J. Math. Kyoto Univ. (JMKYAZ)
32-4 (1992) 763-795

Limit theorems for random difference equations driven by mixing processes

By

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1. Introduction

The purpose of this paper is to study the weak convergence of laws of a sequence of stochastic processes determined through random difference equations driven by stationary mixing processes. As concerns limit theorems for stochastic processes driven by mixing processes, including the case of random ordinary differential equations, there are a lot of studies on the central limit theorem and the diffusion approximation theorem. These results can be found in Khas'minskii [15], Ibragimov-Linnik [8], Kesten-Papanicolaou [14], Ethier-Kurtz [4], Kushner [19], Kunita [16], [17], and many articles in their references. This work is much influenced by these papers while we would like to emphasize that a notable feature of this paper is to develop these works to allow the limit processes to have jumps. In this point, we are strongly motivated by the works of Gnedenko-Kolmogorov [7], and Samur [21], [22].

Let $\{\xi_k^n; n\in \boldsymbol{N}, k\in \boldsymbol{N}^*\}$, where $\boldsymbol{N}=\{1, 2, \cdots\}$ and $\boldsymbol{N}^*=\{0, 1, 2, \cdots\}$, be an array of $\boldsymbol{R}^e$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Throughout this paper, we suppose that $\{\xi_k^n; k\in \boldsymbol{N}^*\}$ is stationary for every $n\in \boldsymbol{N}$. Let $\{F^n(x), G^n(x); n\in \boldsymbol{N}\}$ be a sequence of functions on $\boldsymbol{R}^d$. Then, for each $n\in \boldsymbol{N}$, we determine an $\boldsymbol{R}^d$-valued stochastic process $\{\varphi_k^n; k\in \boldsymbol{N}^*\}$ inductively by

$$(1.1)\qquad \begin{cases} \varphi_0^n = x_0 \in \boldsymbol{R}^d \\ \varphi_k^n - \varphi_{k-1}^n = F^n(\varphi_{k-1}^n)(\xi_k^n - a^n) + (1/n)G^n(\varphi_{k-1}^n) \qquad \text{for } k=1, 2, \cdots \end{cases}$$

where we set $a^n = E[\xi_0^n I_{\{|\xi_0^n|\leq \tau\}}]$ for some positive constant τ and I_A denotes the indicator function of the set A. Further, we define an interpolating process φ^n of $\{\varphi_k^n\}_k$ by

$$(1.2)\qquad \varphi_t^n = \varphi_{[nt]}^n \qquad \text{for } t\in[0, \infty),$$

where $[t]$ denotes the integer part of t. Then, $\{\varphi^n\}_{n\in \boldsymbol{N}}$ is regarded as a sequence of random variables with values in the space $\boldsymbol{D}_d = \boldsymbol{D}([0, \infty); \boldsymbol{R}^d)$ of càdlàg (right continuous with left hand limits) functions. As usual, we equip the space $\boldsymbol{D}_d$ with the Skorohod J_1-topology.

The problem we would like to discuss in this paper is to show the weak convergence of laws of $\{\varphi^n\}_n$ determined by (1.1) and (1.2) to a jump-diffusion (=strongly

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Communicated by Prof. S. Watanabe, February 4, 1991

Markovian càdlàg process). For this purpose, we will require several assumptions on (1.1). Loosely speaking, they are stated as follows.

1) The family of random variables $\{\xi_k^n; k\in \boldsymbol{N}^*\}$ in (1.1) is not necessarily independent but has certain mixing property, such as the strongly mixing property or the uniformly mixing one.

2) The random variables $\{\xi_k^n; n\in \boldsymbol{N}, k\in \boldsymbol{N}^*\}$ satisfy the condition ensuring the weak convergence of the sums $\sum_{k=1}^{[nt]}\{\xi_k^n-a^n\}$.

3) The coefficients $F^n(x)$ and $G^n(x)$ tend to $F(x)$ and $G(x)$ in suitable function spaces, respectively.

Then, we will show that the processes $\{\varphi^n\}_n$ of (1.2) converge in the sense of laws on $\boldsymbol{D}_d$ to a solution of stochastic differential equation of jump type:

$$\tag{1.3} \begin{aligned}\varphi_t=x_0+\int_{(0,t]}F(\varphi_{u-})dB_u+\int_{(0,t]}\{C(\varphi_{u-})+G(\varphi_{u-})\}du \\ +\int_{(0,t]}\int_{\{|z|\leq\tau\}}F(\varphi_{u-})z\tilde{N}(dudz)+\int_{(0,t]}\int_{\{|z|>\tau\}}F(\varphi_{u-})zN(dudz),\end{aligned}$$

where $B.$ is an e-dimensional centered Brownian motion, $N(dudz)$ is a Poisson random measure on $(0,\infty)\times\boldsymbol{R}^e$, and $\tilde{N}(dudz)$ denotes the compensated measure. Also, $C(x)$ is a correction function arising from the dependence of $\{\xi_k^n\}_k$ and the derivatives of the function $F(x)$. See Theorem 2.8 and a series of other theorems for the precise statement.

In the above, note that the processes φ^n are not necessarily Markovian while the solution of (1.3) is a jump-diffusion. Therefore, we can say that our problem is *the jump-diffusion approximation* for $\{\varphi^n\}_n$. Also, the above jump-diffusion approximation contains limit theorems for the sums of random variables, which are studied in the classic textbook [7], and [21]. Indeed, put $F^n(x)\equiv I$ (the $e\times e$ identity matrix) and $G^n(x)\equiv b(\in\boldsymbol{R}^e)$ in (1.1). Then φ_t^n of (1.2) is reduced to the sum $\sum_{k=1}^{[nt]}\{\xi_k^n-a^n\}+([nt]/n)b$ and we know from (1.3) that the limit process φ_t is represented as

$$\tag{1.4} \varphi_t=B_t+bt+\int_{(0,t]}\int_{\{|z|\leq\tau\}}z\tilde{N}(dudz)+\int_{(0,t]}\int_{\{|z|>\tau\}}zN(dudz).$$

Hence, φ_t is a Lévy process and the right hand side of (1.4) is exactly the Lévy-Itô decomposition. Therefore, in the class of finite dimensional stationary processes, our result includes the results in [7], [21], and [22]. The precise discussion will be given in the next section. On the other hand, in the previous paper Fujiwara [5], the jump-diffusion approximation for $\{\varphi^n\}_n$ determined by (1.1) and (1.2) has been studied under rather restricted conditions. We will also see that the results in this paper improve the previous ones.

Section 2 will be devoted to the case where $\{\xi_k^n; k\in\boldsymbol{N}^*\}$ in (1.1) is uniformly mixing. Main results are Theorem 2.8 and Theorem 2.12. The proof of them will be given by applying a result essentially shown in Fujiwara-Kunita [6]. It will be stated as Theorem 2.44, which enables us to treat limit theorems in this paper in a unified way. Furthermore, the application of them will be discussed. See Theorem 2.82 and Theorem 2.86.

In Section 3, we will discuss the case of strongly mixing processes. Applying Theorem 2.44 again, we will show a diffusion approximation theorem as a special case where the jump part of (1.3) is degenerate.

The final Section 4 will be devoted to studying the possible application of the theorems established in the previous sections to the case where $\{\xi_k^n ; k\in \boldsymbol{N}^*\}_n$ is a sequence of Markov chains. In other words, we will try to find a class of Markov chains for which these theorems hold.

2. The case of uniformly mixing processes

First of all, we give precise definition of several mixing properties for the stationary processes $\{\xi_k^n ; k\in \boldsymbol{N}^*\}_n$ in (1.1). Set $\mathcal{F}_k^n=\sigma[\xi_l^n ; l\in \boldsymbol{N}^*, l\leqq k]$, $\mathcal{F}^{n,k}=\sigma[\xi_l^n ; l\in \boldsymbol{N}^*, l\geqq k]$ and define

(2.1) $$\alpha_k^n=\sup_{l\in \boldsymbol{N}^*} \sup\{|P(A\cap B)-P(A)P(B)| ; A\in\mathcal{F}_l^n, B\in\mathcal{F}^{n,l+k}\},$$

(2.2) $$\phi_k^n=\sup_{l\in \boldsymbol{N}^*} \sup\left\{\left|\frac{P(A\cap B)}{P(A)}-P(B)\right| ; A\in\mathcal{F}_l^n, B\in\mathcal{F}^{n,l+k}, P(A)>0\right\},$$

(2.3) $$\psi_k^n=\sup_{l\in \boldsymbol{N}^*} \sup\left\{\left|\frac{P(A\cap B)}{P(A)P(B)}-1\right| ; A\in\mathcal{F}_l^n, B\in\mathcal{F}^{n,l+k}, P(A)P(B)>0\right\},$$

for each $n, k\in \boldsymbol{N}$. Then obviously we have $\alpha_k^n\leqq\phi_k^n\leqq\psi_k^n$, and $\psi_k^n\equiv 0$ if $\{\xi_k^n\}_k$ is independent. The sequence of stationary processes $\{\xi_k^n ; k\in \boldsymbol{N}^*\}_n$ is said to be *strongly mixing* (*α-mixing*), *uniformly mixing* (*ϕ-mixing*), or *ψ-mixing* according as α_k^n, ϕ_k^n, ψ_k^n converges to 0 as $k\to\infty$ for each n, respectively. See Eberlein-Taqqu [3] for various aspects of mixing processes. Throughout this paper, we will deal with the stationary processes with one of the above mixing properties.

Now, the purpose of this section is to establish jump-diffusion approximation theorems for the sequence of stochastic processes $\{\varphi^n ; n\in \boldsymbol{N}\}$ determined by (1.1) and (1.2) when $\{\xi_k^n ; k\in \boldsymbol{N}^*\}$ in (1.1) is uniformly mixing. In order to state our main results, Theorem 2.8 and Theorem 2.12, we introduce the following conditions (U. I)$\sim$(U. III) for $\{\xi_k^n\}_{n,k}$, which are common to them.

(U. I): There exists a Borel measure μ on $\boldsymbol{R}^e\setminus\{0\}$ such that for all $g\in C(\boldsymbol{R}^e)=:$ the set of all bounded continuous functions vanishing on some neighborhood of 0,

(2.4) $$n\int_{R^e} g(z)P(\xi_0^n\in dz)\to\int_{R^e} g(z)\mu(dz) \qquad \text{as } n\to\infty$$

and that

(2.5) $$\int_{R^e\setminus\{0\}} \min\{|z|^2, 1\}\mu(dz)<\infty.$$

(U. II): (1) There exists a real number V_0^{pq} for all $p, q=1, \cdots, e$ such that

(2.6) $$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}|nE[\eta_{0,\delta}^{n,(p)}\eta_{0,\delta}^{n,(q)}]-V_0^{pq}|=0,$$

where $\eta^n_{k,\delta}=\xi^n_{k,\delta}-E[\xi^n_{k,\delta}]$, $\xi^n_{k,\delta}=\xi^n_k I_{\{|\xi^n_k|\leq\delta\}}$, and $\eta^{n,(p)}_k$ denotes the p-th component of η^n_k.

(2) There exists a real number V^{pq}_1 for all $p, q=1, \cdots, e$ such that

$$\lim_{\delta\downarrow 0}\lim_{n\to\infty}\sup|n\sum_{k=1}^{n-1}E[\eta^{n,(p)}_{0,\delta}\eta^{n,(q)}_{k,\delta}]-V^{pq}_1|=0. \tag{2.7}$$

Let ϕ^n_k be the rate function defined by (2.2).

(U. III): $$\bar{\phi}=:\sum_{k=1}^{\infty}\sup_{n\in N}(\phi^n_k)^{1/2}<\infty.$$

Let $C^m(R^d, R^e)$ $(m\in N^*)$ be the set of all C^m-maps from R^d to R^e. For $f\in C^m(R^d, R^e)$, we define the norms $\|\cdot\|_{m*}$ by

$$\|f\|_{m*}=\sup_{x\in R^d}\left\{\frac{|f(x)|}{1+|x|}\right\}+\sum_{1\leq|\alpha|\leq m}\sup_{x\in R^d}|\partial^\alpha_x f(x)|,$$

where $\alpha=(\alpha_1, \cdots, \alpha_d)$ is a multi-index of nonnegative integers, $|\alpha|=\alpha_1+\cdots+\alpha_d$, $\partial^\alpha_x f(x)=(\partial^{|\alpha|}/(\partial x_1)^{\alpha_1}\cdots(\partial x^d)^{\alpha_d})f(x)$. We denote by $C^m_{b*}(R^d, R^e)$ the set of all $f\in C^m(R^d, R^e)$ such that $\|f\|_{m*}<\infty$. The space $C^0_{b*}(R^d, R^e)$ and the norm $\|\ \|_{0*}$ are often denoted by $C_{b*}(R^d, R^e)$ and $\|\ \|_*$, respectively. We also denote by $R^d\otimes R^e$ the set of all $d\times e$ real matrices.

As a condition for the coefficients F^n and G^n in (1.1), we introduce the following:

(C): (1) $F^n\in C^2(R^d, R^d\otimes R^e)$ for all $n\in N$. Further, there exists $F\in C^2_{b*}(R^d, R^d\otimes R^e)$ such that $\lim_{n\to\infty}\|F^n-F\|_{2,K}=0$ for every compact set K in R^d, where $\|F\|_{2,K}=\sum_{|\alpha|\leq 2}\sup_{x\in K}|\partial^\alpha_x F(x)|$.

(2) $G^n\in C^1(R^d, R^e)$ for all $n\in N$, and there exists $G\in C^1_{b*}(R^d, R^e)$ such that $\lim_{n\to\infty}\|G^n-G\|_{1,K}=0$ for every compact set K in R^d.

Here, it should be noticed that Conditions (U. I)$\sim$(U. III) and (C) are not sufficient in general for the weak convergence of the processes $\{\varphi^n\}_n$. Indeed, Theorem 3.2 in Samur [22] tells us that it is necessary to hold that $\lim_{n\to\infty}nP[|\xi^n_0|>\varepsilon, |\xi^n_1|>\varepsilon]=0$ for every $\varepsilon>0$ even if $\{\xi^n_k\}_k$ is 1-dependent and F^n, G^n are constants. Therefore, we need to find a sufficient condition which ensures at least the above property. In the next theorem, we give the condition in terms of the uniform integrability of some class of random variables.

(2.8) Theorem. *Suppose that Conditions (U. I)$\sim$(U. III) and (C) are satisfied. Moreover, we suppose*

(U. IV)$_1$: $\{X^{(n)}_k=:nE[|\xi^n_{k,N}|^2|\mathfrak{F}^n_{k-1}]\,;n, k\in N\}$ *is uniformly integrable for every* $N>0$. *Also, take* $a^n=E[\xi^n_{0,\tau}]$ *in* (1.1) *for arbitrary* $\tau\in C(\mu)=:\{r>0\,;\mu(|z|=r)=0\}$. *Then, the sequence of* D_d*-valued random variables* $\{\varphi^n\}_n$ *determined by* (1.1) *and* (1.2) *converges in law as* $n\to\infty$ *to the unique solution* φ *of the stochastic differential equation* (1.3), *in which*

(i) B_t *is an e-dimensional Brownian motion with the mean* 0 *and the covariance matrix*

$$(V^{pq}=V^{pq}_0+V^{pq}_1+V^{qp}_1)_{p,q=1,\cdots,e}, \tag{2.9}$$

(ii) $C(x)=(C^i(x))_{i=1,\cdots,d}$ *is a function of class* $C^1(R^d, R^d)$ *defined by*

(2.10) $$C^j(x)=\sum_{i=1}^{d}\sum_{p,q=1}^{e} F^{ip}(x)V_1^{pq}\frac{\partial F^{jq}}{\partial x^i}(x),$$

(iii) *$N(dudz)$ is a stationary Poisson random measure with the intensity measure $du\mu(dz)$ and $\tilde{N}(dudz)=N(dudz)-du\mu(dz)$.*

See Ikeda-Watanabe [10] for stochastic integrals based on Poisson random measures and stochastic differential equations of jump type such as (1.3).

(2.11) Remark. Let $\{\xi_k^n\}_{n,k}$ in Theorem 2.8 be given by $\xi_k^n=\xi^n\circ\theta^k$ for $k\in \boldsymbol{Z}=\{\cdots, -1, 0, 1, \cdots\}$, where $\{\xi^n\}_n$ is a sequence of $\boldsymbol{R}^e$-valued random variables defined on $(\Omega, \mathcal{F}, P)$ and $\theta:\Omega\mapsto\Omega$ is a bimeasurable, bijective mapping such that $P\circ\theta^{-1}=P$. Suppose that $\{\xi_k^n\}_k$ is a uniformly mixing process with the rate function ϕ_k^n defined by

$$\phi_k^n=\sup_{l\in Z}\sup\left\{\left|\frac{P(A\cap B)}{P(A)}-P(B)\right| ; A\in\mathcal{F}_l^n, B\in\mathcal{F}^{n,l+k}, P(A)>0\right\}$$

wher $\mathcal{F}_k^n=\sigma[\xi_l^n ; l\in \boldsymbol{Z}, l\leqq k]$ and $\mathcal{F}^{n,k}=\sigma[\xi_l^n ; l\in \boldsymbol{Z}, l\geqq k]$. In this case, Condition (U. IV)$_1$ is simplified as

(U. IV)$_1^*$: $\{X^{(n)}=: nE[|\xi_{1,N}^n|^2|\mathcal{F}_0^n] ; n\in \boldsymbol{N}\}$ is uniformly integrable for every $N>0$.

In fact, if we note that $E[|\xi_{l,N}^n|^2|\mathcal{F}_0^n]=E[|\xi_{1,N}^n|^2|\mathcal{F}_0^n]\circ\theta^{k-1}$, it is clear that (U.IV)$_1^*$ implies (U.IV)$_1$.

