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Fundamental solution of the Cauchy problem for a Schrödinger pseudo-differential operator(*)

By

Daniela MARI

In this paper we shall construct a fundamental solution of the Cauchy problem for pseudo-differential operator of Schrödinger type by means of Fourier integral operators of infinite order.

In [4] D. Fujiwara considers the Schrödinger operator

$$\hbar D_t-\hbar^2\Delta_x+V(t, x)$$

($\hbar$ is a small positive parameter), and obtains a fundamental solution of the Cauchy problem when the real potential $V(t, x)$ satisfies suitable conditions. At first he constructs a sequence of approximate fundamental solutions in the frame of L^2-theory of oscillatory integral trasformations; afterwards studies the convergence of iterated integrals of Feynman type constructed by using those solutions.

In [8] H. Kitada extends the result to a pseudo-differential operator with symbol $h(t, x, \xi)$ satisfying on $[0, T]\times\boldsymbol{R}^{2n}$

$$|\partial_\xi^\alpha D_x^\beta h(t, x, \xi)|\leqq\begin{cases} c_{\alpha,\beta}(1+|x|+|\xi|)^{2-|\alpha+\beta|} & (|\alpha+\beta|\leqq 1)\\ c_{\alpha,\beta} & (|\alpha+\beta|\geqq 2).\end{cases}$$

Also Kitada uses approximate fundamental solutions but he finds them as Fourier integral operators.

A further paper on this subject from a more general point of view is due to H. Kitada-H. Kumano-go [9].

Also the question of the well-posedness of the Cauchy problem has been studied for differential operators of Schrödinger type and even for more general non Kowalevskian differential operators. In fact J. Takeuchi ([13]-[14]), S. Mizohata ([11]-[12]), W. Ichinose ([5]-[6]), give sufficient conditions and necessary conditions to the well-posedness in the frame of L^2 and H^∞ spaces.

In this paper we construct a fundamental solution of the Cauchy problem for the pseudo-differential operator of Schrödinger type

$$P(t, x, D_t, D_x)=D_t+\Delta_x+A(t, x, D_x) \tag{0.1}$$

where $t\in[0, T]$, $x\in\boldsymbol{R}^n$, $\Delta_x=\sum_{j=1}^n\partial_{x_j}^2$

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and $A(t, x, D_x)$ is a complex valued pseudo-differential operator of order $p\in[0, 1[$ with symbol $a(t, x, \xi)\in C([0, T], C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^n))$ satisfying for some $A>0$:

$$(0.2)\qquad \sup_{\substack{x\in\boldsymbol{R}^n\\ t\in[0,T]}}|\partial_\xi^\alpha D_x^\beta a(t, x, \xi)|<A^{|\alpha+\beta|+1}(\alpha+\beta)!\,\langle\xi\rangle^{p-|\alpha+\beta|}\forall\alpha,\ \beta\in\boldsymbol{Z}_+^n,\ \forall\xi\in\boldsymbol{R}^n\quad |\xi|\gg1.$$

The result is obtained by using a parametrix that is constructed as a Fourier integral operator of infinite order on $\mathcal{D}_{L^2}^{\{\sigma\}}(\mathcal{D}_{L^2}^{\{\sigma\}'})$, $\sigma\in[1, 1/p[$, with phase function $\phi(t, x, \xi)=x\cdot\xi+|\xi|^2t$.

The Fourier integral operators of infinite order have been studied by L. Cattabriga-L. Zanghirati [3] on Gevry classes, whereas the spaces $\mathcal{D}_{L^2}^{\{\sigma\}}$ and $\mathcal{D}_{L^2}^{\{\sigma\}'}$ have been considered for example by K. Taniguchi in [15] where he considers pseudo-differential and Fourier integral operators of finite order. Recently R. Agliardi [1] has studied Fourier integral operators with phase of order 1 and amplitude of infinite order on $\mathcal{D}_{L^2}^{\{\sigma\}}$ and its dual.

This paper is organized as follows. The first section contains the main definitions and notation; in section 2 is produced the calculus for Fourier integral operators of infinite order on $\mathcal{D}_{L^2}^{\{\sigma\}}(\mathcal{D}_{L^2}^{\{\sigma\}'})$. Indeed in section 3 the parametrix of the Cauchy problem for the operator (0.1) is constructed by solving transport equations; then the fundamental solution is determined.

The author is studying the possibility to extend the result here obtained to a more general operator than (0.1).

1. Main notation and definitions

For $x, \xi\in\boldsymbol{R}^n$ we set $\langle x\rangle^2=1+|x|^2=1+\sum_{j=1}^n x_j^2$, $x\cdot\xi=\sum_{j=1}^n x_j\xi_j$. We write $D_t=-i\partial_t$, $D_x=(D_1, \cdots, D_n)$ $D_j=-i\partial_{x_j}$ and $D_x^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$, $\alpha\in\boldsymbol{Z}_+^n$.

We recall the definitions of the function spaces which later we shall use mainly (for more details see [15] and [7]). We omit the domain because $\boldsymbol{R}^n$ is meant.

