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Boundary variation and quasiconformal maps of Riemann surfaces

Dedicated to Professor Nobuyuki Suita on his sixtieth birthday

By

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Introduction and main results

Recently, the author has developed in [6] a method to obtain second variational formulas for fundamental quantities on Riemann surfaces under quasiconformal deformation. For instance, in the case of values of Green's functions, we have obtained Theorem 1 below. (See [6] for more details. Also see [4], [5] and [7].)

Let R_0 be an arbitrary Riemann surface, and $B(R_0)$ be the complex Banach space consisting of all Beltrami differentials on R_0 with the norm $\|\mu\|_\infty = \mathrm{ess.sup}_{p\in R}|\mu|(p)$. We consider a real 1-parameter family $\{\mu(t):\, t\in I\}$ in $B(R_0)$ such that

$$\mu(0)\equiv 0 \quad \text{and} \quad \|\mu(t)\|_\infty < 1 \qquad \text{for every } t,$$

and

$$\mu(t) \qquad \text{is smooth on } I.$$

(For the sake of simplicity, we consider in this paper smooth functions, which means ones of C^∞-class, only.) Let f_t be a quasiconformal mapping of R_0 to another R_t with the complex dilatation $\mu(t)$ for every t.

Fix a point p on R_0, and suppose that R_0 admits the Green's functions. Fix a simply connected neighborhood U_p of p in R_0 and a conformal mapping Z_p of U_p onto the unit disk $B=\{|z|<1\}$ such that $Z_p(p)=0$. Further assume that $\mu(t)=0$ on U_p for every t.

Let $g_t(\cdot,\, p)=g(\cdot,\, p_t;\, R_t)$ be a Green's function on R_t with the pole $p_t=f_t(p)$ for every t. Then we can define Robin's constant $\gamma(t)$ at p_t on R_t by setting

$$\gamma(t)=\lim_{z\to 0} g_t(f_t\circ(Z_p)^{-1}(z),\, p)+\log|z|$$

for every t, and can show the following

Theorem 1 ([6, Theorem 1′]). *Set* $\phi_{t,p} = -{*}dg_t(\cdot,\, p)+i\cdot dg_t(\cdot,\, p)$ *and* $\Phi_p(t)=\phi_{t,p}\circ f_t-\phi_{0,p}$, *where and in the sequel,* $\phi\circ f$ *is the pull-back of* ϕ *by* f. *Then*

$$\text{(0-1)} \qquad \frac{d^2\gamma}{dt^2}(0)=\frac{-1}{2\pi}\cdot\mathrm{Re}\iint_{R_0}\Big(\frac{d^2\mu}{dt^2}(0)\cdot\phi_{0,p}\wedge{*}\phi_{0,p}+\frac{d\Phi_p}{dt}(0)\wedge{*}\frac{d\Phi_p}{dt}(0)\Big).$$

Also note that, the same argument as in [6] gives the polarized version of Theorem 1 as follows.

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Fix another q on $R_0-\{p\}$. Also fix a simply connected neighborhood U_q of q in R_0 and a conformal mapping Z_q of U_q onto the unit disk $B=\{|z|<1\}$ such that $Z_q(q)=0$. As before, we also assume that $\mu(t)=0$ on U_q for every t.

Let $g_t(\cdot, q)$ be a Green's function on R_t with the pole $q_t=f_t(q)$ for every t. Then we can show, by the same argument as in [6], the following

Theorem 2. *Set* $\phi_{t,q}=-{}^*dg_t(\cdot, q)+i\cdot dg_t(\cdot, q)$ *and* $\Phi_q(t)=\phi_{t,q}\circ f_t-\phi_{0,q}$. *Then*

$$\dot{g}(p, q)\left(=\frac{dg(p_t, q_t ; R_t)}{dt}\Big|_{t=0}\right)=\frac{-1}{2\pi}\mathrm{Re}\iint_R \frac{d\mu}{dt}(0)\cdot\phi_p\wedge{}^*\phi_q, \tag{0-2}$$

and

$$\ddot{g}(p, q)\left(=\frac{d^2g(p_t, q_t ; R_t)}{dt^2}\Big|_{t=0}\right) \tag{0-3}$$

$$=\frac{-1}{2\pi}\cdot\mathrm{Re}\left(\frac{d^2\mu}{dt^2}(0)\cdot\phi_{0,p}\wedge{}^*\phi_{0,q}+\frac{d\Phi_p}{dt}(0)\wedge\frac{{}^*d\Phi_q}{dt}(0)\right).$$

Now the purpose of this paper is to show that variational formulas under quasiconformal deformation imply some general ones under boundary variation. As a consequence, formulas such as above contain, in particular, the classical Garabedian-Schiffer's formula under Hadamard variation (cf. [1] and [2]), as a special case, and some of Yamaguchi's formulas (cf. [8] and [9]) as well.

We start with definition of boundary variation of a Riemann surface. Let R be an arbitrary Riemann surface contained in another Riemann surface, say S. We assume that the relative boundary ∂R of R in S consists of a finite number of smooth simple closed curves, which we will denote by $\{\Gamma_j\}_{j=1}^n$.

