)$ is compact, there exist a positive real number $\varepsilon < 1$, a positive integer p, and pairs $\{V_i, \xi_i\}_{i=1,\dots,p}$ of coordinates neighbourhoods and local coordinates for M such that

(1) $\xi_i(V_i) = \{x \in \mathbf{R}^n;\ |x| < 1\}$,

(2) f_1 maps $\{\exp(\sqrt{-1}r);\ r \in I_i = [2\pi(i-1)/p, 2\pi i/p]\}$ into V_i,

(3) $\xi_i(f_1(\exp(\sqrt{-1}2\pi i/p)) = 0$,

(4) the open subset $W_i = \xi_i^{-1}\{|x| < \varepsilon\}$ of M is contained in $V_i \cap V_{i+1}$, where $V_{p+1} = V_1$.

Define an open subset U of $M(L_k)$ by

$$U := \bigcap_{i=1}^{p} \{g \in M(L_k);\ g(e^{\sqrt{-1}I_i}) \subset V_i,\ g(e^{2\pi i\sqrt{-1}/p}) \in W_i\}.$$

By assumption on f_1, the intersection $M(f_0) \cap U$ is not empty. For an element f_2 of this intersection, put

$$g_i(s) := \xi_i^{-1}((1-s)\xi_i(f_2(e^{2\pi i\sqrt{-1}/p})) + s\xi_i(f_1(e^{2\pi i\sqrt{-1}/p}))) \text{ for } 0 \leqq s \leqq 1,\ i = 1, \dots, p.$$

Since each V_i is homeomorphic to $\{|x| < 1\}$, there exists a continuous map h_i: $[0, 1] \times e^{\sqrt{-1}I_i} \to V_i$ such that

$$h_i(0, *) = f_2, \qquad h_i(1, *) = f_1 \qquad \text{on } \exp(\sqrt{-1}I_i),$$
$$h_i(*, e^{2\pi(i-1)\sqrt{-1}/p}) = g_{i-1},$$
$$h_i(*, e^{2\pi i\sqrt{-1}/p}) = g_i.$$

Putting $h(s, t) := h_i(s, t)$ for $s \in [0, 1]$, $t \in I_i$, we see that f_1 is homotopic with f_2, and so $f_1 \in M(f_2) = M(f_0)$. Q.E.D.

Lemma 2.4. *If $f_1 \in M(L_k)$ is homotopic with f_0, then there exists a continuous map F: $[0, 1] \to M(L_k)$ such that*

$$F(0) = f_0 \quad \text{and} \quad F(1) = f_1.$$

Proof. By assumption, we can take a continuous map f: $[0, 1] \times S^1 \to M$ such that $f(0, *) = f_0$, $f(1, *) = f_1$. We may assume this f is of class C^k. Then the function

$$F(s)(t) := f(s, t) \qquad \text{for } s \in [0, 1],\ t \in S^1,$$

gives a path connecting f_0 and f_1 in $M(L_k)$. Q.E.D.

By these two lemmas, we see that $M(L_k)$ is decomposed into disjoint union of arcwise connected, closed subsets of the form $M(f)$, $f \in M(L_k)$. Thus, we have the following theorem.

Theorem 2.5. *Let $k = 0, 1, 2, \ldots, \infty$, and let M be a connected C^k-manifold.*

i) *For any $f \in M(L_k)$, $M(f)$ is the connected component containing f. Further, $M(f)$ is arcwise connected.*

ii) *There exists a bijective correspondence between the fundamental group $\pi_1(M)$ and the set of connected components of $M(L_k)$. In particular, $M(L_k)$ is connected if and only if M is connected and simply connected.*

2.2. Lie group structure on a loop group. In this subsection, we take an n-dimensional Lie group G as the manifold M in the preceding subsections, and give a Banach Lie group structure to $G(L_k)$, making use of the above results.

Let $\mathfrak{g}$ be the Lie algebra of G.

Taking, as F in Lemma 2.1 ii), the product map $G \times G \to G$ and inversion map $G \to G$, and bracket product on $\mathfrak{g}$ respectively, we see that $G(L_k)$ is a topological group, and that $\mathfrak{g}(L_k)$ is a Banach Lie algebra in a natural way.

Let $\exp: \mathfrak{g} \to G$ be the exponential map. Since exp is a local diffeomorphism, $\Psi(\exp): \mathfrak{g}(L_k) \to G(L_k)$ is a local homeomorphism (see Lemma 2.1). Moreover, if $f, f' \in \mathfrak{g}(L_k)$ commute with each other, we have from definition

$$\Psi(\exp)(f + f') = \Psi(\exp)(f) \cdot \Psi(\exp)(f').$$

In particular, $\Psi(\exp)(tf)$ $(t \in \mathbf{R})$ is a 1-parameter group in $G(L_k)$. In this sense, we can say that $\Psi(\exp)$ satisfies the property for the exponential on $\mathfrak{g}(L_k)$, and so we write exp for $\Psi(\exp)$.

Let $\alpha: G \times \mathfrak{g} \ni (g, x) \to \mathrm{Ad}\,(g)x \in \mathfrak{g}$ be the adjoint action of G on $\mathfrak{g}$. By the same reason for exp, we use the same symbol $\mathrm{Ad}\,(g)x$ for $\Psi(\alpha)(g, x)$ also in case $g \in G(L_k)$, and $x \in \mathfrak{g}(L_k)$. This new Ad defines a continuous homomorphism from $G(L_k)$ into $GL(n, L_k)$ $(\subset L(\mathfrak{g}(L_k), \mathfrak{g}(L_k))^\times)$ and further it satisfies

Lemma 2.6. $g(\exp x)g^{-1} = \exp\,(\mathrm{Ad}\,(g)x)$ for ${}^\forall g \in G(L_k)$, ${}^\forall x \in \mathfrak{g}(L_k)$.

We define $\beta: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$\beta(x, y) := \mathrm{Ad}\,(\exp x)y \qquad \text{for } x, y \in \mathfrak{g}.$$

By Lemma 1.7, $\Psi(\beta)$ is a C^∞-map from $\mathfrak{g}(L_k) \times \mathfrak{g}(L_k)$ into $\mathfrak{g}(L_k)$. Applying Corollary 1.7.1 to the formula

$$\frac{d}{dt}\beta(tx, y) = (\mathrm{ad}\,x)(\beta(tx, y)) \qquad \text{for } {}^\forall x, y \in \mathfrak{g},$$

we have

$$\frac{d}{dt}\Psi(\beta)(tx, y) = (\mathrm{ad}\,x)(\Psi(\beta)(tx, y)) \qquad \text{for } {}^\forall x, y \in \mathfrak{g}(L_k).$$

Since $\Psi(\beta)(x, y) = \mathrm{Ad}\,(\exp x)y$, we see that the infinitesimal generator of the 1-parameter group $\mathrm{Ad}\,(\exp tx)$ is equal to $\mathrm{ad}\,x$. On the other hand, the same is true for the 1-parameter group $\exp\,(t(\mathrm{ad}\,x)) = \sum_{p=0}^\infty \frac{1}{p!} t^p (\mathrm{ad}\,x)^p$. Therefore, it holds that

Lemma 2.7. Ad $(\exp x) = \exp(\mathrm{ad}\, x)$ *for* $^{\forall}x \in \mathfrak{g}(L_k)$.

Now, we can prove the main result of this section as follows.

Theorem 2.8. *Let G be a finite-dimensional Lie group, $\mathfrak{g}$ the Lie algebra of G, and $k = 0, 1, 2, \ldots$. For a sufficiently small neighbourhood V of 1 in G, there exist a unique Lie group structure on $G(L_k)$ such that its local coordinates at 1 is given by $(V(L_k), \log)$, where $\log = \Psi(\exp^{-1})$ on $V(L_k)$.*

Moreover, for another finite-dimensional Lie group G', open subset U of G, and C^{j+k}-map f from U into G', the map $\Psi(f)\colon U(L_k) \to G'(L_k)$ is of class C^j, and further if $U = G$ and f is a Lie group homomorphism, so is $\Psi(f)$.