For $\sigma\gg1$ we denote by $\mathcal{E}_b^{\{\sigma\}}$ the space of all $f\in\mathcal{E}^\infty$ such that for some $A>0$

$$\sup_{x\in\boldsymbol{R}^n}|D^\alpha f(x)|<A^{|\alpha|+1}\alpha!^\sigma\qquad\forall\alpha\in\boldsymbol{Z}_+^n.$$

For $\varepsilon>0$, $\sigma\geqq1$ we define

$$\mathcal{S}_{\sigma,\varepsilon}=\{u\in\mathcal{S}\,;\ \exp(\varepsilon\langle\xi\rangle^{1/\sigma})\tilde{u}(\xi)\in\mathcal{S}\}$$

where $\tilde{u}$ is the Fourier transform of u.

$\mathcal{S}_{\sigma,\varepsilon}$ is a Frechet space with the seminorms

$$|u|_{\mathcal{S}_{\sigma,\varepsilon};k}=\sup_{\xi\in\boldsymbol{R}^n}\sup_{|\alpha|+p=k}|\langle\xi\rangle^p\partial_\xi^\alpha(\exp(\varepsilon\langle\xi\rangle^{1/\sigma})\tilde{u}(\xi)|\qquad k=0, 1, \cdots$$

Furthermore we consider

$$\mathcal{D}_{L^2,\varepsilon}^{\{\sigma\}}=\{f\in L^2\,;\ \exp(\varepsilon\langle\xi\rangle^{1/\sigma})\tilde{f}(\xi)\in L^2\}$$

which is an Hilbert space with the norm

$$\|f\|_{\mathcal{D}^{\{\sigma\}'}_{L^2,\varepsilon}}=\|\exp(\varepsilon\langle\xi\rangle^{1/\sigma})\tilde{f}(\xi)\|_{L^2}.$$

If $\mathcal{D}^{\{\sigma\}'}_{L^2,\varepsilon}$ is the dual space of $\mathcal{D}^{\{\sigma\}}_{L^2,\varepsilon}$, we can see that it is the space of the ultradistributions u such that $\exp(-\varepsilon\langle\xi\rangle^{1/\sigma})\tilde{u}(\xi)\in L^2$. So if we denote even for $\varepsilon<0$ with $\mathcal{D}^{\{\sigma\}}_{L^2,\varepsilon}$ the space of the ultradistributions u such that $\exp(\varepsilon\langle\xi\rangle^{1/\sigma})\tilde{u}(\xi)\in L^2$, we can write

$$\mathcal{D}^{\{\sigma\}'}_{L^2,\varepsilon}=\mathcal{D}^{\{\sigma\}}_{L^2,-\varepsilon} \qquad \text{for} \quad \varepsilon>0.$$

Finally we define

$$\mathcal{D}^{\{\sigma\}}_{L^2}=\varinjlim_{\varepsilon\to 0+}\mathcal{D}^{\{\sigma\}}_{L^2,\varepsilon} \qquad \mathcal{D}^{\{\sigma\}'}_{L^2}=\varprojlim_{\varepsilon\to 0+}\mathcal{D}^{\{\sigma\}'}_{L^2,\varepsilon}.$$

Now we recall the definitions of symbol of infinite order of Gevrey type and relative formal series (for details see [16]).

For $\sigma\geqq 1$ we denote by $S^{\infty,\sigma}$ the space of all functions $a(x,\xi)\in C^\infty$ satisfying there exists $A>0$ and $\forall h>0$ there exists $C_h>0$ such that

$$\sup_{x\in\boldsymbol{R}^n}|\partial_\xi^\alpha D_x^\beta a(x,\xi)|<C_hA^{|\alpha+\beta|}\alpha!\,\beta!^\sigma\langle\xi\rangle^{-|\alpha|}\exp(h\langle\xi\rangle^{1/\sigma}) \tag{1.1}$$
$$\forall\alpha,\ \beta\in\boldsymbol{Z}_+^n,\ \forall\xi\in\boldsymbol{R}^n \qquad |\xi|\gg 1.$$

We define $\sum_{j\geqq 0}a_j(x,\xi)$ formal series of symbols in $S^{\infty,\sigma}$ if $a_j\in C^\infty$ satisfy there exists $A>0$ and $\forall h>0$ there exists $C_h>0$ such that

$$\sup_{x\in\boldsymbol{R}^n}|\partial_\xi^\alpha D_x^\beta a_j(x,\xi)|<C_hA^{|\alpha+\beta|}\alpha!(\beta!\,j!)^\sigma\langle\xi\rangle^{-|\alpha|-j}\exp(h\langle\xi\rangle^{1/\sigma})$$
$$\forall\alpha,\ \beta\in\boldsymbol{Z}_+^n,\ \forall\xi\in\boldsymbol{R}^n \qquad |\xi|\gg 1.$$

Let $\sum_{j\geqq 0}a_j$ and $\sum_{j\geqq 0}b_j$ formal series of symbols; we say that $\sum a_j\sim\sum b_j$ if there exists $A>0$ and $\forall h>0$ there exists $C_h>0$ such that

$$\sup_{x\in\boldsymbol{R}^n}|\partial_\xi^\alpha D_x^\beta\sum_{j<s}(a_j(x,\xi)-b_j(x,\xi)|<C_hA^{|\alpha+\beta|+s}\alpha!(\beta!\,s!)^\sigma\langle\xi\rangle^{-|\alpha|-s}\exp(h\langle\xi\rangle^{1/\sigma})$$
$$\forall\alpha,\ \beta\in\boldsymbol{Z}_+^n, \qquad |\xi|\gg 1.$$

