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Equations of evolution on the Heisenberg group I

By

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1. Introduction

Theory of nilpotent Lie groups and its irreducible unitary representations become a powerful tools in the study of linear partial differential operators. (Folland, Helffer-Nourrigat, Rockland, Rothschild, Rothschild-Stein, etc.) The major concern of these works is to study the hypoellipticity or local solvability of linear partial differential operator. We believe that the same spirit is effective in investigating the Cauchy problem for the equations of evolution.

For the fundamental solution of operators on the Heisenberg group, there are also many works. B. Gaveau studied the heat equation and A.L. Nachman investigated the wave equation. Contrary to these works, we are concerned with the well-posedness for the Cauchy problem for the operators of higher order on the Heisenberg group. We hope that this becomes a model case for more general differential operators with multiple characteristics.

In this paper, we shall limit ourselves to treating the parabolic case. Let us consider the operators of higher order on the Heisenberg group $\boldsymbol{H}^n$.

$$P=\partial_t^m+\sum_{j=1}^{m} A_j\partial_t^{m-j},$$

where A_j are the homogeneous right invariant differential operators of order pj on $\boldsymbol{H}^n$. ($p\in\boldsymbol{N}$)

Roughly speaking, our main result is formulated as follows. If for any non-trivial irreducible unitary representation π of $\boldsymbol{H}^n$, $\pi(P)$ satisfies "parabolic" conditions, then the Cauchy problem

$$\begin{cases} Pu=f \\ \operatorname{supp}_t u\subset[0, \infty) \end{cases} \tag{1.1}$$

is well-posed: i.e. for any positive number T and any $f\in C_0^\infty((-T, T)\times\boldsymbol{H}^n)$ with support contained in $[0, T)\times\boldsymbol{H}^n$, there is a solution $u(x, t)\in C^\infty((-T, T)\times\boldsymbol{H}^n)$ of (1.1) and this solution is unique in the Sobolev space subordinated to $\boldsymbol{H}^n$.

2. Statement of results

We recall some notion on the Lie group. (c.f. Rockland [R]) Let G be a simply-connected nilpotent Lie group, with Lie algebra $\mathcal{G}$ and (complexified) universal envelop-

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ing algebra $\mathcal{U}(\mathcal{G})$. Since G is simply-connected, the exponential map $\exp: \mathcal{G}\to G$ is a diffeomorphism. We identify $\mathcal{G}$ with the right-invariant real vector fields on G by associating to $X\in\mathcal{G}$ the vector fields, still denote X, defined by

$$(X\varphi)(x)=\frac{d}{dt}\varphi(\exp tX\cdot x)|_{t=0}\,, \qquad \varphi\in C^{\infty}(G)$$

This identification extends uniquely to an isomorphism between the algebra $\mathcal{U}(\mathcal{G})$ and the algebra of all right-invariant differential operators on G (with complex coefficients). If $X_1, \cdots, X_N$ form an ordered basis for $\mathcal{G}$, then every right-invariant differential operator on G can be expressed uniquely in the form

$$P=\sum_{|\alpha|\le m} a_\alpha X_1^{\alpha_1}\cdots X_N^{\alpha_N}\,, \qquad a_\alpha\in \boldsymbol{C}$$

by the Birkhoff-Poincaré-Witt theorem. If P is a differential operator on G, then by P^t we denote the formal transpose of $\mathcal{G}$ with respect to Haar measure which is the image under exp of Lebegue measure on $\mathcal{G}$, and by P^* the formal adjoint of P. Especially, if $X\in\mathcal{G}$, then $X^t=-X$.

If π is a unitary representation of G on the Hilbert space H, then $v\in H$ is called a C^∞-vector for π if the map $x\mapsto\pi(x)v$ from G to H is C^∞. The C^∞-vectors form a vector subspace of H, which we denote by H_∞. The representation π determines a Lie algebra representation π of $\mathcal{G}$ as linear maps: $H_\infty\to H_\infty$ defined by

$$\pi(X)v=\frac{d}{dt}\pi(\exp tX)v|_{t=0}\,, \qquad X\in\mathcal{G},\ v\in H_\infty\,.$$

This extends uniquely to a representation of the algebra $\mathcal{U}(\mathcal{G})$ as linear maps: $H_\infty\to H_\infty$. If π is irreducible, then there is a unitary-equivalence taking H to $L^2(\boldsymbol{R}^n)$ for some n (possibly 0) and taking H_∞ to $\mathcal{S}(\boldsymbol{R}^n)$, the Schwartz space. If π is a unitary representation of G on H, then π determines a representation of the algebra $L^1(G)$ as bounded operators on H by

$$\pi(f)v=\int_G \pi(y)v\cdot f(y)dy\,, \qquad v\in H$$

and if π is irreducible, $\pi(f)$ is a compact operator for $f\in L^1(G)$.

Let $r\mapsto\delta_r$ be a homomorphism from $\boldsymbol{R}^+$, the multiplicative group of positive real numbers, into Aut $\mathcal{G}$, the group of automorphism of $\mathcal{G}$ of the form $\delta_+=\exp(\log r)A)$, where $A:\mathcal{G}\to\mathcal{G}$ is a semisimple linear transformation with positive eigenvalues, $\gamma_1, \cdots, \gamma_N$. We then call $\{\delta_r\}$ a group of dilations for G and we say that $P\in\mathcal{U}(\mathcal{G})$ is homogeneous of degree k if $\delta_r(P)=r^kP$ for every r. Taking an ordered basis $X_1, \cdots, X_N$ for $\mathcal{G}$ of eigenvectors of A, we see that $\delta_r(X_i)=r^{\gamma_i}X_i$. We denote by $\mathcal{U}_k(\mathcal{G})$ the set of all homogeneous right invariant operator $P\in\mathcal{U}(\mathcal{G})$ of degree k.