Proof. Thanks to Lemma 1.8, the second assertion follows from the first one.

Let W be a neighbourhood of 1 in G such that $\log = \exp^{-1}\colon W \to \exp^{-1}(W)$ is a diffeomorphism, and let V be an open neighbourhood of 1 in G such that

$$V^{-1} = V, \qquad \text{and } VV \subset W.$$

It is enough to prove the following properties (1) ~ (4):

(1) there exists an open neighbourhood U_1 of 1 in $G(L_k)$, such that $U_1 U_1 \subset V(L_k)$, and

$$\log(U_1) \times \log(U_1) \ni (x_1, x_2) \to \log(e^{x_1} e^{x_2}) \in \log(V(L_k))$$

is a C^∞-map,

(2) the map

$$\log(V(L_k)) \ni x \to \log(e^{-x}) \in \log(V(L_k))$$

is a C^∞-map,

(3) for any $g \in V(L_k)$, there exists an open neighbourhood U_2 of 1 in $G(L_k)$ such that $U_2 \cup gU_2 \subset V(L_k)$ and

$$\log(U_2) \ni x \to \log(ge^x) \in \log(V(L_k))$$

is a C^∞-map,

(4) for any $g \in G(L_k)$, there exists an open neighbourhood U_3 of 1 in $G(L_k)$ such that $U_3 \cup gU_3 g^{-1} \subset V(L_k)$ and

$$\log(U_3) \ni x \to \log(ge^x g^{-1}) \in \log(V(L_k))$$

is a C^∞-map.

For an open neighbourhood W_1 of 1 in G such that $W_1 W_1 \subset V$, put $U_1 = W_1(L_k)$, then (1) holds.

The map in (2) is multiplication by -1, whence (2) is true.

For $g \in V(L_k)$, $g(S^1)$ is compact subset of V, and hence there exists an open neighbourhood W_2 of 1 in G such that $W_2 \cup g(S^1) W_2 \subset V$. Put $U_2 = W_2(L_k)$. The map in (3) is the restriction of

$$\log(V(L_k)) \times \log(V(L_k)) \ni (x, y) \to \log(e^x e^y) \in \log(W(L_k)),$$

which is of class C^∞ by Lemma 1.7.

Let g be an arbitrary element of $G(L_k)$. Since $G(L_k)$ is a topological group, there exists an open neighbourhood U_3 of 1 in $G(L_k)$ such that $U_3 \cup gU_3g^{-1} \subset V(L_k)$. The map in (4) is the restriction of the continuous linear map $\mathrm{Ad}(g)$ by Lemma 2.6, and so it is of class C^∞. Q.E.D.

Thanks to the second assertion of this theorem, we can replace log with an arbitrary local coordinates ϕ as follows.

Corollary 2.8.1. *Let G, $\mathfrak{g}$, and G' be the same as in Theorem 2.8. For any coordinate neighbourhood (V, ϕ) of 1 in G, there exists a unique Lie group structure on $G(L_k)$ whose coordinate neighbourhood of 1 is given by $(V(L_k), \Psi(\phi))$. This structure is independent of the choice of (V, ϕ).*

By definition, $V(L_k) = V(L_0) \cap G(L_k)$ and $\Psi(\phi)$ on $V(L_k)$ is the restriction of $\Psi(\phi)$ on $V(L_0)$. In this sense, we may say that the differentiable structure of $G(L_k)$ is uniform with respect to k, and we have

Corollary 2.8.2. *Let G, $\mathfrak{g}$, and G' be the same as in Theorem 2.8. For any coordinate neighbourhood (V, ϕ) of 1 in G, there exists a unique Fréchet Lie group structure on $G(L_\infty)$ whose coordinate neighbourhood of 1 is given by $(V(L_\infty), \Psi(\phi))$. This structure is independent of the choice of (V, ϕ).*

Moreover, for any open subset U of G, and any C^∞-map f from U into G', $\Psi(f)\colon U(L_\infty) \to G'(L_\infty)$ is of class C^∞.

§ 3. A complete affine Lie algebra

In this section, we introduce the completed version of a usual affine Lie algebra.

First, we prepare notations and basic results for a finite-dimensional complex simple Lie group G with Lie algebra $\mathfrak{g}$ after [8].

Then, we give a continuous 2-cocycle on the Banach Lie algebra $\mathfrak{g}(L_k)$, and construct the corresponding 1-dimensional central extension of $\mathfrak{g}(L_k)$, which is again a Banach Lie algebra, and is the main object in this article.

Note that $\mathfrak{g}(L_k)$ is given by replacing the Laurent polynomial ring in the case of usual affine Lie algebra with the Banach algebra L_k.

3.1. Preliminaries for a finite-dimensional complex simple Lie group. We fix the notations for the rest of this paper.

Let G be a finite-dimensional connected and simply connected complex simple Lie group, H a Cartan subgroup of G, B a Borel subgroup containing H, and $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{b}$ their Lie algebras respectively. Let $\mathfrak{n}$ be the nil-radical of $\mathfrak{b}$ and $U = \exp(\mathfrak{n})$ the corresponding unipotent radical of B.

Denote by Δ the root system of $(\mathfrak{g}, \mathfrak{h})$, and Δ_+ the set of positive roots corresponding to $\mathfrak{b}$, $\Pi = \{\alpha_1, \dots, \alpha_l\}$ the set of simple roots in Δ_+. For each $\alpha \in \mathfrak{h}^*$ $(= \mathrm{Hom}_{\mathbf{C}}(\mathfrak{h}, \mathbf{C}))$, we define subspace $\mathfrak{g}^\alpha$ by

$$\mathfrak{g}^\alpha := \{x \in \mathfrak{g};\ [h, x] = \alpha(h)x \quad \text{for } {}^\forall h \in \mathfrak{h}\}\,.$$

For a Chevalley basis $x_\alpha \in \mathfrak{g}^\alpha$ ($\alpha \in \Delta$), $h_1, \ldots, h_l \in \mathfrak{h}$, put

$$U^\alpha := \{\exp(sx_\alpha); s \in \mathbf{C}\},$$

$$e_i := x_{\alpha_i}, \qquad f_i := x_{-\alpha_i} \qquad \text{for } i = 1, \ldots, l.$$

Under these notations, we have the following well-known results.

Proposition 3.1 [8]. i) *The normalizer N of H in G is generated by the elements*

$$w_\alpha(s) := \exp(sx_\alpha)\exp(-s^{-1}x_{-\alpha})\exp(sx_\alpha)$$

with $\alpha \in \Delta$, and $s \in \mathbf{C}^\times$.

ii) *For any $\alpha \in \Delta$, $s \in \mathbf{C}^\times$, the element*

$$h_\alpha(s) := w_\alpha(1)^{-1}w_\alpha(s)$$

belongs to H, and the map

$$(s_1, \ldots, s_l) \to h_{\alpha_1}(s_1)\ldots h_{\alpha_l}(s_l)$$

defines a Lie group isomorphism of $(\mathbf{C}^\times)^l$ onto H.

We can define an involutive antilinear antiautomorphism on $\mathfrak{g}$ by

$$e_i^* := f_i, \qquad h_i^* := h_i \qquad \text{for } i = 1, \ldots, l.$$

For this involution and the Killing form $B(\cdot, \cdot)$ on $\mathfrak{g}$, unique up to scalar multiples as a non-degenerate, invariant, and symmetric bilinear form on $\mathfrak{g}$, put

$$B_0(x, y) := B(x, y^*) \qquad \text{for } x, y \in \mathfrak{g}.$$

This sesquilinear form on $\mathfrak{g}$ is Hermitian and positive definite.