2. Calculus for certain Fourier integral operators of infinite order

We consider Fourier integral operators of infinite order on the spaces $\mathcal{D}^{\{\sigma\}}_{L^2}(\mathcal{D}^{\{\sigma\}'}_{L^2})$ with amplitude

$$p(t;x,\xi)\in C([0,T],C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^n)) \qquad \text{belonging to } S^{\infty,\sigma} \tag{2.1}$$

and phase

$$\begin{cases}\phi(t;x,\xi)=x\cdot\xi+|\xi|^2t\\ x,\ \xi\in\boldsymbol{R}^n \quad t\in[0,T_0] \quad 0<T_0<1/2.\end{cases} \tag{2.2}$$

For these p the integral $\int|p(t;x,\xi)\tilde{u}(\xi)|d\xi$ is convergent for any $t\in[0,T_0]$ both if $u\in\mathcal{S}_{\sigma,\varepsilon}$ and if $u\in\mathcal{D}^{\{\sigma\}}_{L^2}$. Thus we can define $\forall t\in[0,T_0]$ the operator

$$(P(t)u)(x)=\int\exp(ix\cdot\xi+|\xi|^2t)p(t;x,\xi)\tilde{u}(\xi)d\xi \tag{2.3}$$

$d\xi=(2\pi)^{-n}d\xi$, both on $\mathcal{S}_{\sigma,\varepsilon}$ and on $\mathcal{D}^{\{\sigma\}}_{L_2}$.

We give now some preliminary lemmas and then we shall study the action of the operator (2.3) on the above mentioned spaces.

Lemma 2.1. *Let $a_j(x)\in C^\infty$, $j=0, \cdots, n$, and we assume that for some C_0, $A_0\geqq 0$*

$$\sup_{x\in \boldsymbol{R}^n} |D_x^\alpha a_0(x)|\leqq nC_0A_0^{|\alpha|+1}(|\alpha|+1)!^\sigma \quad \forall\alpha\in \boldsymbol{Z}_+^n$$

$$\sup_{x\in \boldsymbol{R}^n} |D_x^\alpha a_j(x)|\leqq C_0A_0^{|\alpha|}|\alpha|!^\sigma \qquad j=1, \cdots, n .$$

Set $L=\sum_{j=1}^n a_j(x)\partial_{x_j}+a_0(x)$. Then $\forall u\in C^\infty$ such that

$$\sup_{x\in \boldsymbol{R}^n} |D_x^\alpha u(x)|\leqq CA^{|\alpha|}|\alpha|!^\sigma \qquad \forall\alpha\in \boldsymbol{Z}_+^n,$$

we have $\forall l\in \boldsymbol{N}$, $\forall\alpha\in \boldsymbol{Z}_+^n$

$$\sup_{x\in \boldsymbol{R}^n} |D_x^\alpha L^l u(x)|\leqq C(2nC_0)^l\{(A+A_0)^2/A\}^l(A+A_0)^{|\alpha|}(|\alpha|+l)!^\sigma$$

Proof. By induction on l.

Lemma 2.2. *Let $\phi(t, x, \xi)$ be as (2.2) and $\tau\in]2T_0, 1[$. We consider the operator*

$$L=L(t, x, \xi, \partial_\xi)=\sum_{j=1}^n \partial_{\xi_j}\Big(\frac{\partial_{\xi_j}\phi}{|\nabla_\xi\phi|^2}\Big)$$

for any $x, \xi\in \boldsymbol{R}^n$ with $|x|\geqq\tau\langle\xi\rangle$ and $t\in[0, T_0]$. Then $\forall u\in C^\infty$ such that

$$\sup_{\xi\in \boldsymbol{R}^n} |D_\xi^\alpha u(\xi)|<CA^{|\alpha|}\alpha!^\sigma \qquad \forall\alpha\in \boldsymbol{Z}_+^n$$

we have $\forall l\in \boldsymbol{N}$

$$|L^l u(\xi)|<C\tilde{C}_{\tau,A}|x|^{-l}l!^\sigma \qquad \forall x, \xi\in \boldsymbol{R}^n \text{ with } |x|\geqq\tau\langle\xi\rangle \text{ and } t\in[0, T_0], \tag{2.4}$$

where

$$\tilde{C}_{\tau,A}=\frac{2n(1+\sqrt{3})\tau}{\tau-2T_0}\Big(A+\frac{4(1+\sqrt{3})}{\tau-2T_0}\Big)^2\frac{1}{A}$$

Proof. We notice that

$$|\nabla_\xi\phi|>\Big(1-\frac{2T_0}{\tau}\Big)|x| \qquad \forall x, \xi\in \boldsymbol{R}^n, |x|\geqq\tau\langle\xi\rangle, t\in[0, T_0].$$

If we denote

$$L=\sum_{j=1}^n \frac{\partial_{\xi_j}\phi}{|\nabla_\xi\phi|^2}\partial_{\xi_j}+\sum_{j=1}^n \partial_{\xi_j}\Big(\frac{\partial_{\xi_j}\phi}{|\nabla_\xi\phi|^2}\Big)=\sum_{j=1}^n a_j(t, x, \xi)\partial_{\xi_j}+a_0(t, x, \xi),$$

we prove that for the same x, ξ, t

$$|\partial_\xi^\alpha a_j(t, x, \xi)|\leqq 4^{|\alpha|}\Big(\frac{(1+\sqrt{3})\tau}{\tau-2T_0}\Big)^{|\alpha|+1}\frac{1}{|x|^{|\alpha|+1}}\alpha! \qquad \forall\alpha\in \boldsymbol{Z}_+^n.$$

Then by applying Lemma 2.1 we obtain (2.4).