Recall that the $2n+1$-dimensional Heisenberg algebra, h_n, is the Lie algebra with generators X_i, Y_j, $i=1, \cdots, n$, Z satisfying the commutation relations

$$[X_i, Y_j]=\delta_{i,j}Z\,, \qquad [X_i, Z]=[Y_i, Z]=0\,.$$

The Heisenberg group $\boldsymbol{H}^n$ is the unique simply-connected Lie group having h_n are its

Lie algebra. The group H^n has a group of dilations $\{\delta_r\}$ defined by

$$\delta_r(X_i)=rX_i, \qquad \delta_r(Y_i)=rY_i, \qquad \delta_r(Z)=r^2Z .$$

We shall use exponential coordinates

$$(x', x'', x_0)\in \boldsymbol{R}^{2n+1}\longmapsto \exp(x'X+x''Y-x_0Z).$$

Hereafter, as a set we identify the group H^n with its corresponding Lie algebra h_n. Then $H^n=R^n\oplus R^n\oplus R$ and we let (x', x'', x_0) denote the components of a vector x in H^n. Then, the bracket operation is given by

$$[x, y]=(0, 0, \langle x'', y'\rangle-\langle x', y''\rangle)$$

and the formula for multiplication is

$$x\cdot y=x+y+\frac{1}{2}[x, y] .$$

The group convolution takes the form

$$(u*v)(x)=\int_{H^n}u(xy^{-1})v(y)dy ,$$

where dy is the standard Lebesgue measure. Considered as right-invariant vector fields on H^n, the element of the basis $\{X, Y, Z\}$ are given by

$$Z=\frac{\partial}{\partial x_0}, \qquad X_j=\frac{\partial}{\partial x_j'}-\frac{x_j''}{2}\frac{\partial}{\partial x_0}, \qquad Y_j=\frac{\partial}{\partial x_j''}+\frac{x_j'}{2}\frac{\partial}{\partial x_0}, \qquad (j=1, \cdots, n).$$

There are two classes of irreducible unitary representations, as follows from the Stone-von Neuman theorem:

(1) A family of 1-dimensional representations which map Z to 0. They are parametrized by $(\xi, \eta)\in R^{2n}$, and are given by

$$\pi_{(\xi,\eta)}(x', x'', x_0)=e^{i(x'\xi+x''\eta)}, \qquad (\xi, \eta)\in R^{2n}$$

i.e.

$$\pi_{(\xi,\eta)}(X_i)=\sqrt{-1}\xi_i,\ \pi_{(\xi,\eta)}(Y_i)=\sqrt{-1}\eta_i,\ \pi_{(\xi,\eta)}(Z)=0 .$$

(2) A family parametrized by $\lambda\in R\setminus\{0\}$ acting on $L^2(R^n)$ which map Z to a non-zero scalar. They are given by

$$[\pi_\lambda(x', x'', x_0)v](t)=e^{i\lambda(\langle x'', s\rangle-x_0+\langle x', x''\rangle/2)}v(s+x') \qquad \text{for } v\in L^2(R^n).$$

i.e.

$$\pi_i(X_i)=\frac{\partial}{\partial s_i}, \qquad \pi_\lambda(Y_i)=\sqrt{-1}\lambda s_i, \qquad \pi_\lambda(Z)=\sqrt{-1}\lambda .$$

Now, we consider the operators of evolution on H^n

$$P=\partial_t^m+\sum_{j=1}^m A_j\partial_t^{m-i}, \qquad A_j\in \mathcal{U}_{pj}(H^n),$$

where p is a positive integer. For $\zeta\in C$, $\xi\in R^n$, let us introduce the following two generalized symbols of P according to two families of irreducible representations on H^n.

$$p_m(\zeta, \xi)=(i\zeta)^m+\sum_j^m \pi_{(\xi', \xi'')}(A_j)(i\zeta)^{m-j}$$

and

$$\mathcal{P}(\zeta, \lambda)=(i\zeta)^m+\sum_j^m \pi_\lambda(A_j)(i\zeta)^{m-j} : \mathcal{S}(\boldsymbol{R}^n) \rightarrow \mathcal{S}(\boldsymbol{R}^n).$$

We introduce the "parabolic" conditions on these generalized symbols:

(P-1) For every root ζ_j of $p_m(\zeta, \xi)=0$, there is a positive constant δ such that

$$\operatorname{Im} \zeta_j \geqq \delta\langle\xi\rangle^p \quad \text{for } \xi \neq 0$$

and

(P-2) for any ζ and λ with $\operatorname{Im} \zeta \leqq 0$, $|\lambda|=1$, the operator $\mathcal{P}$ and $\mathcal{P}^* : \mathcal{S}(\boldsymbol{R}^n) \rightarrow \mathcal{S}(\boldsymbol{R}^n)$ are injective.

We note that condition (P-1) also can be written as follows: for any ζ and ξ with $\operatorname{Im} \zeta \leqq 0$, $|\xi| \neq 0$, the operator of multiplication by $p_m(\zeta, \xi)$, $\boldsymbol{C} \rightarrow \boldsymbol{C}$, is injective.

