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Structure of solutions for the Lewy type equations

Dedicated to the late Professor Hans Lewy

By

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§1. Introduction

Let L be an real analytic vector field in R^3. It is called a differential operator of Lewy type if it enjoys the following properties:

(a.1) *L, $\bar{L}$ and $[L, \bar{L}]$ are linearly independent.*
(a.2) *$^{\exists}$real analytic functions X and Y such that*
(i) $X=\bar{X}$, (ii) $L(Y)=0$, (iii) $L(X+iY\bar{Y})=0$ *and* (iv) $dX\wedge d\bar{Y}\wedge dY\neq 0$.

Suppose L be a differential operator of Lewy Type. Taking a suitable analytic change of coordinates, one can transform L to this: $a(x)\{1/2(\partial/\partial x_1-i(\partial/\partial x_2))+i(x_1-ix_2)\partial/\partial x_3\}$, where $a(x)$ denotes a nonvanishing real analytic function. The operator $1/2(\partial/\partial x_1-i(\partial/\partial x_2))+i(x_1-ix_2)\partial/\partial x_3$ is the celebrated H. Lewy one ([3]); it is hereafter denoted by L_x. Then the equation of Lewy type $Lu=f$ is reduced to the Lewy equation $L_x u=f$. Sato [6] and Greiner-Kohn-Stein [2] gave the micro-local solvability conditions and the local solvability ones for the Lewy equation, respectively. In Greiner-Kohn-Stein [2], the following results are included: Let $\mathcal{D}$ and $\partial\mathcal{D}$ denote $\{(z_1, z_2)\in C^2 ; \operatorname{Im} z_2>|z_1|^2\}$ and $\{(z_1, z_2)\in C^2 ; \operatorname{Im} z_2=|z_1|^2\}$, respectively. $R^3=\{(x_1, x_2, x_3)\}$ is identified with $\partial\mathcal{D}$ by $z_1=x_1+ix_2$ and $z_2=x_3+i(x_1^2+x_2^2)$. Let Ω be an open set in R^3.

[G-K-S-I] Let $f\in\mathcal{E}'(\Omega)$, then the Lewy equation $L_x u=f$ has a solution $u\in\mathcal{E}'$ in a neighborhood of a point P in Ω if and only if $C_b(f)$ given by $C_b(f)=C(f)(z_1, z_2)|_{\partial\mathcal{D}}$ is real analytic in a neighborhood of P, where $C(f)(z_1, z_2)$ is the Cauchy-Szegö Integral defined by

$$C(f)(z_1, z_2)=\int_{\partial\mathcal{D}} S(z_1, z_2 ; w_1, w_2) f\, d\sigma_w ,$$

where $S(z_1, z_2 ; w_1, w_2)=1/\{\pi(i(\bar{w}_2-z_2)-2\bar{w}_1 z_1)\}^2$ and $d\sigma_w=d\operatorname{Re} w_1 d\operatorname{Im} w_1 d\operatorname{Re} w_2$.

[G-K-S-II] $-L_x\bar{L}_x\cdot K=K\cdot(-L_x\bar{L}_x)=I-C_b$ when acting on $\mathcal{E}'$, where the operator K is defined by

$$K(f)=f*\frac{\{\log(|z_1|^2-ix_3)-\log(|z_1|^2+ix_3)\}}{2\pi^2(|z_1|^2-ix_3)}$$

provided that the convolution is with respect to the Heisenberg group.

[G-K-S-III] Suppose that the condition of [G-K-S-I] for f is satisfied. If f belongs

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to one of the spaces (see Folland-Stein [1] for definition) $S_k^p(\Omega, \text{loc})$, $\Gamma_\alpha(\Omega, \text{loc})$, $L^\infty(\Omega, \text{loc})$ or $C^\infty(\Omega)$, then one can find a u which belongs to $S_{k+1}^p(\omega, \text{loc})$, $\Gamma_{\alpha+1}(\omega, \text{loc})$, $\Gamma_1(\omega, \text{loc})$ or $C^\infty(\omega)$, respectively, where ω denotes an neighborhood ($\subset\Omega$) of P.

In this article, nevertheless, we investigate the structure of classical solutions for the Lewy equation from a different viewpoint: in [5], the author gave a characterization that the solvability of the Mizohata equation is reduced to that of the Cauchy Problem for the Cauchy-Riemann equation; the same is valid for the Lewy equation; that is, the solvability of the Lewy equation can be reduced to that of the Cauchy problem for the Cauchy-Riemann equation; more precisely, for the Cauchy-Riemann equation with two (or three) parameters, which is assured by the property of "Heisnberg group" attached to the Lewy operator; as a result, it seems that it has become far more obvious why the Cauchy-Szegö kernel appears in solving the Lewy equation; and moreover, we have obtained delicate terms which are not in [2] or [6].

Now we shall state our results. First, denoting through by k a positive integer or ∞, we state the following

Theorem I. *Every C^k solution u of the homogeneous Lewy equation $L_x u=0$ in a neighborhood of $P(x_1^0, x_2^0, x_3^0)\in\Omega$ can be expressed in the following fashion: $^\exists\rho$: a constant>0;*

$$u(x_1, x_2, x_3)=\sum_{n=0}^{\infty} w_1^n\cdot h_n(w_2) \qquad \text{in } \omega_\rho,$$

where $w_1=x_1-ix_2-(x_1^0-ix_2^0)$, $w_2=|w_1|^2+i(x_3-x_3^0-2x_2^0x_1+2x_1^0x_2)$, $\omega_\rho=\{(x_1, x_2, x_3);\ |w_1|^2<\rho, |\mathrm{Im}\, w_2|<\rho\}$ and $h_n(z)$ $(n=0, 1, 2, \cdots)$ are holomorphic in a complex domain $D_\rho=\{z\in C; 0<\mathrm{Re}\, z<\rho, |\mathrm{Im}\, z|<\rho\}$, furthermore $h_m(z)$ $(m=0, 1, 2, \cdots, k)$ are continuous in $\{z\in C; 0\leqq\mathrm{Re}\, z<\rho, |\mathrm{Im}\, z|<\rho\}$. (The summation is uniformly convergent on compact sets in ω_ρ.)

Remark. It follows that every C^1 solution of the homogeneous Lewy equation is "a function" of the independent two solutions w_1 and w_2.