Proposition 2.1. *If $u\in\mathcal{S}_{\sigma,\varepsilon}$ then Pu defined by (2.3) is in $\mathcal{S}$.*

Proof. By letting $v(y)=\mathcal{F}^{-1}(\exp(\varepsilon\langle\xi\rangle^{1/\sigma})\tilde{u}(\xi))(y)$ and by fixing $\tau\in]2T_0, 1[$ turns out

$$|\langle x\rangle^j\partial_\xi^\alpha(Pu)(x)|\leqq I_1+I_2$$

with

$$I_1=\Big|\Sigma\binom{\alpha}{\alpha'}\iint_{|x-y|\leq\tau\langle\xi\rangle}\langle x\rangle^j\langle D_y\rangle^{|\alpha|+n+1}v(y)\exp(i\phi(t, x, \xi)-iy\cdot\xi)$$

$$\cdot\partial_x^{\alpha-\alpha'}p(t, x, \xi)\max(-\varepsilon\langle\xi\rangle^{1/\sigma})\langle\xi\rangle^{-|\alpha|+|\alpha'|-n-1}dyd\xi\Big|$$

$$I_2=\Big|\sum_{\alpha'\leq\alpha}\binom{\alpha}{\alpha'}\iint_{|x-y|\geq\tau\langle\xi\rangle}\langle x\rangle^j\langle D_y\rangle^{|\alpha|+n+1}v(y)\exp(i\phi(t, x, \xi)-iy\cdot\xi)$$

$$\cdot L_\xi^l(\partial_\xi^{\alpha-\alpha'}p(t, x, \xi)\exp(-\varepsilon\langle\xi\rangle^{1/\sigma})\langle\xi\rangle^{-|\alpha|+|\alpha'|-n-1})dyd\xi\Big|$$

where

$$L_\xi=\sum_{j=1}^n\partial_{\xi_j}\Big(\frac{\partial_{\xi_j}\phi-y_j}{|\nabla_\xi\phi-y|^2}\Big).$$

It is easy to prove that

$$I_1\leqq C_{\alpha,j}|v|_{S_{|\alpha|+j+n+1}}$$

and since

$$|\partial_\xi^\beta(\partial_x^{\alpha-\alpha'}p(t, x, \xi)\exp(-\varepsilon\langle\xi\rangle^{1/\sigma})\langle\xi\rangle^{-n-1})|\leqq C_{\alpha,\beta}\beta!\langle\xi\rangle^{-n-1}$$

in view of Lemma 2.2, by choosing $l=j+n+1$ and by using $\langle x\rangle^j\leqq\sqrt{2}^j\langle y\rangle^j\langle x-y\rangle^j$, we obtain again

$$I_2\leqq C'_{\alpha,j,\tau}|v|_{S_{|\alpha|+j+n+1}}.$$

Theorem 2.1. *If P is defined by* (2.3), *then* $\forall\varepsilon>0$ *there exists* $\delta_\varepsilon>0$ *such that* $\forall\delta\in]0, \delta_\varepsilon]$, *$P$ is a continuous map from $\mathcal{S}_{\sigma,\varepsilon}$ to $\mathcal{S}_{\sigma,\delta}$ and from $\mathcal{D}_{L^2,\varepsilon}^{\{\sigma\}}$ to $\mathcal{D}_{L^2,\delta}^{\{\sigma\}}$.*

Proof. Since $u\in\mathcal{S}_{\sigma,\varepsilon}$ implies $Pu\in\mathcal{S}$,

$$|\partial_\xi^{\alpha'}\widetilde{Pu}(\xi)|=\Big|\iint\exp(ix\cdot(\eta-\xi)+i|\eta|^2t)(-ix)^{\alpha'}p(t, x, \eta)\tilde{u}(\eta)d\eta dx\Big|$$

$$\leqq\iint\langle\xi-\eta\rangle^{-k-j}|\langle D_x\rangle^{k+j}[\langle x\rangle^{-|\alpha|-n-1}(-ix)^{\alpha'}$$

$$\cdot\langle D_\eta\rangle^{|\alpha|+n+1}(p(t, x, \eta)\tilde{u}(\eta)\exp(i|\eta|^2t))]|d\eta dx$$

with arbitrary $k>0$. Hence, if we notice that

$$|\partial_\eta^\gamma\exp(i|\eta|^2t)|\leqq C_{h,\sigma}^{|\gamma|}\gamma!^\sigma\exp(h\langle\eta\rangle^{1/\sigma})\qquad\forall h>0$$

and take $\varepsilon'>0$ arbitrarly, we have

$$|\langle\xi\rangle^j\partial_\xi^\alpha(\exp(\delta\langle\xi\rangle^{1/\sigma})\widetilde{Pu}(\xi))|\leqq C_{j,\alpha,\varepsilon',\varepsilon,h,\sigma}^*|u|_{S_{\sigma,\varepsilon};|\alpha|+2n+2+j}$$

$$\cdot\exp(\delta\langle\xi\rangle^{1/\sigma})\int\langle\eta\rangle^{-n-1}\exp\Big(-\frac{\sigma}{e}\langle\xi-\eta\rangle^{1/\sigma}\Big)\exp((\varepsilon'-\varepsilon+h)\langle\eta\rangle^{1/\sigma})d\eta.$$