For every Γ_j, fix a tubelar neighborhood U_j of Γ_j in S and a locally conformal mapping (or equivalently, a holomorphic immersion) φ_j of U_j into $\boldsymbol{C}$. Here we assume that U_j are mutually disjoint. Further, we consider a family $\{\gamma_j(\zeta, \varepsilon) ; \zeta\in\Gamma_j\}_{-1<\varepsilon<1}$, of smooth immersions of Γ_j into $\boldsymbol{C}$ such that γ_j is also smooth as a map of $\Gamma_j\times(-1, 1)$, and that $\gamma_j(\zeta, 0)=\varphi_j(\zeta)$ for every j.

Then for every $\varepsilon\in[-\eta, \eta]$ with a suficiently small positive η, $\Gamma_{j,\varepsilon}=\varphi_j^{-1}\circ\gamma_j(\cdot, \varepsilon)$ is well-defined and is a simple closed curve in S freely homotopic to Γ_j (in $R\cup(\bigcup_{j=1}^n U_j)$) for every j. Denote by R_ε the subsurface of S which is surrounded by $\{\Gamma_{j,\varepsilon}\}$ and corresponds to $R=R_0$.

Thus we have a family $\{R_\varepsilon\}_{-\eta<\varepsilon<\eta}$ of Riemann surfaces, which we call *a generalized boundary variation of* R (with respect to $\{\Gamma_{j,\varepsilon}\}$ or to $\{\gamma_j(\cdot, \varepsilon)\}$).

Here note that we can reconstruct the above variation in a more traditional manner. Since ∂R is compact, we can find a (sufficiently small) positive constant η such that $\Gamma_{j,\varepsilon}$ can be parametrized in the following form;

$$\varphi_j(\zeta)+c(\zeta, \varepsilon)\cdot\boldsymbol{n}(\zeta) \qquad (\zeta\in\Gamma_j)$$

for every $\varepsilon\in(-\eta, \eta)$, where $c(\zeta, \varepsilon)$ is a real smooth function such that $c(\cdot, 0)=0$ and $\boldsymbol{n}(\zeta)$ is the outer unit normal vector of $\varphi_j(\Gamma_j)$ at $\varphi_j(\zeta)$ in $\boldsymbol{C}$.

Thus in the above definition, we may prescribe a smooth function $c(\zeta, \varepsilon)$ on $\partial R\times(-\eta, \eta)$ such that $c(\cdot, 0)=0$, instead of $\{\Gamma_{j,\varepsilon}\}$.

Remark. When R is a bounded domain in $\boldsymbol{C}$ with compact smooth boundary ∂R and $c(\zeta, \varepsilon)=\varepsilon\cdot\rho(\zeta)$ with a smooth function ρ on ∂R, the corresponding boundary variation is nothing but the classical Hadamard variation of R.

In this paper, we consider only the case of values of Green's functions, for other cases can be discussed by using exactly the same argument (cf. [4] and [6]).

As in Theorem 2, fix two distinct points p and q on R. We may assume that p and q are contained in all R_{ε}. Let $g_{\varepsilon}(\cdot, p)$ and $g_{\varepsilon}(\cdot, q)$ are Green's functions on R_{ε} with the poles p and q, respectively. Also we denote g_0 simply by g in the sequel. Then we recognize that Theorem 2 implies the following formulas, whose proofs will be given in § 1.

Theorem 3.

(0-4)
$$\dot{g}(p, q)\left(=\frac{dg_{\varepsilon}(p, q)}{d\varepsilon}\bigg|_{\varepsilon=0}\right)=\frac{1}{2\pi}\int_{\partial R}\dot{c}\frac{\partial g(\cdot, p)}{\partial n}\cdot\frac{\partial g(\cdot, q)}{\partial n}ds,$$

and

(0-5)
$$\ddot{g}(p, q)\left(=\frac{d^2g_{\varepsilon}(p, q)}{d\varepsilon^2}\bigg|_{\varepsilon=0}\right)$$
$$=-\frac{1}{\pi}\iint_R d\dot{g}(\cdot, p)\wedge{}^*d\dot{g}(\cdot, q)+\frac{1}{2\pi}\int_{\partial R}(\ddot{c}-\kappa\dot{c}^2)\frac{\partial g(\cdot, p)}{\partial n}\cdot\frac{\partial g(\cdot, q)}{\partial n}ds,$$

where $\kappa(\zeta)$ and $s=s(\zeta)$ are the curvature of $\varphi_j(\Gamma_j)$ at $\varphi_j(\zeta)$ and an arc length parameter on $\varphi_j(\Gamma_j)$, respectively, provided that $\zeta\in\Gamma_j$, and we write $\partial c(\zeta, \varepsilon)/\partial\varepsilon|_{\varepsilon=0}$ and $\partial^2c(\zeta, \varepsilon)/\partial\varepsilon^2|_{\varepsilon=0}$ simply as $\dot{c}(\zeta)$ and $\ddot{c}(\zeta)$, respectively.