Next we have obtained the following another expression which corresponds to [G-K-S-I]:

Theorem II. *Given $f(x)=f(x_1, x_2, x_3)\in C^1(R^3)$, then the Lewy equation $L_x u(x)=f(x)$ has a C^1 solution $u(x)$ in a neighborhood of $P(x_1^0, x_2^0, x_3^0)\in R^3$ if and only if there is a positive constant $c(\neq|x_3^0|)$ such that the function $\boldsymbol{A}f(x)$ given by*

$$Af(x)=\int_\omega S(x;y)f(y)\,dy_1dy_2dy_3$$

$$\equiv\lim_{\varepsilon\to 0}\int_\omega S_\varepsilon(x;y)f(y)\,dy_1dy_2dy_3,$$

$x\in\Omega_c\equiv\{x\in R^3;\ |x_k-x_k^0|<c,\ k=1, 2, 3\}$, is Lipschitz continuous in Ω_c and extends holomorphically in x_3 to the complex domain $\{x_3\in C; \mathrm{Re}\, x_3\neq x_3^0\pm c\}$ uniformly for x_1 and for x_2, where $S(x;y)\equiv 1/\{\pi[x_1^2+x_2^2+y_1^2+y_2^2-2(x_1+ix_2)(y_1-iy_2)+i(y_3-x_3)]\}^2$, $S_\varepsilon(x;y)\equiv 1/\{\pi[\varepsilon^2+x_1^2+x_2^2+y_1^2+y_2^2-2(x_1+ix_2)(y_1-iy_2)+i(y_3-x_3)]\}^2$, $\omega\equiv\{(y_1, y_2, y_3)\in R^3; (y_1-x_1)^2$

$+(y_2-x_2)^2<c_1^2, |y_3-x_3^0-2x_2y_1+2x_1y_2|<c\}$ *and* c_0 *denotes a positive constant such that* $c_0 \leqq \min\{c, c/[2(2c+|x_1^0|+|x_2^0|)]\}$.

Notice that $S(x; y)$ is just the Cauchy-Szegö kernel: $S(x; y)=S(z_1, z_2; w_1, w_2)$ where $z_1=x_1+ix_2$, $z_2=x_3+i|z_1|^2$, $w_1=y_1+iy_2$ and $w_2=y_3+i|w_1|^2$. Therefore we get the following

Theorem III. *Let* $f(x)\in C^1(R^3)\cap L^2(R^3)$. *Assume* $Af(x)$ *satisfies the above condition. Then* $C_b(f)\equiv C_bf(x)$ *is real analytic in a neighborhood of* P; *that is,*

$$C_bf(x)=\int_{R^3} S(x; y)f(y)dy_1dy_2dy_3$$

$$\equiv \lim_{\varepsilon\to 0}\int_{R^3} S_\varepsilon(x; y)f(y)dy_1dy_2dy_3$$

is real analytic in a neighborhood of P.

Notation. For a function $g(x)=g(x_1, x_2, x_3)$, $g^\#(y_1, y_2, y_3; x_1, x_2)$ denotes $g(x_1+y_1, x_2+y_2, y_3+2x_2y_1-2x_1y_2)$.

We note that

$$Af(x)=\frac{1}{2\pi^2}\int_J \frac{f^\#(\sqrt{\xi}\cos\theta, \sqrt{\xi}\sin\theta, \eta; x_1, x_2)}{\{\xi+i(\eta-x_3)\}^2}d\xi d\eta d\theta$$

$$\equiv \frac{1}{2\pi^2}\lim_{\varepsilon\to 0}\int_J \frac{f^\#(\sqrt{\xi}\cos\theta, \sqrt{\xi}\sin\theta, \eta; x_1, x_2)}{\{\varepsilon^2+\xi+i(\eta-x_3)\}^2}d\xi d\eta d\theta,$$

where $J\equiv\{(\xi, \eta, \theta); 0\leqq\xi\leqq c_0, x_3^0-c\leqq\eta\leqq x_3^0+c, 0\leqq\theta\leqq 2\pi\}$.

Next we obtain the following theorem which, in a sense, corresponds to [G-K-S-II].

Theorem IV. *Let* $f(x)\in C^k(R^3)$. *For* $\forall$*positive constants* c *and* c_0 *it holds that*

$$L_x\left\{\frac{-1}{2\pi^2}\int_0^{c_0^2}d\xi\int_{x_3^0-c}^{x_3^0+c}\frac{Hf(\xi, \eta; x_1, x_2)}{\xi+i(\eta-x_3)}d\eta\right\}=f(x)-Af(x)-Rf(x),$$

where

$$Hf(\xi, \eta; x_1, x_2)\equiv\int_0^{2\pi}\frac{f^\#(\sqrt{\xi}\cos\theta, \sqrt{\xi}\sin\theta, \eta; x_1, x_2)}{\sqrt{\xi}\exp(-i\theta)}d\theta,$$

and $Rf(x)$ *is a certain function which belongs to* $C^k(D_c)$ *and extends holomorphically in* x_3 *to the complex domain* $\{x_3\in C; \mathrm{Re}\, x_3\neq x_3^0\pm c\}$ *uniformly for* x_1 *and for* x_2. *Here* D_c *denotes* $\{(x_1, x_2, x_3)\in R^3; |x_3-x_3^0|<c\}$.

Now our main theorem is as follows:

Theorem V. *Let* $f(x)\in C^k(R^3)$. *Assume* $f(x)$ *satisfies the solvabilıty condition in Theorem II. Then every* C^m *solution* $u(x)$ *of the Lewy equation in a neighood of* $P(x_1^0, x_2^0, x_3^0)$ *can be expressed in the following fashion, provided* $m\in\{1, 2, \cdot, \cdot, \cdot, k\}$:

$$u(x)=n(x)+C_Lf(x)+R_L\{Af(x)+Rf(x)\},$$

where $n(x)$ *denotes a* C^m *solution of the homogeneous Lewy equation which has a form* *stated in Theorem I and* $C_L f(x)$ *denotes*

$$\frac{-1}{2\pi^2}\int_0^{c_0^2} d\xi \int_{x_3^0-c}^{x_3^0+c} \frac{Hf(\xi, \eta\,;\, x_1, x_2)}{\xi+i(\eta-x_3)}\, d\eta\,,$$

and $R_L\{Af(x)+Rf(x)\}$ *is given by*

$$R_L\{Af(x)+Rf(x)\}=\frac{-1}{\pi}\int_{S_\delta} \frac{\{Af(\xi^P, \eta^P, \tau^P)+Rf(\xi^P, \eta^P, \tau^P)\}}{[\xi-i\eta-\{x_1-x_1^0-i(x_2-x_2^0)\}]}\, d\xi d\eta\,,$$

where ξ^P, η^P, τ^P *and* S_δ *denotes* $\xi+x_1^0$, $\eta+x_2^0$, $-ix_3+\xi^2+\eta^2-x_1^2-x_2^2+(x_1^0)^2+(x_2^0)^2+2x_1^0\xi+2x_2^0\eta$ *and* $\{(\xi, \eta)\in R^2\,;\, \xi^2+\eta^2<\delta^2\}$, *respectively; provided* δ *is a suitably choosen positive constant such that the above integral can be defined.*

Remark V.1. $C_L f(x)\in C^{k+1}$, $R_L\{Af(x)+Rf(x)\}\in C^k$, *if* $f(x)\in C^k$.

We mention that Treves ([7] and [8]) gave an integral representation of solutions of "solvable" linear PDEs.

Finally, about the proofs; first Theorem I is proved in §2. Next we prove Theorem IV in §3. To prove necessity part of Theorem II, we prepare the following "key Lemma" which is proved in §4:

Lemma A. *Given* $f(x)\in C^1(R^3)$, *assume the Lewy equation* $L_x u(x)=f(x)$ *has a* C^1 *solution* $u(x)$ *in a neihgborhood of* P. *Then, there is a positive constant* $c(\neq |x_3^0|)$ *such that, taking a constant* c_0 *such that* $0<c_0<\min\{c, c/[2(2c+|x_1^0|+|x_2^0|)]\}$, *it holds that*

$$f(x)=L_x\Big\{\mathrm{P.V.}\frac{1}{\pi i}\int_{x_3^0+c}^{x_3^0-c} u(x_1, x_2, \eta)/(\eta-x_3)d\eta\Big\}$$
$$+L_x\Big\{\frac{1}{2\pi^2 i}\int_\ell \frac{\Theta u(\xi, \eta\,;\, x_1, x_2)}{\{\xi+i(\eta-x_3)\}}\, d(\xi+i\eta)\Big\}+L_x\{C_L f(x)\}\,,$$

where ℓ *is the oriented path* $ABCD$; $A(0, x_3^0-c)$, $B(c_0^2, x_3^0-c)$, $C(c_0^2 x_3^0+c)$ *and* $D(0, x_3^0+c)$, *and* $\Theta u(\xi, \eta\,;\, x_1, x_2)$ *denotes*

$$\int_0^{2\pi} u^{\#}(\sqrt{\xi}\cos\theta, \sqrt{\xi}\sin\theta, \eta\,;\, x_1, x_2)d\theta\,.$$

Theorem IV and Lemma A explain why the Cauchy-Szegö kernel appears in solving the Lewy equation. The sufficiency of Theorem II follows from Theorem IV and the subsequent Corollary to Proposition B:

Proposition B. *Let* U *be a neighborhood of* (x_1^0, x_2^0) *in* R^2 *and* I *an open interval in* R^1 *containing* x_3^0. *Let* $h(x)\in$ *Lip* $(U\times I)$ *and be continuously differentiable in* x_3. *Taking a suitable positive constant* δ, *the function* $R_{\bar{L}} h(x)$ *given by*

$$R_{\bar{L}} h(x)=\frac{-1}{\pi}\int_{S_\delta} \frac{h(\xi^P, \eta^P, \chi^P)}{[\xi-i\eta-\{x_1-x_1^0-i(x_2-x_2^0)\}]}\, d\xi d\eta \qquad (x\in U_\delta)$$

satisfies the equation

$$L\{R_{\bar{L}} h(x)\}\equiv\Big\{\frac{1}{2}\Big(\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2}\Big)+(x_1-ix_2)\frac{\partial}{\partial x_3}\Big\}R_{\bar{L}} h(x)=h(x) \qquad \text{in } U_\delta\,,$$

where $\chi^P=\xi^2+\eta^2-x_1^2-x_2^2+(x_1^0)^2+(x_2^0)^2+x_3+2x_1^0\xi+2x_2^0\eta$ *and* $U_\delta=\{x\,;\,(x_1-x_1^0)^2+(x_2-x_2^0)^2<\delta^2,\ x_3\in I\}$. ($\xi^P$, η^P *and* S_δ *are the same notations as in Theorem V.*)

Remark B.1. If $h(x)\in C^k$, then $R_L^\sim h(x)\in C^{k+1}$.

Corollary B.1. *Same notation as above. Let* $h(x)\in Lip\ (U\times I)$ *and extend holomorphically in* x_3 *to a complex domain* $\{x_3\in C\,;\,\mathrm{Re}\,x_3\neq x_3^0\pm c\}$ (c*: a positive constant such that* $c\neq|x_3^0|$) *uniformly for* x_1 *and for* x_2. *Then,* $^\exists$*a positive constant* δ *such that the function* $R_Lh(x)$ *satisfies the Lewy equation* $L_x\{R_Lh(x)\}=h(x)$ *in* U_δ.

Theorem V thus results from Theorem IV and Corollary B.1; Proposition B is proved in §5. Finally, though Theorem III results from Theorem II and [G-K-S-I], a direct proof to it is given in §6.