By choosing h, δ such that $\max(0, \varepsilon/2, -\sigma/e)<h<\varepsilon/4$, $0<\delta<\varepsilon/2-h$ and $\varepsilon'\leqq\varepsilon/2$ we get

$$|Pu|_{S_{\sigma,\varepsilon};j+|\alpha|}\leqq C|u|_{S_{\sigma,\varepsilon};j+|\alpha|+2n+2}\qquad C=C(j, \alpha, \sigma, \varepsilon).$$

The first part of the Theorem is so proved. Now we show that $\forall u\in\mathcal{S}_{\sigma,\varepsilon}$ there exists $\delta'_{\varepsilon}>0$ such that $\forall\delta>0$, $\delta<\delta'_{\varepsilon}$ turns out

$$\|Pu\|_{\mathcal{D}^{\{\sigma\}}_{L^2,\delta}}\leqq M_{\varepsilon,\delta}\|u\|_{\mathcal{D}^{\{\sigma\}}_{L^2,\varepsilon}}. \tag{2.5}$$

Then, since $\mathcal{S}_{\sigma,\varepsilon}$ is dense in $\mathcal{D}^{\{\sigma\}}_{L^2,\delta}$, this inequality is true for any $u\in\mathcal{D}^{\{\sigma\}}_{L^2,\varepsilon}$. Similarly to [15], we set

$$q_j(t,x,\xi)=Os-\iint\exp(i(x-y)\cdot(\eta-\xi)+i|\xi|^2t)\exp(\delta\langle\eta\rangle^{1/\sigma})\phi_j(\eta)p(t,y,\xi)\exp(-\varepsilon\langle\xi\rangle^{1/\sigma})dyd\eta$$

where

$$\begin{aligned}&\phi_j(\eta)=\phi(\eta/j),\quad j=0,1,\cdots,\quad\text{and}\quad\phi\in C_0^\infty\\&\phi(\eta)=1\quad\text{for } |\eta|\leqq 1/2,\quad \phi(\eta)=0\quad\text{for } |\eta|\geqq 1.\end{aligned}\tag{2.6}$$

If we prove that for suitable δ $\{q_j\}_j$ is a bounded set in $S^0_{0,0}$ $\forall t\in[0,T_0]$ (as defined by [10]) and denote by $q(t,x,\xi)$ the limit of q_j, in view of Theorem 7.3 ch. 3 [10]

$$q_j(t,x,D_x)v\xrightarrow{\text{in } L^2}q(t,x,D_x)v\qquad\forall v\in L^2\ \forall t\in[0,T_0]$$

and therefore $\|q(t,x,D_x)v\|_{L^2}\leqq M\|v\|_{L^2}$ for the same v, t (Calderon-Vaillancourt theorem). Now, if we denote with $\tilde v(\xi)=\exp(\varepsilon\langle\xi\rangle^{1/\sigma})\tilde u(\xi)$ for any $u\in\mathcal{S}_{\sigma,\varepsilon}$, we have $q(t,x,\xi)v(x)=\mathcal{F}^{-1}(\exp(\delta\langle\xi\rangle^{1/\sigma})\widetilde{Pu}(\xi))(x)$. Therefore turns out (2.5). So it remains to prove the boundedness of $\{q_j\}$ in $S^0_{0,0}$. This is obtained by a slight modification of the proof of the Theorem 1.1 of [1] by choosing $\delta<\min(\varepsilon/2, C_{\sigma,A})$, $C_{\sigma,A}=(\sigma/e)(A/2n)^{1/\sigma}$ where A is the constant as in (1.1).

Corollary 2.1. *The operator P defined by* (2.3) *is a continuous map from $\mathcal{D}^{\{\sigma\}}_{L^2}$ to $\mathcal{D}^{\{\sigma\}}_{L^2}$ that can be extended to a codtinuous operator from $\mathcal{D}^{\{\sigma\}'}_{L^2}$ to $\mathcal{D}^{\{\sigma\}'}_{L^2}$.*

Theorem 2.2. *Let $r(t,x,\xi)\in C([0,T_0], C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^n))$ with the property $\exists B, C>0$, $A, h>0$ and $\forall\alpha\in\boldsymbol{Z}^n_+$ $\exists C_\alpha>0$ such that*

$$\sup_{x\in\boldsymbol{R}^n}|\partial_\xi^\alpha D_x^\beta r(t,x,\xi)|C_\alpha A^{|\beta|}\beta!^\sigma\exp(-h\langle\xi\rangle^{1/\sigma})\forall\xi\in\boldsymbol{R}^n,\ |\xi|\geqq B,\qquad t\in[0,T_0].$$

If ϕ satisfies (2.2), *then the operator*

$$(R(t)u)(x)=\int\exp(i\phi(t,x,\xi))r(t,x,\xi)\tilde u(\xi)d\xi\qquad u\in\mathcal{D}^{\{\sigma\}}_{L^2},\ t\in[0,T_0]$$

extends to a continuous map from $\mathcal{D}^{\{\sigma\}'}_{L^2}$ to $\mathcal{D}^{\{\sigma\}}_{L^2}$.