3.2. Construction of a complete affine Lie algebra. Let $\tilde{\mathfrak{g}}_k = \mathfrak{g}(L_k)$. The bracket product $[\cdot, \cdot]_0$ in $\tilde{\mathfrak{g}}_k$ is given by

$$[f, g]_0(s) = [f(s), g(s)] \qquad \text{for } f, g \in \tilde{\mathfrak{g}}_k.$$

We extend the bilinear form $B(\cdot, \cdot)$, the involution $x \to x^*$, and the Hermitian form $B_0(\cdot, \cdot)$, from $\mathfrak{g}$ to $\tilde{\mathfrak{g}}_k$, by

$$B(f, g) := \int_0^1 B(f(e^{2\pi\sqrt{-1}r}), g(e^{2\pi\sqrt{-1}r}))dr$$

$$f^*(s) := f(s)^*$$

$$B_0(f, g) := B(f, g^*) \qquad \text{for } f, g \in \tilde{\mathfrak{g}}_k, \ s \in S^1.$$

Then, $B(\cdot, \cdot)$ is a non-degenerate invariant symmetric bilinear form, $f \to f^*$ is an involutive antilinear antiautomorphism, and $B_0(\cdot, \cdot)$ is an inner product on $\tilde{\mathfrak{g}}_k$.

We may assume the norm $|\cdot|_k$ on $\tilde{\mathfrak{g}}_k \simeq (L_k)^{\dim(\mathfrak{g})}$ is given by

$$|f|_k = \sup_{\substack{s \in S^1 \\ 0 \leqq j \leqq k}} B_0(\partial^j f(s), \partial^j f(s))^{1/2} \qquad \text{for } f \in \tilde{\mathfrak{g}}_k\,,$$

where ∂ is defined in the same way as in 1.1.

From now on, we always assume $k \geqq 1$.

Let us define a bilinear form $Z(\cdot,\cdot)$ on $\tilde{\mathfrak{g}}_k$ by

$$Z(f, g) := B(\partial f, g) \qquad \text{for } f,\ g \in \tilde{\mathfrak{g}}_k\,.$$

We have the following

Lemma 3.2. *$Z(\cdot,\cdot)$ is a continuous 2-cocycle.*

Proof. Continuity of $Z(\cdot,\cdot)$ follows from those of the maps $\partial\colon \tilde{\mathfrak{g}}_k \to \tilde{\mathfrak{g}}_{k-1}$ and $B(\cdot,\cdot)\colon \tilde{\mathfrak{g}}_{k-1} \times \tilde{\mathfrak{g}}_k \to \mathbf{C}$.

Let $f_i \in \tilde{\mathfrak{g}}_k$ $(i = 1, 2, 3)$ and $\tau(r) := \exp(2\pi\sqrt{-1}r)$. By definition, it holds that

$$\begin{aligned}
B(\partial f_1, f_2) &= \frac{1}{2\pi\sqrt{-1}} \int_0^1 \lim_{\delta\to 0} \frac{1}{\delta} B(f_1(\tau(r+\delta)) - f_1(\tau(r)), f_2(\tau(r)))dr \\
&= \frac{1}{2\pi\sqrt{-1}} \lim_{\delta\to 0} \frac{1}{\delta} \left\{ \int_0^1 B(f_1(\tau(r)), f_2(\tau(r-\delta)) - f_2(\tau(r)))dr \right\} \\
&= -\int_0^1 B(f_1(\tau(r)), \partial f_2(\tau(r)))dr = -B(f_1, \partial f_2)\,.
\end{aligned}$$

Hence, $Z(\cdot,\cdot)$ is antisymmetric because $B(\cdot,\cdot)$ is symmetric.

By a similar calculation, we have

$$\begin{aligned}
&B(\partial[f_1, f_2]_0, f_3) + B(\partial[f_2, f_3]_0, f_1) + B(\partial[f_3, f_1]_0, f_2) \\
&\quad = B([\partial f_1, f_2]_0, f_3) + B([f_1, \partial f_2]_0, f_3) + B(\partial[f_2, f_3]_0, f_1) + B(\partial[f_3, f_1]_0, f_2)\,.
\end{aligned}$$

Since $B(\cdot,\cdot)$ is invariant, this is equal to

$$B(\partial f_1, [f_2, f_3]_0) + B(\partial f_2, [f_3, f_1]_0) + B(\partial[f_2, f_3]_0, f_1) + B(\partial[f_3, f_1], f_2)\,.$$

Thanks to the above result i) and symmetricity of $B(\cdot,\cdot)$, this sum equals 0. Q.E.D.

Let $\hat{\mathfrak{g}}_k$ be the 1-dimensional central extension of $\tilde{\mathfrak{g}}_k$ corresponding to the 2-cocycle $Z(\cdot,\cdot)$. As a vector space, $\hat{\mathfrak{g}}_k$ is equal to the direct sum $\tilde{\mathfrak{g}}_k + \mathbf{C}c$, where c represents $1 \in \mathbf{C}$. And the bracket product $[\cdot,\cdot]$ on $\hat{\mathfrak{g}}_k$ is given by

$$[f_1 + r_1 c, f_2 + r_2 c] = [f_1, f_2]_0 + Z(f_1, f_2)c \qquad \text{for } f_i \in \tilde{\mathfrak{g}}_k,\ r_i \in \mathbf{C}\ (i = 1, 2)\,.$$

By definition, $Z(\mathfrak{g}, \tilde{\mathfrak{g}}_k) = 0$, whence $\mathfrak{g}$ can be regarded as a subalgebra of $\hat{\mathfrak{g}}_k$.

We define a norm $|\cdot|_k$ on $\hat{\mathfrak{g}}_k$ by

$$|f + rc|_k := \max(|f|_k, |r|) \qquad \text{for } f \in \tilde{\mathfrak{g}}_k,\ r \in \mathbf{C}\,.$$

Thanks to the continuity of $Z(\cdot,\cdot)$, $(\hat{\mathfrak{g}}_k, |\cdot|_k)$ is a Banach Lie algebra, and $\tilde{\mathfrak{g}}_k$ and $\mathbf{C}c$ are closed subspaces of $\hat{\mathfrak{g}}_k$.

We extend the bilinear form $B(\cdot,\cdot)$ to $\hat{\mathfrak{g}}_k$ trivially on the center $\mathbf{C}c$, and *-operation by

$$(f + rc)^* := f^* + \bar{r}c \qquad \text{for } f \in \tilde{\mathfrak{g}}_k,\ r \in \mathbf{C}.$$

The following lemma is easily verified from definition.

Lemma 3.3. i) $[\tilde{\mathfrak{g}}_k, \tilde{\mathfrak{g}}_k]_0 = \tilde{\mathfrak{g}}_k$.
ii) $[\tilde{\mathfrak{g}}_k, \tilde{\mathfrak{g}}_k] = \hat{\mathfrak{g}}_k$, *and hence* $[\hat{\mathfrak{g}}_k, \hat{\mathfrak{g}}_k] = \hat{\mathfrak{g}}_k$.

Remark 3.4. The 2-cocycle $Z(\cdot,\cdot)$ coincides with the usual one (cf., [4], 7.1 and [3], 1.1) on the dense subalgebra $\tilde{\mathfrak{g}} := \mathfrak{g} \otimes \mathbf{C}[\zeta, \zeta^{-1}] \subset \tilde{\mathfrak{g}}_k$, where $\zeta: S^1 \to S^1 \subset \mathbf{C}$ is the identity map. Hence, the Banach Lie algebra $\hat{\mathfrak{g}}_k$ contains the usual affine Lie algebra $\hat{\mathfrak{g}}$ as a dense subalgebra.

§4. An action of a loop group on the corresponding completed affine Lie algebra

As in the preceding section, the completed affine Lie algebra $\hat{\mathfrak{g}}_k$ is a non-trivial 1-dimensional central extension of the Lie algebra $\tilde{\mathfrak{g}}_k$ of C^k-loops in a finite-dimensional complex simple Lie algebra $\mathfrak{g}$.