§2. Proof of Theorem I

Let $u(x)$ satisfy the homogeneous Lewy equation

$$L_xu(x)=0 \tag{2.1}$$

in a neighborhood of P. Set $x^0=(x_1^0,\ x_2^0,\ x_3^0)$. We denote $u(x_1+x_1^0,\ x_2+x_2^0,\ x_3+x_3^0+2x_2^0x_1-2x_1^0x_2)$ by $u(x\bigstar x^0)$. Then $u(x\bigstar x^0)$ is a $C^k(O)$ solution of (2.1) in a neighborhood O of the origin in R^3, where $O=\{x\,;\,x_1^2+x_2^2<\rho,\ |x_3|<\rho\}$ ($^\exists\rho$: a positive constant). Introducing the polar coordinates $x_1+ix_2=r\exp(i\theta)$, (2.1) becomes to

$$\left\{\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\theta}+2ir\frac{\partial}{\partial x_3}\right\}\tilde{u}(r,\ \theta,\ x_3)=0 \tag{2.2}$$

where $\tilde{u}(r,\ \theta,\ x_3)\equiv u(x\bigstar x^0)$. Considering the Fourier series of $\tilde{u}(r,\ \theta,\ x_3)$, from (2.2), we have the following:

$$u(x\bigstar x^0)=\tilde{u}(r,\ \theta,\ x_3)=\sum_{n=-\infty}^{\infty}\tilde{u}_n(r,\ x_3)\exp(in\theta), \tag{2.3}$$

$$\left\{\frac{\partial}{\partial r}+2ir\frac{\partial}{\partial x_3}+\frac{n}{r}\right\}\tilde{u}_n(r,\ x_3)=0 \tag{2.4}$$

in $\{(r,\ x_3)\,;\,0<r<\sqrt{\rho},\ |x_3|<\rho\}\equiv\tilde{O}$.

Notice that

$$\sum_{n=-\infty}^{\infty}\left|\frac{n}{r}\tilde{u}_n(r,\ x_3)\right|^2 \tag{2.5}$$

is uniformly convergent on compact sets in $\tilde{O}$. Now (2.4) are rewritten as follows:

$$\left(\frac{\partial}{\partial t}+i\frac{\partial}{\partial x_3}\right)\{\sqrt{t}^{\,n}\tilde{u}_n(\sqrt{t},\ x_3)\}=0 \qquad \text{in } \tilde{\omega}_\rho, \tag{2.6}$$

where $\tilde{\omega}_\rho\equiv\{(t,\ x_3)\,;\,0<t<\rho,\ |x_3|<\rho\}$.

Therefore (2.6) give the following Lemmas:

Lemma 2.1. $\tilde{u}_n=0$ *for* $^\forall n>0$.

Lemma 2.2. *For* $^\forall n\leqq 0$ $\sqrt{t}^{\,n}\tilde{u}_n(\sqrt{t},\ x_3)$ *is holomorphic in* $t+ix_3\in\tilde{\omega}_\rho$.

On the other hand, we easily obtain the following

Lemma 2.3. $\sqrt{t}^{-n}\tilde{u}_{-n}(\sqrt{t}, x_3)\in C^0$ $(\tilde{\omega}_\rho\cup\{t=0\})$ *for* $n=0, 1, \cdots, k$.

Therefore, putting $h_n(t+ix_3)=\sqrt{t}^{-n}\tilde{u}_{-n}(0\leqq n)$, we have

$$u(x\bigstar x_0)=\sum_{n=0}^{\infty}(x_1-ix_2)^n h_n(x_1^2+x_2^2+ix_3).$$

That is,

$$u(x_1, x_2, x_3)=\sum_{n=0}^{\infty}w_1^n\cdot h_n(w_2) \qquad \text{in } \omega_\rho$$

which converges on compact sets in ω_ρ, by virtue of (2.5). Taking Lemma 2.3 into consideration, we have thus proved Theorem I.

Remark. From H. Lewy [4] it is known that every C^1 solution of (2.1) can locally extends holomorphically in a complex domain

$$\{(z_1, z_2)\in C^2\,;\ \mathrm{Im}\,z_2>|z_1|^2,\ z_1=x_1+ix_2,\ \mathrm{Re}\,z_2=x_3\}.$$

§3. Proof of Theorem IV

First we easily see the following

Lemma 3. *Let* $f(x)\in C^1(R^3)$. *Then,*

$$\frac{\partial}{\partial z_1}f^*(\sqrt{\xi}\cos\theta, \sqrt{\xi}\sin\theta, \eta\,;\,x_1, x_2)$$

$$=\sqrt{\xi}\exp(-i\theta)\Big\{\frac{\partial}{\partial\xi}-\frac{i}{2\xi}\frac{\partial}{\partial\theta}-i\frac{\partial}{\partial\eta}\Big\}f^*(\sqrt{\xi}\cos\theta, \sqrt{\xi}\sin\theta, \eta\,;\,x_1, x_2)$$

$$-i\bar{z}_1\frac{\partial}{\partial\eta}f^*(\sqrt{\xi}\cos\theta, \sqrt{\xi}\sin\theta, \eta\,;\,x_1, x_2),$$

where $\dfrac{\partial}{\partial z_1}=\dfrac{1}{2}\Big(\dfrac{\partial}{\partial x_1}-i\dfrac{\partial}{\partial x_2}\Big)$.

Now, for simplicity, we use the following notations:

$$\alpha=c_0^2,\ \beta=x_3^0+c,\ \gamma=x_3^0-c\,;\ \partial(\xi, \eta)=\frac{\partial}{\partial\xi}-i\frac{\partial}{\partial\eta}\,;$$

$$Hf_\chi(\xi, \eta\,;\,x_1x_2)=\int_0^{2\pi}\frac{f_\chi^*(\sqrt{\xi}\cos\theta, \sqrt{\xi}\sin\theta, \eta\,;\,x_1, x_2)}{\sqrt{\xi}\exp(-i\theta)}d\theta,$$

where χ denotes ξ or η;

$$\Theta f(\xi, \eta\,;\,x_1, x_2)=\int_0^{2\pi}f^*(\sqrt{\xi}\cos\theta, \sqrt{\xi}\sin\theta, \eta\,;\,x_1, x_2)d\theta\,;$$

$$\Theta f_\theta(\xi, \eta\,;\,x_1, x_2)=\int_0^{2\pi}f_\theta^*(\sqrt{\xi}\cos\theta, \sqrt{\xi}\sin\theta, \eta\,;\,x_1, x_2)d\theta.$$