Let us denote by $H_{(\infty)}(\boldsymbol{H}^n)$ the space of functions such that for any multi-index $I=(i_1, \cdots, i_N) \in\{-n,-n+1, \cdots, n\}^N$

$$X^I u=X_{(i_1)} \cdots X_{(i_N)} u \in L^2(\boldsymbol{H}^n),$$

where $X_{(i)}=X_i$ if $i>0$, $=Y_{-i}$, if $i<0$ and $=Z$ if $i=0$. Then we have

Theorem 1. *Suppose that* (P-1) *and* (P-2) *hold. Then for any* $T>0$ *and any positive integer* $l \geqq 2$. *the Cauchy problem*

$$\left\{\begin{array}{l} Pu=f \text{ in } (0, T) \times \boldsymbol{H}^n \\ \partial_t^j u|_{t=0}=g_j \text{ on } \boldsymbol{H}^n, \ 0 \leqq j \leqq m-1 \end{array}\right. \tag{2.2}$$

has a unique solution $u(x, t) \in C^{m+l-2}([0, T] ; H_{(\infty)}(\boldsymbol{H}^n))$ *if* $f \in C^l([0, T] ; H_{(\infty)}(\boldsymbol{H}^n))$ *and* $g_j \in H_{(\infty)}(\boldsymbol{H}^n)$, $j=0, \cdots, m-1$.

This theorem will be proved by using the analysis in [M] and [HN]. But the nature of the Cauchy problem requires us to modify the arguments. We shall introduce the notion of the pseudo-differential operator with a parameter in section 3.

3. Spaces of symbols and pseudo-differential operators with a parameter

We recall the some function spaces, introduced in [M]. For a non-negative integer k, B_k will denote the clossure of $\mathcal{S}(\boldsymbol{H}^n)$ in the norm $\|f\|_k=\sum_{|\alpha| \leq k} \max |D^\alpha f|$. $\mathcal{M}$ is the space of distribution $u \in \mathcal{S}'(\boldsymbol{H}^n)$ such that there exists a positive integer k for which $qu \in B_k'$ for every polynomial $q(x)$ on $\boldsymbol{H}^n$. If $(u_j)_1^\infty$ is a sequence in $\mathcal{M}$ and $u \in \mathcal{M}$, then we say that u_j tends to u on $\mathcal{M}$ if and only if one can find k so that qu_j, $qu \in B_k'$ for all j and $qu_j \rightarrow qu$ weakly in B_k' when q is a polynomial.

Then, the following basic properties were proved. (Lemma 2.2, 2.3 in [M])

(1) If $u_j \in \mathcal{M} \rightarrow u \in \mathcal{M}$, then

$$v \longrightarrow v * u_j \quad \text{and} \quad v \longrightarrow u_j * v$$

are continuous maps on $\mathcal{S}(\boldsymbol{H}^n)$ for j fixed, and

$$v*u_j \longrightarrow v*u, \quad u_j*v \longrightarrow u*v \qquad \text{in } \mathcal{S}(\boldsymbol{H}^n)$$

when $j\to\infty$ and $v\in\mathcal{S}(\boldsymbol{H}^n)$.

(2) Let $u\in\mathcal{M}$. Then, one can find a sequence $(u_j)_1^\infty$ in $\mathcal{S}(\boldsymbol{H}^n)$ tending to u in $\mathcal{M}$.

(3) Let u and $(u_j)_1^\infty$ be as above. Let $\lambda\in\boldsymbol{R}$ and $f\in\mathcal{S}(\boldsymbol{R}^n)$. Then, $\pi_\lambda(u_j)f$ is continuous in $\mathcal{S}(\boldsymbol{H}^n)$. The limit is independent of the choice of the sequence $(u_j)_1^\infty$ and defines a continuous linear map $\pi_\lambda(u)$ on $\mathcal{S}(\boldsymbol{R}^n)$.

(4) If u and v are in $\mathcal{M}$, then $u*v\in\mathcal{M}$ and

$$\pi_\lambda(u*v)=\pi_\lambda(u)\pi_\lambda(v)\,.$$

(5) If $u\in\mathcal{M}$ and $\pi_\lambda(u)=0$ for every $\lambda\in\boldsymbol{R}$, then $u=0$.

Let us denote the dual variable of t by ζ and let $\|\xi\|=|\xi'|+|\xi''|+1$, $\|\bar{\xi}\|=\|\xi\|+|\lambda|^{1/2}$, $\lambda=\xi_{2n+1}$ and $\bar{\xi}=(\xi, \lambda)$. When k and l are non-negative integers, we set

$$e_{k,l}(\bar{\xi})=\{(1+|\lambda|)/(1+|\lambda|+\|\xi\|^2+|\zeta|^{2/p})\}^{k-\min(k,l)}$$

and

$$\Gamma=\{\zeta\in\boldsymbol{C}\,;\,\mathrm{Im}\,\zeta\leqq 0\}\,.$$

We introduce some symbol classes:

Definition 3.1. $\mu\in\boldsymbol{R}$ and k is a non-negative integer, then the class $S_\Gamma^{\mu;k}$ consists of the functions $a(\zeta, \xi, \lambda)$ such that

1) $a(\zeta_0, \xi, \lambda)\in C^\infty(\boldsymbol{H}^n)$ for any fixed ζ_0;

2) for any multi-index α of dimension $2n+1$, there exists a positive constant C such that

$$|D_{\bar{\xi}}^{\alpha}a(\zeta, \xi, \lambda)|\leqq Ce_{k,\alpha_{2n+1}}(\bar{\xi}, \zeta)(1+|\zeta|^{1/p}+|\lambda|^{1/2}+\|\xi\|)^{\mu-\|\bar{\alpha}\|}$$

for $\zeta\in\Gamma$, $\xi\in\boldsymbol{R}^n$, $\lambda\in\boldsymbol{R}$, where $\|\bar{\alpha}\|=|\alpha'|+|\alpha''|+2\alpha_{2n+1}$.