In the present section, we define—and describe explicitly—an extension of the adjoint action on $\tilde{\mathfrak{g}}_k$ of the corresponding loop group $\tilde{G}_k$ to that on $\hat{\mathfrak{g}}_k$.

We further extend the action on $\hat{\mathfrak{g}}_k$ to that on a Lie algebra $\hat{\mathfrak{g}}_k^e$, called the extended affine Lie algebra, after the terminology in [2]. $\hat{\mathfrak{g}}_k^e$ is a semi-direct sum of $\hat{\mathfrak{g}}_k$ and the "derivation" ∂ on $\hat{\mathfrak{g}}_k$, and is the simplest Kac-Moody algebra of infinite-dimension.

4.1. A 1-cocycle for the adjoint action of loop group $\tilde{G}_k$. Let k be a positive integer. Let $\tilde{G}_k = G(L_k)$ be the group of C^k-loops in the finite-dimensional connected and simply connected complex simple Lie group G, and let Ad_0 be the adjoint action of $\tilde{G}_k$ on the loop algebra $\tilde{\mathfrak{g}}_k$, which is denoted in §2 simply by Ad. Put

$$\mathrm{Aut}\,(\tilde{\mathfrak{g}}_k) := \{T \in L(\tilde{\mathfrak{g}}_k, \tilde{\mathfrak{g}}_k)^\times;\ T[x, y] = [Tx, Ty] \qquad \text{for } {}^\forall x,\ y \in \tilde{\mathfrak{g}}_k\}.$$

We define $\mathrm{Aut}\,(\hat{\mathfrak{g}}_k)$ in the same way. The group $\mathrm{Ad}_0\,(\tilde{G}_k)$ is a subgroup of $\mathrm{Aut}\,(\tilde{\mathfrak{g}}_k)$.

At first, we calculate how deforms the 2-cocycle $Z(\cdot,\cdot)$ by the Ad_0-action of $\tilde{G}_k$.

Lemma 4.1. *For each g in $\tilde{G}_k$, there exists a unique element z_g in $\tilde{\mathfrak{g}}_{k-1}$ such that*

$$Z(\mathrm{Ad}_0\,(g)x, \mathrm{Ad}_0\,(g)y) = Z(x, y) + B(z_g, [x, y]_0) \qquad \textit{for } {}^\forall x,\ y \in \tilde{\mathfrak{g}}_k.$$

Proof. The uniqueness of z_g is clear if it exists, because $B(\cdot, \cdot)$ is non-degenerae.

Since the set of g satisfying the lemma forms a subgroup of $\tilde{G}_k$, it is enough to prove the assertion for members of a generating set of $\tilde{G}_k$. We can take $\exp(fz)$, $z \in \mathfrak{g}$, $f \in L_k$, as generators of $\tilde{G}_k$, because G is assumed to be simply connected (cf., Theorem 2.5 ii)).

For an element $g = \exp(fz)$, expanding $\mathrm{Ad}_0(\exp(fz))$ into the series $\sum_m \frac{1}{m!} f^m(\mathrm{ad}_0\, z)^m$ and using the invariance of $B(\cdot, \cdot)$, we have the equality

$$B(\partial(\mathrm{Ad}_0(\exp(fz))x), \mathrm{Ad}_0(\exp(fz))y) = B([(\partial f)z, x]_0 + \partial x, y)\,.$$

Again by invariance of $B(\cdot, \cdot)$, the right hand side of this equality equals $Z(x, y) + B((\partial f)z, [x, y]_0)$. And so the assertion of the lemma is valid with $z_g = (\partial f)z$.

Q.E.D.

Corollary 4.1.1 (of Proof).

i) $z_{\exp(fz)} = (\partial f)z$ *for* $^\forall f \in L_k$, $^\forall z \in \mathfrak{g}$.

ii) *The mapping* $\tilde{G}_k \ni g \to z_g \in \tilde{\mathfrak{g}}_{k-1}$ *is a normalized* 1-*cocycle for the* Ad_0-*action of* $\tilde{G}_k$, *that is, it holds that*

$$z_1 = 0\,, \qquad z_{g^{-1}} = -\mathrm{Ad}_0(g)z_g \qquad \textit{for } ^\forall g \in \tilde{G}_k\,,$$

$$z_{g_1 g_2} = z_{g_2} + \mathrm{Ad}_0(g_2^{-1})z_{g_1} \qquad \textit{for } ^\forall g_1, g_2 \in \tilde{G}_k\,.$$

Corollary 4.1.2. *The mapping* $\tilde{G}_k \ni g \to z_g \in \tilde{\mathfrak{g}}_{k-1}$ *is of class* C^∞.

Proof. Thanks to Corollary 4.1.1 ii), it is enough to prove the assertion in a neighbourhood of 1 in $\tilde{G}_k$.

Let $x_1, \ldots, x_n$ be a basis of $\mathfrak{g}$. For a sufficiently small $\varepsilon > 0$, the pair of the set

$$U := \{e^{f_1 x_1} \ldots e^{f_n x_n}; f = (f_1, \ldots, f_n) \in (L_k)^n, |f|_k < \varepsilon\}\,,$$

and the mapping

$$U \ni e^{f_1 x_1} \ldots e^{f_n x_n} \to (f_1, \ldots, f_n) \in (L_k)^n$$

is a local coordinate of 1 in $\tilde{G}_k$.

For $g = e^{f_1 x_1} \ldots e^{f_n x_n}$, by Corollary 4.1.1, it holds the equality

$$z_g = (\partial f_n)x_n + \mathrm{Ad}_0(e^{-f_n x_n})((\partial f_{n-1})x_{n-1}) + \cdots + \mathrm{Ad}_0(e^{-f_n x_n} \ldots e^{-f_2 x_2})((\partial f_1)x_1)\,.$$

Since the mapping $\partial\colon L_k \to L_{k-1}$ is continuous and linear, it is of class C^∞. Moreover, Ad_0 is a C^∞-mapping from $\tilde{G}_k$ into $GL(\tilde{\mathfrak{g}}_{k-1})$. Hence, the above equality implies the assertion. Q.E.D.

4.2. An extension of Ad_0-action of $\tilde{G}_k$ to $\hat{\mathfrak{g}}_k$. Now, we can extend the adjoint action of $\tilde{G}_k$ on $\tilde{\mathfrak{g}}_k$ to the central extension $\hat{\mathfrak{g}}_k$.

For each $g \in \tilde{G}_k$, define the linear operator $\mathrm{Ad}(g)$ on $\hat{\mathfrak{g}}_k$ by

$$\mathrm{Ad}(g)(x + rc) := \mathrm{Ad}_0(g)x + (B(z_g, x) + r)c \qquad \text{for } x \in \tilde{\mathfrak{g}}_k,\ r \in \mathbf{C},$$

This definition gives actually an action of $\tilde{G}_k$ on $\tilde{\mathfrak{g}}_k$ as follows.

Theorem 4.2. *$\mathrm{Ad}: g \to \mathrm{Ad}(g)$ is a group-homomorphism of $\tilde{G}_k$ into* $\mathrm{Aut}(\hat{\mathfrak{g}}_k)$ *of class C^∞.*

Proof. Let g be an arbitrary element in $\tilde{G}_k$. By Lemma 4.1, it holds that, for any $x, y \in \tilde{\mathfrak{g}}_k$,

$$\begin{aligned}
[\mathrm{Ad}(g)x, \mathrm{Ad}(g)y] &= [\mathrm{Ad}_0(g)x, \mathrm{Ad}_0(g)y]_0 + Z(\mathrm{Ad}_0(g)x, \mathrm{Ad}_0(g)y)c \\
&= \mathrm{Ad}_0(g)[x, y]_0 + Z(x, y)c + B(z_g, [x, y])c \\
&= \mathrm{Ad}(g)\{[x, y]_0 + Z(x, y)c\} = \mathrm{Ad}(g)[x, y].
\end{aligned}$$

Hence, $\mathrm{Ad}_0(g)$ is an element of $\mathrm{Aut}(\hat{\mathfrak{g}}_k)$.