Proof. If $0<\varepsilon<h$ let u satisfying $\exp(-\varepsilon\langle\xi\rangle^{1/\sigma})\tilde u(\xi)=\tilde v(\xi)\in\mathcal{S}$. By denoting with

$$b_j(t,x,\xi)=Os-\iint\exp(i(x-y)\circ(\eta-\xi)+i|\xi|^2t)\exp(\delta\langle\eta\rangle^{1/\sigma})\phi_j(\eta)r(t,y,\xi)\exp(\varepsilon\langle\xi\rangle^{1/\sigma})dyd\eta$$

where $\phi_j(\eta)$ are the functions of (2.6), in a similar way as Theorem 2.1 we prove that b_j is a bounded subset of $S^0_{0,0}$ so that if b is the limit of b_j

$$\|b(t,x,D_x)v\|_{L^2}\leqq M\|v\|_{L^2}\qquad\forall v\in L^2,\ t\in[0,T_0].$$

Now we notice that $\mathcal{F}^{-1}(\exp(\delta\langle\xi\rangle^{1/\sigma})\widetilde{Ru}(\xi))(x)=b(t,x,D_x)v(x)$ so that turns out

$\|Ru\|_{\mathcal{D}^{\{\sigma\}}_{L^2,\delta}} \leqq M\|u\|_{\mathcal{D}^{\{\sigma\}}_{L^2,-\varepsilon}} \quad \forall\delta<\delta_\varepsilon$.

Theorem 2.3. *Let*

$$(P_1(t)u)(x)=\int \exp(ix\cdot\xi)p_1(t, x, \xi)\tilde{u}(\xi)d\xi$$

$$(P_2(t)u)(x)=\int \exp(ix\cdot\xi+i|\xi|^2t)p_2(t, x, \xi)\tilde{u}(\xi)d\xi$$

where $u\in\mathcal{D}^{\{\sigma\}}_{L^2}$, $t\in[0, T]$, $0<T_0<1/2$ *and*

$$p_1(t, x, \xi),\ p_2(t, x, \xi)\in C([0, T_0], \mathcal{S}^{\infty,\sigma}(\boldsymbol{R}^n\times\boldsymbol{R}^n)).$$

Then $\forall t\in[0, T_0]$ *there exists an operator* $Q(t)$ *defined on* $\mathcal{D}^{\{\sigma\}}_{L^2}$ *by*

$$(Q(t)u)(x)=\int \exp(ix\cdot\xi+i|\xi|^2t)q(t, x, \xi)\tilde{u}(\xi)d\xi$$

such that

$$P_1(P_2u)=Qu+Ru$$

where R *is a continuous map from* $\mathcal{D}^{\{\gamma\}'}_{L^2}$ *to* $\mathcal{D}^{\{\sigma\}}_{L^2}$ *and* $q(t, x, \xi)\sim\sum_{j\geq 0} q_j(t, x, \xi)$ *uniformly with respect to* t,

$$q_j(t, x, \xi)=\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_\xi^\alpha p_1(t, x, \xi)D_x^\alpha p_2(t, x, \xi).$$

Proof. The theorem is proved by arguing in a similar way as the analogous theorem in [1].

We shall need to consider formal series of symbols of the following type:
Let $p_j(x, \xi)\in C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ such that

(2.7) there exists $A>0$ and $\forall\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$\sup_{x\in\boldsymbol{R}^n}|\partial_\xi^\alpha D_x^\beta p_j(x, \xi)|\leqq CA^{|\alpha+\beta|+j}(\alpha+\beta+j)!\langle\xi\rangle^{-|\alpha+\beta|-j}\exp(\varepsilon\langle\xi\rangle^{1/\sigma})$$

$$\forall\alpha,\ \beta\in\boldsymbol{Z}_+^n\ \forall\xi\in\boldsymbol{R}^n \qquad |\xi|\gg 1.$$

Definition 2.1. Let $\sum_{j\geq 0} p_j$ and $\sum_{j\geq 0} q_j$ formal series of symbols of previous type. We say that $\sum p_j\sim\sum q_j$ if there exists $A>0$ and $\forall\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$\sup_{x\in\boldsymbol{R}^n}|\partial_\xi^\alpha D_x^\beta\sum_{j<s}(p_j(x, \xi)-q_j(x, \xi))|\leqq C_\varepsilon A^{|\alpha+\beta|+s}(\alpha+\beta+s)!\langle\xi\rangle^{-|\alpha+\beta|-s}\exp(\varepsilon\langle\xi\rangle^{1/\sigma})$$

$$\cdot\forall\alpha,\ \beta\in\boldsymbol{Z}_+^n\ \forall\xi\in\boldsymbol{R}^n \qquad |\xi|\gg 1.$$

As in [16] we can prove that

Proposition 2.2. *For every* $\sum_{j\geq 0} p_j$, p_j *satisfying* (2.7), *there exists* $p\in C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ *such that*

(2.8)
$$\sup_{x\in\boldsymbol{R}^n}|\partial_\xi^\alpha D_x^\beta p(x, \xi)|\leqq C_\varepsilon A^{|\alpha+\beta|}(\alpha+\beta)!\langle\xi\rangle^{-|\alpha+\beta|}\exp(\varepsilon\langle\xi\rangle^{1/\sigma})$$

$$\forall\alpha,\ \beta\in\boldsymbol{Z}_+^n\ \forall\xi\in\boldsymbol{R}^n \qquad |\xi|\ll 1$$

and p *is equivalent to* $\sum p_j$ *as in definition* 2.1.