We set $S_\Gamma^{\mu;\infty}=\bigcap_{k\geqq 0}S_\Gamma^{\mu;k}$ and $S_\Gamma^{\infty;k}=\bigcup_\mu S_\Gamma^{\mu;k}$. Then

$$S_\Gamma^{\mu;k}\cdot S_\Gamma^{\mu';k'}\subset S_\Gamma^{\mu+\mu';k+k'} \quad S_\Gamma^{\mu;k}\subset S_\Gamma^{\mu;0}$$

for any μ and k.

Many of statements about (λ)-pseudo-differential operator without parameter ζ (c.f. A. Melin [M]) can also be proved for the case with a parameter ζ. First, the theory of asymptotic summation carries over to symbols depending on a parameter. We say that $\sum_0^\infty a_j$ is a formal symbol if $a_j\in S_\Gamma^{\mu;k_j}$, $\{k_j\}_0^\infty$ is an unbounded increasing sequence of non-negative integers and the following inequalities hold: for some positive constant C and any multi-index $\bar{\alpha}$,

$$|D_{\bar{\xi}}^{\alpha}a_j(\zeta, \xi, \lambda)|\leqq Ce_{k_j,\alpha_{2n+1}}(\bar{\xi}, \zeta)(1+|\zeta|^{1/p}+|\lambda|^{1/2}+\|\xi\|)^{\mu-\|\alpha\|}$$

for any $\xi\in\boldsymbol{H}^n$, $\zeta\in\Gamma$. Then

Lemma 1. *Assume that* $\sum_0^\infty a_j$ *is a formal symbol. Then one can find* $a\in S_\Gamma^{\mu;k_0}$ *such that*

$$a-\sum_{0}^{N-1} a_j \in S_{\Gamma}^{\mu, kN}, \qquad N=1, 2, \cdots .$$

Moreover, a is uniquely determined modulo $S_{\Gamma}^{\mu, \infty}$ *and we write* $a \sim \sum_0^\infty a_j$.

The proof of this lemma is almost verbatim repetitions of the arguments in [M]. We only state the important change in the proof. The role of $1+\|x\|$ is now played by $1+|\zeta|^{1/p}+\|\xi\|$ and the cut-off function in Lemma 3.4 in [M] $\psi_j(\xi)=\psi(\varepsilon_j\xi/|\lambda|^{1/2})$ is replaced by

$$\psi_j(\zeta, \xi, \lambda)=\psi(\varepsilon_j\zeta/|\lambda|^{1/2}, \varepsilon_j\xi'/|\lambda|^{1/2}, \varepsilon_j\xi''/|\lambda|^{1/2}).$$

Here ψ is in $C^\infty(\boldsymbol{R}^{2n+1})$ for any fixed $\zeta \in \Gamma$ such that

$$\psi=1 \qquad \text{when } |\zeta|^{1/p}+|\xi'|+|\xi''| \geqq 2$$

and

$$\psi=0 \qquad \text{when } |\zeta|^{1/p}+|\xi'|+|\xi''| \leqq 1 .$$

Then, we may choose the sequence $\{\varepsilon_j\}$ such that the sum $a=\sum_0^\infty \psi_j a_j$ converges in C^∞ and satisfies the desired propertes.

Let us denote

$$\mathcal{F}u(\xi, \lambda)=\int_{H^n} e^{i(\langle \xi'', x'\rangle+\langle \xi', x''\rangle-\lambda x_{2n+1})}u(x)dx$$

and

$$\mathrm{Op}_\lambda(a)f(x')=(2\pi)^{-n}\iint e^{i\langle x'-y', \eta'\rangle}a\Big(\zeta, \lambda\Big(\frac{x'+y'}{2}\Big), \eta'\Big)f(y')dy'd\eta'.$$

The following lemma is also valid for symbols with a parameter ζ.

Lemma 2. *Assume that* $a \in S_{\Gamma}^{\mu, k}$. *Then* $u=\mathcal{F}^{-1}a$ *is in* $\mathcal{M}$ *for any fixed* $\zeta \in \Gamma$ *and we have*

$$\pi_\lambda(\mathcal{F}^{-1}a)=\mathrm{Op}_\lambda a(\cdot, \lambda)$$

$\lambda \in \boldsymbol{R}$.