Let g' be another element of $\tilde{G}_k$. By definition, we have

$$\begin{aligned}
\mathrm{Ad}(g)\,\mathrm{Ad}(g')x &= \mathrm{Ad}(g)\{\mathrm{Ad}_0(g')x + B(z_{g'}, x)c\} \\
&= \mathrm{Ad}_0(g)\,\mathrm{Ad}_0(g')x + B(z_g, \mathrm{Ad}_0(g')x)c + B(z_{g'}, x)c.
\end{aligned}$$

By invariance of $B(\cdot, \cdot)$ and Corollary 4.1.1 ii), for any $x \in \tilde{\mathfrak{g}}_k$, this equals

$$\mathrm{Ad}_0(gg')x + B(z_{gg'}, x)c = \mathrm{Ad}(gg')x,$$

and so Ad is a group homomorphism.

The smoothness of Ad follows from Corollary 4.1.2. Q.E.D.

For each $x \in \tilde{\mathfrak{g}}_k$, we obtain a 1-parameter group $\mathrm{Ad}(e^{tx})$, $t \in \mathbf{R}$, of operators on $\tilde{\mathfrak{g}}_k$. This 1-parameter group is equal to the usual one defined by the adjoint action of the Lie algebra $\hat{\mathfrak{g}}_k$, that is, there holds that

Lemma 4.3. $\mathrm{Ad}(e^x) = e^{\mathrm{ad}\,x} := \sum_{m \geq 0} \frac{1}{m!}(\mathrm{ad}\,x)^m$ *for* $\forall x \in \tilde{\mathfrak{g}}_k$.

Proof. It is enough to prove that the infinitesimal generator of the 1-parameter group $\mathrm{Ad}(e^{tx})$, $t \in \mathbf{R}$, is equal to $\mathrm{ad}\,x$. For any $y, z \in \tilde{\mathfrak{g}}_k$, by definition of $z_{e^{tx}}$, we have

$$\begin{aligned}
B\left(\frac{d}{dt}(z_{e^{tx}})\bigg|_{t=0}, [y, z]_0\right) &= \frac{d}{dt}B(z_{e^{tx}}, [y, z]_0)\bigg|_{t=0} \\
&= \frac{d}{dt}Z(\mathrm{Ad}_0(e^{tx})y, \mathrm{Ad}_0(e^{tx})z)\bigg|_{t=0} \\
&= Z([x, y]_0, z) + Z(y, [x, z]_0) \\
&= Z(x, [y, z]_0) = B(\partial x, [y, z]_0).
\end{aligned}$$

Since $B(\cdot, \cdot)$ is non-degenerate and $\tilde{\mathfrak{g}}_k$ is generated by the elements of the form $[y, z]$, y, $z \in \tilde{\mathfrak{g}}_K$, it holds that

$$\frac{d}{dt}(z_{e^{tx}})\bigg|_{t=0} = \partial x .$$

Hence, for any $y \in \tilde{\mathfrak{g}}_k$, we have by definition of $\mathrm{Ad}\,(e^{tx})$ on $\hat{\mathfrak{g}}_k$

$$\frac{d}{dt}\mathrm{Ad}\,(e^{tx})y\bigg|_{t=0} = [x, y]_0 + B(\partial x, y)c = [x, y] . \qquad \text{Q.E.D.}$$

Corollary 4.3.1 (of Proof).

$$\frac{d}{dt}(z_{e^{tx}}) = \mathrm{Ad}_0\,(e^{-tx})\partial x \qquad \textit{for } {}^{\forall}x \in \tilde{\mathfrak{g}}_k .$$

Proof. By Corollary 4.1.1 ii), we have

$$\frac{d}{dt}z_{e^{tx}} = \mathrm{Ad}_0\,(e^{tx})\left\{\left(\frac{d}{dt}z_{e^{tx}}\right)\bigg|_{t=0}\right\} .$$

And, as we showed in the proof of Lemma 4.3, $\dfrac{d}{dt}z_{e^{tx}}\bigg|_{t=0}$ equals ∂x. Q.E.D.

4.3. Extended affine Lie algebras $\hat{\mathfrak{g}}_k^e$. Since the 2-cocycle $Z(\cdot, \cdot)$ is ∂-invariant:

$$Z(\partial x, y) + Z(x, \partial y) = 0 \qquad \text{for } {}^{\forall}x,\ y \in \tilde{\mathfrak{g}}_k ,$$

∂ defines, by $\partial c = 0$, a continuous linear map from $\hat{\mathfrak{g}}_k$ into $\hat{\mathfrak{g}}_{k-1}$, denoted by the same symbol ∂, and satisfies the derivation property, that is, it holds that

$$[\partial x, y] + [x, \partial y] = 0 \qquad \text{for } {}^{\forall}x,\ y \in \hat{\mathfrak{g}}_k .$$

We put $\hat{\mathfrak{g}}_k^e := \mathbf{C}\partial + \hat{\mathfrak{g}}_k$, and extend the bracket product on $\hat{\mathfrak{g}}_k$ to a bilinear map $\hat{\mathfrak{g}}_k^e \times \hat{\mathfrak{g}}_k^e \ni (x, y) \to [x, y] \in \hat{\mathfrak{g}}_{k-1}^e$ by

$$[r_1\partial + x_1, r_2\partial + x_2] := r_1\partial x_2 - r_2\partial x_1 + [x_1, x_2] \qquad \text{for } r_1,\ r_2 \in \mathbf{C},\ x_1,\ x_2 \in \hat{\mathfrak{g}}_k .$$

In the same way, we also extend the bracket product $[\cdot, \cdot]_0$ on $\tilde{\mathfrak{g}}_k$ to a bilinear map $\tilde{\mathfrak{g}}_k^e \times \tilde{\mathfrak{g}}_k^e \to \tilde{\mathfrak{g}}_{k-1}^e$, where $\tilde{\mathfrak{g}}_k^e := \mathbf{C}\partial + \tilde{\mathfrak{g}}_k$.

Though $\hat{\mathfrak{g}}_k^e$ is not really a Lie algebra, we call $\hat{\mathfrak{g}}_k^e$ the extended affine Lie algebra after the terminology in [2] for simplicity, because the bilinear map $[\cdot, \cdot]\colon \hat{\mathfrak{g}}_k^e \times \hat{\mathfrak{g}}_k^e \to \hat{\mathfrak{g}}_{k-1}^e$, satisfies the properties of the usual bracket product, antisymmetricity and Jacobi identity. From the same reason, we call "action" the linear map $\mathrm{Ad}\,(g)\colon \hat{\mathfrak{g}}_k^e \to \hat{\mathfrak{g}}_{k-1}^e$ defined below for each $g \in \tilde{G}_k$.

We obtain the relation between the action of ∂ and those of elements in $\tilde{G}_k$ under Ad_0 as follows.

Lemma 4.4. *For each element g in $\tilde{G}_k$, there holds that*

$$\mathrm{Ad}_0\,(g) \circ \partial \circ \mathrm{Ad}_0\,(g^{-1}) = \partial + \mathrm{ad}_0\,(z_{g^{-1}}) .$$

Proof. Because of the invariance of $B(\cdot, \cdot)$, for any $x, y \in \tilde{\mathfrak{g}}_k$, we have the equalities

$$\begin{aligned} B((\partial + \mathrm{ad}_0 (z_{g^{-1}}))x, y) &= Z(x, y) + B(z_{g^{-1}}, [x, y]_0) \\ &= Z(\mathrm{Ad}_0 (g^{-1})x, \mathrm{Ad}_0 (g^{-1})y) \\ &= B(\partial \, \mathrm{Ad}_0 (g^{-1})x, \mathrm{Ad}_0 (g^{-1})y) \\ &= B((\mathrm{Ad}_0 (g) \circ \partial \circ \mathrm{Ad}_0 (g^{-1}))x, y) , \end{aligned}$$

and this implies the lemma, because $B(\cdot, \cdot)$ is non-degenerate. Q.E.D.