In the next theorem, instead of the uniform integrability condition (U.IV)$_1$, we assume the ϕ-mixing property.

(2.12) Theorem. *Suppose that Condition (U.I)$\sim$(U.III) and (C) are satisfied. Moreover, suppose the following*

(U. IV)$_2$: $$\sup_{n\in N}\phi_1^n<\infty,$$

where ϕ_k^n is the rate function defined by (2.3). *Then the conclusion of Theorem 2.8 still holds.*

Let us mention the connection between this theorem and the preceding works. Let $\{\xi_k^n\}_{n,k}$ be an array of 1-dimensional, independent, and identically distributed random variables satisfying Conditions (U.I) and (U.II)-(1). Then, by Theorem 1 of §25 in Gnedenko-Kolmogorov [7], it is known that a sequence of processes $\{\varphi^n\}_n$ defined by

(2.13) $$\varphi_t^n=\sum_{k=1}^{[nt]}\{\xi_k^n-E[\xi_{0,\tau}^n]\}+([nt]/n)b$$

converges in the sense of finite dimensional distributions to the Lévy process of (1.4) with the characteristics (V_0, b, μ). Furthermore, we know from Samur [21] and [22] that $\{\varphi^n\}_n$ of (2.13) converges in law to the Lévy process of (1.4) with the characteristics (V, b, μ) if $\{\xi_k^n\}_{n,k}$ satisfies Conditions (U.I)$\sim$(U.III) and (C). Therefore, Theorem 2.12 is regarded as an extension of them to the case where the limit process is a solution of a stochastic differential equation driven by Lévy process because the processes $\{\varphi^n\}_n$ of (2.13) are obtained by putting $F^n(x)\equiv I$ (the $e\times e$ identity matrix and

$G^n(x)\equiv b$ ($\in \boldsymbol{R}^e$) in (1.1).

Also, in the previous paper Fujiwara [5], the same assertion as Theorem 2.12 is shown under extra conditions:

$$\sup_{n\in N} nE[|\xi_{0,N}^n|^2]<\infty \qquad \text{for all } N>0,$$

$$\sum_{k=1}^{\infty} \sup_{n\in N} (\phi_k^n)^{1/2}<\infty,$$

and more restricted regularity condition on the coefficients F^n. See Theorem 1 and Theorem 3 in [5]. Therefore, Theorem 2.12 is an improvement of the results in [5].

(**2.14**) **Remark.** Recently, limit theorems for a sequence of semimartingales have been studied by many authors. See [11], [12], and [20] for the weak convergence of stochastic integrals based on semimartingales. Furthermore, see [13], [18], and [23] for that of solutions of stochastic differential equations driven by semimartingales. We can regard the processes $\{\varphi^n\}_n$ in Theorem 2.8 and Theorem 2.12 as a sequence of semimartingales. But they do not seem to satisfy at least conditions given in [13] or [23] in general.

We will prove Theorem 2.8 and Theorem 2.12 in a unified way. To this end, we introduce a technical condition (U. IV)* as follows.

Let $\{X_t^n\}_n$ be a sequence of 1-dimensional càdlàg processes. We say that $\{X^n\}_n$ satisfies *Condition* (*CT& UI*) if $\{X^n\}_n$ is *C-tight*, that is to say, $\{X^n\}_n$ is tight in $\boldsymbol{D}_1$ and any weak limit law is supported in the space $C([0, \infty), \boldsymbol{R}^1)$ and if, further, $\{X_t^n\}_n$ is uniformly integrable for every $t\geqq 0$.

For each $n\in \boldsymbol{N}$, $0<\delta<N$, define nondecreasing càdlàg processes X_t^n and $Y_t^n(\delta)$ by

$$X_t^n=\sum_{k=1}^{[nt]} E[|\eta_{k,N}^n|^2|\mathcal{F}_{k-1}^n], \tag{2.15}$$

$$Y_t^n(\delta)=\sum_{k=1}^{[nt]} E[|\xi_{k,N}^{n,\delta}|^2|\mathcal{F}_{k-1}^n], \tag{2.16}$$

where $\xi_{k,N}^{n,\delta}=\xi_k^n\cdot I_{\{\delta<|\xi_k^n|\leq N\}}$.

(**U. IV**)*: (1) $\{X^n\}_n$ of (2.15) satisfies Condition (CT & UI) for every N.
(2) $\{Y^n(\delta);\ n\in \boldsymbol{N}, 0<\delta\leqq N\}$ of (2.16) satisfies

$$\lim_{\theta\downarrow 0} \sup_{\delta\leq N} \limsup_{n\to\infty} P[W_T(Y^n(\delta), \theta)>\varepsilon]=0, \tag{2.17}$$

for alll $\varepsilon>0$, where $W_T(\varphi, \theta)$ denotes the modulus of continuity defined by $W_T(\varphi, \theta)= \sup\{|\varphi_t-\varphi_s|;\ t-s\leqq\theta,\ s\leqq t\leqq T\}$ for $\varphi\in \boldsymbol{D}_d$, and

$$\lim_{K\uparrow\infty} \sup_{\delta\leq N} \limsup_{n\to\infty} E[Y_t^n(\delta);\ Y_t^n(\delta)>K]=0, \tag{2.18}$$

for each t.

Next lemma gives us a useful characterization of C-tightness of a sequence of càdlàg processes.

(**2.19**) **Lemma.** *Let* $\{X^n\}_n$ *be a sequence of càdlàg processes. Then the following* (*a*)

$\sim$(c) are equivalent.

(*a*) $\{X^n\}_n$ *is C-tight.*

(*b*) $\{X^n\}_n$ *satisfies that for all* $T>0$ *and* $\delta>0$

$$\lim_{K\to\infty}\limsup_{n\to\infty} P[\sup_{t\leq T}|X_t^n|>K]=0, \tag{2.20}$$

$$\lim_{\theta\downarrow 0}\limsup_{n\to\infty} P[W_T(X^n, \theta)>\delta]=0. \tag{2.21}$$

(*c*) $\{X^n\}_n$ *is tight and it satisfies that for all* $T, \delta>0$

$$\limsup_{n\to\infty} P[\sup_{t\leq T}|\Delta X_t^n|>\delta]=0, \tag{2.22}$$

where $\Delta X_t^n=X_t^n-X_{t-}^n$.

Proof. See Proposition 3.26 in [11, p. 315]. □

For nondecreasing càdlàg processes X_t and Y_t, we say that X_t is strongly majorized by Y_t and denote by $X_t \ll Y_t$ if (Y_t-X_t) is also nondecreasing. Then it is clear from Lemma 2.10 that $\{X^n\}_n$ satisfies Condition (CT & UI) if there exists $\{Y^n\}_n$ such that $X_t^n \ll Y_t^n$ for all n and that $\{Y^n\}_n$ satisfies (CT & UI). We will often use this property without mentioning.

Next results, Lemma 2.23 and Lemma 2.26, enable us to treat Theorem 2.8 and Theorem 2.12 in the same framework.

(2.23) **Lemma.** *Condition* $(U.IV)_1$ *implies Condition* $(U.IV)^*$.

To prove this lemma, we prepare a lemma which gives us a characterization of uniform integrability.

(2.24) **Lemma.** *Let* $\{X^n\}_n$ *be a family of integrable random variables. Then the following are equivalent.*

(*a*) $\{X^n\}_n$ *is uniformly integrable.*

(*b*) *There exists a positive, increasing convex function* $G(x)$ *defined on* $[0, \infty)$ *such that* $\lim_{x\to+\infty}\{G(x)/x\}=+\infty$ *and that* $\sup_n E[G(|X^n|)]<\infty$.

Proof. See Theorem 19 in Dellacherie-Meyer [1, Chapitre I, p. 34]. □

Proof of Lemma 2.23. Define a càdlàg process U_t^n by $U_t^n=(1/n)\sum_{k=1}^{[nt]}X_k^{(n)}=\sum_{k=1}^{[nt]}E[|\xi_{k,N}^n|^2|\mathcal{F}_{k-1}^n]$. Then we have $X_t^n \ll 2\{U_t^n+E[U_t^n]\}$ and $\sup_{\delta\leq N} Y_t^n(\delta) \ll U_t^n$. Therefore, it suffices to show that $\{U^n\}_n$ satisfies (CT & UI).

By $(\mathrm{U.IV})_1$ and by Lemma 2.24, there exists a positive, increasing, convex function G on $[0, \infty)$ such that

$$\lim_{x\to+\infty}\{G(x)/x\}=\infty, \qquad \sup_{n,k} E[G(X_k^{(n)})]<\infty. \tag{2.25}$$

Then by the convexity of G, it is clear that $\sup_n E[G(U_t^n)]<\infty$ for each t, which implies the uniform integrability of $\{U_t^n\}_n$. We next show the tightness of $\{U^n\}_n$. Set $\mathcal{F}_t^n=\mathcal{F}_{[nt]}^n$. Let σ and τ be arbitrary $\{\mathcal{F}_t^n\}$-stopping times such that $\sigma\leqq\tau\leqq T$ and that $\tau-\sigma\leqq r$ and let C be arbitrary positive number. Since

$$E[U_\tau^n-U_\delta^n]$$

$$=E[(1/n)\sum_{k=[n\sigma]+1}^{[n\tau]} X_k^{(n)}]$$

$$=(1/n)E[\sum_{k=[n\sigma]+1}^{[n\tau]} X_k^{(n)} I_{\{X_k^{(n)}\leq C\}}]+(1/n)E[\sum_{k=[n\sigma]+1}^{[n\tau]} X_k^{(n)} I_{\{X_k^{(n)}>C\}}]$$

$$\leqq CE[([n\tau]-[n\sigma])/n]+(1/n)\sum_{k=1} E[X_k^{(n)} I_{\{X_k^{(n)}>C\}}]$$

$$\leqq C(r+(1/n))+([nT]/n)\sup_{n,k} E[X_k^{(n)} I_{\{X_k^{(n)}>C\}}],$$

we have

$$\lim_{r\downarrow 0}\limsup_{n\to\infty}\sup\{E[U_\tau^n-U_\sigma^n]\,;\,\sigma\leqq\tau\leqq T,\ \tau-\sigma\leqq r\}\leqq T\sup_{n,k} E[X_k^{(n)} I_{\{X_k^{(n)}>C\}}].$$

Since Condition (U.IV)$_1$ implies that the last term converges to 0 as $C\uparrow\infty$, we see from the theorem of Aldous that $\{U^n\}_n$ is tight in $\boldsymbol{D}_1$. See Theorem 4.5 in [11, p. 320]. Finally, me show that $\{U_t^n\}_n$ satisfies (2.22). Let G be the function given above. Since

$$P[\sup_{t\leq T}\Delta U_t^n>\delta]=P[\max_{1\leq k\leq[nT]}(1/n)X_k^{(n)}>\delta]$$
$$\leqq[nT]\sup_k P[X_k^{(n)}>n\delta]$$
$$\leqq[nT]\sup_k P[G(X_k^{(n)})>G(n\delta)]$$
$$\leqq\frac{[nT]}{n\delta}\cdot\frac{n\delta}{G(n\delta)}\sup_{n,k} E[G(X_k^{(n)})],$$

(2.22) follows from (2.25). Thus, we see from Lemma 2.19 that $\{U^n\}_n$ is C-tight. □

(2.26) Lemma. *Conditions (U.I), (U.II), and (U.IV)$_2$ imply Condition (U.IV)*.*

Proof. By the definition of ϕ_1^n, we can see that for every n, $k\in\boldsymbol{N}$ and $\delta<N$

$$E[|\eta_{k,N}^n|^2|\mathcal{F}_{k-1}^n]\leqq(\sup_n\phi_1^n+1)E[|\eta_{0,N}^n|^2],$$

and that

$$E[|\xi_{k,N}^{n,\delta}|^2|\mathcal{F}_{k-1}^n]\leqq(\sup_n\phi_1^n+1)E[|\xi_{0,N}^{n,\delta}|^2].$$

Since (U.I) and (U.II) imply that $A=:\sup_n nE[|\eta_{0,N}^n|^2]$ is finite, X_t^n of (2.15) is strongly majorized by the deterministic process $A(\sup_n\phi_1^n+1)\times([nt]/n)$. Hence, it is obvious that (U.IV)*-(1) is satisfied. Similarly, we can show that $\{Y_t^n(\delta)\}_{n,\delta}$ of (2.16) satisfies (U.IV)*-(2) if we note the property.

$$\sup_{\delta\leq N}\limsup_{n\to\infty} nE[|\xi_{0,N}^{n,\delta}|^2]\leqq\sup_{\delta\leq N}\int_{\{\delta\leq|z|\leq N\}}|z|^2\mu(dz)\leqq\int_{\{|z|\leq N\}}|z|^2\mu(dz)<\infty.\ \square$$

By Lemma 2.23 and Lemma 2.26, it is immediate that both Theorem 2.8 and Theorem 2.12 follow from the next result.

(2.27) Theorem. *Suppose that Conditions (U.I)$\sim$(U.III), (U.IV)*, and (C) are satisfied. Then the conclusion of Theorem 2.8 holds.*

In the following, we will give a proof of Theorem 2.27. For this purpose, we apply a limit theorem for stochastic processes determined by random difference equations of general form. Since it can be deduced from Theorem 2.1 of Fujiwara-Kunita [6] by making a little improvement in the conditions and the proof, we restrict ourselves to stating the assertion.

For each $n\in \boldsymbol{N}$, let $\{f_k^n(\cdot)\,;\,k\in \boldsymbol{N}^*\}$ be a $\boldsymbol{C}^2(\boldsymbol{R}^d, \boldsymbol{R}^d)$-valued stochastic process with parameter k defined on a probability spoce $(\Omega, \mathcal{F}, P)$. We define a sequence of the sub σ-fields $\{\mathcal{F}_k^n\,;\,k\in \boldsymbol{N}^*\}_{n\in \boldsymbol{N}}$ of $\mathcal{F}$ by $\mathcal{F}_k^n=\sigma[f_0^n, f_1^n, \cdots, f_k^n]$. Let $\{g_k^n(\cdot)\,;\,n\in \boldsymbol{N}, k\in \boldsymbol{N}^*\}$ be a sequence of deterministic functions of class $\boldsymbol{C}^1(\boldsymbol{R}^d, \boldsymbol{R}^d)$. Associated with the sequence $\{f_k^n, g_k^n\}_n$, we consider the following stochastic difference equation:

$$
\begin{cases} \varphi_0^n=x_0\in \boldsymbol{R}^d \\ \varphi_k^n-\varphi_{k-1}^n=f_k^n(\varphi_{k-1}^n)+g_k^n(\varphi_{k-1}^n), \qquad k=1, 2, \cdots, \end{cases} \tag{2.28}
$$

and define a sequence of càdlàg processes $\{\varphi^n\}_n$ by (1.2). We now put the following conditions (A. I)$\sim$(A. V) on $\{f_k^n, g_k^n\}$.

To make the notations simple, we often use the following abbreviations. For ε, $M>0$, we set

$$
f_{k,M}^n(x)=f_k^n(x)I_{(\|f_k^n\|*\leq M)}, \qquad f_{k,M}^{n,\varepsilon}(x)=f_k^n(x)I_{(\varepsilon<\|f_k^n\|*\leq M)},
$$
$$
\bar{f}_{k,M}^n(x)=E[f_{k,M}^n(x)], \qquad \tilde{f}_{k,M}^n(x)=f_{k,M}^n(x)-\bar{f}_{k,M}^n(x).
$$

We denote by $f_k^{n,(i)}$ the i-th component of f_k^n.
For each $n\in \boldsymbol{N}$, let $\mathcal{A}^n$ be the set of all real-valued càdlàg processes A_t^n satisfying

(i) A^n is $\{\mathcal{F}_{[nt]}^n\,;\,t\geqq 0\}$-adapted,
(ii) $t\mapsto A_t^n$ is nondecreasing,
(iii) $E[A_t^n]<\infty$ for each $t\in[0, \infty)$.

For $A^n\in \mathcal{A}^n$, it follows from Doob-Meyer's decomposition theorem that there exists a unique predictable process $A^{n,p}$ of class $\mathcal{A}^n$ such that $A_t^n-A_t^{n,p}$ is a martingale. We call the process $A^{n,p}$ the *compensator* of A^n.

We also denote by $\boldsymbol{S}_d$ the set of all $d\times d$ real, symmetric, nonnegative definite matrices.

(A. I): For every compact set K in $\boldsymbol{R}^d$ and for every $T>0$, and $\varepsilon>0$,

$$
\lim_{n\to\infty}\max_{0\leq k\leq [nT]} P[\sup_{x\in K}|f_k^n(x)|>\varepsilon]=0.
$$

(A. II): (1) For every compact set K in $\boldsymbol{R}^d$ and for every positive constants M, T, there exists a sequence of stochastic processes $\{D^n\in \mathcal{A}^n\}_n$ satisfying the following properties (i), either (ii) or (ii)$'$:

(i) For all $s\leqq t\leqq t'\leqq T$,

$$
\sum_{|\alpha|\leq 2,\,|\beta|\leq 1}\sum_{k=[ns]+1}^{[nt]}\sup_{x,y\in K}|E[\sum_{l=k+1}^{[nt']}\partial_y^\alpha \tilde{f}_{l,M}(y)|\mathcal{F}_k^n]||\partial_x^\beta \tilde{f}_{k,M}(x)| \tag{2.29}
$$
$$
+\sum_{|\beta|\leq 1}\sum_{k=[ns]+1}^{[nt]}\sup_{x\in K}|\partial_x^\beta \tilde{f}_{k,M}(x)|^2
$$
$$
\leqq D_t^n-D_s^n.
$$

(ii) The sequence of compensators $\{D^{n,p}\}_n$ of $\{D^n\}_n$ satisfies Condition (CT & UI).

(ii)′ $\{D^n\}_n$ itself satisfies Condition (CT & UI).