For a and $b \in S_{\Gamma}^{\infty, k}$, we set

$$a\# b=\mathcal{F}(\mathcal{F}^{-1}a * \mathcal{F}^{-1}b)$$

which is defined as an element in $\mathcal{S}'(\boldsymbol{H}^{n*})$ with a parameter ζ. This is the symbol of operator product i.e.

$$\mathrm{Op}_\lambda(a\# b)=\mathrm{Op}_\lambda(a)\,\mathrm{Op}(b)$$

if $a\# b \in S_{\Gamma}^{\mu, k}$, which follows from the next fact. (c.f. Proposition 3.6 in [M])

Proposition 1. *Assume that* $a \in S_{\Gamma}^{\mu, k}$ *and* $b \in S_{\Gamma}^{\mu', k'}$. *Then* $a\# b \in S_{\Gamma}^{\mu+\mu', k+k'}$ *and*

$$a\# b \sim \sum_{0}^{\infty} \{(i\lambda\sigma(D_\xi, D_\eta)/2)^k a(\zeta, \xi, \lambda)b(\zeta, \eta, \lambda)/k!\}|_{\xi=\eta},$$

where σ *is the bilinear form* $\langle x'', y'\rangle-\langle x', y''\rangle$ *on* $\boldsymbol{H}^n$.

4. Parametrices for parabolic operators

Let P be the operator satisfying the all assumptions in Theorem 1 and

$$\hat{u}(\zeta, \xi)=\int e^{-i\zeta t}\mathcal{F}u(\xi, t)dt .$$

Then, from the property (4) for π_λ and Lemma 2, it follows that

$$\pi_\lambda(P^t)=\pi_\lambda(P\delta)=\mathrm{Op}_\lambda(p(\zeta, \cdot, \lambda)) ,$$

where, the symbol $p=P\delta$ is a quasi-homogeneous polynomial in ζ and ξ of degree m. Here we have used the fact that for $j=1, \cdots, m$, A_j is a right invariant operator on $\boldsymbol{H}^n$.

The hypothesis (P-1) and the quasi-homogeneity of p mean that $p(\zeta, \xi, \lambda)\neq 0$ if $\zeta\in\Gamma$ and $\lambda=0$, which implies

Lemma 3. *One can find positive constants C and C_α such that if $\zeta\in\Gamma$, $\lambda\geqq 1$ and $\|\xi\|^2\geqq C|\lambda|$, then*

$$|p(\zeta, \xi, \lambda)|\geqq C^{-1}\{|\zeta|+(1+|\xi|+|\lambda|^{1/2})^p\}^m$$

and

$$|\partial_\xi^\alpha(1/p)|\leqq C_\alpha\{|\zeta|^{1/p}+(1+|\xi|+|\lambda|^{1/2})\}^{pm-|\alpha|}$$

for any multi-index α.

Proof. For some $c>0$, if $|\xi|>c|\lambda|^{1/2}$, then all roots ζ_j, $j=1, \cdots, m$ of the equation $p(\zeta, \xi, \lambda)=0$ satisfy the inequalities:

$$\mathrm{Im}\,\zeta_j(\xi, \lambda)\geqq\delta(1+|\xi|+|\lambda|^{1/2})^p , \qquad \delta>0 .$$

Since

$$(\mathrm{Re}\,\zeta-\mathrm{Re}\,\zeta_j)^2\geqq\varepsilon\,(\mathrm{Re}\,\zeta)^2-\frac{\varepsilon}{1-\varepsilon}(\mathrm{Re}\,\zeta_j)^2 \qquad \text{for any } 1>\varepsilon>0 .$$

if we take ε is sufficiently small, we have

$$|\zeta-\zeta_j|\geqq\delta'(1+|\zeta|^{1/p}+|\xi|+|\lambda|^{1/2})^p , \qquad \delta'>0 .$$

From this, the assertions follows immeadiately.

Proposition 2. *Suppose that P satisfies* (P-1). *Then one can find $q\in S_\Gamma^{-pm,0}$ such that*

$$p\#q-1\in S_\Gamma^{0,\infty} , \qquad q\#p-1\in S_\Gamma^{0,\infty} .$$

Proof. It follows from Lemma 3 that one can find a function $\psi(\zeta, \xi)$ in $C^\infty(\boldsymbol{R}^{2n})$ for each $\zeta\in\Gamma$ such that for some positive number R and r,

$$\psi(\zeta, \xi)=1 \qquad \text{for } |\zeta|^{1/p}+|\xi|>R$$

$$\psi(\zeta, \xi)=0 \qquad \text{for } |\zeta|^{1/p}+|\xi|<r$$

and the following inequality holds with a positive constant C

$$|p(\zeta, \bar{\xi})| \geqq C(1+|\zeta|+\|\bar{\xi}\|^{p})^{m}$$

if $\Psi(\zeta, \bar{\xi})=\phi(\zeta/\lambda^{1/2}, \xi/\lambda^{1/2})\neq 0$. Set $g(\zeta, \bar{\xi})=\Psi(\zeta, \bar{\xi})/p(\zeta, \bar{\xi})$. Then $g\in S_{\Gamma}^{-pm,0}$ and $pg-1=\Psi-1\in S_{\Gamma}^{0,\infty}$. By Proposition 1 we see that $p\#g=1-h$, where h is in $S_{\Gamma}^{0,1}$. Then from Lemma 1 it follows that for some q_0 in $S_{\Gamma}^{0,0}$,

$$q_0 \sim \sum_{0}^{\infty} h^{\#,j},$$

where $h^{\#,j}=h\#h\cdots\#h$ (j times). Therefore, the first assertion is satisfied if we take $q=g\#q_0$. The second assertion with another $q'\in S_{\Gamma}^{-pm,0}$ is proved by the similar way. Since a standard argument tells us that $q-q'\in S_{\Gamma}^{0,\infty}$, the proof is complete.