Remark. (i) By setting $p=q$ in the right hand sides of the above formulas, we have the formulas for Robin's constants. Furthermore, we can derive similar formulas for other quantities such as the energy functions of period reproducers. See [4] and [6].

(ii) When $c(\zeta, \varepsilon)=\varepsilon\cdot\rho(\zeta)$, then $\ddot{c}(\zeta)$ vanishes identically, and hence the formulas in Theorem 3 reduces to the classical Garabedian-Schiffer's ones. (Cf. [2] and [1, p 560].)

Finally, we note that the data family $\{\Gamma_{j,\varepsilon}\}$, or equivalently, $\{c(\zeta, \varepsilon)\}$ in the definition of boundary variation can be represented also by using an implicit function. We will discuss such a formulation and state formulas of Yamaguchi's type (cf. for instance, [8]), whose proofs will be given in § 2.

Let $\mathcal{D}$ be an unbranched domain spread over $\boldsymbol{C}\times B$, where $B=\{\varepsilon\in\boldsymbol{C}\,|\,|\varepsilon|<1\}$. (In particular, ε is a complex parameter.) Assume that there are another domain $\tilde{\mathcal{D}}$ spread over $\boldsymbol{C}\times B$ and a real smooth function $\psi(z, \varepsilon)$ which satisfies the followings:

(i) For every $\varepsilon\in B$, $\mathcal{D}_{\varepsilon}=\{(z, \varepsilon)\in\mathcal{D}\}$ is a relatively compact subsurface of $\tilde{\mathcal{D}}_{\varepsilon}=\{(z, \varepsilon)\in\tilde{\mathcal{D}}\}$ (, and hence $\mathcal{D}\subset\tilde{\mathcal{D}}$).

(ii) $\mathcal{D}=\{(z, \varepsilon)\in\tilde{\mathcal{D}}\,|\,\psi(z, \varepsilon)<0\}$, and $\partial\mathcal{D}=\{(z, \varepsilon)\in\tilde{\mathcal{D}}\,|\,\psi(z, \varepsilon)=0\}$.

(iii) For every $\varepsilon\in B$, $(\partial\psi/\partial z)\neq 0$ on $\partial\mathcal{D}_{\varepsilon}$.

Then we may rewrite the formula (0.5) in Theorem 3 as follows.

Theorem 4. *Fix p and q in $\mathcal{D}_0$. Then*

$$\frac{\partial^2}{\partial\varepsilon\partial\bar{\varepsilon}}g_\varepsilon(p,q)\Big|_{\varepsilon=0}=-\frac{1}{\pi}\mathrm{Re}\iint_{\mathcal{D}_0}d\Big(\frac{\partial g_\varepsilon(\cdot,p)}{\partial\bar{\varepsilon}}\Big|_{\varepsilon=0}\Big)\wedge{}^*\overline{d\Big(\frac{\partial g_\varepsilon(\cdot,q)}{\partial\bar{\varepsilon}}\Big|_{\varepsilon=0}\Big)} \tag{0.6}$$

$$-\frac{1}{4\pi}\int_{\partial\mathcal{D}_0}K_R(\cdot)\frac{\partial g(\cdot,p)}{\partial n}\cdot\frac{\partial g(\cdot,q)}{\partial n}ds$$

where

$$K_R=(\psi_{\varepsilon\bar{\varepsilon}}|\psi_z|^2-\mathrm{Re}\,\psi_\varepsilon(|\psi_z|^2)_{\bar{\varepsilon}}+|\psi_\varepsilon|^2\psi_{z\bar{z}})\cdot|\psi_z|^{-3}.$$

We further rewrite the above formula to a complex version such as considered by H. Yamaguchi. Here we state a formula only for the case of Robin's constant. The general case can be treated by the same argument and by polarization.

Corollary (cf. [8]). *Under the same circumstance as in Theorem 4, assume that $p=q$. Let $\gamma(\varepsilon)$ be Robin's constant at p on $\mathcal{D}_\varepsilon$ for every ε. Then*

$$\frac{\partial^2\gamma}{\partial\varepsilon\partial\bar{\varepsilon}}(0)=-\frac{2}{\pi}\Big\|\frac{\partial^2 g_\varepsilon(\cdot,p)}{\partial\bar{\varepsilon}\partial z}dz\Big\|^2_{\mathcal{D}_0}-\frac{1}{4\pi}\int_{\partial\mathcal{D}_0}K_C(\cdot)\Big(\frac{\partial g(\cdot,p)}{\partial n}\Big)^2ds \tag{0.7}$$

where K_C is the Levi's curvature of the boundary, namely,

$$K_C=(\psi_{\varepsilon\bar{\varepsilon}}|\psi_z|^2-2\,\mathrm{Re}\,\psi_\varepsilon\psi_{\bar{z}}\psi_{z\bar{\varepsilon}}+|\psi_\varepsilon|^2\psi_{z\bar{z}})\cdot|\psi_z|^{-3}.$$

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§1. Proof of Theorem 3

The main trick in our proof of Theorem 3 is to interpret boundary variation as variation under quasiconformal deformation. (Compare with [3].)