For each $g \in \tilde{G}_k$, we extend the operator $\mathrm{Ad}\,(g)$ on $\hat{\mathfrak{g}}_k$ to the linear map from $\hat{\mathfrak{g}}_k^e$ into $\hat{\mathfrak{g}}_{k-1}^e$, by

$$\mathrm{Ad}\,(g)\partial := \partial + z_{g^{-1}} - \tfrac{1}{2}B(z_g, z_g)c ,$$

Lemma 4.5. *Let* $k \geqq 2$.
i) *For any* $g \in \tilde{G}_k$, *it holds that*

$$[\mathrm{Ad}\,(g)x, \mathrm{Ad}(g)y] = \mathrm{Ad}\,(g)[x, y] \qquad \textit{for } {}^\forall x,\ y \in \hat{\mathfrak{g}}_k^e .$$

ii) *Moreover, letting* g' *be another element of* $\tilde{G}_k$, *there holds the equality*

$$\mathrm{Ad}\,(g)\,\mathrm{Ad}\,(g')x = \mathrm{Ad}\,(gg')x \qquad \textit{for } {}^\forall x \in \hat{\mathfrak{g}}_k^e .$$

Proof. It is enough to prove the two equalities
(1) $[\mathrm{Ad}\,(g)\partial, \mathrm{Ad}\,(g)x] = \mathrm{Ad}\,(g)\partial x$ for ${}^\forall x \in \tilde{\mathfrak{g}}_k$,
(2) $\mathrm{Ad}\,(g)\,\mathrm{Ad}\,(g')\partial = \mathrm{Ad}\,(gg')\partial$.
The left hand side of (1) equals

$$[\partial + z_{g^{-1}}, \mathrm{Ad}_0 (g)x] = \partial\, \mathrm{Ad}_0 (g)x + [z_{g^{-1}}, \mathrm{Ad}_0 (g)x]_0 + Z(z_{g^{-1}}, \mathrm{Ad}_0 (g)x)c .$$

This is equal, by Corollary 4.1.1 ii) and Lemma 4.4, to

$$\begin{aligned} &\partial(\mathrm{Ad}_0 (g)x) + \mathrm{Ad}_0 (g)[-z_g, x]_0 + Z(z_{g^{-1}}, \mathrm{Ad}_0 (g)x)c \\ &= \mathrm{Ad}_0 (g)(\partial x + [z_g, x]_0) + \mathrm{Ad}_0 (g)[-z_g, x]_0 + Z(z_{g^{-1}}, \mathrm{Ad}_0(g)x)c \\ &= \mathrm{Ad}_0 (g)\partial x + B(\partial z_{g^{-1}}, \mathrm{Ad}_0 (g)x)c . \end{aligned}$$

By invariance of $B(\cdot, \cdot)$, Corollary 4.1.1 ii), and Lemma 4.4,

$$\begin{aligned} &= \mathrm{Ad}_0 (g)\partial x + B(\mathrm{Ad}_0 (g)(\partial x + [z_g, x]_0), \mathrm{Ad}_0\, z_g)c \\ &= \mathrm{Ad}_0 (g)\partial x + B(\partial x, z_g)c + B([z_g, x]_0, z_g)c = \mathrm{Ad}\,(g)\partial x , \end{aligned}$$

and thus (1) holds.

The left hand side of (2) is equal, by Corollary 4.1.1 ii), to

$$\mathrm{Ad}\,(g)\{\partial + z_{g'^{-1}} - \tfrac{1}{2}B(z_{g'}, z_{g'})c\} = \mathrm{Ad}\,(g)\partial + \mathrm{Ad}\,(g)z_{g'^{-1}} - \tfrac{1}{2}B(z_{g'}, z_{g'})c .$$

By definition of $\mathrm{Ad}(g)$

$$= \partial + z_{g^{-1}} - \tfrac{1}{2}B(z_g, z_g)c + \mathrm{Ad}_0(g)(z_{g'^{-1}}) + B(z_g, z_{g'^{-1}})c - \tfrac{1}{2}B(z_{g'}, z_{g'})c$$

$$= \partial + z_{(gg')^{-1}} + B(z_g, z_{g'^{-1}})c - \tfrac{1}{2}\{B(z_g, z_g) + B(z_{g'}, z_{g'})\}c\,.$$

On the other hand, the right hand side of (2) is equal, by invariance and symmetricity of $B(\cdot, \cdot)$ and again by Corollary 4.1.1 ii), to

$$\begin{aligned}&\partial + z_{(gg')^{-1}} - \tfrac{1}{2}B(z_{gg'}, z_{gg'})c\\ &\quad = \partial + z_{(gg')^{-1}} - \tfrac{1}{2}\{B(z_g, z_g) + B(z_{g'}, z_{g'}) + 2B(\mathrm{Ad}_0(g'^{-1})z_g, z_{g'})\}c\\ &\quad = \partial + z_{(gg')^{-1}} - \tfrac{1}{2}\{B(z_g, z_g) + B(z_{g'}, z_{g'})\}c + B(z_g, z_{g'-1})c\,,\end{aligned}$$

and so (2) holds. Q.E.D.

Since this new Ad-action stabilizes all the central elements in $\hat{\mathfrak{g}}_k^e$, it defines an action of $\tilde{G}_k$ on $\tilde{\mathfrak{g}}_k^e = \hat{\mathfrak{g}}_k^e/\mathbf{C}c$. Obviously, this new action on $\tilde{\mathfrak{g}}_k^e$ is an extension of Ad_0 on $\tilde{\mathfrak{g}}_k$. So, we denote it by the same symbol Ad_0. By definition, we have

Lemma 4.6. $\mathrm{Ad}_0(g)\partial = \partial + z_{g^{-1}}$ *for* $\forall g \in \tilde{G}_k$.

§5. Weyl group of the completed affine Lie algebra

In this section, as an application of the results in §4, we calculate explicit forms of the normalizers and centralizers, in the loop group $\tilde{G}_k$, of the Cartan subalgebras of the affine Lie algebras $\hat{\mathfrak{g}}_k$ and $\hat{\mathfrak{g}}_k^e$. We see that, in both cases, the quotient groups of the normalizers by the centralizers are canonically isomorphic to the usual affine Weyl group.

5.1. Cartan subalgebras of $\hat{\mathfrak{g}}_k$ and $\hat{\mathfrak{g}}_k^e$. The dense subalgebra $\hat{\mathfrak{g}}^e := \hat{\mathfrak{g}} + \mathbf{C}\partial$ of $\hat{\mathfrak{g}}_k^e$, is one of Kac-Moody algebras of affine type with tier number 1 (see [4], 7.1), and its Cartan subalgebra $\hat{\mathfrak{h}}^e$ is given by

$$\hat{\mathfrak{h}}^e := \mathfrak{h} + \mathbf{C}c + \mathbf{C}\partial\,.$$

We denote by $\hat{\mathfrak{h}}$ the intersection of $\hat{\mathfrak{h}}^e$ with $\hat{\mathfrak{g}} = [\hat{\mathfrak{g}}^e, \hat{\mathfrak{g}}^e]$. Clearly, it holds that

$$\hat{\mathfrak{h}} = \mathfrak{h} + \mathbf{C}c\,.$$

Moreover, we see that $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}^e$ are maximal abelian subalgebras of $\hat{\mathfrak{g}}_k$ and $\hat{\mathfrak{g}}_k^e$ respectively.

We define one more commutative Lie algebra $\mathfrak{h}^e$ by

$$\mathfrak{h}^e := \mathfrak{h} + \mathbf{C}\partial\,.$$

Then $\mathfrak{h}^e$ is an abelian subalgebra of $\tilde{\mathfrak{g}}_k^e$.