By the quasi-homogeneity, we see that

$$\mathcal{P}(\zeta, \lambda)=|\lambda|^{pm/2}\mathcal{P}(\zeta/|\lambda|^{p/2}, \pm 1)=|\lambda|^{pm/2}\mathcal{P}_{\pm}(\zeta/|\lambda|^{p/2}).$$

For $s\in \boldsymbol{R}$, $\zeta\in\Gamma$ and $\mu>0$, let us denote by $H_{s,\zeta,\mu}(\boldsymbol{R}^n)$ the space of distributions $u\in \mathcal{S}'(\boldsymbol{R}^n)$ for which $\Lambda^s u\in L^2(\boldsymbol{R}^n)$, where Λ^s means that

$$\Lambda^s u=\{1+|\zeta|^{2/p}+(|D_x|^2+|x|^2\mu^2)\}^{s/2}u.$$

and we write $\|\Lambda^s u\|_{L^2(\boldsymbol{R}^n)}=\|u\|_{s,\zeta,u}$. Then, since $\mathrm{Op}_1(q)$ is bounded from $H_{s,\zeta,1}(\boldsymbol{R}^n)$ to $H_{s+pm,\zeta,1}(\boldsymbol{R}^n)$ and for $r\in S_{\Gamma}^{0,\infty}$, $\mathrm{Op}_1(r)$ is bounded on $H_{s,\zeta,1}(\boldsymbol{R}^n)$, Proposition 2 implies

Proposition 3. *Suppose that* (P-1) *is satisfied. Then for* $s\in\boldsymbol{R}$, *there is a constant* C *such that*

$$\|u\|_{s+pm,\gamma,1}\leqq C\{\|\mathcal{P}_{\pm}(\gamma)u\|_{s,\gamma,1}+\|u\|_{s,\gamma,1}\}$$

for any $u\in\mathcal{S}(\boldsymbol{R}^n)$, $\gamma\in\Gamma$.

By Proposition 2, we have

Lemma 4. *Let* $s\in\boldsymbol{R}$. *If* $u\in H_{s,\gamma}(\boldsymbol{R}^n)$ *and* $\mathcal{P}_{\pm}(\gamma)u=0$, $\gamma\in\Gamma$. *Then*, $u\in\mathcal{S}(\boldsymbol{R}^n)$.

We need the continuity proprety of the estimates.

Lemma 5. *Suppose that for* $\gamma_0\in\Gamma$ *the inequality*

$$\|u\|_{s+pm,\gamma_0,1}\leqq C\|\mathcal{P}_{\pm}(\gamma_0)u\|_{s,\gamma_0,1} \tag{4.1}$$

holds with some positive constant C. *if* $u\in\mathcal{S}(\boldsymbol{R}^n)$. *Then, there are a neighborhood* U *of* γ_0 *and a positive constant* C' *such that*

$$\|u\|_{s+pm,\gamma,1}\leqq C'\|\mathcal{P}_{\pm}(\gamma)u\|_{s,\gamma,1}$$

holds for any $\gamma\in U\cap\Gamma$.

Proof. Since $(D_x^2+|x|^2)$ is self-adjoint, Λ^s is equivalent to $(1+|\zeta|^{s/p}+(D_x^2+|x|^2)^{s/2})$ and we can write

$$\mathcal{P}_{\pm}(\gamma)-\mathcal{P}_{\pm}(\gamma')=\sum_{finite,\,|\alpha|\leq m-1} C_\alpha(\gamma, \gamma')T_\alpha,$$

where $T_\alpha=T_{\alpha_1}\cdots T_{\alpha_N}$, $T_j=x_{-j}$ if $j<0$, $=D_{x_j}$ if $j>0$ and $C_\alpha(\gamma-\gamma')=O(|\gamma-\gamma'|)$. From these observations, the assertion follows.

Theorem 2. *Suppose that* (P-1) *and* (P-2) *are satisfied. Then, for* $s\in\boldsymbol{R}$, *there are constants* C *and* C' *such that*

$$\|u\|_{s+pm,\zeta,|\lambda|}\leqq C\|\mathcal{P}(\zeta,\ \lambda)u\|_{s,\zeta,|\lambda|}$$

and

$$\|u\|_{s+pm,\zeta,|\lambda|}\leqq C'\|\mathcal{P}^{*}(\zeta,\ \lambda)u\|_{s,\zeta,|\lambda|}$$

for any $u\in\mathcal{S}(\boldsymbol{R}^{n})$, $\zeta\in\Gamma$ *and* $|\lambda|\geqq 1$.

Proof. From Proposition 3, it follows that there is a positive constant C such that

$$\|u\|_{s+pm,\gamma,1}\leqq C\|\mathcal{P}_{\pm}(\gamma)u\|_{s,\gamma,1}$$

if any $u\in\mathcal{S}(\boldsymbol{R}^{n})$ and $|\gamma|$ is sufficiently large in Γ. As for bounded γ, we have

Lemma 6. *For any* $\gamma\in\Gamma$, *there is a positive constant* C_{γ} *such that*

$$\|u\|_{s+mp,\gamma,1}\leqq C_{\gamma}\|\mathcal{P}_{\pm}(\gamma)u\|_{s,\gamma,1}\,.$$

Proof of Lemma 6. The hypothesis (P-2) and the estimate in Proposition 3 tell us that the inequality (4.1) holds for any $\gamma_{0}\in\Gamma$. In fact, if for some $\gamma\in\Gamma$, (4.1) does not holds, then for any N there is $u_{N}\in\mathcal{S}(\boldsymbol{R}^{n})$ such that

$$\|u_{N}\|_{s+pm,\gamma,1}\geqq N\|\mathcal{P}_{\pm}(\gamma)u_{N}\|_{s,\gamma,1}\quad\text{and}\quad\|u_{N}\|_{s+pm,\gamma,1}=1\,.$$