(2) For every compact set K in $\boldsymbol{R}^d$ and for every positive constants $\varepsilon\leqq M$, T, there exists a sequence of stochastic processes $\{E^n(\varepsilon)\in\mathcal{A}^n\}$ satisfying the following properties (i)~(iii):

(i)

$$\sum_{|\beta|\leq 1}\sum_{k=1}^{[nt]}\sup_{x\in K}|\partial_x^\beta f_{k,M}^{n,\varepsilon}(x)|^2\ll E_t^n(\varepsilon). \tag{2.30}$$

(ii) For every $\kappa>0$,

$$\lim_{\theta\downarrow 0}\sup_{0\leq\varepsilon\leq M}\limsup_{n\to\infty}P[W_T(E^{n,p}(\varepsilon),\theta)>\kappa]=0. \tag{2.31}$$

(iii) For every $t\geqq 0$,

$$\lim_{K\uparrow\infty}\sup_{0\leq\varepsilon\leq M}\limsup_{n\to\infty}E[E_t^{n,p}(\varepsilon)\,;\,E_t^{n,p}(\varepsilon)>K]=0. \tag{2.32}$$

(3) For every compact set K and for every positive constants $\varepsilon<M$, there exists a sequence of deterministic nondecreasing functions $\{D^n\}_n$ satisfying the following properties (i) and (ii).

(i)

$$\sum_{|\beta|\leq 1}\sum_{k=1}^{[nt]}\Big\{E[\sup_{x\in K}|\partial_x^\beta f_{k,M}^{n,\varepsilon}(x)|]+\sup_{x\in K}|\partial_x^\beta g_k^n(x)+\partial_x^\beta \bar{f}_{k,\varepsilon}^n(x)|\Big\}\ll \bar{D}_t^n. \tag{2.33}$$

(ii) For every $T>0$

$$\sup_{n\in N}\bar{D}_T^n<\infty,\qquad\text{and}\qquad \lim_{\theta\downarrow 0}\limsup_{n\to\infty}W_T(\bar{D}^n,\theta)=0. \tag{2.34}$$

(A. III): (1) There exists a Borel measure $\nu(df)$ on $\boldsymbol{C}_{b*}(\boldsymbol{R}^d,\boldsymbol{R}^d)$ satisfying the following properties (i)~(ii):

(i) There exists some $\gamma\in C(\nu)=:\{r>0\,;\,\nu(\|f\|_*=r)=0\}$ such that

$$\int_{(\|f\|_*\leq\gamma)}\|f\|_{1*}^2\nu(df)<\infty,\qquad\text{and}\qquad \nu(\|f\|_*>\gamma)<\infty. \tag{2.35}$$

(ii) For every M, $\varepsilon\in C(\nu)$ and for every bounded continuous function h on $\boldsymbol{R}^d$

$$\lim_{n\to\infty}E[\sup_{x\in K}|E[\sum_{k=[ns]+1}^{[nt]}h(f_k^n(x))I_{(\varepsilon<\|f_k^n\|_*\leq M)}|\mathcal{F}_{[ns]}^n] \tag{2.36}$$
$$-(t-s)\int_{C_{b*}(R^d,R^d)}h(f(x))I_{(\varepsilon<\|f\|_*\leq M)}\nu(df)|]=0.$$

(2) There exists a function $a_0\in\boldsymbol{C}_{b*}^2(\boldsymbol{R}^d,\boldsymbol{S}_d)$ such that for every compact set K and $s\leqq t$

$$\lim_{\varepsilon\in C(\nu)\downarrow 0}\limsup_{n\to\infty}E[\sup_{x\in K}|E[\sum_{k=[ns]+1}^{[nt]}\tilde{f}_{k,\varepsilon}^{n,(i)}(x)\tilde{f}_{k,\varepsilon}^{n,(j)}(x)|\mathcal{F}_{[ns]}^n]-(t-s)a_0^{ij}(x)|]=0. \tag{2.37}$$

(3) There exist functions $a_1\in\boldsymbol{C}_{b*}^2(\boldsymbol{R}^d,\boldsymbol{R}^d\otimes\boldsymbol{R}^d)$ and $c\in\boldsymbol{C}_{b*}^1(\boldsymbol{R}^d,\boldsymbol{R}^d)$ such that for every compact set $K, M>0$, and $s\leqq t$

$$\lim_{n\to\infty}E[\sup_{x\in K}|E|\sum_{k=[ns]+1}^{[nt]-1}\sum_{l=k+1}^{[nt]}\tilde{f}_{k,M}^{n,(i)}(x)\tilde{f}_{l,M}^{n,(j)}(x)|\mathcal{F}_{[ns]}^n]-(t-s)a_1^{ij}(x)|]=0, \tag{2.38}$$

and

(2.39) $$\lim_{n\to\infty} E[\sup_{x\in K}|E[\sum_{k=[ns]+1}^{[nt]-1}\sum_{l=k+1}^{[nt]}\sum_{j=1}^{d}((\partial/\partial x^j)\tilde{f}_{k,M}^{n,(i)})(x) \times \tilde{f}_{l,M}^{n,(j)}(x)\,|\,\mathcal{F}_{[ns]}^{n}]-(t-s)c^i(x)|]=0,$$

respectively.

(4) There exists a function $b\in C_{b*}^1(\boldsymbol{R}^d, \boldsymbol{R}^d)$ such that for every compact set K and $s\leqq t$

(2.40) $$\lim_{n\to\infty}\sup_{x\in K}|\sum_{k=[ns]+1}^{[nt]}\{g_k^n(x)+\bar{f}_{k,\gamma}^n(x)\}-(t-s)b(x)|=0,$$

where γ is as in (1)-(i).

(**A. IV**): For every compact set K, $s\leqq t$, $M>0$, it holds that

(1)

(2.41) $$\lim_{n\to\infty}\sup_{s\in[0,t]}\sum_{|\alpha|\leq 2}E[\sup_{x\in K}|E[\sum_{k=[ns]+1}^{[nt]}\partial_x^\alpha\tilde{f}_{k,M}^n(x)\,|\,\mathcal{F}_{[ns]}^n]\,|]=0,$$

(2)

(2.42) $$\lim_{n\to\infty}\sum_{|\alpha|\leq 2}E[\sum_{k=[ns]+1}^{[nt]}\sup_{x,y\in K}\{|E[\sum_{l=k+1}^{[nt]}\partial_y^\alpha\tilde{f}_{l,M}(y)\,|\,\mathcal{F}_k^n]\|\tilde{f}_{k,M}^n(x)|^2\}=0.$$

(**A. V**): For every $t>0$, it holds that

(2.43) $$\lim_{M\to\infty}\limsup_{n\to\infty}P[\sup_{k\leq[nt]}\|f_k^n\|_*>M]=0.$$

(**2.44**) **Theorem.** *Suppose that Conditions* (*A. I*)$\sim$(*A. V*) *are satisfied. Then, the sequence of* $\boldsymbol{D}_d$*-valued random variables* $\{\varphi^n\}_n$ *determined by* (2.28) *and* (1.2) *converges in law to the unique solution of the following stochastic differential equation*:

(2.45) $$\varphi_t=x_0+\int_{(0,t]}\sigma(\varphi_{u-})dB_u+\int_{(0,t]}(b+c)(\varphi_{u-})du +\int_{(0,t]}\int_{(\|f\|_*\leq\gamma)}f(\varphi_{u-})\tilde{N}(dudf)+\int_{(0,t]}\int_{(\|f\|_*>\gamma)}f(\varphi_{u-})N(dudf),$$

where (i) σ *is a Lipschitz continuous function from* $\boldsymbol{R}^d$ *to* $\boldsymbol{R}^d\otimes\boldsymbol{R}^r$ *such that* $\sigma(x)\sigma(x)^*=a_0(x)+\{a_1(x)+a_1(x)^*\}$ *where* a^* *denotes the transpose of the matrix* a,

(ii) B_t *is an r-dimensional standard Brownian motion,*

(iii) $N(dudf)$ *is a stationary Poisson random measure with the intensity measure* $du\nu(df)$.

Before giving a proof of Theorem 2.27, we prepare basic inequalities which hold for uniformly mixing random variables. The following result is shown in the proofs of Lemma VIII. 3.102 in [11] and Lemma 5.6.2 in [17].

(**2.46**) **Lemma.** *Let* $\{\xi_k\}_{k\in N^*}$ *be a stationary, uniformly mixing process with the rate function* ϕ_k. *Suppose that* ξ *is an* L^p, $\mathcal{F}^k$*-adapted random variable for some* $p\in[2,\infty]$, *where* $\mathcal{F}^k=\sigma[\xi_k, \xi_{k+1}, \cdots]$. *Then, it holds that for all* $l<k$

(2.47) $$|E[\xi\,|\,\mathcal{F}_l]-E[\xi]|\leqq 2(\phi_{k-l})^{1/q}\{E[|\xi|^p\,|\,\mathcal{F}_l]^{1/p}+E[|\xi|^p]^{1/p}\},$$

where $\mathcal{F}_l=\sigma[\xi_0, \cdots, \xi_l]$ *and* q *is the conjugate of* p, *that is*, $q=1/(1-1/p)$.

Proof. Let $Q(\omega, d\omega')$ be a regular conditional probability of $P(\cdot|\mathcal{F}_l)$ on $\mathcal{F}^k$. Define a signed measure $\mu(\omega, d\omega')$ by $\mu(\omega, d\omega')=Q(\omega, d\omega')-P(d\omega')|_{\mathcal{F}^k}$, and let $\mu(\omega, d\omega')=\mu^+(\omega, d\omega')-\mu^-(\omega, d\omega')$ be the Jordan-Hahn decomposition of $\mu(\omega, d\omega')$. Then, by the definition of the rate function, we have for all $B\in\mathcal{F}^k$ $|\mu(\omega, B)|\leqq\phi_{k-l}$ $P-a.s.$, which implies that $\mu^+(\omega, \Omega)$, $\mu^-(\omega, \Omega)\leqq\phi_{k-l}$. Hence we have

$$(2.48)\qquad |E[\xi|\mathcal{F}_l]-E[\xi]|^q=\left|\int\xi d\mu\right|^q$$

$$\leqq 2^{q-1}\left\{\left|\int\xi d\mu^+\right|^q+\left|\int\xi d\mu^-\right|^q\right\}$$

$$\leqq 2^{q-1}\left\{\mu^+(\omega, \Omega)\left\{\int|\xi|^p d\mu^+\right\}^{q/p}+\mu^-(\omega, \Omega)\left\{\int|\xi|^p d\mu^-\right\}^{q/p}\right\}$$

$$\leqq 2^q\phi_{k-l}\left\{\int|\xi|^p d|\mu|\right\}^{q/p}.$$

Moreover, since $|\mu|(\omega, d\omega')\leqq Q(\omega, d\omega')+P(d\omega')|_{\mathcal{F}^k}$, we have

$$\left\{\int|\xi|^p d|\mu|\right\}^{q/p}\leqq\{E[|\xi|^p|\mathcal{F}_l]^{1/p}+E[|\xi|^p]^{1/p}\}^q.$$

Combining this with (2.48), we obtain (2.47). □

(2.49) Lemma. *Let* $\{\xi_k\}_k$ *be the same as in Lemma* 2.46. *Suppose that* $\sup_{k,\omega}|\xi_k(\omega)|\leqq C$. *Then we have for all* $m<l<k$

$$(2.50)\qquad |E[\eta_k\eta_l|\mathcal{F}_m]-E[\eta_k\eta_l]|\leqq 16C^2(\phi_{k-l})^{1/2}(\phi_{l-m})^{1/2},$$

where $\eta_k=\xi_k-E[\xi_k]$.

Proof. Applying Lemma 2.46 for $p=\infty$ $(q=1)$, we have

$$(2.51)\qquad |E[\eta_k\eta_l|\mathcal{F}_m]-E[\eta_k\eta_l]|\leqq 16C^2\phi_{l-m}.$$

Similarly, we have

$$(2.52)\qquad |E[\eta_k\eta_l|\mathcal{F}_m]|\leqq E[|E[\eta_k|\mathcal{F}_l]||\eta_l||\mathcal{F}_m]\leqq 8C^2\phi_{k-l},$$

$$(2.53)\qquad |E[\eta_k\eta_l]|\leqq 8C^2\phi_{k-l}.$$

Therefore, by (2.51), (2.52), and (2.53), we obtain

$$|E[\eta_k\eta_l|\mathcal{F}_m]-E[\eta_k\eta_l]|^2$$

$$\leqq|E[\eta_k\eta_l|\mathcal{F}_m]-E[\eta_k\eta_l]|\times\{|E[\eta_k\eta_l|\mathcal{F}_m]|+|E[\eta_k\eta_l]|\}$$

$$\leqq(16C^2)^2\phi_{k-l}\phi_{l-m},$$

which implies (2.50). □

Proof of Theorem 2.27. We first give a proof of Theorem 2.27 under additional assumptions that $\lim_{n\to\infty}\|F^n-F\|_*=0$ and that F is not identically 0. In this case $F_I=\inf_n\|F^n\|_*>0$. For each $n\in\boldsymbol{N}$ and $k\in\boldsymbol{N}^*$, set

$$(2.54)\qquad f_k^n(x)=F^n(x)\xi_k^n, \quad\text{and}\quad g_k^n(x)=(1/n)G^n(x)-F^n(x)a^n,$$

in (2.28). Also, define $\mathcal{F}_k^n$ by $\sigma[\xi_0^n, \cdots, \xi_k^n]$. Then, what we should do is to check that, under Conditions (U.I)～(U.III), (U.IV)*, and (C), f_k^n and g_k^n defined as above satisfy

(A.I)$\sim$(A.V) in Theorem 2.44. In the sequel, let K be an arbitrary compact set in $\boldsymbol{R}^d$ and let ε, M be arbitrary positive numbers such that $\varepsilon<M$.

(Check of (A.I)) By the stationarity of $\{\xi_k^n\}_k$ and (U.I), it is obvious that (A.I) is satisfied.

(Check of (A.II)) (1) For $\varepsilon<M$, we set $F_S=\sup_n\|F^n\|_*$, $F_I=\inf_n\|F^n\|_*$, $\bar{F}=\sup_n\|F^n\|_{2,K}$, $\delta(n)=\varepsilon/\|F^n\|_*$, $\delta=\varepsilon/F_S$, $N(n)=M/\|F^n\|_*$, and $N=M/F_I$. Then, since we see from Lemma 2.46 that

$$\begin{aligned}
&\sup_{y\in K}|E[\sum_{l=k+1}^{[nt']}\partial_y^\alpha \tilde{f}_{l,M}(y)\,|\,\mathcal{F}_k^n]|\\
&\leqq \sup_{y\in K}\sum_{l=k+1}^{[nt']}|E[\partial_y^\alpha F^n(y)\eta_{l,N(n)}^n\,|\,\mathcal{F}_k^n]|\\
&\leqq 2\bar{F}\sum_{l=k+1}^{[nt']}(\phi_{l-k}^n)^{1/2}\{E[|\eta_{l,N(n)}^n|^2\,|\,\mathcal{F}_k^n]^{1/2}+E[|\eta_{0,N(n)}^n|^2]^{1/2}\},
\end{aligned}$$

the left hand side of (2.29) is dominated by

$$\begin{aligned}
(2.55)\qquad & 2(\bar{F})^2(\sum_{k=[ns]+1}^{[nt]}\sum_{l=k+1}^{[nt']}(\phi_{l-k}^n)^{1/2}\{E[|\eta_{l,N(n)}^n|^2\,|\,\mathcal{F}_k^n]^{1/2}\\
&+E[|\eta_{0,N(n)}^n|^2]^{1/2}\}\times|\eta_{k,N(n)}^n|+\sum_{k=[ns]+1}^{[nt]}|\eta_{k,N(n)}^n|^2)\\
&\leqq 2(\bar{F})^2(\sum_{l=1}^{\infty}(\phi_l^n)^{1/2}\sum_{k=[ns]+1}^{[nt]}E[|\eta_{l+k,N(n)}^n\,|\,\mathcal{F}_k^n]\\
&+(\bar{\phi}+1)\times\sum_{k=[ns]+1}^{[nt]}\{|\eta_{k,N(n)}^n|^2+E[|\eta_{0,N(n)}^n|^2]\}).
\end{aligned}$$

Hence if we define $\{D^{n,1}\in\mathcal{A}^n\}_n$ by

$$(2.56)\qquad D_t^{n,1}=\sum_{l=0}^{\infty}(\phi_l^n)^{1/2}\sum_{k=1}^{[nt]}E[\eta_{l+k,N(n)}^n|^2\,|\,\mathcal{F}_k^n]$$

where we set $\phi_0^n\equiv 1$, we see from (2.55) that the left hand side of (2.29) is dominated by $C\{D_t^{n,1}-D_s^{n,1}+E[D_t^{n,1}-D_s^{n,1}]\}$ for some constant C which does not depend on n, s, t. Therefore, (A.II)-(1)-(i) is satisfied if we define $\{D^n\}_n$ in (2.29) by

$$(2.57)\qquad D_t^n=C\{D_t^{n,1}+E[D_t^{n,1}]\}.$$

In the sequel, we prove that the compensators $\{D^{n,1,p}\}_n$ of $\{D^{n,1}\}_n$ satisfy (CT & UI) because it implies that (A.II)-(1)-(ii) is satisfied with $\{D^n\}_n$ of (2.57).

From the definition of compensator, we have

$$(2.58)\qquad D_t^{n,1,p}=\sum_{l=0}^{\infty}(\phi_l^n)^{1/2}X_t^{n,(l)},$$

where $X_t^{n,(l)}$ is a nondecreasing càdlàg process defined by

$$(2.59)\qquad X_t^{n,(l)}=\sum_{k=1}^{[nt]}E[|\eta_{l+k,N(n)}^n|^2\,|\,\mathcal{F}_{k-1}^n].$$

We first show inductively that $\{X^{n,(l)}\}_n$ satisfies Condition (CT & UI) for all $l\in\boldsymbol{N}^*$.