Then

$$\frac{\partial}{\partial z_1}\left\{\frac{-1}{2\pi^2}\int_0^{c_0^2}d\xi\int_{x_3^0-c}^{x_3^0+c}\frac{Hf(\xi,\ \eta\ ;\ x_1,\ x_2)}{\xi+i(\eta-x_3)}d\eta\right\}$$

$$=\frac{\partial}{\partial z_1}\lim_{\varepsilon\to 0}\left\{\frac{-1}{2\pi^2}\int_0^{\alpha}d\xi\int_{\gamma}^{\beta}\frac{Hf(\xi,\ \eta\ ;\ x_1,\ x_2)}{\varepsilon^2+\xi+i(\eta-x_3)}d\eta\right\}$$

$$=\lim_{\varepsilon\to 0}\frac{\partial}{\partial z_1}\left\{\frac{-1}{2\pi^2}\int_0^{\alpha}d\xi\int_{\gamma}^{\beta}\frac{Hf(\xi,\ \eta\ ;\ x_1,\ x_2)}{\varepsilon^2+\xi+i(\eta-x_3)}d\eta\right\}$$

$$=\lim_{\varepsilon\to 0}\left\{\frac{-1}{2\pi^2}\int_0^{\alpha}\int_{\gamma}^{\beta}\frac{\partial(\xi,\ \eta)\Theta f(\xi,\ \eta\ ;\ x_1,\ x_2)}{\varepsilon^2+\xi+i(\eta-x_3)}d\xi d\eta\right.$$

$$+\frac{i}{2\pi^2}\int_0^{\alpha}\int_{\gamma}^{\beta}\frac{\Theta f_{\theta}(\xi,\ \eta\ ;\ x_1,\ x_2)}{2\xi\{\varepsilon^2+\xi+i(\eta-x_3)\}}d\xi d\eta$$

$$\left.+\frac{i\bar{z}_1}{2\pi^2}\int_0^{\alpha}\int_{\gamma}^{\beta}\frac{Hf_{\eta}(\xi,\ \eta\ ;\ x_1,\ x_2)}{\varepsilon^2+\xi+i(\eta-x_3)}d\xi d\eta\right\}$$

$$=\lim_{\varepsilon\to 0}\left\{\frac{-1}{2\pi^2}\int_{\gamma}^{\beta}d\eta\int_0^{2\pi}d\theta\int_0^{\alpha}\frac{f_{\xi}^{\#}(\sqrt{\xi}\cos\theta,\ \sqrt{\xi}\sin\theta,\ \eta\ ;\ x_1,\ x_2)}{\varepsilon^2+\xi+i(\eta-x_3)}d\xi\right.$$

$$+\frac{i}{2\pi^2}\int_0^{\alpha}d\xi\int_0^{2\pi}d\theta\int_{\gamma}^{\beta}\frac{f_{\eta}^{\#}(\sqrt{\xi}\cos\theta,\ \sqrt{\xi}\sin\theta,\ \eta\ ;\ x_1,\ x_2)}{\varepsilon^2+\xi+i(\eta-x_3)}d\eta$$

$$\left.+\frac{i\bar{z}_1}{2\pi^2}\int_0^{\alpha}\int_{\gamma}^{\beta}\frac{Hf_{\eta}(\xi,\ \eta\ ;\ x_1,\ x_2)}{\varepsilon^2+\xi+i(\eta-x_3)}d\xi d\eta\right\}$$

$$=-\frac{1}{2\pi^2}\int_{\gamma}^{\beta}\frac{\Theta f(\alpha,\ \eta\ ;\ x_1,\ x_2)}{\alpha+i(\eta-x_3)}d\eta+\frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_{\gamma}^{\beta}\frac{f(x_1,\ x_2,\ \eta)}{\varepsilon^2+i(\eta-x_3)}d\eta$$

$$+\frac{i}{2\pi^2}\lim_{\varepsilon\to 0}\int_0^{\alpha}\left[\frac{\Theta f(\xi,\ \eta\ ;\ x_1,\ x_2)}{\varepsilon^2+\xi+i(\eta-x_3)}\right]_{\eta=\gamma}^{\eta=\beta}d\xi$$

$$-Af(x)+\lim_{\varepsilon\to 0}\frac{i\bar{z}_1}{2\pi^2}\int_0^{\alpha}\int_{\gamma}^{\beta}\frac{Hf_{\eta}(\xi,\ \eta\ ;\ x_1,\ x_2)}{\varepsilon^2+\xi+i(\eta-x_3)}d\xi d\eta$$

$$=-\frac{1}{2\pi^2}\int_{\gamma}^{\beta}\frac{\Theta f(\alpha,\ \eta\ ;\ x_1,\ x_2)}{\alpha+i(\eta-x_3)}d\eta+\mathrm{P.V.}\ \frac{1}{\pi i}\int_{\gamma}^{\beta}\frac{f(x_1,\ x_2,\ \eta)}{\eta-x_3}d\eta$$

$$+f(x_1,\ x_2,\ x_3)-Af(x_1,\ x_2,\ x_3)$$

$$+\frac{i\bar{z}_1}{2\pi^2}\int_0^{\alpha}\left[\frac{\Theta f(\xi,\ \eta\ ;\ x_1,\ x_2)}{\xi+i(\eta-x_3)}\right]_{\eta=\gamma}^{\eta=\beta}d\xi$$

$$+\frac{i\bar{z}_1}{2\pi^2}\int_0^{\alpha}\int_{\gamma}^{\beta}\frac{Hf_{\eta}(\xi,\ \eta\ ;\ x_1,\ x_2)}{\xi+i(\eta-x_3)}d\xi d\eta\,.$$

On the other hand,

$$\frac{\partial}{\partial x_3}\left\{\frac{-1}{2\pi^2}\int_0^{c_0^2}d\xi\int_{x_3^0-c}^{x_3^0+c}\frac{Hf(\xi,\ \eta\ ;\ x_1,\ x_2)}{\xi+i(\eta-x_3)}-d\eta\right\}$$

$$=\frac{\partial}{\partial x_3}\lim_{\varepsilon\to 0}\left\{\frac{-1}{2\pi^2}\int_0^{\alpha}d\xi\int_{\gamma}^{\beta}\frac{Hf(\xi,\ \eta\ ;\ x_1,\ x_2)}{\varepsilon^2+\xi+i(\eta-x_3)}d\eta\right\}$$

$$=\frac{\partial}{\partial x_3}\lim_{\varepsilon\to 0}\frac{i}{2\pi^2}\int_0^{\alpha}d\xi\int_{\gamma}^{\beta}\{\log[\varepsilon^2+\xi+i(\eta-x_3)]\}_{\eta}Hf(\xi,\ \eta\ ;\ x_1,\ x_2)d\eta$$