5.2. Centralizers in $\tilde{G}_k$ of Cartan subalgebras. Define subgroups $\tilde{Z}$, $\hat{Z}$, $\tilde{Z}^e$, and $\hat{Z}^e$ of $\tilde{G}_k$ by

$$\tilde{Z} := \{g \in \tilde{G}_k;\ \mathrm{Ad}_0(g)h = h \quad \text{for } {}^\forall h \in \mathfrak{h}\}\,,$$

$$\hat{Z} := \{g \in \tilde{G}_k;\ \mathrm{Ad}(g)h = h \quad \text{for } {}^\forall h \in \hat{\mathfrak{h}}\}\,,$$

$$\tilde{Z}^e := \{g \in \tilde{G}_k;\ \mathrm{Ad}_0(g)h = h \quad \text{for } {}^\forall h \in \mathfrak{h}^e\}\,,$$

$$\hat{Z}^e := \{g \in \tilde{G}_k;\ \mathrm{Ad}(g)h = h \quad \text{for } {}^\forall h \in \hat{\mathfrak{h}}^e\}\,.$$

Let $H := \exp \mathfrak{h}$ be the Cartan subgroup of G corresponding to $\mathfrak{h}$, and $\tilde{H}_k := H(L_k) \subset \tilde{G}_k$. We can easily show

Lemma 5.1. $\tilde{Z} = \tilde{H}_k$.

Next, to determine other centralizers, we calculate z_g for $g \in \tilde{H}_k$ as follows. By Proposition 3.1 and Theorem 2.8, we see that

$$(L_k)^\times \times \cdots \times (L_k)^\times \ni (f_1, \dots, f_l) \to h_{\alpha_1}(f_1) \dots h_{\alpha_l}(f_l) \in \tilde{H}_k$$

is a Lie group isomomorphism. Here, for $\alpha \in \Delta$ and $f \in (L_k)^\times$,

$$h_\alpha(f) := w_\alpha(1)^{-1} w_\alpha(f)\,,$$

$$w_\alpha(f) := \exp(fx_\alpha)\exp(-f^{-1}x_{-\alpha})\exp(fx_\alpha)\,,$$

and $h_1, \dots, h_l \in \mathfrak{h}$, $x_\alpha \in \mathfrak{g}^\alpha$ ($\alpha \in \Delta$), are the Chevalley basis of $\mathfrak{g}$ as in § 3. We have

Lemma 5.2. *$z_{h_\alpha(f)} = -f^{-1}(\partial f)h_\alpha$ for ${}^\forall \alpha \in \Delta$, ${}^\forall f \in (L_k)^\times$, where $h_\alpha := [x_\alpha, x_{-\alpha}] \in \mathfrak{h}$.*

Proof. Because h_α, x_α, and $x_{-\alpha}$ form an sl_2-triplet, there hold the equalities

$$\mathrm{Ad}_0(\exp(fx_\alpha))x_{-\alpha} = x_{-\alpha} + fh_\alpha - f^2 x_\alpha\,,$$

$$\mathrm{Ad}_0(\exp(fx_\alpha))h_\alpha = h_\alpha - 2fx_\alpha\,,$$

$$\mathrm{Ad}_0(\exp(-f^{-1}x_\alpha))h_\alpha = h_\alpha - 2f^{-1}x_\alpha\,,$$

$$\mathrm{Ad}_0(\exp(-f^{-1}x_\alpha))x_\alpha = x_\alpha + f^{-1}h_\alpha - f^{-2}x_{-\alpha}\,.$$

Further, by Corollary 4.1.1 i) and Lemma 4.4, we have

$$\mathrm{Ad}_0(\exp(fx_\alpha))\partial = \partial - (\partial f)x_\alpha\,,$$

$$\mathrm{Ad}_0(\exp(-f^{-1}x_{-\alpha}))\partial = \partial + \partial(f^{-1})x_{-\alpha} = \partial - f^{-2}(\partial f)x_{-\alpha}\,.$$

Hence, it holds that

$$\begin{aligned}\mathrm{Ad}_0(w_\alpha(f))\partial &= \mathrm{Ad}_0(\exp(fx_\alpha))\,\mathrm{Ad}_0(\exp(-f^{-1}x_{-\alpha}))(\partial - (\partial f)x_\alpha)\\ &= \mathrm{Ad}_0(\exp(fx_\alpha))(\partial - (\partial f)x_\alpha - f^{-1}(\partial f)h_\alpha)\\ &= \partial - f^{-1}(\partial f)h_\alpha\,,\end{aligned}$$

and so,

$$\begin{aligned}\mathrm{Ad}_0(h_\alpha(f))\partial &= \mathrm{Ad}_0(e^{-x_\alpha}e^{x_{-\alpha}}e^{-x_\alpha})(\partial - f^{-1}(\partial f)h_\alpha)\\ &= \mathrm{Ad}_0(e^{-x_\alpha}e^{x_{-\alpha}})(\partial - f^{-1}(\partial f)h_\alpha - 2f^{-1}(\partial f)x_\alpha)\\ &= \mathrm{Ad}_0(e^{-x_\alpha})(\partial + f^{-1}(\partial f)h_\alpha - 2f^{-1}(\partial f)x_\alpha)\\ &= \partial + f^{-1}(\partial f)h_\alpha .\end{aligned}$$

On the other hand, by Lemma 4.4, $\mathrm{Ad}_0(h_\alpha(f))\partial$ is equal to $\partial + z_{h_\alpha(f)^{-1}} = \partial + z_{h_\alpha(f^{-1})}$. So, we have

$$z_{h_\alpha(f)} = (f^{-1})^{-1}(\partial(f^{-1}))h_\alpha = -f^{-1}(\partial f)h_\alpha . \qquad \text{Q.E.D.}$$

From this lemma, and Corollary 4.1.1 ii), it follows immediately that

Proposition 5.3. *For* $^\forall f_1, \ldots, f_l \in (L_k)^\times$,

$$z_{h_{\alpha_1}(f_1)\ldots h_{\alpha_l}(f_l)} = -f_1^{-1}(\partial f_1)h_1 - \cdots - f_l^{-1}(\partial f_l)h_l .$$

Now, we can describe the centralizers explicitly as follows.

Theorem 5.4.
i) $\hat{Z} = \{h_{\alpha_1}(f_1)\ldots h_{\alpha_l}(f_l); f_1, \ldots, f_l \in \exp(L_k)\}$.
ii) $\tilde{Z}^e = \hat{Z}^e = H$.

Proof. i) Let $g \in \hat{Z}$. By definition, $\mathrm{Ad}(g)h = \mathrm{Ad}_0(g)h + B(z_g, h)c$ for any $h \in \mathfrak{h}$, and so $g \in H(L_k)$ by Lemma 5.1. Hence, g is written as

$$(*) \qquad g = h_{\alpha_1}(f_1)\ldots h_{\alpha_l}(f_l) \qquad \text{for } ^\exists f_1, \ldots, f_l \in (L_k)^\times .$$

Therefore, by Proposition 5.3, we have

$$\hat{Z} = \{h_{\alpha_1}(f_1)\ldots h_{\alpha_l}(f_l); \textstyle\sum_i B(f_i^{-1}(\partial f_i)h_i, h) = 0 \text{ for } ^\forall h \in \mathfrak{h}\} .$$

For any $f \in (L_k)^\times$, the integral

$$\frac{1}{2\pi\sqrt{-1}}\int_0^1 f(e^{2\pi\sqrt{-1}\theta})^{-1}(\partial f)(e^{2\pi\sqrt{-1}\theta})d\theta$$

is equal to 0 if and only if the winding number of f around the origin of $\mathbf{C}$ is equal to 0. And this is equivalent, by Theorem 2.5, to the condition that f is in the connected component of 1 in $(L_k)^\times = \mathbf{C}^\times(L_k)$.