Since we may assume that $N(n)>1$ for all n, we have $|\eta^n_{k,N(n)}|^2 \leqq 4\{|\eta^n_{k,N}|^2+|\xi^{n,1}_{k,N}|^2+E[|\xi^{n,1}_{0,N}|^2]\}$, which implies that $X^{n,(0)}_t \ll 4\{X^n_t+Y^n_t(1)+E[Y_t(1)]$ for $\{X^n\}_n$ of (2.15) and $\{Y^n(1)\}_n$ of (2.16). On the other hand, we see from Lemma 2.19 that $\{Y^n(1)\}_n$ satisfies (CT & UI). Hence, it is clear from (U.IV)* that the assertion holds for $l=0$.

We next consider the case of $l=1$. Associated with $X^{n,(1)}$, define a stochastic process $Z^{n,(1)}$ of class $\mathcal{A}^n$ by

$$Z^{n,(1)}_t=\sum_{k=1}^{[nt]}E[|\eta^n_{k+1,N(n)}|^2|\mathcal{F}^n_k]\,. \tag{2.60}$$

Then it is easy to see that $\{Z^{n,(1)}\}_n$ satisfies (CT & UI) because so does $\{X^{n,(0)}\}_n$. Also, note that $X^{n,(1)}_t=Z^{n,(1),p}_t$, where $Z^{n,(1),p}_t$ denotes the compensator of $Z^{n,(1)}_t$. Let $\sigma\leqq\tau$ be $\{\mathcal{F}^n_t\}$-stopping times bounded by T. Then, by Lenglart's inequality ([11, p.35], we have for arbitrary $\varepsilon, \delta>0$

$$\begin{aligned} &P[X^{n,(1)}_\tau-X^{n,(1)}_\sigma>\delta] \\ &\leqq(1/\delta)\{\varepsilon+E[\sup_{t\leqq T}\Delta Z^{n,(1)}]\}+P[Z^{n,(1)}_\tau-Z^{n,(1)}_\sigma>\varepsilon]\,. \end{aligned} \tag{2.61}$$

Since (CT & UI) for $\{Z^{n,(1)}\}_n$ implies that

$$\begin{aligned} &\lim_{n\to\infty}E[\sup_{t\leqq T}\Delta Z^{n,(1)}_t]=0\,, \\ &\lim_{\theta\downarrow 0}\limsup_{n\to\infty}\sup_{\sigma\leqq\tau\leqq T,\,\tau-\sigma\leqq\theta}P[Z^{n,(1)}_\tau-Z^{n,(1)}_\sigma>\varepsilon]=0\,, \end{aligned} \tag{2.62}$$

we see from (2.61) that

$$\lim_{\theta\downarrow 0}\limsup_{n\to\infty}\sup_{\sigma\leqq\tau\leqq T,\,\tau-\sigma\leqq\theta}P[X^{n,(1)}_\tau-X^{n,(1)}_\sigma>\delta]=0\,,$$

Hence, by the Aldous criterion, we can conclude that $\{X^{n,(1)}\}_n$ is tight.

In order to prove the C-tightness of $\{X^{n,(1)}\}_n$, we need to show that (2.22) holds for $\{X^{n,(1)}\}_n$. Recall Lemma 2.19. Set

$$S^n=\sup_{t\leqq T}\Delta Z^{n,(1)}_t=\max_{1\leqq k\leqq[nT]}E[|\eta^n_{k+1,N(n)}|^2|\mathcal{F}^n_k]\,.$$

Then we have

$$\sup_{t\leqq T}\Delta X^{n,(1)}_t=\max_{1\leqq k\leqq[nT]}E[E[|\eta^n_{k+1,N(n)}|^2|\mathcal{F}^n_k]|\mathcal{F}^n_{k-1}]\leqq\max_{1\leqq k\leqq[nT]}E[S^n|\mathcal{F}^n_{k-1}]\,.$$

Since $\{E[S^n|\mathcal{F}^n_{k-1}]\,;\,k=1,2,\cdots\}$ is a martingale, Doob's inequality implies that for all $\delta>0$

$$\begin{aligned} &P[\sup_{t\leqq T}\Delta X^{n,(1)}_t>\delta] \\ &\leqq P[\max_{1\leqq k\leqq[nT]}E[S^n|\mathcal{F}^n_{k-1}]>\delta] \\ &\leqq(1/\delta)E[E[S^n|\mathcal{F}^n_{[nT]-1}]]=(1/\delta)E[\sup_{t\leqq T}\Delta Z^{n,(1)}_t]\,. \end{aligned}$$

Hence, we see from (2.62) that (2.22) holds for $\{X^{n,(1)}\}_n$.

In order to prove that $\{X^{n,(1)}\}_n$ satisfies (CT & UI), it remains to show that $\{X^{n,(1)}_t\}_n$ is uniformly integrable. For arbitrary $C, M>0$, we have

$$\begin{aligned} E[X^{n,(1)}_t\,;\,X^{n,(1)}_t>C]&=E[Z^{n,(1),p}_t\,;\,Z^{n,(1),p}_t>C] \\ &\leqq E[Z^{n,(1)}_t\,;\,Z^{n,(1)}_t>M]+E[Z^{n,(1)}_t\,;\,Z^{n,(1)}_t\leqq M\,;\,Z^{n,(1),p}_t>C] \\ &\leqq E[Z^{n,(1)}_t\,;\,Z^{n,(1)}_t>M]+(M/C)\sup_n E[Z^{n,(1)}_t]\,. \end{aligned}$$

Hence, by the uniform integrability of $\{Z_t^{n,(1)}\}_n$, we obtain $\lim_{C\to\infty}\sup_n E[X_t^{n,(1)}; X_t^{n,(1)} > C]=0$. Thus, we have shown that $\{X^{n,(1)}\}_n$ satisfies (CT & UI).

Repeating this discussion, we can conclude that $\{X^{n,(l)}\}_n$ satisfies (CT & UI) for all $l\in\boldsymbol{N}$.

We now prove that $\{D_t^{n,1,p}=:\sum_{l=0}^{\infty}(\phi_l^n)^{1/2}X_t^{n,(l)}\}_n$ is C-tight. For arbitrary $\varepsilon>0$, choose l_0 such that $\sup_n\sum_{l=l_0+1}^{\infty}(\phi_l^n)^{1/2}<\varepsilon$. Then, since $\{X^{n,(l)}\}_n$ is C-tight for each l, it is obvious that $\sum_{l=0}^{l_0}(\phi_l^n)^{1/2}X_t^{n,(l)}$ is C-tight. On the other hand, note that for $t\leqq T$

$$\sup_n E[\sum_{l=l_0+1}^{\infty}(\phi_l^n)^{1/2}X_t^{n,(l)}]\leqq \varepsilon\sup_n[nT]E[|\eta_{0,N(n)}^n|^2]=:C_{T,N}\varepsilon\,.$$

Then, we have for all $T>0$ and $\delta>0$

$$\begin{aligned}
&\lim_{\theta\downarrow 0}\ \limsup_{n\to\infty} P[W_T(D^{n,1,p},\theta)>\delta] \\
&\leqq \lim_{\theta\downarrow 0}\ \limsup_{n\to\infty} P[W_T(\sum_{l=0}^{l_0}(\phi_l^n)^{1/2}X^{n,(l)},\theta)>\delta/2] \\
&\quad +\lim_{\theta\downarrow 0}\ \limsup_{n\to\infty} P[W_T(\sum_{l=l_0+1}^{\infty}(\phi_l^n)^{1/2}X^{n,(l)},\theta)>\delta/2] \\
&\leqq 0+\limsup_{n\to\infty} P[\sum_{l=l_0+1}^{\infty}(\phi_l^n)^{1/2}X_T^{n,(l)}>\delta/2] \\
&\leqq \limsup_{n\to\infty}(2/\delta)E[\sum_{l=l_0+1}^{\infty}(\phi_l^n)^{1/2}X_T^{n,(l)}]\leqq(2/\delta)C_{T,N}\varepsilon\,.
\end{aligned}$$

Since ε is arbitrary, we see that $\{D^{n,1,p}\}_n$ satisfies (2.21). On the other hand, since $\sup_n E[D_T^{n,1,p}]<\infty$, it is obvious that $\{D^{n,1,p}\}_n$ satisfies (2.20). Hence, we see from Lemma 2.19 that $\{D^{n,1,p}\}$ is C-tight.

We next show the uniform integrability of $\{D_t^{n,1,p}\}_n$. For arbitrary $\varepsilon>0$, choose l_0 such that $\sup_n\sum_{l=l_0+1}^{\infty}(\phi_l^n)^{1/2}<\varepsilon$, as before. Then we have for all $C>0$

$$\begin{aligned}
(2.63)\qquad & E[D_t^{n,1,p}; D_t^{n,1,p}>C] \\
&\leqq E[\sum_{l=0}^{l_0}(\phi_l^n)^{1/2}X_t^{n,(l)}; \sum_{l=0}^{l_0}(\phi_l^n)^{1/2}X_t^{n,(l)}>C/2] \\
&\quad +E[\sum_{l=0}^{l_0}(\phi_l^n)^{1/2}X_t^{n,(l)}; \sum_{l=l_0+1}^{\infty}(\phi_l^n)^{1/2}X_t^{n,(l)}>C/2] \\
&\quad +E[\sum_{l=l_0+1}^{\infty}(\phi_l^n)^{1/2}X_t^{n,(l)}] \\
&\leqq 2E[\sum_{l=0}^{l_0}X_t^{n,(l)}; \sum_{l=0}^{l_0}X_t^{n,(l)}>C/2]+2E[\sum_{l=l_0+1}^{\infty}(\phi_l^n)^{1/2}X_t^{n,(l)}] \\
&\leqq 2E[\sum_{l=0}^{l_0}X_t^{n,(l)}; \sum_{l=0}^{l_0}X>C/2]+2\varepsilon C_{T,N}\,.
\end{aligned}$$

Since $\{\sum_{l=0}^{l_0}X_t^{n,(l)}\}_n$ is uniformly integrable as we saw before, by (2.63), we obtain

$$\lim_{C\to\infty}\sup_n E[D_t^{n,1,p}; D_t^{n,1,p}>C]\leqq 2\varepsilon C_{T,N}\,.$$

Since $\varepsilon>0$ is arbitrary, we can obtain the uniform integrability of $\{D_t^{n,1,p}\}_n$. By the discussion above, we have shown that $\{D^n\}_n$ of (2.57) satisfies (A. II)-(1)-(ii).

(2) Define a stochastic process $E^n(\varepsilon)$ of class $\mathcal{A}^n$ by

$$E_t^n(\varepsilon)=(\sup_n\|F^n\|_{1,K})\times\sum_{k=0}^{[nt]}|\xi_{k,N}^{n,\delta}|^2, \tag{2.64}$$

where we set $\delta=\varepsilon/F_S$ and $N=M/F_I$ as before. Then it is obvious from (U. IV)*-(2) that (A. II)-(2) is satisfied with $\{E^n(\varepsilon)\}$ of (2.64) because $E_t^{n,p}(\varepsilon)\ll(\sup_n\|F^n\|_{1,K})\times Y_t^n(\delta)$.

(3) Note that we may assume that $\delta<\tau$. Take

$$\bar{D}_t^n=2[nt]E[|\xi_{0,\tau}^{n,\delta}|]\times\sup_n\|F^n\|_{1,K}+([nt]/n)\sup_n\|G^n\|_{1,K}. \tag{2.65}$$

Then, by the stationarity of $\{\xi_k^n\}_k$ it is clear that (2.33) holds. Since (U. I) implies that $\sup_n nE[|\xi_{0,\tau}^{n,\delta}|]<\infty$, it is also clear that (2.34) holds.

(Check of (A. III)) (1) Define a σ-finite measure ν on $\boldsymbol{C}(\boldsymbol{R}^d, \boldsymbol{R}^d)$ by

$$\nu(df)=\mu\circ M^{-1}(df), \tag{2.66}$$

where M is a mapping from $\boldsymbol{R}^e$ to $\boldsymbol{C}(\boldsymbol{R}^d, \boldsymbol{R}^d)$ defined by $M(z)=F(\cdot)z$ for $z\in\boldsymbol{R}^e$. Also, set $\gamma=\tau\|F\|_*$. Note that $\gamma\in C(\nu)=\{r>0; (r/\|F\|_*)\in C(\mu)\}$ because we assume that $\tau\in C(\mu)$. (2.35) is an obvious result from (2.5) and (C). To see (2.36), owing to the polynomial approximation, it is enough to show it when $h(x)=x^m$ for $m\in\boldsymbol{N}^*$. For ε, $M\in C(\nu)$, put $\delta(n)=\varepsilon/\|F^n\|_*$, $N(n)=M/\|F^n\|_*$. Then, by Lemma 2.46, we have

$$\begin{aligned} &E[\sup_{x\in K}|E[\sum_{k=[ns]+1}^{[nt]}\{f_k^n(x)\}^m I_{\{\varepsilon<\|f_k^n\|_*\leq M\}}|\mathcal{F}_{[ns]}^n] \\ &\quad -E[\sum_{k=[ns]+1}^{[nt]}\{f_k^n(x)\}^m I_{\{\varepsilon<\|f_k^n\|_*\leq M\}}]|] \\ &\leq(F_S)^m\sum_{k=[ns]+1}^{[nt]}E[|E[(\xi_{k,N(n)}^{n,\delta(n)})^m-E[(\xi_{k,N(n)}^{n,\delta(n)})^m]|\mathcal{F}_{[ns]}^n]|] \\ &\leq 4(F_S)^m\bar{\phi}E[(\xi_{k,N(n)}^{n,\delta(n)})^{2m}]^{1/2}\to 0 \qquad \text{as } n\to\infty, \end{aligned} \tag{2.67}$$

because $E[(\xi_{0,N(n)}^{n,\delta(n)})^{2m}]\leq E[(\xi_{0,N}^{n,\delta})^{2m}]$. On the other hand, since (U. I) implies that

$$\begin{aligned} \sum_{k=[ns]+1}^{[nt]}E[(\xi_{k,N(n)}^{n,\delta(n)})^m]&\to(t-s)\int z^m I_{\{\delta<|z|\leq N\}}\mu(dz) \\ &=(t-s)\int z^m I_{\{\varepsilon<\|F\|_*|z|\leq M\}}\mu(dz), \end{aligned}$$

we have

$$\lim_{n\to\infty}E[\sup_{x\in K}|\sum_{k=[ns]+1}^{[nt]}E[\{f_k^n(x)\}^m I_{\{\varepsilon<\|f_k^n\|_*\leq M\}}]-(t-s)\int f(x)^m I_{\{\varepsilon<\|f\|_*\leq M\}}\nu(df)|]=0.$$

Combining this with (2.67) we obtain (2.36).

(2) Define $a_0\in \boldsymbol{C}_{b*}^2(\boldsymbol{R}^d, \boldsymbol{S}_d)$ in (2.37) by

$$a_0^{ij}(x)=\sum_{p,q=1}^{e}F^{ip}(x)V_0^{pq}F^{jq}(x), \tag{2.68}$$

where V_0 is the matrix defined by (2.6). Then, it is easy to see that (U. II)-(1) implies (2.37).