$$=\frac{\partial}{\partial x_3}\lim_{\varepsilon\to 0}\frac{i}{2\pi^2}\int_0^\alpha[\log\{\varepsilon^2+\xi+i(\eta-x_3)\}Hf(\xi,\eta;x_1,x_3)]_{\eta=\gamma}^{\eta=\beta}d\xi$$

$$-\frac{\partial}{\partial x_3}\lim_{\varepsilon\to 0}\frac{i}{2\pi^2}\int_0^\alpha d\xi\int_\gamma^\beta\log\{\varepsilon^2+\xi+i(\eta-x_3)\}Hf_\eta(\xi,\eta;x_1,x_2)d\eta$$

$$=\frac{1}{2\pi^2}\int_0^\alpha[Hf(\xi,\eta;x_1,x_2)\{\xi+i(\eta-x_3)\}^{-1}]_{\eta=\gamma}^{\eta=\beta}d\xi$$

$$-\frac{1}{2\pi^2}\int_0^\alpha\int_\gamma^\beta\frac{Hf_\eta(\xi,\eta;x_1,x_2)}{\xi+i(\eta-x_3)}d\xi d\eta\,.$$

Therefore we have:

$$L_x\left\{\frac{-1}{2\pi^2}\int_0^{c_0^2}d\xi\int_{x_3^0-c}^{x_3^0+c}\frac{Hf(\xi,\eta;x_1,x_2)}{\xi+i(\eta-x_3)}d\eta\right\}$$

$$=f(x)-Af(x)-\frac{1}{2\pi^2}\int_\gamma^\beta\frac{\Theta f(\alpha,\eta;x_1,x_2)}{\alpha+i(\eta-x_3)}d\eta$$

$$+\text{P.V.}\frac{1}{\pi i}\int_\gamma^\beta\frac{f(x_1,x_3,\eta)}{\eta-x_3}d\eta+\frac{i\bar z_1}{2\pi^2}\int_0^\alpha\left[\frac{\Theta f(\xi,\eta;x_1,x_2)}{\xi+i(\eta-x_3)}\right]_{\eta=\gamma}^{\eta=\beta}d\xi$$

$$+\frac{i\bar z_1}{2\pi^2}\int_0^\alpha[Hf(\xi,\eta;x_1,x_2)\{\xi+i(\eta-x_3)\}^{-1}]_{\eta=\gamma}^{\eta=\beta}d\xi$$

Therefore it is proved that $Rf(x)$ defined by

$$Rf(x)=L_x\left\{\frac{-1}{2\pi^2}\int_0^{c_0^2}d\xi\int_{x_3^0-c}^{x_3^0+c}\frac{Hf(\xi,\eta;x_1,x_2)}{\xi+i(\eta-x_3)}d\eta\right\}-f(x)+Af(x)$$

belongs to $C^k(D_c)$ and extends holomorphically in x_3 to the complex domain $\{x_3\in C;\ \mathrm{Re}\,x_3\neq x_3^0\pm c\}$ uniformly for x_1 and for x_2. This completes the proof of Theorem IV.

§4. Proof of Lemma A

Let $u(x)$ be a C^1 solution of the Lewy equation

$$L_x u(x)=f(x) \tag{4.1}$$

in a neighborhood Ω of P. We may assume $\Omega=\{x;|x_j-x_j^0|<2c,\ j=1,2,3\}$, where c is a suitably chosen positive constant such that $c\neq|x_3^0|$. Set $\Omega_c=\{x;|x_j-x_j^0|<c\}$. Take ${}^\forall(x_1,x_2,x_3)$ in Ω_c. Let (x_1,x_2) be fixed for a while. Taking a positive constant c_0 such that $c_0\leqq\min.(c,c/[2(2c+|x_1^0|+|x_2^0|)])$, we see $u^\#(y_1,y_2,x_3;x_1,x_2)\in C^1(\{y_1^2+y_2^2<c_0^2\}\times\{|x_3-x_3^0|<c\})$. Then from (4.1) we have:

$$\left\{\frac{1}{2}\left(\frac{\partial}{\partial y_1}-i\frac{\partial}{\partial y_2}\right)+i(y_1-iy_2)\frac{\partial}{\partial x_3}\right\}u^\#(y_1,y_2,x_3;x_1,x_2)=f^\#(y_1,y_2,x_3;x_1,x_2)\,. \tag{4.3}$$

Set $y_1+iy_2=r\exp(i\theta)$. Then (4.3) becomes to

$$\left\{\frac{1}{2r}\frac{\partial}{\partial r}-\frac{i}{2r^2}\frac{\partial}{\partial\theta}+i\frac{\partial}{\partial x_3}\right\}u^\#(r,\theta,x_3;x_1,x_2)=\{r\exp(-i\theta)\}^{-1}\tilde f^\#(r,\theta,x_3;x_1,x_2)\,, \tag{4.4}$$

where $\tilde g^\#(r,\theta,x_3;x_1,x_2)\equiv g^\#(r\cos\theta,r\sin\theta,x_3;x_1,x_2)$, where g denotes u or f. Consequently, from (4.4), we have:

$$\left(\frac{\partial}{\partial r^2}+i\frac{\partial}{\partial x_3}\right)\Theta u(r^2, x_3; x_1, x_2)=Hf(r^2, x_3; x_1, x_2) \tag{4.5}$$

Putting $t=r^2$ $(r=\sqrt{t})$, (4.5) is rewritten as follows:

$$\left(\frac{\partial}{\partial t}+i\frac{\partial}{\partial x_3}\right)\Theta u(t, x_3; x_1, x_2)=Hf(t, x_3; x_1, x_2) \tag{4.6}$$

in $\mathscr{D}=\{(t, x_3); 0<t<c_0^2, |x_3-x_3^0|<c\}$.