Fix a sufficiently small positive constant η so that

$$F_j:(\zeta,\tau)\longrightarrow\varphi_j(\zeta)+\tau\cdot n(\zeta)$$

gives a smooth immersion of $W_j=\Gamma_j\times(-\eta,\eta)$ into $\varphi_j(U_j)$ for every j. Also we may assume that $\phi_j=\varphi_j^{-1}\circ F_j$ is well-defined and gives a smooth homeomorphism of W_j into S such that $\phi_j(\zeta,0)=\zeta$. Further, fix a smooth non-decreasing function $\chi(\tau)$ on $\boldsymbol{R}$ such that $0\leq\chi\leq1$, $\chi=0$ in a neighborhood of $(-\infty,-\eta]$ and $\chi=1$ in a neighborhood of $[0,+\infty)$.

Now define a map f_ε of R into S by setting

$$(\phi_j)^{-1}\circ f_\varepsilon\circ\phi_j(\zeta,\tau)=(\zeta,\chi(\tau)\cdot(c(\zeta,\varepsilon)+\tau)+(1-\chi(\tau))\cdot\tau)$$

$$=(\zeta,\tau+\chi(\tau)c(\zeta,\varepsilon))$$

on $W_j'=\Gamma_j\times(-\eta,0)$ $(\subset\phi_j^{-1}(R\cap U_j))$ for every j, and by setting $f_\varepsilon=id$ on $R-\bigcup_{j=1}^n\phi_j(W_j)$. Then we have the following

Lemma 1. *For every ε with sufficiently small $|\varepsilon|$, the above map f_ε is a quasiconformal map of R onto R_ε.*

Proof. If $(\max_{-\eta<\tau<0}|\chi'(\tau)|)(\max_{\partial R}|c(\zeta, \varepsilon)|)<1$, which holds for every ε with sufficiently small $|\varepsilon|$, it is clear from the definition that f_ε is a homeomorphism of R onto R_ε.

In the sequel of this proof, fix j and we will drop all subscript j. And we will show that f_ε is quasiconformal on $\psi(W)\cap R$ for every ε with sufficiently small $|\varepsilon|$, which is enough to show the assertion.

Parametrize Γ by a smooth function $\zeta=\zeta(u)$; $0\leqq u\leqq 1$, and in the sequel, we denote $\varphi(\zeta(u))$, $c(\zeta(u), \tau)$, $\kappa(\zeta(u))$ and $\boldsymbol{n}(\zeta(u))$ simply by $\varphi(u)$, $c(u, \tau)$, $\kappa(u)$ and $\boldsymbol{n}(u)$, respectively. Then F considered as a map of $(u, \tau)\in[0, 1]\times(-\eta, \eta)$ has the form;

$$z=F(u, \tau)=\varphi(u)+\tau\cdot\boldsymbol{n}(u) \qquad (0\leqq u\leqq 1, \ -\eta<\tau<\eta).$$

Also it is clear that

$$\boldsymbol{n}(u)=-i\cdot\frac{\varphi'(u)}{|\varphi'(u)|},$$

Hence, regarding z as a local parameter, and u and τ also as functions of z, arround any point of $\psi(W)\cap R$, we can represent $\tilde{f}_\varepsilon=\varphi\circ f_\varepsilon\circ\varphi^{-1}$ locally as

$$\tilde{f}_\varepsilon(z)=z-i\cdot\chi(\tau)\cdot c(u, \varepsilon)\cdot\varphi'(u)/|\varphi'(u)|. \tag{1-1}$$

Recall that the curvature κ of Γ is defined by the following equation.

$$(\varphi'(u)/|\varphi'(u)|)'=i\cdot\kappa(u)\cdot\varphi'(u).$$

Since

$$z_u\left(=\frac{\partial z}{\partial u}\right)=(1+\tau\kappa)\varphi',$$

and

$$z_\tau\left(=\frac{\partial z}{\partial\tau}\right)=-i\varphi'/|\varphi'|,$$

a simple computation shows that

$$u_z\circ F=\frac{\overline{\varphi'}}{2(1+\tau\kappa)}/|\varphi'|^2, \qquad u_{\bar z}\circ F=\frac{\varphi'}{2(1+\tau\kappa)}/|\varphi'|^2.$$

and

$$\tau_z\circ F=\frac{i}{2}\frac{\overline{\varphi'}}{|\varphi'|}, \qquad \tau_{\bar z}\circ F=-\frac{i}{2}\frac{\varphi'}{|\varphi'|}.$$

Hence we have

$$(\tilde{f}_\varepsilon)_z\circ F=1+\frac{1}{2}\chi'c+\frac{1}{2(1+\tau\kappa)}(\kappa\chi c-i\chi c_u/|\varphi'|), \tag{1-2}$$

and

$$(\tilde{f}_\varepsilon)_{\bar z}\circ F=\left(-\frac{1}{2}\chi'c+\frac{1}{2(1+\tau\kappa)}(\kappa\chi c-i\chi c_u/|\varphi'|)\right)\left(\frac{\varphi'}{|\varphi'|}\right)^2. \tag{1-3}$$