3. Main results

In this section we consider the operator of Schrödinger type:

$$P(t, x, D_t, D_x)=D_t+\Delta_x+A(t, x, D_x)$$

denoted with (0.1) in Introduction, where A is a pseudodifferential operator with symbol satisfying (0.2). A simple example of function of this type is obviously $a(t, \xi)=\varphi(t)\langle\xi\rangle^p$ where φ is a continuous function in $[0, T]$ but $a(t, x, \xi)=\varphi(t)\langle\xi\rangle^p \exp(-|x|^2/|\xi|^2)$ also satisfies (0.2).

We prove

Theorem 3.1. *Let P the operator* (0.1) *satisfying* (0.2). *Then* $\forall s, t\in[0, T_0]$ $0<T_0<1/2$, *there exists* $e(t, s, x, \xi)\in C^S([0, T_0]^2; C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^n))$ *such that for any* $\varepsilon>0$

$$\sup_{x\in\boldsymbol{R}^n}|\partial_\xi^\alpha D_x^\beta e(t, s, x, \xi)|\leqq C_{\varepsilon,\sigma}\tilde{A}^{|\alpha+\beta|}(\alpha+\beta)!\langle\xi\rangle^{-|\alpha+\beta|}\exp(\varepsilon\langle\xi\rangle^{1/\sigma})$$

for some $C, \tilde{A}>0$, $\forall t, s\in[0, T_0]$, $\forall\alpha, \beta\in\boldsymbol{Z}_+^n$, $\forall\xi\in\boldsymbol{R}^n$ $|\xi|\gg1$, $\sigma\in[1, 1/p[$. *Furthermore the operator defined by*

$$(Eu)(x)=\int\exp(i|\xi|^2(t-s)+ix\cdot\xi)e(t, s, x, \xi)\tilde{u}(\xi)d\xi \qquad \forall u\in\mathcal{D}_{L^2}^{\{\sigma\}}$$

satisfies

$$\begin{cases} PE(t, s)=R(t, s) \\ E(s, s)=I \end{cases}$$

where R is a continuous map from $\mathcal{D}_{L^2}^{\{\sigma\}\prime}$ *to* $\mathcal{D}_{L^2}^{\{\sigma\}}$ $\forall s, t\in[0, T_0]$ *and I is the identity operator.*

Proof. We construct a sequence of functions $\{e_h(t, s, x, \xi)\}_{h\geqq0}$ such that $\forall h\geqq0$ $e_{2h+1}(t, s, x, \xi)=0$ and $e_{2h}(t, s, x, \xi)$ are solutions of the following Cauchy problems:

(C_0):

$$\begin{cases} \partial_t e_0(t, s, x, \xi)-2(\xi\cdot\nabla_x e_0)(t, s, x, \xi)+ia(t, x, \xi)e_0(t, s, x, \xi)=0 \\ e_0(s, s)=1 \end{cases}$$

(C_h):

$$\begin{cases} \partial_t e_{2h}(t, s, x, \xi)-2(\xi\cdot\nabla_x e_{2h})(t, s, x, \xi)+ia(t, x, \xi)e_{2h}(t, s, x, \xi)=-id_h(t, s, x, \xi) \\ e_{2h}(s, s)=0 \end{cases}$$

with

$$d_h(t, s, x, \xi)=\Delta_x e_{2h-2}(t, s, x, \xi)+\sum_{k=1}^{h}\sum_{|\gamma|=k}\frac{1}{\gamma!}\partial_\xi^\gamma a(t, x, \xi)D_x^\gamma e_{2h-2k}(t, s, x, \xi).$$

We shall prove that $\sum_{h\geq0}e_h(t, s, x, \xi)$ is a formal series as in definition 2.1 $\forall t, s\in[0, T_0]$, so by Proposition 2.2 we can conclude that there exists $e(t, s, x, \xi)\in C^1([0, T_0], C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^n))$ satisfying (2.8) such that

$$e(t, s, x, \xi) \sim \sum_{h \geqq 0} e_h(t, s, x, \xi).$$

Moreover for (C_0) (C_h)

$$e(s, s, x, \xi)=1$$

$$D_t e(t, s, x, \xi)+|\xi|^2 e(t, s, x, \xi)+q(t, s, x, \xi)=0$$

where $q(t, s, x, \xi) \sim \sum q_j(t, s, x, \xi)$ and

$$q_j(t, s, x, \xi)=\sum_{k=0}^{j} \sum_{|\gamma|=k} \frac{1}{\gamma !} \partial_\xi^\gamma(-|\xi|^2+a(t, x, \xi)) D_x^\gamma e_{j-k}(t, s, x, \xi).$$