(3) Define $a_1 \in C^2_{b*}(\boldsymbol{R}^d, \boldsymbol{R}^d \otimes \boldsymbol{R}^e)$ in (2.38) by

$$a_1^{ij}(x) = \sum_{p,q=1}^{e} F^{ip}(x) V_1^{pq} F^{jq}(x)\,, \tag{2.69}$$

where V_1 is the matrix defined by (2.7). Then, by Lemma 2.49, we have

$$\begin{aligned}\sum_{k<l}{}' E[\,|E[\eta^n_{k,\delta(n)}\eta^n_{l,\delta(n)}\,|\,\mathcal{F}^n_{[ns]}] - E[\eta^n_{k,\delta(n)}\eta^n_{l,\delta(n)}]\,|\,] &\leqq \sum_{k<l}{}'(\phi^n_{k-l})^{1/2}(\phi^n_{l-[ns]})^{1/2} \times 16\delta(n)^2 \\ &\leqq 16\bar{\phi}^2(\varepsilon/F_I)^2 \to 0 \qquad \text{as } \varepsilon \downarrow 0\,,\end{aligned}$$

where $\sum_{k<l}{}'$ denotes the summation over (k, l) such that $[ns]+1 \leqq k < l \leqq [nt]$. Hence (U. II)-(2) implies that

$$\lim_{\varepsilon \in C(\nu) \to 0} \limsup_{n\to\infty} E[\sup_{x\in K} |E[\sum_{k=[ns]+1}^{[nt]-1} \sum_{l=k+1}^{[nt]} \tilde{f}^{n,(i)}_{k,\varepsilon}(x) \tilde{f}^{n,(j)}_{l,\varepsilon}(x)\,|\,\mathcal{F}^n_{[ns]}] - (t-s)a_1(x)^{ij}\,|\,] = 0\,. \tag{2.70}$$

Therefore, to show (2.38), it remains to prove that for each $\varepsilon < M$

$$\limsup_{n\to\infty} E[\sup_{x\in K} |E[\sum_{k=[ns]+1}^{[nt]-1} \sum_{l=k+1}^{[nt]} \{\tilde{f}^{n,(i)}_{k,M}(x)\tilde{f}^{n,(j)}_{l,M}(x) - \tilde{f}^{n,(i)}_{k,\varepsilon}(x)\tilde{f}^{n,(j)}_{l,\varepsilon}(x)\}\,|\,\mathcal{F}^n_{[ns]}]\,|\,] = 0\,. \tag{2.71}$$

But it is an immediate consequence from the facts that we have

$$\limsup_{n\to\infty} \sum_{k<l}{}' E[\,|E[\eta^{n,\delta(n)}_{k,N(n)}\eta^n_{l,N(n)}\,|\,\mathcal{F}^n_{[ns]}]\,|\,] = 0\,, \tag{2.72}$$

and

$$\limsup_{n\to\infty} \sum_{k<l}{}' E[\,|E[\eta^{n,\delta(n)}_{k,N(n)}\eta^n_{l,\delta(n)}\,|\,\mathcal{F}^n_{[ns]}]\,|\,] = 0\,. \tag{2.73}$$

We give a proof of (2.72) only because (2.73) is similarly shown. By Lemma 2.46, we have

$$\begin{aligned}&\sum_{k<l}{}' [E[\,|E[\eta^{n,\delta(n)}_{k,N(n)}\eta^n_{l,N(n)}\,|\,\mathcal{F}^n_{[ns]}]\,|\,] \\ &\leqq \sum_{k<l}{}' E[\,|E[\eta^{n,\delta(n)}_{k,N(n)}\eta^n_{l,N(n)}\,|\,\mathcal{F}^n_k]\,|\,] \\ &\leqq \sum_{k<l}{}' E[\,|\eta^{n,\delta(n)}_{k,N(n)}| \times 2(\phi^n_{l-k})^{1/2}\{E[\,|\eta^n_{l,N(n)}|^2\,|\,\mathcal{F}^n_k]^{1/2} + E[\,|\eta^n_{0,N(n)}|^2]^{1/2}\}] \\ &\leqq 2\sum_{k<l}{}' (\phi^n_{l-k})^{1/2}\{E[\,|\xi^{n,\delta}_{k,N}|\,E[\,|\eta^n_{l,N(n)}|^2\,|\,\mathcal{F}^n_k]^{1/2}] + 3E[\,|\xi^{n,\delta}_{0,N}|\,]E[\,|\eta^n_{0,N(n)}|^2]^{1/2}\} \\ &\leqq 2\sum_{k=[ns]+1}^{[nt]} \sum_{l=1}^{\infty} (\phi^n_l)^{1/2} E[\,|\xi^{n,\delta}_{k,N}|\,E[\,|\eta^n_{l+k,N(n)}|^2\,|\,\mathcal{F}^n_k]^{1/2}] \\ &\quad + 6\bar{\phi}([nt]-[ns])E[\,|\xi^{n,\delta}_{0,N}|\,]E[\,|\eta^n_{0,N(n)}|^2]^{1/2}\,.\end{aligned} \tag{2.74}$$

Since it is clear that the second term converges to 0 as $n\to\infty$, we show that so does the first term. For arbitrary $\zeta > 0$, take l_0 such that $\sup_n \sum_{l=l_0+1}^{\infty} (\phi^n_l)^{1/2} < \zeta$. Then we have

$$\begin{aligned}&\sum_{k=[ns]+1}^{[nt]} \sum_{l=l_0+1}^{\infty} (\phi^n_l)^{1/2} E[\,|\xi^{n,\delta}_{k,N}|\,E[\,|\eta^n_{l+k,N(n)}|^2\,|\,\mathcal{F}^n_k]^{1/2}] \\ &\leqq \zeta([nt]-[ns])E[\,|\xi^{n,\delta}_{0,N}|^2]^{1/2} E[\,|\eta^n_{0,N(n)}|^2]^{1/2} \leqq C_{T,N}\zeta\,.\end{aligned} \tag{2.75}$$

On the other hand, set $Z_t^{n,(l)}=\sum_{k=1}^{[nt]} E[|\eta_{k+l,N(n)}^n|^2|\mathcal{F}_k^n]$. Then, for $l\leqq l_0$ and the for arbitrary $\zeta>0$, we have

$$(2.76)\qquad \sum_{k=[ns]+1}^{[nt]} E[|\xi_{k,N}^{n,\delta}|E[|\eta_{l+k,N(n)}^n|^2|\mathcal{F}_k^n]^{1/2}]$$

$$=\sum_{k=[ns]+1}^{[nt]} E[|\xi_{k,N}^{n,\delta}|(\Delta Z_{(k/n)}^{n,(l)})^{1/2}]$$

$$\leqq\zeta([nt]-[ns])E[|\xi_{0,N}^{n,\delta}|]$$

$$+\sum_{k=[ns]+1}^{[nt]} E[|\xi_{k,N}^{n,\delta}|^2]^{1/2}E[\Delta Z_{(k/n)}^{n,(l)}\,;\,\sup_{t\leq T}\Delta Z_t^{n,(l)}>\zeta]^{1/2}$$

$$\leqq\zeta([nt]-[ns])E[|\xi_{0,N}^{n,\delta}|]$$

$$+\{([nt]-[ns])E[|\xi_{0,N}^{n,\delta}|^2]\}^{1/2}E[Z_T^{n,(l)}\,;\,\sup_{t\leq T}\Delta Z_t^{n,(l)}>\zeta]^{1/2}\,.$$

Here, note that $\{Z^{n,(l)}\}_n$ satisfies (CT & UI) because so does $\{X^{n,(l)}\}_n$ of (2.59). So, we have $\limsup_{n\to\infty}E[Z_T^{n,(l)}\,;\,\sup_{t\leq T}\Delta Z_t^{n,(l)}>\zeta]=0$, which implies that the right hand side of (2.76) converges to 0 as $n\to\infty$. Therefore we obtain

$$(2.77)\qquad \lim_{n\to\infty}\sum_{k=[ns]+1}^{[nt]}\sum_{l=1}^{l_0}(\phi_l^n)^{1/2}E[|\xi_{k,N}^{n,\delta}|E[|\eta_{l+k,N(n)}^n|^2|\mathcal{F}_k^n]^{1/2}]=0\,.$$

Combining this with (2.75) we get (2.72).

Next, define $c\in C_{b*}^1(\boldsymbol{R}^d, \boldsymbol{R}^d)$ in (2.39) by

$$(2.78)\qquad c^i(x)=\sum_{j=1}^{d}\sum_{p,q=1}^{e}\frac{\partial F^{ip}(x)}{\partial x^j}V_1^{pq}F^{jq}(x)\,.$$

Then, (2.39) follows by the same way as in showing (2.38).

(4) We will show that (2.40) holds with $\gamma=\tau\|F\|_*$ and $b(x)=G(x)$. Since $\bar{f}_{k,\gamma}^n(x)=F^n(x)E[\xi_0^nI_{(|\zeta_0^n|\leq\tau\|F\|_*/(\|F^n\|_*))}]$ and $F^n(x)a^n=F^n(x)E[\xi_0^nI_{(|\xi_0^n|\leq\tau)}]$, we have

$$\sup_{x\in K}|\sum_{k=[ns]+1}^{[nt]}\{g_k^n(x)+\bar{f}_{k,\gamma}^n(x)\}-(t-s)b(x)$$

$$\leqq\sup_{x\in K}|(1/)\times\sum_{k=[ns]+1}^{[nt]}G^n(x)-(t-s)G(x)|$$

$$+F_S[nt]E[|\xi_0^n||I_{(|\xi_0^n|\leq\tau)}-I_{(|\xi_0^n|\leq\tau\|F\|_*/(\|F^n\|_*))}|]\,.$$

The first term converges to 0 by (C). We next consider the second term. Let $\{\varepsilon_k\,;\,k\in\boldsymbol{N}\}$ be a sequence of positive numbers such that $1\pm\varepsilon_k\in C(\mu)$ for all k and that $\lim_{k\to\infty}\varepsilon_k=0$. Then for each ε_k, there exists $n_0\in\boldsymbol{N}$ such that $|\|F\|_*/(\|F^n\|_*)-1|<\varepsilon_k$ for all $n\geqq n_0$. Hence, we have

$$\limsup_{n\to\infty}nE[|\xi_0^n\|I_{(|\xi_0^n|\leq\tau)}-I_{(|\xi_0^n|\leq\tau\|F\|_*/(\|F^n\|_*))}|]$$

$$\leqq\limsup_{n\to\infty}nE[|\xi_0^n|I_{(\tau(1-\varepsilon_k)<|\xi_0^n|\leq\tau(1+\varepsilon_k))}]$$

$$=\int|z|I_{(\tau(1-\varepsilon_k)<|z|<\tau(1+\varepsilon_k))}\mu(dz)\,.$$

Since the last term converges to $\int|z|I_{(|z|=\tau)}\mu(dz)=0$ as $k\to\infty$, we get the conclusion.

(Check of (A.IV)) (1) (2.41) is an immediate consequence from the facts that $\bar{\phi}<\infty$ and that $\sup_n nE[|\eta_{0,N(n)}^n|^2]<\infty$.

(2) Since we have

$$(2.89)\quad \sum_{k<l}{}' E[|E[\eta^n_{l,N(n)}|\mathcal{F}^n_k]||\eta^n_{k\,N(n)}|^2]$$

$$\leqq \sum_{k<l}{}' E[2(\phi^n_{l-k})^{1/2}\{E[|\eta^n_{l,N(n)}|^2|\mathcal{F}^n_k]^{1/2}+E[|\eta^n_{0,N(n)}|^2]^{1/2}\}|\eta^n_{k,N(n)}|^2],$$

we can show that the last term converges to 0 as $n\to\infty$ by the same way as in the proof of (2.72). (2.42) follows from this immediately.

(Check of (A.V)) First, note that

$$P[\sup_{k\leq[nt]}\|f^n_k\|_*>M]=P[\sup_{k\leq[nt]}|\xi^n_k|>M/(\|F^n\|_*)]$$

$$\leqq[nt]P[|\xi^n_0|>M/F_S]\leqq[nt]P[|\xi^n_0|>N'],$$

for all $N'\in C(\mu)$ such that $N'<M/F_S$. Since (U.I) implies that $nP[\xi^n_0|>N']\to\mu(|z|>N')$ as $n\to\infty$, it is clear that (2.43) holds.

Thus we have proved Theorem 2.27 under the additional conditions stated in the first paragraph of the proof of Theorem 2.27. We next consider the general case. But we restrict ourselves to giving the idea of the proof.

Since it seems difficult to apply Theorem 2.44 directly to the processes $\{\varphi^n\}_n$ determined by (1.1), we introduce the localized and truncated prcesses as follows. For each $L>0$, let $r_L(x)$ be a smooth function such that $0\leqq r_L(x)\leqq 1$, $r_L(x)=1$ on $\{|x|\leqq L\}$, and that $r_L(x)=0$ on $\{|x|\geqq L+1\}$. Set $F^n_L(x)=r_L(x)F^n(x)$, $G^n_L(x)=r_L(x)G^n(x)$. Let $L>0$, $N\in C(\mu)$ be fixed. Then, associated with (1.1), we consider the following stochastic difference equation:

$$(2.80)\quad \begin{cases}\varphi^{n,N,L}_0=x_0\\ \varphi^{n,N,L}_k-\varphi^{n,N,L}_{k-1}=F^n_L(\varphi^{n,N,L}_{k-1})(\xi^n_{k,N}-a^n)+(1/n)G^n_L(\varphi^{n,N,L}_{k-1}).\end{cases}$$

Define a sequence of càdlàg processes $\{\varphi^{n,N,L}\}_n$ by $\varphi^{n,N,L}_t=\varphi^{n,N,L}_{[nt]}$. We apply Theorem 2.44 to this $\{\varphi^{n,N,L}\}_n$. Then, by the same way as in the first case, we can see that $\varphi^{n,N,L}$ converges in law to the process $\varphi^{N,L}$ which is the unique solution of the stochastic differential equation:

$$(2.81)\quad \varphi^{N,L}_t=x_0+\int_{(0,t]}F_L(\varphi^{N,L}_{u-})dB_u+\int_{(0,t]}(G_L+G_L)(\varphi^{N,L}_{u-})du$$

$$+\int_{(0,t]}\int_{(|z|\leq\tau)}F_L(\varphi^{N,L}_{u-})z\tilde{N}(dudz)+\int_{(0,t]}\int_{(\tau<|z|\leq N)}F_L(\varphi^{N,L}_{u-})zN(dudz),$$

where $B.$, $N(dudz)$, $\tilde{N}(dudz)$ are the same as in Theorem 2.27, and we define C_L by (2.10) for F_L instead of F. Furthermore, we can remove the restriction on N and L by the similar way in the proof of Theorem 2.1 in Fujiwara-Kunita [6]. Thus we have completed the proof of Theorem 2.27. □

At the final stage of this section, we give two consequences from Theorem 2.8 and Theorem 2.12 in the case where ξ^n_k is of the form $\xi^n_k=\xi_k/n^{1/\alpha}$ for some $\alpha\in(0,2]$. As we will see below, conditions required for $\{\xi_k\}_k$ are much simpler than conditions in Theorem 2.8 or Theorem 2.12.

(2.82) Theorem. *Let $\{\xi_k;\ k\in N^*\}$ be a stationary, uniformly mixing process with the rate function ϕ_k satisfying $\sum_{k=1}^{\infty}\phi_k^{1/2}<\infty$. Suppose that ξ_0 is an L^2-function. Set $\xi_k^n=\xi_k/\sqrt{n}$, $a^n=E[\xi_{0,N}^n]$ for any $N\in(0,\infty]$. Further, suppose that Condition (C) holds. Then, the sequence of stochastic processes $\{\varphi^n\}_n$ determined by (1.1) and (1.2) converges in law to the unique solution φ of the following stochastic differential equation:*

$$\varphi_t=x_0+\int_{(0,t]}F(\varphi_u)dB_u+\int_{(0,t]}\{C(\varphi_u)+G(\varphi_u)\}du\,, \tag{2.83}$$

where $B.$ and $C(\cdot)$ are the same as in Theorem 2.8, respectively, in which

$$V_0^{pq}=E[(\xi_0^{(p)}-E[\xi_0^{(p)}])(\xi_0^{(q)}-E[\xi_0^{(q)}])]\,, \tag{2.84}$$

$$V_1^{pq}=\sum_{l=1}^{\infty}E[|\xi_0^{(p)}-E[\xi_0^{(p)}])(\xi_l^{(q)}-E[\xi_l^{(q)}])] \qquad \text{for } p,q=1,\cdots,e\,. \tag{2.85}$$

Proof. We will prove this theorem by checking conditions in Theorem 2.8. Since $\lim_{n\to\infty}nP[|\xi_0^n|>\delta]=0$ for all $\delta>0$, (U.I) is satisfied with $\mu\equiv 0$. It is also clear that (U.II) is satisfied with V_0 of (2.84) and V_1 of (2.85). We next show that Condition (U.IV)$_1$ is satisfied. Since $|\xi_0|^2$ itself is uniformly integrable, by Lemma 2.24, there exists a positive, convex, increasing function G on $[0,\infty)$ such that $\lim_{x\uparrow\infty}(G(x)/x)=+\infty$ and that $E[G(|\xi_0|^2)]<\infty$. Then Jensen's inequality implies that

$$\begin{aligned}\sup_{n,k}E[G(nE[|\xi_{k,N}^n|^2|\mathcal{F}_{k-1}^n])]&=\sup_{n,k}E[G(E[|\xi_k|^2I_{(|\xi_k|\le\sqrt{n}N)}|\mathcal{F}_{k-1}])]\\&\le\sup_{n,k}E[E[G(|\xi_k|^2I_{(|\xi_k|\le\sqrt{n}N)})|\mathcal{F}_{k-1}]]\\&\le\sup_{k}E[G(|\xi_k|^2)]=E[G(|\xi_0|^2)]<\infty\,,\end{aligned}$$

where $\mathcal{F}_{k-1}=\sigma[\xi_0,\cdots,\xi_{k-1}]$. Therefore, again by Lemma 2.24, we see that (U.IV)$_1$ is satisfied. Thus, we have completed the proof. □

(2.86) Theorem. *Let $\{\xi_k;\ k\in N^*\}$ be a stationary, 1-dimensional, uniformly mixing process with the rate function ϕ_k satisfying $\sum_{k=1}^{\infty}\phi_k^{1/2}<\infty$. Suppose that there exists some $\alpha\in(0,2)$ and nonnegative constants C_+, C_- such that*

$$\lim_{x\to+\infty}x^{\alpha}P[\xi_0>x]=C_+,\qquad \lim_{x\to-\infty}|x|^{\alpha}P[\xi_0<x]=C_-\,. \tag{2.87}$$

Define $\{\xi_k^n;\ n\in N,\ k\in N^*\}$ in (1.1) by

$$\xi_k^n=\frac{\xi_k}{n^{1/\alpha}}\,. \tag{2.88}$$

Suppose that $\{\xi_k^n\}$ of (2.88) satisfies Condition (U.IV)$_i$ ($i=1$ or 2) and Condition (C) with $e=1$ is satisfied. Then, the sequence of càdlàg processes $\{\varphi^n\}_n$ determined by (1.1) and (1.2) converges in law to the unique solution of stochastic differential equation:

$$\begin{aligned}\varphi_t=x_0&+\int_{(0,t]}G(\varphi_{u-})du\\&+\int_{(0,t]}\int_{(|z|\le\tau)}F(\varphi_{u-})z\tilde{N}_{\alpha}(dudz)+\int_{(0,t]}\int_{(|z|>\tau)}F(\varphi_{u-})zN_{\alpha}(dudz)\,,\end{aligned} \tag{2.89}$$

where $N_{\alpha}(dudz)$ is Poisson random measure with the intensity measure

$$du\cdot\{C_+I_{\{z>0\}}+C_-I_{\{z<0\}}\}\frac{\alpha}{|z|^{1+\alpha}}dz\,.$$

Proof. By Theorem 2.8 or Theorem 2.12, all that we have to do is only to check Conditions (U.I) and (U.II). It is immediate from (2.87) that (U.I) is satisfied with

$$\mu(dz)=\{C_+I_{\{z>0\}}+C_-I_{\{z<0\}}\}\frac{\alpha}{|z|^{1+\alpha}}dz\,.$$

We next show that (U.II) is satisfied with $V_0=V_1=0$. To this aim, it is sufficient to prove $\lim_{\delta\downarrow 0}\lim_{n\to\infty}nE[|\xi^n_{0,\delta}|^2]=0$. But this easily follows from (2.87) because for sufficiently large C we have

$$nE[|\xi^n_{0,\delta}|^2]=n^{1-(2/\alpha)}E[\xi_0^2 I_{\{|\xi_0|\leq n^{1/\alpha}\delta\}}]\leqq n^{1-(2/\alpha)}\int_0^{\delta n^{1/\alpha}}2xP[|\xi_0|>x]dx$$

$$=2n^{1-(2/\alpha)}\left\{\int_0^C xP[|\xi_0|>x]dx+\int_0^{\delta n^{1/\alpha}}xP[|\xi_0|>x]dx\right\}$$

$$\sim 2n^{1-(2/\alpha)}\{C+((C_++C_-)/(2-\alpha))\{(n^{1/\alpha}\delta)^{2-\alpha}-C^{2-\alpha}\}\}$$

$$\sim(2(C_++C_-)/(2-\alpha))\delta^{2-\alpha}\to 0,\qquad \text{as}\quad \delta\downarrow 0\,,$$

where $f(n)\sim g(n)$ means that $\lim_{n\to\infty}(f(n)/g(n))=1$. □

3. The case of strongly mixing processes

In this section, we will discuss the same problem as in the previous section when $\{\xi^n_k\}_k$ in (1.1) is strongly mixing. Let $\{\xi^n_k\,;\,k\in N^*\}_n$ be a sequence of strongly mixing processes with the rate function α^n_k defined by (2.1). Corresponding to Conditions (U.IV)$_1$ and (U.III) in Theorem 2.8, we introduce the following.