Since c and c_u tend to 0 uniformly on Γ as $|\varepsilon|$ tends to 0, we conclude that f_ε is a quasiconformal map with the complex dilatation $\mu(\varepsilon)=\mu_\varepsilon(z)d\bar z/dz$ which satisfies that

$$\mu_\varepsilon \circ F=\frac{-\chi' c(1+\tau\kappa)+\kappa\chi c-i\chi c_u/|\varphi'|}{(2+\chi' c)(1+\tau\kappa)+\kappa\chi c-i\chi c_u/|\varphi'|}\left(\frac{\varphi'}{|\varphi'|}\right)^2$$

on $W'=[0, 1]\times(-\eta, 0)$ for every ε with sufficiently small $|\varepsilon|$. q.e.d.

(i) Now we will derive (0-4) from (0-2). Since

$$\frac{d\mu}{d\varepsilon}=f_{\bar{z}\varepsilon}/f_z-f_{\bar{z}}f_{z\varepsilon}/(f_z)^2$$

with $f=\tilde{f}_\varepsilon$, (1-2) and (1-3) gives that

$$\frac{d\mu}{d\varepsilon}(0)=\left(-\frac{1}{2}\chi'\dot{c}+\frac{\chi}{2(1+\tau\kappa)}(\kappa\dot{c}-i\dot{c}_u/|\varphi'|)\right)\left(\frac{\varphi'}{|\varphi'|}\right)^2\cdot\frac{d\bar{z}}{dz}.$$

And since $\phi=i(dg+i^*dg)=2ig_z dz$ for $g(\cdot)=g_0(\cdot, p)$ or $g(\cdot)=g_0(\cdot, q)$, where z is a generic local parameter on R, the formula (0-2) can be rewritten as

$$\dot{g}(p, q)=\frac{-1}{2\pi}\sum_{j=1}^{n}\mathrm{Re}\iint_{\varphi(W_j')}\left(-\frac{1}{2}\chi'\dot{c}+\frac{\chi}{2(1+\tau\kappa)}(\kappa\dot{c}-i\dot{c}_u/|\varphi_j'|)\right)$$
$$\times(\varphi_j'/|\varphi_j'|)^2(2i(g_p)_z)(2i(g_q)_z)2dx\,dy.$$

Here and in the sequel, we set $g_p(\cdot)=g_0(\cdot, p)$, $g_q(\cdot)=g_0(\cdot, q)$ and $z=x+iy$. Recall that $(g_p)_z$ and $(g_q)_z$ can be considered as smooth functions on $\varphi_j(U_j)$.

Again fix j and drop all subscript j. Also we set

$$I_1=\frac{-1}{2\pi}\mathrm{Re}\iint_{\varphi(W')}-\frac{1}{2}\chi'\dot{c}\left(\frac{\varphi'}{|\varphi'|}\right)^2(-8)(g_p)_z(g_q)_z dx\,dy,$$

and

$$I_2=\frac{-1}{2\pi}\mathrm{Re}\iint_{\varphi(W')}\chi\frac{(\kappa\dot{c}-i\dot{c}_u/|\varphi'|)}{2(1+\tau\kappa)}\left(\frac{\varphi'}{|\varphi'|}\right)^2(-8)(g_p)_z(g_q)_z dx\,dy,$$

where $W'=[0, 1]\times(-\eta, 0)$ as before. Next, we let $\chi(\tau)$ tend to

$$\begin{aligned}\chi_0(\tau)&=0 \quad \text{for } -\infty<\tau<0\\ &=1 \quad \text{for } 0\leqq\tau<+\infty\end{aligned}$$

monotonically. Then I_2 clearly converges to 0. Moreover, since $\int_{-\eta}^{0}\chi' d\tau=1$ and since the Jacobian $\partial(x, y)/\partial(\tau, u)$ of F is equal to $\mathrm{Im}\,\bar{z}_\tau z_u=(1+\tau\kappa)|\varphi'|$, we can see that

$$I_1=\frac{-1}{2\pi}\mathrm{Re}\iint_{W'}-\frac{1}{2}\chi'\dot{c}\left(\frac{\varphi'}{|\varphi'|}\right)^2(g_p)_z(g_q)_z(-8)(1+\tau\kappa)|\varphi'|\,d\tau du$$

converges to

$$\frac{-1}{2\pi}\mathrm{Re}\int_\Gamma-\frac{1}{2}\dot{c}\left(\frac{\varphi'}{|\varphi'|}\right)^2(g_p)_z(g_q)_z(-8)ds,$$

as χ tends to χ_0. Since, for $g=g_p$ or $g=g_q$, we have

$$g_z=g_\tau\tau_z+g_u u_z=g_\tau\tau_z=\left(\frac{i}{2}\frac{\overline{\varphi'}}{|\varphi'|}\right)g_\tau \tag{1-4}$$

on Γ, and since $\partial/\partial\tau=\partial/\partial n$, we obtain the formula (0-4).