On the other hand, $\exp(L_k)$ is an connected open subgroup of $(L_k)^\times$, and so coincides with the connected component of 1.

Therefore, i) follows from the definition of $B(\cdot, \cdot)$ on $\tilde{\mathfrak{g}}_k$, and the two facts that $\{h_1, \ldots, h_l\}$ is a basis of $\mathfrak{h}$ and that $B(\cdot, \cdot)$ is non-degenerate on $\mathfrak{h} \times \mathfrak{h}$.

ii) Let g be an element in $\tilde{Z}^e$. By the same reason for $\hat{Z}$, g is of the form $(*)$. Since g centralizes ∂, z_g is equal to 0 by definition. Hence, by Proposition 5.3, it holds that $-f_i^{-1}(\partial f_i) = 0$ for $^\forall i = 1, \ldots, l$, and so $\partial f_i = 0$ for $^\forall i$. This means that all the f_i's are constant functions, and hence g belongs to

H. Therefore, $\tilde{Z}^e$ is contained in H. The inclusion $\hat{Z}^e \subset H$ is proved in the same manner, and the converse inclusions are obvious. Q.E.D.

5.3. Normalizers of Cartan subalgebras. We denote by $\tilde{N}$ (resp. $\hat{N}$, $\tilde{N}^e$, and $\hat{N}^e$) the normalizers in $\tilde{G}_k$ of $\mathfrak{h}$ (resp. $\hat{\mathfrak{h}}$, $\mathfrak{h}^e$, and $\hat{\mathfrak{h}}^e$):

$$\tilde{N} := \{g \in \tilde{G}_k;\ \mathrm{Ad}_0(g)\mathfrak{h} \subset \mathfrak{h}\},$$

$$\hat{N} := \{g \in \tilde{G}_k;\ \mathrm{Ad}(g)\hat{\mathfrak{h}} \subset \hat{\mathfrak{h}}\},$$

$$\tilde{N}^e := \{g \in \tilde{G}_k;\ \mathrm{Ad}_0(g)\mathfrak{h}^e \subset \mathfrak{h}^e\},$$

$$\hat{N}^e := \{g \in \tilde{G}_k;\ \mathrm{Ad}(g)\hat{\mathfrak{h}}^e \subset \hat{\mathfrak{h}}^e\}.$$

Theorem 5.5. *Let N be the normalizer of $\mathfrak{h}$ in G as in* 3.1. *Then,*

i) $\tilde{N} = \hat{N} = N.\tilde{H}_k$,

ii) $\tilde{N}^e = \hat{N}^e = N.\{h_{\alpha_1}(C_1\zeta^{n_1})\dots h_{\alpha_l}(C_l\zeta^{n_l});\ n_1, \dots, n_l \in \mathbf{Z},\ C_1, \dots, C_l \in \mathbf{C}^\times\}$.

Proof. i) Obviously, $N.\tilde{H}_k$ normalizes $\mathfrak{h}$ and $\hat{\mathfrak{h}}$. And, since $\mathrm{Ad}(g)x - \mathrm{Ad}_0(g)x \in \mathbf{C}c \subset \hat{\mathfrak{h}}$ for any g and x, it is clear that $\tilde{N} = \hat{N}$.

Let $g \in \tilde{N}$. Then, for any $h \in \mathfrak{h}$, $\mathrm{Ad}_0(g)h$ belongs to $\mathfrak{h}$ and, in particular, $\mathrm{Ad}_0(g)h$ is a constant map. Hence, it holds that

$$\mathrm{Ad}(g(s))h = \mathrm{Ad}(g(1))h \qquad \text{for } {}^\forall s \in S^1$$

in $\mathfrak{g}$, and so $\mathrm{Ad}_0(g(1)^{-1}g)$ centralizes $\mathfrak{h}$. Therefore, $g_1 := g(1)^{-1}g$ belongs to $\tilde{H}_k$ by Lemma 5.1, whence $g = g(1)g_1$ is an element of $N.\tilde{H}_k$. So, $\tilde{N} \subset N.\tilde{H}_k$.

ii) By the same reason as for i), we see that $\tilde{N}^e = \hat{N}^e$.

Let $g \in \tilde{N}^e$. By definition of $\mathrm{Ad}_0(g)$ on $\tilde{\mathfrak{g}}_k^e$, $\mathrm{Ad}_0(g)$ normalizes $\mathfrak{h}$. Hence, by i), g is written as

$$g = g'.h_{\alpha_1}(f_1)\dots h_{\alpha_l}(f_l)$$

with $g' \in N$, $f_1, \dots, f_l \in (L_k)^\times$. And so, by Corollary 4.1.1 and Proposition 5.3, there holds that

$$\mathrm{Ad}_0(g)\partial = \partial + f_1^{-1}(\partial f_1)h_{\alpha_1} + \cdots + f_l^{-1}(\partial f_l)h_{\alpha_l}.$$

Since this belongs to $\mathfrak{h}^e$, there exist $n_1, \dots, n_l \in \mathbf{C}$ such that $f_i^{-1}(\partial f_i) = n_i$ for ${}^\forall i = 1, \dots, l$. These differential equations are easily solved as

$$f_i(e^{2\pi\sqrt{-1}\theta}) = C_i \exp(2\pi\sqrt{-1}n_i\theta)$$

with some constants $C_i \in \mathbf{C}^\times$ ($i = 1, \dots, l$). Hence, $n_i \in \mathbf{Z}$, because f_i are functions on S^1. Therefore, g belongs to the rightest side of ii).

Conversely, every element of the rightest side of ii) obviously normalizes $\mathfrak{h}^e$ and $\hat{\mathfrak{h}}^e$. Q.E.D.

Now, we consider the quotient groups $\hat{N}/\hat{Z}$ and $\hat{N}^e/\hat{Z}^e$.

By the above theorems, $\hat{N}^e \cap \hat{Z}$ equals to $H = \hat{Z}^e$. Moreover, every element in $\hat{N}$ is congruent with an element in $\hat{N}^e$ modulo $\hat{Z}$. Hence, $\hat{N}^e/\hat{Z}^e$ is isomorphic to $\hat{N}/\hat{Z}$ canonically.

Put $\hat{W} := \hat{N}^e/\hat{Z}^e$ and

$$T := \{h_{\alpha_1}(C_1\zeta^{n_1})\dots h_{\alpha_l}(C_l\zeta^{n_l});\, n_1, \dots, n_l \in \mathbf{Z},\, C_1, \dots, C_l \in \mathbf{C}^\times\}\,.$$

Since $N \cap T = H = \hat{Z}^e$ and N normalizes T, we have $\hat{W} = (N/H) \ltimes (T/H)$. From the proof of Theorem 5.5 ii), we see that the mapping $T \ni g \to \mathrm{Ad}_0(g)\partial - \partial$ gives an isomorphism of T/H onto the coroot lattice of $(\mathfrak{g}, \mathfrak{h})$. Thus, we get the following theorem.

Theorem 5.6. *The quotient groups $\hat{N}/\hat{Z}$ and $\hat{N}^e/\hat{Z}^e$ are both isomorphic to the affine Weyl group $\hat{W} = W \ltimes \check{Q}$ canonically, where W and $\check{Q}$ are the Weyl group and the coroot lattice of $(\mathfrak{g}, \mathfrak{h})$ respectively.*

Remark 5.7. In the case $k = \infty$, in [7], the Lie algebra $\mathfrak{h} \otimes_{\mathbf{C}} L_\infty$ and the group $\hat{Z} = \exp(\mathfrak{h} \otimes_{\mathbf{C}} L_\infty)$ play essential roles to realize the basic representation of the affine Lie algebra. In [1], the dense subalgebra $\mathfrak{h} \otimes_{\mathbf{C}} \mathbf{C}[\zeta, \zeta^{-1}]$ appears in the same context. The fact that $\hat{Z}$ appears as the centralizer of the Cartan subalgebra, seems to have a close relation with their works.

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