(S. I): $$\sup_{n\in N}E[|\sqrt{n}\,\xi^n_0|^{2+\delta}]<\infty \qquad \text{for some}\quad \delta>0\,.$$

(S. II): $$\bar{\alpha}=:\sum_{k=1}^{\infty}\sup_{n\in N}(\alpha^n_k)^{1/p}<\infty$$

for some $p>0$ such that $(1/p)\in(0,\,\delta/2+\delta)\cap(0,\,1/2]$, where δ is the positive number given in (S.I).

Then we have the following result.

(3.1) **Theorem.** *Suppose that Conditions* (*S.I*), (*S.II*), (*U.II*), *and* (*C*) *are satisfied. We take* $a^n=E[\xi^n_0]$ *in* (1.1). *Then, the sequence of stochastic processes* $\{\varphi^n\}_n$ *determined by* (1.1) *and* (1.2) *converges in law to the unique solution* φ *of the stochastic differential equation* (2.83) *where* $B.$ *and* $C(\cdot)$ *are the same as in Theorem* 2.8.

(3.2) **Remark.** It should be noticed in the conclusion of this theorem that the limit process φ is restricted to be continuous owing to Condition (S.I).

We will prove this theorem by applying Theorem 2.44 as in the proof of Theorem 2.27. We first prepare basic inequalities which hold for strongly mixing random variables.

(3.3) Lemma. *Let* $\{\xi_k\}_{k\in N^*}$ *be a stationary, strongly mixing process with the rate function* α_k. *Suppose that* ξ *is an* L^p, $\mathcal{F}^k$*-adapted random variable for some* $p\in[1, \infty]$, *where* $\mathcal{F}^k=\sigma[\xi_k, \xi_{k+1}, \cdots]$. *Then, it holds that for all* $l<k$

$$E[|E[\xi|\mathcal{F}_l]-E[\xi]|^r]^{1/r}\leqq 2(2^{1/r}+1)(\alpha_{k-l})^{1/q}E[|\xi|^p]^{1/p}, \tag{3.4}$$

where $\mathcal{F}_l=\sigma[\xi_0, \cdots, \xi_l]$ *and* $(1/p)+(1/q)=1/r$.

Proof. See Lemma VIII. 3.102 in Jacod-Shiryaev [11, p.456]. □

(3.5) Lemma. *Let* $\{\xi_k\}_k$ *be the same as in Lemma* 3.3. *Suppose that* $\sup_{k,\omega}|\xi_k(\omega)|\leqq C$. *Then we have for all* $m<l<k$

$$E[|E[\eta_k\eta_l|\mathcal{F}_m]-E[\eta_k\eta_l]|]\leqq 24C^2(\alpha_{k-l})^{1/2}(\alpha_{l-m})^{1/2}, \tag{3.6}$$

where $\eta_k=\xi_k-E[\xi_k]$.

Proof. This lemma can be proved by the similar way for Lemma 2.49. □

Proof of Theorem 3.1. As in the proof of Theorem 2.27, we will prove this theorem under the assumption that $\lim_{n\to\infty}\|F^n-F\|_*=0$ and that F is not identically 0. Define f_k^n and g_k^n in (2.28) by (2.54), in which we take $a^n=E[\xi_0^n]$. In the sequel, we show one by one that Conditions (A. I)~(A. V) in Theorem 2.44 are satisfied. But we will only check (A. II)-(1), (A. III)-(1), and (A. IV)-(2) because the others can be checked by applying Lemma 3.3 and Lemma 3.5 in a similar manner for the proof of Theorem 2.27.

We will use the notations, such as $\xi_{k,N}^n$ the $\eta_{l,N(n)}^n$, as in the previous section.

(Check of (A. II)) (1) Since we have

$$\sup_{y\in K}|E[\sum_{l=k+1}^{[nt']}\partial_y^\alpha\tilde{f}_{l,M}^n(y)|\mathcal{F}_k^n]|\leqq\|F^n\|_{2,K}\sum_{l=k+1}^{[nt']}|E[\eta_{l,N(n)}^n|\mathcal{F}_k^n]|,$$

$$\sup_{x\in K}|\partial_x^\beta\tilde{f}_{l,M}^n(x)|\leqq\|F^n\|_{2,K}|\eta_{l,N(n)}^n|,$$

the left hand side of (2.29) is dominated by

$$\begin{aligned}\bar{F}^2(&\sum_{k=[ns]+1}^{[nt]}\sum_{l=k+1}^{[nt']}|E[\eta_{l,N(n)}^n|\mathcal{F}_k^n]||\eta_{k,N(n)}^n| \\ &+\sum_{k=[ns]+1}^{[nt]}2\{|\xi_{k,N(n)}^n|^2+E[|\xi_{0,N(n)}^n|^2]\}) \\ \leqq(\bar{F})^2((1/n)\times&\sum_{k=[ns]+1}^{[nt]}\sum_{l=1}^{[nT]}|E[\sqrt{n}\,\eta_{l+k,N(n)}^n|\mathcal{F}_k^n]||\sqrt{n}\,\eta_{k,N(n)}^n| \\ &+(1/n)\times\sum_{k=[ns]+1}^{[nt]}2\{|\sqrt{n}\,\xi_{k,N(n)}^n|^2+E[|\sqrt{n}\,\xi_{0,N(n)}^n|^2]\})\end{aligned} \tag{3.7}$$

where $\bar{F}=\sup_n\|F^n\|_{2,K}$. Hence, if we define a nondecreasing càdlàg process D_t^n by the right hand side of (3.7) for $s=0$, then it is clear that Condition (i) is satisfied. We next show that (ii)′ is satisfied. Since D_t^n defined above is the sum of $D_t^{n,1}$ and $D_t^{n,2}$, where

$$D_t^{n,1}=\bar{F}^2((1/n)\times\sum_{k=1}^{[nt]}\sum_{l=1}^{[nT]}|E[\sqrt{n}\,\eta_{l+k,N(n)}^n|\mathcal{F}_k^n]||\sqrt{n}\,\eta_{k,N(n)}^n|), \tag{3.8}$$

$$D_t^{n,2}=\bar{F}^2((1/n)\times\sum_{k=1}^{[nt]}2\{|\sqrt{n}\,\xi_{k,N(n)}^n|^2+E[|\sqrt{n}\,\xi_{0,N(n)}^n|^2]\}), \tag{3.9}$$

it is sufficient to show that $\{D^{n,i}\}_n(i=1, 2)$ satisfies (CT & UI), respectively. We first consider $\{D^{n,1}\}_n$. For n, k, set

$$K^n(k)=\sum_{l=1}^{[nT]}|E[\sqrt{n}\,\eta_{l+k,N(n)}^n|\mathcal{F}_k^n]||\sqrt{n}\,\eta_{k,N(n)}^n|\ . \tag{3.10}$$

Then, since $D_t^{n,1}=\bar{F}^2\times(1/n)\sum_{k=1}^{[nt]}K^n(k)$, we see that $\{D^{n,1}\}_n$ satisfies (CT & UI) if $\{K^n(k);\ n\in\boldsymbol{N},\ k\in\boldsymbol{N}^*\}$ is uniformly integrable. See the proof of Lemma 2.23. But it can be shown by the similar way in the proof of (4.39) in [6] that for some $\varepsilon\in(0, \delta/2)$

$$\sup_{n,k}\|K^n(k)\|_{(1+\varepsilon)}<\infty\ , \tag{3.11}$$

where $\|\cdot\|_r$ denotes the L^r-norm with respect to $P(d\omega)$. Hence, $\{K^n(k);\ n, k\}$ is uniformly integrable.

We next consider $\{D^{n,2}\}_n$. Since $D_t^{n,2}\ll(1/n)\sum_{k=1}^{[nt]}2\{|\sqrt{n}\,\xi_{k,N}^n|^2+E[|\sqrt{n}\,\xi_{0,N}^n|^2]\}$ and since $\{|\sqrt{n}\,\xi_{k,N}^n|^2;\ n, k\}$ is uniformly integrable by (S.I), it is easy to see that $\{D^{n,2}\}_n$ satisfies (CT & UI).

Thus we have checked Condition (A.II)-(1).

(Check of (A.III)) We show that (1) is satisfied with $\nu(df)\equiv 0$. In fact, in (2.36), we have

$$\begin{aligned}
&E[\sup_{x\in K}|E[\sum_{k=[ns]+1}^{[nt]}h(f_k^n(x))I_{(\varepsilon<\|f_k^n\|_*\le M)}|\mathcal{F}_{[ns]}^n]|]\\
&\le\|h\|_\infty\times\sum_{k=[ns]+1}^{[nt]}P[\varepsilon<\|F^n\|_*|\xi_k^n|\le M]\le\|h\|_\infty\sum_{k=1}^{[nt]}P[|\xi_0^n|>\varepsilon/F_S]\\
&\le\|h\|_\infty T\Big(\frac{F_S}{\varepsilon}\Big)^{2+\delta}E[|\sqrt{n}\,\xi_0^n|^{2+\delta}]\times n^{-(\delta/2)}\to 0 \qquad\text{as}\quad n\to\infty\ ,
\end{aligned}$$

which implies the conclusion.

(Check of (A.IV)) (2) As in the check of (A.II)-(1), it is sufficient to show that

$$\lim_{n\to\infty}\sum_{k=1}^{[nt]}\sum_{l=k+1}^{[nt]}E[|E[\eta_{l,N(n)}^n|\mathcal{F}_k^n]||\eta_{k,N}^n|^2]=0\ , \tag{3.12}$$

because $\sup_n\|F^n\|_{2,K}<\infty$ for each compact set K. By Hölder's inequality and by Lemma 3.3, we have

$$\begin{aligned}
E[|E[\eta_{l,N(n)}^n|\mathcal{F}_k^n]||\eta_{k,N(n)}^n|^2]&\le\|E[\eta_{l,N(n)}^n|\mathcal{F}_k^n]\|_r\|\eta_{k,N(n)}^n\|_{2q}^2\\
&\le C(\alpha_{l-k}^n)^{1/p'}\|\xi_0^n\|_{q'}\|\xi_{0,N}^n\|_{2q}^2\ ,
\end{aligned}$$

where C is a positive number, $(1/r)+(1/q)=1$, and $(1/p')+(1/q')=1/r$. We now note that for arbitrary $\varepsilon\in(0, \delta)$

$$\|\xi_{0,N}^n\|_{2q}^2=E[|\xi_{0,N}^n|^{q(1-\varepsilon)}|\xi_{0,N}^n|^{q(1+\varepsilon)}]^{1/q}$$

$$\leqq N^{(1-\varepsilon)}E[|\xi_{0,N}^{n}|^{q(1+\varepsilon)}]^{1/q}\leqq N^{(1-\varepsilon)}\|\xi_{0}^{n}\|_{q(1+\varepsilon)}^{(1+\varepsilon)}.$$

Hence we have

$$\sum_{k=1}^{[nt]}\sum_{l=k+1}^{[nt]}E[|E[\eta_{l,N(n)}^{n}|\mathcal{F}_{k}^{n}]\|\eta_{k,N(n)}^{n}|^{2}] \tag{3.13}$$

$$\leqq\sum_{k=1}^{[nt]}\sum_{l=k+1}^{[nt]}C(\alpha_{l-k}^{n})^{1/p'}\|\xi_{0}^{n}\|_{q'}\times N^{(1-\varepsilon)}\|\xi_{0}^{n}\|_{q(1+\varepsilon)}^{(1+\varepsilon)}$$

$$\leqq CtN^{(1-\varepsilon)}\times\sup_{n}\left(\sum_{k=1}^{\infty}(\alpha_{k}^{n})^{1/p'}\right)\|\sqrt{n}\,\xi_{0}^{n}\|_{q'}\|\sqrt{n}\,\xi_{0}^{n}\|_{q(1+\varepsilon)}^{(1+\varepsilon)}\times n^{-(\varepsilon/2)}.$$

Further, if we take $q'=2+\delta$, $q=(2+\delta)/(1+\varepsilon)$, then $1/p'=1-(1/q')-(1/q)=(\delta-\varepsilon)/(2+\delta)$. Hence we can take $\varepsilon_0\in(0,\delta)$ so that p' coincides with p in (S.II). Therefore the left hand side of (3.13) is dominated by

$$CtN^{(1-\varepsilon_0)}\times\sup_{n}\left(\sum_{k=1}^{\infty}(\alpha_{k}^{n})^{1/p}\right)\|\sqrt{n}\,\xi_{0}^{n}\|_{(2+\delta)}^{2+\varepsilon_0}\times n^{-(\varepsilon_0/2)},$$

which converges to 0 as $n\to\infty$. Thus we have checked Condition (A.IV)-(2). By the discussion above, we have completed the proof of Theorem 3.1. □

4. Application to the case of Markov chains

In the previous sections, we have established several jump-diffusion approximation theorems when $\{\xi_k^n; k\in\boldsymbol{N}^*\}_n$ in (1.1) is a sequence of mixing processes. Since the mixing property is sometimes induced by the ergodicity of Markov chain (i.e., Markov process with a discrete time-parameter), we will study, in this section, a class of Markov chains to which these theorems can be applied.

Let $\{\xi_k^n; n\in\boldsymbol{N}, k\in\boldsymbol{N}^*\}$ be an array of $\boldsymbol{R}^e$-valued random variables in (1.1) and set

$$\zeta_k^n=\sqrt{n}\,\xi_k^n, \tag{4.1}$$

for all n, k. We suppose that for each $n\in\boldsymbol{N}$ $\{\zeta_k^n; k\in\boldsymbol{N}^*\}$ is a stationary Markov chain on $\boldsymbol{R}^e$ with the k-step transition probability $P_k^n(\zeta', d\zeta)$ and an invariant probability measure $\Lambda^n(d\zeta)$.

For these sequences of measures $\{P_k^n(\zeta', d\zeta); k\in\boldsymbol{N}\}_n$ and $\{\Lambda^n(d\zeta)\}_n$, we introduce the following conditions (M.I)$\sim$(M.VII).

Associated with Λ^n, we define a Borel measure $\mu^n(dz)$ by

$$\mu^n(dz)=n\Lambda^n(\sqrt{n}\,dz). \tag{4.2}$$

(M.I): There exists a Borel measure μ on $\boldsymbol{R}^e\setminus\{0\}$ such that for all $g\in\mathcal{C}(\boldsymbol{R}^e)$ (=the set of all bounded continuous functions vanishing on some neighborhood of 0),

$$\int_{\boldsymbol{R}^e}g(z)\mu^n(dz)\to\int_{\boldsymbol{R}^e}g(z)\mu(dz)\qquad\text{as}\quad n\to\infty \tag{4.3}$$

and (2.5) holds.

(M.II): Λ^n converges weakly to a probability measure Λ on $\boldsymbol{R}^e$ satisfying $\int_{\boldsymbol{R}^e}|\zeta|^2\Lambda(d\zeta)<\infty$.

(M.III): For each p, $q=1,\cdots,e$, there exists a real number W^{pq} such that

(4.4) $$\lim_{N\uparrow\infty}\limsup_{\delta\downarrow 0}\limsup_{n\to\infty}\left|\int_{R^e}\zeta^{(p)}\zeta^{(q)}I_{(N<|\zeta|\le\sqrt{n}\delta)}\Lambda^n(d\zeta)-W^{pq}\right|=0\,.$$

(M. IV): $P_1^n(\zeta', d\zeta)$ converges weakly to a transition probability $P_1(\zeta', d\zeta)$ uniformly on any compact set in ζ'-space.

(M. V): For every $n\in\boldsymbol{N}$, $\int_{R^e}f(\zeta)P_1^n(\zeta', d\zeta)$ is bounded continuous in ζ' if so does f.

(M. VI): There exist $k_0\in\boldsymbol{N}$, $\varepsilon_0\in(0, 1]$, and a family of probability measures $\{\Gamma^n\,;\,n\in\boldsymbol{N}\}$ on $\boldsymbol{R}^e$ such that

(4.5) $$P_{k_0}^n(\zeta', d\zeta)\geqq\varepsilon_0\Gamma^n(d\zeta)\,,$$

for all n and $\zeta'\in\boldsymbol{R}^e$.