(ii) Next we will derive (0-5) from (0-3). First note that

$$\frac{d^2\mu}{d\varepsilon^2}=(f_{\bar z\varepsilon\varepsilon}/f_z-2f_{\bar z\varepsilon}f_{z\varepsilon}/(f_z)^2+O(\varepsilon))\frac{d\bar z}{dz}$$

with $f=\tilde f_\varepsilon$. Hence

$$\frac{d^2\mu}{d\varepsilon^2}(0)=(\ddot f_{\bar z}-2\dot f_z\dot f_{\bar z})\frac{d\bar z}{dz}.$$

Next, for $g=g_\varepsilon(\cdot,p)$ or $g=g_\varepsilon(\cdot,q)$, we have

$$\dot\phi=\left.\frac{d\Phi}{d\varepsilon}\right|_{\varepsilon=0}=2i(\dot g_z+g_{zz}\dot f+g_z\dot f_z)dz+2ig_z\dot f_{\bar z}d\bar z,$$

which we write as $\dot\phi^{1,0}+\dot\phi^{0,1}$ (cf. [6]). Then since

$$\iint_R\dot\phi_p\wedge{}^*\dot\phi_q=\iint_R(i\dot\phi_p^{1,0}\wedge\dot\phi_q^{0,1}-i\dot\phi_p^{0,1}\wedge\dot\phi_q^{1,0}),$$

the formula (0-3) implies that

$$\begin{aligned}\ddot g(p,q)=&\frac{-1}{2\pi}\mathrm{Re}\Big\{\iint_R\ddot f_{\bar z}\cdot(-8)(g_p)_z(g_q)_z dx\,dy\\&+2i\iint_R(2i(g_p)_z\dot f_{\bar z}d\bar z)\wedge(2i(g_q)_z\dot f_z dz)\\&+\iint_R-i(2i(g_q)_z\dot f_{\bar z}d\bar z)\wedge(2i((\dot g_p)_z+(g_p)_{zz}\dot f+(g_p)_z\dot f_z)dz)\\&+\iint_R-i(2i(g_p)_z\dot f_{\bar z}d\bar z)\wedge(2i((\dot g_q)_z+(g_q)_{zz}\dot f+(g_q)_z\dot f_z)dz)\Big\}\\=&\frac{-1}{2\pi}\mathrm{Re}\iint_R\ddot f_{\bar z}\cdot(-8)(g_p)_z(g_q)_z dx\,dy\\&+\frac{-1}{2\pi}\mathrm{Re}\iint_R-8((\dot g_p)_z+(g_p)_{zz}\dot f)(g_q)_z\dot f_{\bar z}dx\,dy\\&+\frac{-1}{2\pi}\mathrm{Re}\iint_R-8((\dot g_q)_z+(g_q)_{zz}\dot f)(g_p)_z\dot f_{\bar z}dx\,dy,\end{aligned}$$

where we write $\dot g(\cdot,p)$ and $\dot g(\cdot,q)$ simply as $\dot g_p$ and $\dot g_q$, respectively.

Now we write the right hand side of the above equation as $I_3+I_4+I_5$. As before, we let χ tend to χ_0. Then, since (1-3) implies that

$$\ddot f_{\bar z}=\Big(-\frac{1}{2}\chi'\ddot c+\frac{\chi}{2(1+\tau\kappa)}(\kappa\ddot c-i\ddot c_u/|\varphi'|)\Big)\Big(\frac{\varphi'}{|\varphi'|}\Big)^2$$

as a function of (u,τ), we can conclude by the same argument as in (i) that I_3 tends to

$$\tag{1-5} \frac{1}{2\pi}\int_{\Gamma}\dot{c}(\partial g_p/\partial n)(\partial g_q/\partial n)ds\,,$$

and that I_4 tends to

$$\lim_{\chi\to\chi_0}\frac{-1}{2\pi}\mathrm{Re}\iint_R -4\Big((\dot{g}_p)_z+(g_p)_{zz}\Big(-i\chi\dot{c}\frac{\varphi'}{|\varphi'|}\Big)(g_q)_z\Big(-\frac{1}{2}\chi'\dot{c}\Big)\Big(\frac{\varphi'}{|\varphi'|}\Big)^2 2dxdy\Big)$$

$$=\lim_{\chi\to\chi_0}\frac{-1}{4\pi}\mathrm{Re}\iint_R -4(g_q)_z(-2(\dot{g}_p)_z\chi'\dot{c}-(g_p)_{zz}z_\tau(\chi^2)'\dot{c}^2)\Big(\frac{\varphi'}{|\varphi'|}\Big)^2 dxdy$$

$$=\frac{1}{2\pi}\int_{\Gamma}(g_q)_\tau(\dot{g}_p)_\tau\dot{c}\,ds+\frac{1}{4\pi}\int_{\Gamma}(g_q)_\tau(g_p)_{\tau\tau}\dot{c}^2 ds\,.$$

(Here note that, since $(g_p)_{z\bar{z}}=0$ on $R-\{p\}$, $(g_p)_{zz}z_\tau=(g_p)_{z\tau}=(i/2)(\overline{\varphi'}/|\varphi'|)(g_p)_{\tau\tau}$ on Γ by (1-4).) We write the right hand side of this equation as I_6+I_7.