By Theorem 2.3 we have the result. Now, by letting

$$v_{2h}(t, s, x, \xi)=e_{2h}(t, s, x-2\xi t, \xi) \qquad \forall h \geqq 0$$

we can prove inductively analogous estimates to the one proved in [2]

$$\sup_{x \in \boldsymbol{R}^n} |\partial_\xi^\alpha D_x^\beta v_{2h}(t, s, x, \xi)|$$

$$\leqq C_A^{|\alpha+\beta|+2h}(|\alpha+\beta|+2h)!\langle\xi\rangle^{-|\alpha+\beta|-2h} \exp(C_{T_0}\langle\xi\rangle^p|t-s|) \sum_{i=1}^{|\alpha+\beta|+\max(0,4h-1)} \langle\xi\rangle^{pi} \frac{|t-s|^i}{i!}$$

$$\forall |\xi| \geqq B>0 \qquad \forall t, s \in [0, T_0].$$

By using

$$D_\xi^\alpha D_x^\beta e_{2h}(t, s, x, \xi)=\sum_{\delta'+\delta''=\alpha} \binom{\alpha}{\delta'} D_x^{\beta+\delta'} D_\xi^{\delta''} v(t, s, x+2\xi t, \xi)(2t)^{|\delta'|}$$

we obtain

$$\sup_{x \in \boldsymbol{R}^n} |\partial_\xi^\alpha D_x^\beta e_{2h}(t, s, x, \xi)|$$

$$\leqq C_A^{*|\alpha+\beta|+2h}(|\alpha+\beta|+2h)!\langle\xi\rangle^{-|\alpha+\beta|-2h} \exp(C_{T_0}\langle\xi\rangle^p|t-s|) \sum_{i=1}^{|\alpha+\beta|+\max(0,4h-1)} \langle\xi\rangle^{pi} \frac{|t-s|^i}{i!}.$$

Since $\sigma \in [1, 1/p[$ from these estimates we can conclude that $\sum_{h \geqq 0} e_h(t, s)$ is a formal series of symbols such that for some $\tilde{C}_A$, $B_1>0$ and $\forall \varepsilon>0$

$$\sup_{x \in \boldsymbol{R}^n} |\partial_\xi^\alpha D_x^\beta e_h(t, s, x, \xi)| \leqq \tilde{C}_{A,\varepsilon,\sigma}^{|\alpha+\beta|+h}(|\alpha+\beta|+h)!\langle\xi\rangle^{-|\alpha+\beta|-h} \exp(\varepsilon\langle\xi\rangle^{1/\sigma})$$

$$\forall \alpha, \beta \in \boldsymbol{Z}_+^n \ \forall t, s \in [0, T_0], \forall \xi \in \boldsymbol{R}^n, \qquad |\xi|>B_1.$$

Hence we get the result.

Theorem 3.2. *Let $E(t, s)$ be as in Theorem 3.1. Then there exists a continuous map F from $C([0, T_0], \mathcal{D}_{L^2}^{\{\sigma\}'})$ to $C([0, T_0], \mathcal{D}_{L^2}^{\{\sigma\}})$ such that*

$$\tilde{E}(t, s)=E(t, s)+\int_s^t E(t, \tau) F(\tau, s) d\tau$$

is a fundamental solution for the Cauchy problem for the operator (3.1). *Consequently, if $g \in \mathcal{D}_{L^2}^{\{\sigma\}}$, $f \in C([0, T_0], \mathcal{D}_{L^2}^{\{\sigma\}})$ (resp. $g \in \mathcal{D}_{L^2}^{\{\sigma\}'}$ $f \in C([0, T^0], \mathcal{D}_{L^2}^{\{\sigma\}'})$) then*

$$u(t, \cdot)=\tilde{E}(t, s) g+\int_s^t(\tilde{E}(t, \tau) f(\tau, \cdot)) d\tau$$

is a solution of the Cauchy problem

$$\begin{cases} Pu=f \\ u(s, s)=g \end{cases}$$

and $u\in C^1([0, T_0], \mathscr{D}_{L^2}^{\{\sigma\}})$ (*resp.* $C^1([0, T_0], \mathscr{D}_{L^2}^{\{\sigma\}'})$).

Proof. We can regard the operator $R(t, s)$ as a pseudo-differential operator $\tilde{R}(t, s)$ with symbol $\tilde{r}(t, s, x, \xi)=\exp(i|\xi|^2(t-s))r(t, s, x, \xi)$ which has the same properties as the symbol $r(t, s, x, \xi)$ of $R(t, s)$. Then in a similar way as [2] we can prove that there exists $F(t, s)$ solution of

$$\tilde{R}(t, s)=-F(t, s)-\int_s^t \tilde{R}(t, \tau)F(\tau, s)d\tau$$

and F continuously maps $C([0, T_0], \mathscr{D}_{L^2}^{\{\sigma\}'})$ to $C([0, T_0], \mathscr{D}_{L^2}^{\{\sigma\}})$. By using Theorem 3.1 we have so

$$P\tilde{E}(t, s)=0.$$

DIPARTIMENTO DI MATEMATICA
DELL' UNIVERSITÀ DI FERRARA,
VIA MACHIAVELLI, 35
44100 FERRARA

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