(M. VII): For every $N>0$, there exists $p\in(1, \infty)$ such that

(4.6) $$\sup_n\int_{R^e}\left\{\int_{R^e}|\zeta|^2I_{(|\zeta|\le\sqrt{n}N)}P_1^n(\zeta', d\zeta)\right\}^p\Lambda^n(d\zeta')<\infty\,.$$

Then we have the following result.

(**4.7**) **Theorem.** *Suppose that Conditions* (*M. I*)~(*M. VII*) and (*C*) *are satisfied. Then the conclusion of Theorem* 2.8 *holds, in which the matrices* V_0 *and* V_1 *in* (2.9) *are given by*

(4.8) $$V_0^{pq}=\int_{R^e}\left(\zeta^{(p)}-\int_{R^e}\zeta^{(p)}\Lambda(d\zeta)\right)\left(\zeta^{(q)}-\int_{R^e}\zeta^{(q)}\Lambda(d\zeta)\right)\Lambda(d\zeta)+W^{pq}\,,$$

(4.9) $$V_1^{pq}=\int_{R^e}\left(\zeta'^{(p)}-\int_{R^e}\zeta'^{(p)}\Lambda(d\zeta')\right)\int_{R^e}\zeta^{(q)}S(\zeta', d\zeta)\Lambda(d\zeta')\,,$$

respectively, where

(4.10) $$S(\zeta', d\zeta)=\sum_{k=1}^{\infty}\{P_k(\zeta', d\zeta)-\Lambda(d\zeta)\}\,,$$

and $P_k(\zeta', d\zeta)$ *denotes the k-step transition probability defined inductively by* $P_k(\zeta', A)=\int_{R^e}P_{k-1}(\zeta'', A)P_1(\zeta', d\zeta'')$.

(**4.11**) **Remark.** By (4.28) in the proof below, it is assured that the right hand side of (4.10) converges uniformly in ζ'.

In order to prove Theorem 4.7, we first prepare a general result on a relationship between the Markov property and the uniformly mixing property.

(**4.12**) **Lemma.** *Let* $\{\zeta_k\,;\,k\in\boldsymbol{N}^*\}$ *be a stationary Markov chain with the k-step transition probability* $P_k(\zeta', d\zeta)$ *and let* $\Lambda(d\zeta)$ *be an invariant probability measure of* $\{\zeta_k\}_k$. *Then it holds that*

(4.13) $$\phi_k=(1/2)\Lambda\text{-ess.}\sup_{\zeta'}\|P(\zeta', \cdot)-\Lambda(\cdot)\|_{\mathrm{var}}\,,$$

where $\|\cdot\|_{\mathrm{var}}$ *denotes the total variation norm defined by* $\|Q\|_{\mathrm{var}}=\sup\{|\int_{R^e}fdQ|\,;\,f$ *is* *continuous*, $\sup_\zeta |f(\zeta)| \leqq 1\}$.

Proof. We first note the relation:

$$\phi_k = \sup_{l\in N^*} \sup\{\|E[I_B|\mathcal{F}_l]-E[I_B]\|_\infty;\ B\in\mathcal{F}^{l+k}\},$$

where $\mathcal{F}_l=\sigma[\zeta_0, \cdots, \zeta_l]$, $\mathcal{F}^k=\sigma[\zeta_k, \zeta_{k+1}, \cdots]$, and $\|\cdot\|_\infty$ denotes the essential supremum norm with respect to P. See the proof of (17.2.10) in Ibragimov-Linnik [8]. Now, let B be an arbitrary set of $\mathcal{F}^{l+k}$. Then, by the Markov property of $\{\zeta_k\}_k$, there exists a measurable function $h_B(\zeta)$ such that $E[I_B|\mathcal{F}_{l+k}]=h_B(\zeta_{l+k})$ and that $0\leqq h_B(\zeta)\leqq 1$. We fix $\omega\in\Omega$ and denote by $(\mu^+-\mu^-)$ the Jordan-Hahn decomposition of the signed measure $\{P_k^n(\zeta_l(\omega), d\zeta)-\Lambda(d\zeta)\}$. Then we have

$$\begin{aligned}
2|E[I_B|\mathcal{F}_l](\omega)-E[I_B]| \\
&=2|E[E[I_B|\mathcal{F}_{l+k}]|\mathcal{F}_l](\omega)-E[I_B]| \\
&=2|E[h_B(\zeta_{l+k})|\mathcal{F}_l](\omega)-E[h_B(\zeta_{l+k})]| \\
&=\left|\int h_B(\zeta)\{P_k(\zeta_l(\omega), d\zeta)-\Lambda(d\zeta)\}\right|+\left|\int(1-h_B(\zeta))\{P_k(\zeta_l(\omega), d\zeta)-\Lambda(d\zeta)\}\right| \\
&\leqq \int h_B(\zeta)\{\mu^+(d\zeta)+\mu^-(d\zeta)\}+\int(1-h_B(\zeta))\{\mu^+(d\zeta)+\mu^-(d\zeta)\} \\
&=\|\mu\|_{\mathrm{var}}=\|P_k(\zeta_l(\omega), \cdot)-\Lambda(\cdot)\|_{\mathrm{var}}\leqq \Lambda\text{-ess.}\sup_{\zeta'}\|P(\zeta', \cdot)-\Lambda(\cdot)\|_{\mathrm{var}},
\end{aligned}$$

which implies that

$$\phi_k\leqq(1/2)\Lambda\text{-ess.}\sup_{\zeta'}\|P(\zeta', \cdot)-\Lambda(\cdot)\|_{\mathrm{var}}. \tag{4.14}$$

On the other hand, note that for arbitrary Borel set B_0 we have

$$|P_k(\zeta_l, B_0)-\Lambda(B_0)|=|E[I_{\{\zeta_{l+k}\in B_0\}}|\mathcal{F}_l]-E[I_{\{\zeta_{l+k}\in B_0\}}]|\leqq\phi_k,$$

because $\{\zeta_{l+k}\in B_0\}\in\mathcal{F}^{l+k}$. Since $\|P-Q\|_{\mathrm{var}}=2\sup_{A\in\mathcal{B}}|P(A)-Q(A)|$ if P and Q are probability measures, we get $(1/2)\Lambda\text{-ess.}\sup_{\zeta'}\|P(\zeta', \cdot)-\Lambda(\cdot)\|_{\mathrm{var}}\leqq\phi_k$. Therefore, combining this with (4.14), we obtain (4.13). □

Proof of Theorem 4.7. This theorem is proved by applying Theorem 2.8. For simplicity, we will check Conditions (U.I)~(U.III) and (U.IV)$_1$ only in the case of $e=1$.

Concerning (U.I), we have nothing to do because (M.I) is only a translation of (U.I).

Let ϕ_k^n be the rate function of (2.2) for $\{\xi_k^n\}_k$. In order to check (U.III), it suffices to show that

$$\phi_k^n\leqq C\rho^k \tag{4.15}$$

for all n and k, where we set $\rho=(1-\varepsilon_0)^{1/k_0}$ and $C=(1-\varepsilon_0)^{-1}$.

The proof is based on a result in Doeblin's ergodic theory for Markov chains. Following Deuschel-Stroock [2], Exercise 4.1.48-(ii), it is stated as follows.

Let $(E, \mathcal{F})$ be a measurable space and $P(x, dy)$ be a transition probability on $(E, \mathcal{F})$ with the property $P(x, dy)\geqq\alpha Q(dy)$ for some $\alpha\in(0, 1]$ and a probability Q on $(E, \mathcal{F})$. Then it holds that

$$\|\nu_1 P_k - \nu_2 P_k\|_{\mathrm{var}} \leqq 2(1-\alpha)^k .$$

for all probability measures ν_1, ν_2 on $(E, \mathcal{F})$ and for all $k \in \boldsymbol{N}$, where $\nu P_k(A) = \int_E P_k(x, A) \nu(dx)$ and $P_k(x, dy)$ denotes the k-step transition probability. Furthermore, there is a unique invariant measure μ on $(E, \mathcal{F})$ such that $\|\nu P_k - \mu\|_{\mathrm{var}} \leqq 2(1-\alpha)^k$ for all $k \in \boldsymbol{N}$ and probability measure ν.

We apply this result with $P(\zeta', d\zeta) = P^n_{k_0}(\zeta', d\zeta)$, $Q = \Gamma^n$, $\alpha = \varepsilon_0$, $\nu_1 = \delta_{\{\zeta'\}}$, and $\nu_2 = \Lambda^n$, where $\delta_{\{a\}}$ denotes the Dirac measure on $\{a\}$. Then we see from Condition (M.VI) that Λ^n is the unique invariant probability measure and that

$$\|P^n_{mk_0}(\zeta', \cdot) - \Lambda^n(\cdot)\|_{\mathrm{var}} \leqq 2(1-\varepsilon_0)^m ,$$

for all $\zeta' \in \boldsymbol{R}^e$, and m, $n \in \boldsymbol{N}$. Furthermore, since $\sup_{\zeta'} \|P^n(\zeta', \cdot) - \Lambda^n(\cdot)\|_{\mathrm{var}}$ is nonincreasing in k, it holds that for all $l = 1, \cdots, k_0 - 1$,

$$\begin{aligned}(4.16)\qquad \sup_{\zeta'} \|P^n_{mk_0+l}(\zeta', \cdot) - \Lambda^n(\cdot)\|_{\mathrm{var}} &\leqq \sup_{\zeta'} \|P^n_{mk_0}(\zeta', \cdot) - \Lambda^n(\cdot)\|_{\mathrm{var}} \\ &\leqq 2(1-\varepsilon_0)^m = 2\{(1-\varepsilon_0)^{1/k_0}\}^{mk_0+l-l} \\ &\leqq 2(1-\varepsilon_0)^{-1}\{(1-\varepsilon_0)^{1/k_0}\}^{mk_0+l} = 2C\rho^{mk_0+l} .\end{aligned}$$

Hence, we get (4.15) from Lemma 4.12.

Let $\mathcal{F}^n_k$ be the σ-field $\sigma[\zeta^n_0, \cdots, \zeta^n_k]$. Then we have

$$nE[|\xi^n_{k,N}|^2 | \mathcal{F}^n_{k-1}] = E[|\zeta^n_k I_{\{|\zeta^n_k| \leq \sqrt{n}N\}}|^2 | \mathcal{F}^n_{k-1}] = \int_{R^e} |\zeta|^2 I_{\{|\zeta| \leq \sqrt{n}N\}} P^n(\zeta^n_{k-1}, d\zeta) .$$

Hence, (M.VII) clearly implies $(\mathrm{U.IV})_1$.

Finally, we check (U.II). To this end, we first show that (2.6) holds with V_0 of (4.8). Note that

$$\begin{aligned} nE[|\eta^n_{0,\delta}|^2] &= nE[|\xi^n_{0,\delta} - E[\xi^n_{0,\delta}]|^2] \\ &= E[|\zeta^n_0 I_{\{|\zeta^n_0| \leq \sqrt{n}\delta\}}|^2] - E[\zeta^n_0 I_{\{|\zeta^n_0| \leq \sqrt{n}\delta\}}]^2 \\ &= \int |\zeta|^2 I_{\{|\zeta| \leq \sqrt{n}\delta\}} \Lambda^n(d\zeta) - \left\{\int \zeta I_{\{|\zeta| \leq \sqrt{n}\delta\}} \Lambda^n(d\zeta)\right\}^2 . \end{aligned}$$

Since Conditions (M.II) and (M.III) clearly imply that

$$(4.17)\qquad \lim_{\delta \downarrow 0} \limsup_{n \to \infty} \left| \int |\zeta|^2 I_{\{|\zeta| \leq \sqrt{n}\delta\}} \Lambda^n(d\zeta) - \left\{\int |\zeta|^2 \Lambda(d\zeta) + W^{11}\right\} \right| = 0 ,$$

$$(4.18)\qquad \lim_{n \leq \infty} \int \zeta I_{\{|\zeta| \leq \sqrt{n}\delta\}} \Lambda^n(d\zeta) = \int \zeta \Lambda(d\zeta)$$

for every $\delta > 0$, we get the conclusion.

To prove (2.7) for V_1 of (4.9), it is sufficient to show that for every $\delta > 0$

$$(4.19)\qquad \lim_{n \to \infty} n \sum_{k=1}^{n-1} E[\eta^n_{0,\delta} \eta^n_{k,\delta}] = \int_{R^e} \zeta' \int_{R^e} \zeta S(\zeta', d\zeta) \Lambda(d\zeta') .$$

Let ε be an arbitrary positive number and take $k_1 \in \boldsymbol{N}$ such that $\sum_{k=k_1+1}^{\infty} (C\rho^k)^{1/2} < \varepsilon$. We set $\zeta^n_{k,\delta} = \zeta^n_k I_{\{|\zeta^n_k| \leq \sqrt{n}\delta\}}$ for simplicity. Then for $n > k_1 + 1$ we have

(4.20) $$C^n=:\ n\sum_{k=1}^{n-1}[\eta_{\delta,0}^n\eta_{k,0}^n]=n\sum_{k=1}^{n-1}E[\xi_{0,\delta}^n\eta_{k,\delta}^n]=\sum_{k=1}^{n-1}[\zeta_{0,\delta}^n(\zeta_{k,\delta}^n-E[\zeta_{0,\delta}^n])]$$

$$=\sum_{k=1}^{k_1}E[\zeta_{0,\delta}^n(\zeta_{k,\delta}^n-E[\zeta_{0,\delta}^n])]+\sum_{k=k_1+1}^{n-1}E[\zeta_{0,\delta}^n(\zeta_{k,\delta}^n-E[\zeta_{0,\delta}^n])]$$

$$=:\ \sum_{k=1}^{k_1}I_{1,k}^n+I_2^n.$$

First, we consider the second term I_2^n. Since $\{\zeta_k^n\,;\,k\in N^*\}_n$ is, as we have seen before, a sequence of uniformly mixing processes and the rate function ϕ_k^n satisfies (4.15), we have

(4.21) $$|I_2^n|\leqq 2\sum_{k=k_1+1}^{n-1}(\phi_k^n)^{1/2}E[|\zeta_{0,\delta}^n|^2]\leqq 2\lambda^2\varepsilon\ ,$$

where we set $\lambda=:\ \sup_n E[|\zeta_{0,\delta}^n|^2]^{1/2}$. Notice that λ is finite because of (4.17).

We next show that for every k.

(4.22) $$I_{1,k}^n\to I_{1,k}=:\ \int\zeta'\int\zeta\{P_k(\zeta',d\zeta)-\Lambda(d\zeta)\}\Lambda(d\zeta')\qquad\text{as}\quad n\to\infty\ .$$

For arbitary $N>0$, let $\rho_N(x)$ be a nonincreasing, continuous function defined on $\{x\geqq 0\}$ such that $\rho_N(x)=1$ if $0\leqq x\leqq N$ and that $\rho_N(x)=0$ if $x\geqq N+1$. Using it, devide $I_{1,k}^n$ into the sum $\sum_{i=1}^3 J_{i,N}^n$, where we set

$$J_{1,N}^n=E[\zeta_{0,\delta}^n\rho_N(|\zeta_0^n|)\{\zeta_{k,\delta}^n\rho_N(|\zeta_k^n|)-E[\zeta_{0,\delta}^n\rho_N(|\zeta_0^n|)]\}]\ ,$$

$$J_{2,N}^n=E[\zeta_{0,\delta}^n\rho_N(|\zeta_0^n|)\{\zeta_{k,\delta}^n(1-\rho_N(|\zeta_k^n|))-E[\zeta_{0,\delta}^n(1-\rho_N(|\zeta_0^n|))]\}]\ ,$$

$$J_{3,N}^n=E[\zeta_{0,\delta}^n(1-\rho_N(|\zeta_0^n|))\{\zeta_{k,\delta}^n-E[\zeta_{0,\delta}^n]\}]\ .$$

Then we can see that

(4.23) $$J_{1,N}^n=\int(\zeta' I_{(|\zeta'|\leq\sqrt{n}\delta)}\rho_N(|\zeta'|)$$
$$\times\int\zeta I_{(|\zeta|\leq\sqrt{n}\delta)}\rho_N(|\zeta|)\{P_k^n(\zeta',d\zeta)-\Lambda^n(d\zeta)\})\Lambda^n(d\zeta')$$
$$\to J_{1,N}=:\ \int\zeta'\rho_N(|\zeta'|)\int\zeta\rho_N(|\zeta|)\{P_k(\zeta',d\zeta)-\Lambda(d\zeta)\}\Lambda(d\zeta')\qquad\text{as}\quad n\to\infty$$

and that for $i=2,3$

(4.24) $$\lim_{N\to\infty}\limsup_{n\to\infty}|J_{i,N}^n|=0\ .$$

Indeed, (M.IV) and (M.V) imply that for every $k\in N$ $P_k^n(\zeta',d\zeta)$ converges weakly to $P_k(\zeta',d\zeta)$ uniformly on any compact set in ζ'-space and that $\int f(\zeta)P_k(\zeta',d\zeta)$ is bounded continuous in ζ' if so does f. Therefore, combining these properties with (M.II), it is clear that (4.23) holds.