Recall that the formula (0-4) implies that $\dot{g}(\cdot, q)$ is the harmonic function on R which is the solution of the Dirichlet problem for the boundary value

$$-\dot{c}(\cdot)(\partial g(\cdot, q)/\partial\tau)$$

on Γ. Since $*d\dot{g}_p=-(\dot{g}_p)_s d\tau+(\dot{g}_p)_\tau ds=(\dot{g}_p)_\tau ds$ on Γ, we conclude that

$$\tag{1-6} I_6=-\frac{1}{2\pi}\int_{\Gamma}(\dot{g}_q)(\dot{g}_p)_\tau ds=-\frac{1}{2\pi}\int_{\Gamma}(\dot{g}_q)*d(\dot{g}_p)$$

$$=-\frac{1}{2\pi}\iint_R d\dot{g}(\cdot, q)\wedge *d\dot{g}(\cdot, p)\,.$$

Again, since $(g_p)_{z\bar{z}}=0$ on $R-\{p\}$, a simple computation shows that

$$(g_p)_{\tau\tau}\tau_z\tau_{\bar{z}}+(g_p)_\tau(\tau_z)_u u_{\bar{z}}=\frac{1}{4}(g_p)_{\tau\tau}+(g_p)_\tau\Big(\frac{1}{2}\kappa\overline{\varphi'}\Big)\frac{\varphi'}{2}/|\varphi'|^2$$

$$=\frac{1}{4}(g_p)_{\tau\tau}+\frac{1}{4}\kappa(g_p)_\tau=0$$

on Γ. Hence we conclude that

$$\tag{1-7} I_7=\frac{-1}{4\pi}\int_{\Gamma}\kappa\dot{c}^2(g_q)_\tau(g_p)_\tau ds\,.$$

Similarly, we can show that

$$\tag{1-8} I_5=-\frac{1}{2\pi}\iint_R d\dot{g}(\cdot, q)\wedge *d\dot{g}(\cdot, p)-\frac{1}{4\pi}\int_{\Gamma}\kappa\dot{c}^2(g_q)_\tau(g_p)_\tau ds\,.$$

Thus we conclude the formula (0-5) by (1-5), (1-6), (1-7) and (1-8).

§2. Proofs of Theorem 4 and Corollary

Proof of Theorem 4. It is easily seen that there is a (sufficiently small) positive η such that every $\partial\mathscr{D}_\varepsilon$ can be parametrized in the form as

$$z+c(z,\ \varepsilon)\cdot n(z) \qquad (z\in\partial\mathcal{D}_0)$$

with a suitable smooth function $c(z,\ \varepsilon)$ for every ε with $|\varepsilon|<\eta$. Hence, to prove Theorem 4 from the formula (0.5), it suffices to show that

$$\kappa|c_\varepsilon|^2-c_{\varepsilon\bar\varepsilon}=\frac{1}{2}K_R \tag{2.1}$$

on $\partial\mathcal{D}_0$ (, which also implies that K_R is independent of the choice of ψ.).

First, let $\varphi(u)$ be a smooth parametrization of $\partial\mathcal{D}_0$ as before. Then $\psi(\varphi(u),\ 0)\equiv 0$. In particular,

$$\psi_u=\psi_z\varphi'+\psi_{\bar z}\overline{\varphi'}=0\,.$$

Since $\psi_\tau(=\partial\psi/\partial n)=\psi_z z_\tau+\psi_{\bar z}\bar z_\tau=-2i\psi_z\varphi'/|\varphi'|>0$ by the assumption (ii), we conclude that

$$\varphi'/|\varphi'|=i\psi_{\bar z}/|\psi_z|\,. \tag{2.2}$$

Furthemore, a simple computation shows that

$$\kappa=(\psi_{z\bar z}|\psi_z|^2-\mathrm{Re}\{\psi_{zz}(\psi_{\bar z})^2\})/|\psi_z|^3\,.$$

Next

$$\psi(\varphi(u)-ic(u,\ \varepsilon)\varphi'(u)/|\varphi'(u)|,\ \varepsilon)=0$$

for every sufficiently small ε. Hence, differentiating with respect to ε and using (2-2), we have

$$c_\varepsilon=-\psi_\varepsilon/(2|\psi_z|)\,,$$

and hence

$$\kappa c_\varepsilon c_{\bar\varepsilon}=(\psi_\varepsilon\psi_{\bar\varepsilon}\psi_{z\bar z}-\psi_\varepsilon\psi_{\bar\varepsilon}\,\mathrm{Re}\{\psi_{zz}(\psi_{\bar z})^2\}/|\psi_z|^2)/(4|\psi_z|^3)\,.$$

Finally, another simple computation shows that

$$c_{\varepsilon\bar\varepsilon}=(-2\psi_{\varepsilon\bar\varepsilon}|\psi_z|^2-\psi_\varepsilon\psi_{\bar\varepsilon}\psi_{z\bar z}-\psi_\varepsilon\psi_{\bar\varepsilon}\,\mathrm{Re}\{\psi_{zz}(\psi_{\bar z})^2\}/|\psi_z|^2+2\,\mathrm{Re}\{(|\psi_z|^2)_{\bar\varepsilon}\psi_\varepsilon\})/(4|\psi_z|^3)\,.$$

Thus we obtain (2-1), which proves Theorem 4. q.e.d.