(4.6) gives the following:

$$\Theta u(t, x_3; x_1, x_2)=\frac{1}{2\pi i}\int_{\partial\mathscr{D}}\frac{\Theta u(\xi, \eta; x_1, x_2)}{\xi+i\eta-(t+ix_3)}d\zeta-\frac{1}{\pi}\int_{\mathscr{D}}\frac{Hf(\xi, \eta; x_1, x_2)}{\xi+i\eta-(t+ix_3)}d\xi d\eta, \quad \zeta\equiv\xi+i\eta. \tag{4.7}$$

Letting $t\to 0$ and noticing that $\Theta u(0, x_3; x_1, x_2)=2\pi u(x)$, we thus obtain:

$$u(x)=\mathrm{P.V.}\frac{1}{\pi i}\int_{x_3^0+c}^{x_3^0-c}\frac{u(x_1, x_2, \eta)}{\eta-x_3}d\eta+\frac{1}{2\pi^2 i}\int\frac{\Theta u(\xi, \eta; x_1, x_2)}{\xi+i(\eta-x_3)}d\zeta+C_L f(x) \quad \text{for } {}^\forall x\in\Omega_c. \tag{4.7}$$

We thus have the conclusion of Lemma A by operating L_x to the both handsides of (4.7).

§ 5. Proof of Proposition B

First we prove Proposition B in case of $x_1^0=x_2^0=0$; that is, we shall prove that

$$R_{\tilde{L}}h(x)=-\pi^{-1}\int_{S_\delta}\frac{h(\xi^P, \eta^P, \chi^P)}{\xi-i\eta-(x_1-ix_2)}d\xi d\eta$$

is a C^1 solution of

$$\tilde{L}\{R_{\tilde{L}}h(x)\}=h(x) \tag{5.1}$$

in U_δ, by taking a suitable positive constant δ. In the actual case, ξ^P, η^P, χ^P, S_δ and U_δ denotes ξ, η, $\xi^2+\eta^2-x_1^2-x_2^2+x_3$, $\{(\xi, \eta)\in R^2; \xi^2+\eta^2<\delta^2\}$ and $\{x\in R^3; x_1^2+x_2^2<\delta^2, x_3\in I\}$, respectively.

Now first we choose a positive constant δ so that the above integral can be defined. Then we easily obtain the following Lemmas:

Lemma 5.1. *Putting $z_1(\equiv x_1+ix_2)=r\exp(i\theta)$, it holds that*

$$\int_{|\zeta|<|z_1|}h(\xi, \eta, |\zeta|^2-|z_1|^2+x_3)/(\bar{\zeta}-\bar{z}_1)d\xi d\eta$$

$$=-\sum_{n=0}^{\infty}r^{-n-1}\exp\{i(n+1)\theta\}\int_0^r\tau^{n+1}h_n(\tau, r, x_3)d\tau,$$

where $h_n(\tau, r, x_3)\equiv\int_0^{2\pi}h(\tau\cos\phi, \tau\sin\phi, \tau^2-r^2+x_3)e^{-in\phi}d\phi$ $(\zeta=\xi+i\eta)$.

Lemma 5.2. *Same notation as above.*

$$\int_{|\delta>|\zeta|>|z_1||} h(\xi, \eta, |\zeta|^2-|z_1|^2+x_3)/(\bar{\zeta}-\bar{z}_1)d\xi d\eta$$

$$=\sum_{n=0}^{\infty} r^n \exp\{-in\theta\}\int_r^\delta \tau^{-n}h_{-n-1}(\tau, r, x_3)d\tau .$$

By these Lemmas we have:

$$\pi R_{\tilde{L}}h(x)=\sum_{n=0}^{\infty} r^{-n-1}\exp\{i(n+1)\theta\}\int_0^r \tau^{n+1}h_n(\tau, r, x_3)d\tau$$

$$-\sum_{n=0}^{\infty} r^n \exp\{-in\theta\}\int_r^\delta \tau^{-n}h_{-n-1}(\tau, r, x_3)d\tau .$$

Therefore, we see

$R_{\tilde{L}}h(x)\in C^1(U_\delta)$ and

$$\tilde{L}(R_{\tilde{L}}h(x))=\frac{e^{-i\theta}}{2\pi}\left(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\theta}+2r\frac{\partial}{\partial x_3}\right)R_{\tilde{L}}h(r\cos\theta, r\sin\theta, x_3)$$

$$=(2\pi)^{-1}\sum_{n=0}^{\infty}\{h_n(r, r, x_3)e^{in\theta}+h_{-n-1}(r, r, x_3)e^{-i(n+1)\theta}\}$$

$$=h(r\cos\theta, r\sin\theta, x_3)=h(x_1, x_2, x_3).$$

Finally, in case of $(x_1^0, x_2^0)\neq(0, 0)$, the proof is as follows:

We put $h^*(t)=h(t_1+x_1^0, t_2+x_2^0, t_3+2x_1^0t_1+2x_2^0t_2)$, where $t=(t_1, t_2, t_3)$. Then, by virtue of the result proved above, $\exists$a positive constant δ so that

$$R_{\tilde{L}_t}h^*(t) \text{ satisfies } L_t\{R_{\tilde{L}_t}h^*(t)\}=h^*(t) \quad \text{in } U_\delta,$$

where $\tilde{L}_t\equiv\frac{1}{2}\left(\frac{\partial}{\partial t_1}-i\frac{\partial}{\partial t_2}\right)+(t_1-it_2)\frac{\partial}{\partial t_3}$.

Set $x_j=t_j+x_j^0$ $(j=1, 2)$ and $x_3=t_3+2x_1^0t_1+2x_2^0t_2$. Then $R_{\tilde{L}_t}h^*(t)\equiv R_{\tilde{L}}h(x)$ is a C^1 solution of $\tilde{L}\{R_{\tilde{L}}h(x)\}=h(x)$ in U_δ. This completes the proof.