Since the imbedding $H_{s,\gamma,1}\subset H_{s',\gamma,1}$ is compact if $s>s'$ (cf. Proposition 25.4 in [S]), there is a subsequence $u_{N_{j}}$ such that

$$\mathcal{P}_{\pm}(\gamma)u_{N_{j}}\longrightarrow 0\ \text{in}\ H_{s,\gamma,1}\quad\text{and}\quad u_{N_{j}}\longrightarrow u\ \text{in}\ H_{s,\gamma,1}\,.$$

Hence Proposition 3 implies that $u_{N_{j}}$ tends to u in $H_{s+pm,\gamma,1}$, which shows that $\|u\|_{s+pm,\gamma,1}=1$ and $\mathcal{P}_{\pm}(\gamma)u=0$. This is a contradiction to (P-2) by Lemma 4.

Therefore, by Lemma 5 and 6 we obtain the assertion for $\mathcal{P}$. The same argument for $\mathcal{P}^{*}$ completes the proof of Theorem 2.

Theorem 3. *Suppose that* (P-1) *is satisfied. Then for* $s\in\boldsymbol{R}$, *there are positive constants* R, C *and* C' *such that*

$$\|u\|_{s+pm,\zeta,|\lambda|}\leqq C\|\mathcal{P}(\zeta,\ \lambda)u\|_{s,\zeta,|\lambda|}$$

and

$$\|u\|_{s+pm,\zeta,|\lambda|}\leqq \dot{C}'\|\mathcal{P}^{*}(\zeta,\ \lambda)u\|_{s,\zeta,|\lambda|}$$

for any $u\in\mathcal{S}(\boldsymbol{R}^{n})$, $\zeta\in\Gamma\cap\{|\mathrm{Im}\,\zeta|>R\}$ *and* $|\lambda|\leqq 1$.

Proof. The same consideration as in the proof of Proposition 3 shows that the inequality

$$\|u\|_{s+pm,\gamma,|\lambda|}\leqq C\{\|\mathcal{P}(\zeta,\ \lambda)u\|_{s,\zeta,|\lambda|}+\|u\|_{s,\zeta,|\lambda|}\}$$

for any $u\in\mathcal{S}(\boldsymbol{R}^{n})$, $\gamma\in\Gamma$ holds if $|\lambda|\leqq 1$. Only change is to replace the cut-off function

$\Psi(\zeta, \bar{\xi})$ by $\phi(\zeta, \bar{\xi})$ in the argument in the proof of Proposition 2.

4. Proof of theorem 1

We introduce the 'Sobolev' norm attached to the Heisenberg group as follows: for positive integer N,

$$\|u\|_{(N)}=\sum_{|I|\leq N}\|X^I u\|_{L^2(H^n)},$$

where $|I|=\mu_{i_1}+\cdots+\mu_{i_N}$, $\mu_i=1$ if $i\neq 0$ and $\mu_i=2$ if $i=0$. Let denote the Laplace transform of P with respect to the time variable t by Q_ζ. We recall the Plancherel theorem on the Heisenberg group:

$$\int_{H^n}|f(x)|^2dx=\int_{R\setminus\{0\}}\mathrm{tr}\,(\pi_\lambda(f)\pi_\lambda(f)^*)|\lambda|^n d\mu$$

for $f\in L^1\cap L^2(\boldsymbol{H}^n)$, where $d\mu$ is Lebesgue measure on $\boldsymbol{R}$. Then from this, it follows that Theorem 2 and 3 imply that for some positive constants C and C',

$$|\zeta|^m\|u\|_{(0)}+\|u\|_{(N+mp)}\leqq C\|Q_\zeta u\|_{(N)}$$

$$|\zeta|^m\|u\|_{(0)}+\|u\|_{(N+mp)}\leqq C'\|Q_\zeta^* u\|_{(N)}$$

for $u\in H_{(\infty)}(\boldsymbol{H}^n)$. Let $\Gamma_R=\Gamma\cap\{\mathrm{Im}\,\zeta<-R\}$. Since from the above inequalities, if $\zeta\in\Gamma_R$, Q_ζ is the continuous, one to one and onto operator from $H_{(\infty)}$ to $H_{(\infty)}$, there exists the inverse operator Q_ζ^{-1} of Q_ζ, which depends holomorphically on the dual variable ζ of t in the interior of Γ and for any positive integer N there are an positive integer L and positive constant C such that

$$\|Q_\zeta^{-1}u\|_{(N)}\leqq C\|u\|_{(L)}$$

for any $u(x)\in H_{(\infty)}(\boldsymbol{H}^n)$ and any $\zeta\in\Gamma_R$. Here, we have used the following lemma:

Lemma 7. *Let X and Y be Banach spaces. Suppose that A is a bounded linear operator from X to Y with the bounded inverse A^{-1}. Then if the linear bounded operator B from X to Y satisfies*

$$\|A-B\|<\|A\|^{-1},$$

then B has also bounded inverse B^{-1}, which satisfies

$$B^{-1}=A^{-1}\sum_0^\infty\{(A-B)A^{-1}\}^k.$$

Let $w\in C^\infty(\boldsymbol{R}, H_{(\infty)}(\boldsymbol{H}^n))$ such that

$$\partial_t^j w|_{t=0}=g_j, \qquad j=0,\cdots,m-1$$

and for $f_0=Pw-f$,

$$\partial_t^k f_0|_{t=0}=0, \qquad k=0,\cdots,l.$$

Put $v=u-w$. Then, to solve (2.2), it suffices to consider the Cauchy problem

$$(4.2)\qquad \begin{cases} Pv=\tilde{f}_0 \\ \partial_t^j v|_{t=0}=0 \\ j=0, \cdots, m-1, \end{cases}$$

where $\tilde{f}_0=\chi(t)f_0(t)$ if $t\geqq 0$ and $=0$ if $t<0$. Here, χ is a real-valued function in $C^\infty(\boldsymbol{R})$ such that

$$\chi(t)=0 \quad \text{if } t>2T/3 \text{ and } =1 \text{ if } t<T/3.$$

Let us denote the Laplace transform of $\tilde{f}_0$ by

$$\mathcal{L}g(\zeta, x)=\int_{\boldsymbol{R}} e^{-it\zeta}g(t, x)dt\,.$$

Since $\tilde{f}_0$ is in $C^l((-T, T), H_\infty(\boldsymbol{H}^n))$ with support contained in $[0, 2T/3]$, the solution u of the Cauchy problem (4.2) is given by

$$u(t, x)=\frac{1}{2\pi}\int_{\mathrm{Im}\,\zeta=-R} e^{i\zeta\cdot t}Q_\zeta^{-1}\mathcal{L}\tilde{f}_0(\zeta, x)d\zeta\,,$$

which is in $C^{m+l-2}([-T, T], H_{(\infty)}(\boldsymbol{H}^n))$ if R is chosen large enough. (cf. [C] or [Fa]) This finished the proof.

5. Remarks

In this section, we shall discuss on the condition (P-2) of Theorem 1. This condition is not necessary for the Cauchy problem for P to be well-posed, but we have

Theorem 4. *Suppose that the Cauchy problem* (2.2) *is uniquely solvable in the space* $C^\infty([0, T), H_{(\infty)}(\boldsymbol{H}^n))$. *Then for any* γ *in the interior of* Γ,

$$\{u\in\mathcal{S}(\boldsymbol{R}^n)\,;\, \mathcal{P}_\pm(\gamma)u=0\}=\{0\}\,.$$

Proof. We may only consider the '+' case. Suppose that for some $\zeta_0\in\mathrm{Int}(\Gamma)$, there is a function $v\neq 0$ satisfying $\mathcal{P}_+^*(\zeta_0)v=0$. We define a smooth function u by the following way. For $\phi\in C_0^\infty(\boldsymbol{H}^n)$,

$$\langle u_\lambda(t, \cdot), \phi\rangle=(\pi_\lambda(\phi)v, v)_H e^{it\zeta_0\lambda^{p/2}}\,,$$

which satisfies the equation $Pu=0$ because if X is a right invariant vector field on $\boldsymbol{H}^n$,

$$\pi_\lambda(X^t\phi)v=\pi_\lambda(X)\pi_\lambda(\phi)v\,.$$

It is easily seen that this solution violate the inequality: for any positive integer N there is a constant C such that

$$\max_{0\leq t\leq T}\|u(t, \cdot)\|_{L^2}\leqq C\sum_{j=0}^{m-1}\sum_{\langle I\rangle=N}\|\partial_t^j X^I u(0, \cdot)\|\,.$$

This is a contradiction to the well-posedness of the Cauchy problem (2.2).

To clarify the meaning of the condition (P-2), we give some examples.

Example 1 ($m=1$, $n=1$).

$$P=\partial_t-a(X_1^2+Y_1^2)+cZ\ ,$$

where a is a real positive number c is a complex constant. Then the necessary and sufficient condition for the Cauchy problem (2.2) to be well-posed is

$$|\mathrm{Im}\ c|\leqq a\ .$$

Example 2 ($m=1$, $n=1$).

$$P=\{\partial_t-a(X_1^2+Y_1^2)\}\{\partial_t-b(X_1^2+Y_1^2)\}+cZ^2\ ,$$

where a and b are positive constants and c is a complex constant. Then the necessary and sufficient condition for the Cauchy problem (2.2) to be well-posed is

$$|\mathrm{Im}\ c|\leqq(a+b)(ab-\mathrm{Re}\ c)^{1/2}\ .$$

These are proved by the similar argument in [O]. In fact, as for the latter case, it is seen that when $\mathrm{Im}\,\zeta<0$,

$$(i\zeta+a)(i\zeta+b)-c=0$$

is equivalent to

$$\{2\,\mathrm{Im}\,\zeta-(a+b)\}^2\{(\mathrm{Im}\,\zeta-a)(\mathrm{Im}\,\zeta-b)-\mathrm{Re}\ c\}-(\mathrm{Im}\ c)^2=0\ .$$

In these examples, the condition (P-2) is equivalent to
(The case Example 1)

$$|\mathrm{Im}\ c|<a\ ,$$

and
(The case Example 2)

$$|\mathrm{Im}\ c|<(a+b)(ab-\mathrm{Re}\ c)^{1/2}\ ,$$

respectively.

Finally, we mention some related works. On the Euclidean space, I.G. Petrowsky [P] considered the p-parabolic operators with variable coefficients (c.f. [Mz]) and for the operator of Example 1, a related result was given by K. Igari (Example 2 of [I]).

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