Proof of Corollary. The proof relies on the fact that the curvatures K_R and K_C are independent of the choice of ψ (cf. [9]).

By noting this fact, we take as another ψ such a function which is coincident with $-g_\varepsilon(z,\ p)$ in a neighborhood of $\partial\mathcal{D}_0$. In the sequel, We write $g_\varepsilon(z,\ p)$ simply as $g(z,\ \varepsilon)$. Then Theorem 4 gives that

$$\begin{aligned}\frac{d^2\gamma}{d\varepsilon d\bar\varepsilon}(0)=&-\frac{1}{\pi}\|dg_{\bar\varepsilon}\|^2_{\mathcal{D}_0}\\&+\frac{1}{4\pi}\int_{\partial\mathcal{D}_0}(g_{\varepsilon\bar\varepsilon}|g_z|^2-\mathrm{Re}\,g_\varepsilon(|g_z|^2)_{\bar\varepsilon}+|g_\varepsilon|^2g_{z\bar z})\cdot|g_z|^{-3}\cdot\left(\frac{\partial g}{\partial n}\right)^2ds\,.\end{aligned} \tag{2.3}$$

Here it is clear that

$$\|dg_{\bar\varepsilon}\|^2_{\mathcal{D}_0}=\|g_{\bar\varepsilon z}dz\|^2_{\mathcal{D}_0}+\|g_{\bar\varepsilon\bar z}d\bar z\|^2_{\mathcal{D}_0}\,. \tag{2.4}$$

Moreover, since $g_{\bar\varepsilon}$ is harmonic on $\mathcal{D}_0$ (cf. (0.4)), Green's formula gives that

$$\|g_{\bar{\varepsilon}\bar{z}}d\bar{z}\|^2_{\mathscr{D}}=\mathrm{Re}\iint_{\mathscr{D}_0}g_{\bar{\varepsilon}\bar{z}}d\bar{z}\wedge g_{\varepsilon z}{}^*dz$$

$$=\mathrm{Re}\Big(-\iint_{\mathscr{D}_0}g_\varepsilon(g_{\bar{z}})_{z\bar{z}}d\bar{z}\wedge{}^*dz+\int_{\partial\mathscr{D}_0}g_\varepsilon g_{\bar{\varepsilon}\bar{z}}\cdot id\bar{z}\Big)$$

$$=\mathrm{Re}\int_{\partial\mathscr{D}_0}g_\varepsilon g_{\bar{\varepsilon}\bar{z}}\cdot id\bar{z}.$$

Here $dz=\varphi'(u)du=(\varphi'/|\varphi'|)ds=-i(g_{\bar{z}}/|g_z|)ds$. Hence we conclude by (1.4) that

$$\|g_{\bar{\varepsilon}\bar{z}}d\bar{z}\|^2_{\mathscr{D}_0}=-\mathrm{Re}\int_{\partial\mathscr{D}_0}g_\varepsilon g_{\bar{\varepsilon}\bar{z}}g_z|g_z|^{-1}ds$$
$$=-\frac{1}{4}\int_{\partial\mathscr{D}_0}\mathrm{Re}\,g_\varepsilon g_{\bar{\varepsilon}\bar{z}}g_z|g_z|^{-3}\Big(\frac{\partial g}{\partial n}\Big)^2ds. \tag{2.5}$$

Similarly, we can show that

$$\|g_{\bar{\varepsilon}z}dz\|^2_{\mathscr{D}_0}=-\frac{1}{4}\int_{\partial\mathscr{D}_0}\mathrm{Re}\,g_\varepsilon g_{z\bar{\varepsilon}}g_{\bar{z}}|g_z|^{-3}\Big(\frac{\partial g}{\partial n}\Big)^2ds. \tag{2.6}$$

Thus, (2.3), (2.4), (2.5) and (2.6) gives that

$$\frac{\partial^2\gamma}{\partial\varepsilon\partial\bar{\varepsilon}}(0)=-\frac{2}{\pi}\|g_{\bar{\varepsilon}z}dz\|^2_{\mathscr{D}_0}$$
$$+\frac{1}{4\pi}\int_{\partial\mathscr{D}_0}(g_{\varepsilon\bar{\varepsilon}}|g_z|^2-2\,\mathrm{Re}\,g_\varepsilon g_{\bar{z}}g_{z\bar{\varepsilon}}+|g_\varepsilon|^2g_{z\bar{z}})\cdot|g_z|^{-3}\cdot\Big(\frac{\partial g}{\partial n}\Big)^2ds.$$

Recalling that K_C is independent of the choice of ψ, we conclude the assertion.

q.e.d.

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