§6. Proof of Theorem III

Suppose $Af(x)$ satisfies the condition of Theorem II. Then $C_bf(x)$ satisfies the same condition; that is, $C_bf(x)$ belongs to Lip (Ω_c) and extends holomorphically in x_3 to the compex domain $\{x_3\in R^3; \operatorname{Re} x_3\neq x_3^0\pm c\}$ uniformly for x_1 and for x_2. Set $M=\{(x_1, x_2, x_3, x_4)\in R^4; x_4=x_1^2+x_2^2\}$, $x_4^0=(x_1^0)^2+(x_2^0)^2$ and $P^*=(x_1^0, x_2^0, x_3^0, x_4^0)$. We can extend $C_bf(x)$ to the function $C_bf(x_1, x_2, x_3+i(x_4-x_1^2-x_2^2))$ which is defined in a neighborhood $\mathcal{O}$ of $P^*\in R^4$, where $C_bf(x_1, x_2, w)$ is holomorphic with respect to w. Let us denote $C_bf(x_1, x_2, x_3+i(x_4-x_1^2-x_2^2))$ by $C_b^*(f)$. We note $C_b^*(f)|_M=C_bf(x_1, x_2, x_3)\equiv C_b(f)$. It is known that $C_b(f)$ is the boundary function of

$$C(f)(z_1, z_2)\equiv\int_{\{\operatorname{Im} w_2=|w_1|^2\}}\frac{f(y_1, y_2, y_3)}{[\pi\{i(\bar{w}_2-z_2)-2\bar{w}_1z_1\}]^2}dy_1dy_2dy_3$$

which is holomorphic in $\mathcal{M}_+\equiv\{(z_1, z_2)\in C^2; \operatorname{Im} z_2>|z_1|^2\}$, where $(w_1, \operatorname{Re} w_2)=(y_1+iy_2, y_3)$.

Now let $L_{\mathcal{M}}$ denote the tangential Cauchy-Riemann operator $\partial/\partial\bar{z}_1-2iz_1(\partial/\partial\bar{z}_2)$ to $\partial\mathcal{M}_+$ which is identified with M. We denote the inner product

$$\int_{\mathcal{M}} f\bar{g}\,dy_1dy_2dy_3 \text{ in } L^2(\mathcal{M}) \text{ by } (f, g).$$

Then we easily see the following:

Lemma 6.1. *Let $u, v\in C^1(U)$, where U is an open set in R^4. Suppose $u=v$ on $U\cap\mathcal{M}$, then $L_M u=L_L v$ on $U\cap\mathcal{M}$.*

Corollary 6.2. *Let $u\in C^1(\mathcal{M})$. Suppose $u^\dagger$ is an $C^1(R^4)$ extension of u, then $L_{\mathcal{M}}u^\dagger=\bar{L}_x u$ on $\mathcal{M}$.*

Since $C_b^\dagger(f)|_{\mathcal{M}}=C_b(f)$ is the boundary function of $C(f)(z_1, z_2)$, by virtue of Corollary 6.2, it follows that

$$(6.1)\qquad (\bar{L}_x\{C_b(f)\}, v)=0 \qquad \text{for } {}^\forall v\in C_0^\infty(\mathcal{M}).$$

On the other hand, we see the following

Lemma 6.3. *$(\bar{L}_x\{C_b(f)\}, v)=(L_{\mathcal{M}}\{C_b^\dagger f\}, v)$ for ${}^\forall v\in C_0^\infty(\mathcal{M})$.*

Proof. If $C_b^\dagger f\in C^1$, by virtue of Corollary 6.2, $(L_{\mathcal{M}}\{C_b^\dagger f\}, v)=(\bar{L}_x\{C_b^\dagger f\}, v)=(\bar{L}_x\{C_b(f)\}, v)$. In the actual case, we have only to approximate $C_b^\dagger f$ by appropriate C^1 functions.

We thus obtain $(L_{\mathcal{M}}\{C_b^\dagger f\}, v)=0$ for ${}^\forall v\in C_0^\infty(\mathcal{M})$. Here, $(L_M\{C_b^\dagger f\}, v)=(\partial/\partial\bar{z}_1\{C_b^\dagger f\}, v)-2i(\bar{z}_1(\partial/\partial\bar{z}_2)\{C_b^\dagger f\}, v)=(\partial/\partial\bar{z}_1\{C_b^\dagger f\}, v)$. Therefore $\partial/\partial\bar{z}_1\{C_b^\dagger f\}=0$ in distribution sense. Therefore $C_b^\dagger f$ is holomorphic in z_1 and in z_2 in a neighborhood of $P^\dagger$ which is identified with P^*; and hence, we conclude that $C_b(f)\equiv C_b f(x)$ is real analytic in a neighborhood of P. This completes the proof of Theorem III.

Appendix

We here consider C^1 solutions of the homogeneous Lewy equation (H.L) $L_x u=0$ in a neighborhood of P. Any holomorphic function of w_1 and w_2 is a solution of (H.L); but, of course, the converse is not true. As is already remarked, however, every C^1 solution is "a function" of them. Suppose that there exist two independent solutions W_1 and W_2, namely, such that rank $\partial(W_1, W_2)/\partial(x_1, x_2, x_3)=2$ satisfying (H.L), such that every C^1 solution is a holomorphic function of W_1 and W_2. Then we easily see that every C^1 solution is a holomorphic function of w_1 and w_2. Therefore it is not true that there exist two independent solutions W_1 and W_2 of (H.L) such that every C^1 solution is a holomorphic function of them. From Theorem I we see the following

Proposition A. *Let u be a C^1 solution of (H.L) in a neighborhood of P. Then u is a holomorphic function of w_1 and w_2 if and only if ${}^\exists$a positive constant c such that*

$$\int_0^{2\pi} u(x_1^0+r\cos\theta,\ x_2^0+r\sin\theta,\ x_3)e^{-in\theta}d\theta$$

is real analytic in $x_3\in\{x_3^0-c<x_3<x_3^0+c\}$ *for* $^\forall n=0, 1, 2, \cdots$ *and* $^\forall r\in\{0<r<c\}$.

Now let A be an real analytic vector field in R^n. Suppose that it enjoys the following properties:

(A.1) A, $\bar{A}$, A_j are linearly independent, where $A_{j-1}\equiv[A, A_j]$ $(j=0, 1, 2, \cdots, n-3)$ and $A_0\equiv[A, \bar{A}]$.

(A.2) $^\exists X_1, {}^\exists X_2, \cdots, {}^\exists X_{n-1}\in C^\omega$ sucht hat $X_j(j=1, 2, \cdots, n-1)$ are complex-valued solutions of $AX_j=0$.

(A.3) $\operatorname{rank}\partial(X_1, X_2, \cdots, X_{n-1})/\partial(t_1, t_2, \cdots, t_n)=n-1$, where $t_j(j=1, 2, \cdots, n)$ denote coordinates variables.

We propose questions: under what conditions can every C^1 solution of $Au=0$ be a holomorphic function of the $X_1, X_2, \cdots, X_{n-1}$? Is it true that every C^1 solution of $Au=0$ is "a function" of them?

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