To prove (4.24) for $i=2$, it is sufficient to show that

(4.25) $$\lim_{N\to\infty}\limsup_{n\to\infty}\ E[|\zeta_{0,\delta}^n|]E[|\zeta_{0,\delta}^n|(1-\rho_N(|\zeta_0^n|))]=0\ .$$

(4.26) $$\lim_{N\to\infty}\limsup_{n\to\infty}\ E[|\zeta_{0,\delta}^n|\,|\zeta_{k,\delta}^n|(1-\rho_N(|\zeta_k^n|))]=0\ .$$

Since $\lambda=\sup_n E[|\zeta_{0,\delta}^n|^2]^{1/2}$ is finite, (4.25) is immediate from (M. II) because

$$\lim_{N\to\infty}\limsup_{n\to\infty}P[|\zeta_0^n|\geqq N]=\lim_{N\to\infty}\limsup_{n\to\infty}\Lambda^n(|\zeta|\geqq N)\leqq\lim_{N\to\infty}\Lambda(|\zeta|\geqq N)=0\,.$$

We next prove (4.26). Set $X_k^n=E[|\zeta_{k,\delta}^n|^2|\mathcal{F}_{k-1}^n]$ for $n, k\in\boldsymbol{N}$. Then, we have

$$\begin{aligned}(4.27)\qquad & E[|\zeta_{0,\delta}^n||\zeta_{k,\delta}^n|(1-\rho_N(|\zeta_k^n|))]\\ &\leqq E[|\zeta_{0,\delta}^n|E[|\zeta_{k,\delta}^n|(1-\rho_N(|\zeta_k^n|))|\mathcal{F}_{k-1}^n]]\\ &\leqq\lambda\times E[E[|\zeta_{k,\delta}^n|(1-\rho_N(|\zeta_k^n|))|\mathcal{F}_{k-1}^n]^2]^{1/2}\\ &\leqq\lambda\times E[X_k^nE[(1-\rho_N(|\zeta_k^n|))|\mathcal{F}_{k-1}^n]]^{1/2}\\ &\leqq\lambda\times\{CP[|\zeta_0^n|\geqq N]+\sup_{n,k}E[X_k^n\,;\,X_k^n>C]\}^{1/2}\,,\end{aligned}$$

for arbitrary $C>0$. Since $\{X_k^n\,;\,n, k\in N\}$ is uniformly integrable as we have seen before, it is easy to see that (4.26) holds.

Similarly, we can show that (4.24) holds for $i=3$.

Now, let $\{\zeta_k\,;\,k\in\boldsymbol{N}^*\}$ be the Markov process determined by the transition probabilities $\{P_k(\zeta', d\zeta)\}_k$ and the initial distribution $\Lambda(d\zeta)$. Letting $n\to\infty$ in (4.16), it follow from (M. II) and (M. IV) that

$$(4.28)\qquad (1/2)\sup_{\zeta'}\|P_k(\zeta',\cdot)-\Lambda(\cdot)\|_{\mathrm{var}}\leqq C\rho^k\,.$$

Moreover, note that $\Lambda(d\zeta)$ is the unique invariant probability measure of $\{\zeta_k\}_k$. Hence, we see from Lemma 4.12 that $\{\zeta_k\}_k$ is a uniformly mixing process with the rate function ϕ_k satisfying $\phi_k\leqq C\rho^k$. Therefore, if we set

$$(4.29)\qquad J_{2,N}=\int\zeta'\rho_N(|\zeta'|)\int\zeta(1-\rho_N(|\zeta|))\{P_k(\zeta', d\zeta)-\Lambda(d\zeta)\}\Lambda(d\zeta')\,,$$

and

$$(4.30)\qquad J_{3,N}=\int\zeta'(1-\rho_N(|\zeta'|))\int\zeta\{P_k(\zeta', d\zeta)-\Lambda(d\zeta)\}\Lambda(d\zeta')\,,$$

we can easily see by the similar way for $\{J_{i,N}^n\,;\,i=2, 3\}$ that for $i=2, 3$

$$(4.31)\qquad \lim_{N\to\infty}|J_{i,N}|=0\,.$$

Since $|I_{1,k}^n-I_{1,k}|\leqq|J_{1,N}^n-J_{1,N}|+|J_{2,N}^n|+|J_{3,N}^n|+|J_{2,N}|+|J_{3,N}|$, (4.22) follows from (4.23), (4.24), and (4.31).

We now complete the proof of (4.19). Set

$$C=\int\zeta'\int\zeta S(\zeta', d\zeta)\Lambda(d\zeta')\,,\qquad I_2=\sum_{k=k_1+1}^{\infty}\int\zeta'\int\zeta\{P_k(\zeta', d\zeta)-\Lambda(d\zeta)\}\Lambda(d\zeta')\,.$$

Then, by the similar way for (4.21), we can show that

$$(4.32)\qquad |I_2|\leqq 2\int|\zeta|^2\Lambda(d\zeta)\times\varepsilon\,.$$

Therefore, by (4.21), (4.22), and (4.32), we obtain

$$\limsup_{n\to\infty}|C^n-C|\leqq\limsup_{n\to\infty}\{\sum_{k=1}^{k_1}|I_{1,k}^n-I_{1,k}|+|I_2^n|+|I_2|\}\leqq\text{constant}\times\varepsilon\,.$$

Since ε is arbitrary, we get (4.19). Thus, we have completed the proof of Theorem 4.7. □

We now consider Theorem 4.7 in some special cases. First, suppose that $\{\xi_k^n\}_k$ is independent for all n. Then, we only need Conditions (M. I)~(M. III) because it is easy to see that Conditions (M. IV)~(M. VII) are satisfied. In this case, the assertion of Theorem 4.7 holds with $\mu(dz)$, V_0 of (4.8), $V_1=0$, and $\phi_k^n\equiv 0$.

Next, instead of Condition (M. VII), we suppose a stronger condition as follows.

(M. VII)*: For every $n\in\boldsymbol{N}$, there exists a (ζ',ζ)-measurable function $P^n(\zeta',\zeta)$ such that $P_1^n(\zeta',d\zeta)=P^n(\zeta',\zeta)\Lambda^n(d\zeta)$ and $\sup_{n,\zeta',\zeta}|P^n(\zeta',\zeta)|\leqq L$ for some $L>0$.

In this case, we can show that $\{\xi_k^n\}_{n,k}$ satisfies Condition $(\mathrm{U.IV})_2$ in Theorem 2.12, that is to say, it is a sequence of ϕ-mixing processes and that the rate function ϕ_k^n defined by (2.3) satisfies $\phi_k^n\leqq(L+1)C\rho^k$ for all n, k where C and ρ are the same as in (4.15). Therefore, we see that the assertion of Theorem 2.12 is valid if Conditions (M. I)~(M. VI), (M. VII)*, and (C) are satisfied.

Here, let us give examples satisfying Condition (M. I)~(M. III), and so on.

(4.33) Example. Let ξ_0 be a random variable with the property (2.87) for some $\alpha\in(0,2)$ and nonnegative constants C_+, C_-. For each $n\in\boldsymbol{N}$, we define a probability measure Λ^n by the law of of $n^{(1/2-1/\alpha)}\xi_0$. Then, as pointed out in the proof of Theorem 2.86, Condition (M. I) and (M. III) are satisfied with

$$\mu(dz)=\{C_+I_{\{z>0\}}+C_-I_{\{z<0\}}\}\frac{\alpha}{|z|^{1+\alpha}}dz$$

and $W=0$, respectively. Also Condition (M. II) is satisfied with $\Lambda(d\zeta)=\delta_{\{0\}}$.

(4.34) Example. Let $\{\Lambda^n\}_n$ be a sequence of probability measures on $\boldsymbol{R}^e$ satisfying Conditions (M. I)~(M. III). Furthermore, for some $m\in\boldsymbol{N}$, let $\{p_l^n(\zeta);\ l=1,\cdots,m\}$ be a sequence of continuous functions on $\boldsymbol{R}^e$ with the following properties (i)~(iii).

(i) $\int_{\boldsymbol{R}^e}p_l^n(\zeta)\Lambda^n(d\zeta)=0$ for all n, l.

(ii) For each l, p_l^n converges to a function p_l uniformly on any compact set on $\boldsymbol{R}^e$.

(iii) $|p_l^n(\zeta)|\leqq 1$ for all n, l, and ζ.

Put $P^n(\zeta',\zeta)=1+\beta\sum_{l=1}^m p_l^n(\zeta')p_l^n(\zeta)$ for some $\beta\in(0,1/m)$, and we denote by $\{\zeta_k^n;\ k\in\boldsymbol{N}^*\}$ a Markov chain determined by $P_1^n(\zeta',d\zeta)=P^n(\zeta',\zeta)\Lambda^n(d\zeta)$ for each n. Then Theorem 4.7 holds for the sequence of Markov chain $\{\xi_k^n;\ k\in\boldsymbol{N}^*\}_n$ determined by the relation (4.1).

Next result shows that if Condition (M. III) is slightly strengthened, then the limit process can not have any jumps.

(4.35) Theorem. *Suppose that Conditions (M. I), (M. II), (M. IV)~(M. VI), and (C) are satisfied. Moreover, we suppose the following.*

(M. III)*:
$$\lim_{N\uparrow\infty}\limsup_{n\to\infty}\int_{\boldsymbol{R}^e}|\zeta|^2I_{\{|\zeta|>N\}}\Lambda^n(d\zeta)=0\,.$$

Then the conclusion of Theorem 2.8 holds with $\mu(dz)\equiv 0$, (4.36) $$V_0^{pq}=\int_{R^e}\left\{\zeta^{(p)}-\int_{R^e}\zeta^{(p)}\Lambda(d\zeta)\right\}\left\{\zeta^{(q)}-\int_{R^e}\zeta^{(q)}\Lambda(d\zeta)\right\}\Lambda(d\zeta),$$

and the matrix V_1 *of* (4.9).

Proof. We first prove that (2.4) holds with $\mu\equiv 0$. Let $\delta<M$ be arbitrary positive numbers. Since

$$\int|z|^2(\rho_M(|z|)-\rho_\delta(|z|))\mu^n(dz)=\int|\zeta|^2(\rho_{\sqrt{n}M}(|\zeta|)-\rho_{\sqrt{n}\delta}(|\zeta|))\Lambda^n(d\zeta)$$
$$\leqq\int|\zeta|^2(1-\rho_{\sqrt{n}\delta}(|\zeta|))\Lambda^n(dz),$$

(M. I) and (M. III)* imply that $\int|z|^2(\rho_M(|z|)-\rho_\delta(|z|))\mu(dz)=0$. Since δ and M are arbitrary, $\mu(dz)$ must be zero.

Similarly, it is shown that (4.4) holds with $W^{pq}=0$ for all $p, q=1, \cdots, e$. It is also clear that (M. III)* implies $(\text{U. IV})_1$. To complete the proof, it is necessary to check (U. II). In view of the proof of Theorem 4.7, we need only show that (4.24) holds for $i=2, 3$. But, owing to (M. III)*, we can easily show it. □

Finally, we find a class of Markov chains for which Theorem 3.1 holds.

For each $n\in\boldsymbol{N}$, let $\{T_k^n;\ k\in\boldsymbol{N}^*\}$ be a Markov semigroup and suppose that it has an invariant probability measure Λ^n. Then let us say that the sequence of semigroups $\{T_k^n;\ k\in\boldsymbol{N}^*\}_n$ is *uniformly hypercontractive* if there exists $k_0\in\boldsymbol{N}$ such that

(4.37) $$\|T_{k_0}^n\|_{L^2(\Lambda^n)\to L^4(\Lambda^n)}\leqq 1,$$

for all $n\in\boldsymbol{N}$ where $\|\cdot\|_{L^2(\Lambda^n)\to L^4(\Lambda^n)}$ denotes the operator norm from the space $L^2(\Lambda^n)$ to $L^4(\Lambda^n)$.

As in the previous theorems, we suppose that $\{\zeta_k^n;\ k\in\boldsymbol{N}^*\}_n$ of (4.1) is a sequence of Markov chains with its transition probability $P_k^n(\zeta', d\zeta)$ and an invariant probability measure $\Lambda^n(d\zeta)$. Our final result is stated as follows.

(4.38) Theorem. *Suppose that* $\{\Lambda^n\}_n$ *converges weakly to a probability measure* Λ *and that*

(4.39) $$\sup_n\int_{R^e}|\zeta|^{2+\delta}\Lambda^n(d\zeta)<\infty \qquad \text{for some}\quad \delta>0.$$

Moreover, we suppose that Conditions (M. IV) and (M. V) are satisfied and that the sequence of semigroups $\{T_k^n;\ k\in\boldsymbol{N}^*\}_n$ *determined by* $P_k^n(\zeta', d\zeta)$ *is uniformly hypercontractive. In addition, if Condition (C) is satisfied, then the conclusion of Theorem* 3.1 *holds with the matrices* V_0 *of* (4.36) *and* V_1 *of* (4.9).

Proof. Since (4.39) is a translation of Condition (S. I), we show that (S. II) is satisfied. By Lemma 5.5.11 in Deuschel-Stroock [2], we see that the uniform hypercontraction property implies that

(4.40) $$\|T_{k_0}^n f-\bar{f}\|_{L^2(\Lambda^n)}\leqq(1/\sqrt{3})\|f\|_{L^2(\Lambda^n)},$$

for all bounded measurable functions f, where we set $\bar{f}=\int_{R^c} f(\zeta)\Lambda^n(d\zeta)$ and $\|\cdot\|_{L^p(\Lambda^n)}$ denotes the $L^p(\Lambda^n)$-norm. Further, from (4.40), it holds that for all $m\in N$

$$\|T^n_{mk_0}f-\bar{f}\|_{L^2(\Lambda^n)}=\|T^n_{k_0}((T^n_{(m-1)k_0})f-\bar{f})-\overline{(T^n_{(m-1)k_0}f-\bar{f})}\|_{L^2(\Lambda^n)}$$
$$\leqq(1/\sqrt{3})\|T^n_{(m-1)k_0}f-\bar{f}\|_{L^2(\Lambda^n)},$$

which implies that for all $k\in N$

$$\|T^n_kf-\bar{f}\|_{L^2(\Lambda^n)}\leqq\sqrt{3}\{(1/\sqrt{3})^{1/k_0}\}^k\|f\|_{L^2(\Lambda^n)}. \tag{4.41}$$

On the other hand, by the definition (2.1) of the strongly mixing rate function, it holds that

$$\alpha^n_k\leqq\sup_{\|f\|_{L^\infty(\Lambda^n)}\leqq 1}\|T^n_kf-\bar{f}\|_{L^1(\Lambda^n)}, \tag{4.42}$$

for all n, k. Therefore, by (4.41) and (4.42), we have $\alpha^n_k\leqq\sqrt{3}\{(1/\sqrt{3})^{1/k_0}\}^k$ for all n, k, which implies that (S.II) is satisfied. Applying Lemma 3.3, it can be shown by a similar way in the proof of Theorem 4.7 and Theorem 4.35 that (U.II) is satisfied with the matrices V_0 of (4.36) and V_1 of (4.9). So we omit the proof. □

Acknowledgment. The author would like to express his sincere gratitude to Professor Hiroshi Kunita for his constructive suggestions and constant encouragement. He also would like to thank Professor Shinzo Watanabe for his encouragement.

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References

[1] C. Dellacherie and P.A. Meyer, Probabilités et potentiel, Hermann, 1975.
[2] J.D. Deuschel and D.W. Stroock, Large deviations, Academic Press, 1989.
[3] E. Eberlein and M.S. Taqqu ed., Dependence in probability and statistics, Birkhauser, 1986.
[4] S.N. Ethier and T.G. Kurtz, Markov processes, characterization and convergence, John Wiley & Sons, 1986.
[5] T. Fujiwara, On the jump-diffusion approximation of stochastic difference equations driven by a mixing sequence, J. Math. Soc. Japan, **42**-3 (1990), 353-376.
[6] T. Fujiwara and H. Kunita, Limit theorems for stochastic difference-differential equations, Nagoya Math. J., **127** (1992), 83-116.
[7] B.V. Gnedenko and A.N. Kolmogorov, Limit distribtions for sums of independent random variables, Addison-Wesley, 1954.
[8] I.A. Ibragimov and Yu. V. Linnik, Independent and stationary sequences of random variables, Wolters-Noardhoff Publishing Groningen, 1971.
[9] M. Iizuka and H. Matsuda, Weak convergence of discrete time non-Markovian processes related to selection models in population genetics, J. Math. Biology, **15** (1982), 107-127.
[10] N. Ikeda and S. Watanabe, Stochastic differential equations and diffusion processes, North-Holland/Kodansha, 1981.
[11] J. Jacod and A.N. Shiryaev, Limit theorems for stochastic processes, Springer-Verlag, 1987.

[12] Y. Kasahara and S. Watanabe, Limit theorems for point processes and their functionals, J. Math. Soc. Japan, **38**-3 (1986), 543-574.
[13] Y. Kasahara and K. Yamada, Stability theorem for stochastic differential equations with jumps, Stoch. Processes Appl., **38** (1991), 13-32.
[14] H. Kesten and G.C. Papanicolaou, A limit theorem for turbulent diffusion, Comm. Math. Phys., **65** (1979), 97-128.
[15] R.Z. Khas'minskii, A limit theorem for the solution of differential equations with random right-hand sides, Theory Prob. Appl., **11** (1966), 390-406.
[16] H. Kunita, Stochastic flows and application, Tata Institute of Fundamental Research, Springer, 1986.
[17] H. Kunita, Stochastic flows and stochastic differential equations, Cambridge Univ. Press, 1990.
[18] T.G. Kurtz and P. Protter, Weak limit theorems for stochastic integrals and stochastic differential equations, Ann. Prob., **19** (1991), 1035-1070.
[19] H.J. Kushner, Approximation and weak convergence methods for random processes, with applications to stochastic system theory, MIT Press, 1984.
[20] R.S. Liptser and A.N. Shiryaev, Theory of martingales, Kluwer Academic Publishers, 1989.
[21] J.D. Samur, Convergence of sums of mixing triangular arrays of random vectors with stationary rows, Ann. Prob., **12** (1984), 390-426.
[22] J.D. Samur, On the invariance principle for stationary φ-mixing triangular arrays with infinitely divisible limits, Prob. Th. and Rel. Fields, **75** (1987), 245-259.
[23] L. Słomiński, Stability of strong solutions of stochastic differential equations, Stoch. Processes Appl., **31** (1989), 173-202.
[24] H. Watanabe, Diffusion approximations of some difference equations, II, Hiroshima Math. J., **14** (1984), 15-34.