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The Brown-Peterson cohomology of $BSO(6)$

By

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1. Introduction

The complex cobordism of the classifying space of the n-th orthogonal group was computed by W.S. Wilson [5], which is the simplest possible result that we can expect. We review it here:

$$\begin{aligned} MU^*BO(n) &\cong MU^*BU(n)/(c_1-c_1^*, \cdots, c_n-c_n^*) \\ &\cong MU^*[[c_1, \cdots, c_n]]/(c_1-c_1^*, \cdots, c_n-c_n^*), \end{aligned}$$

where c_k is the Conner-Floyd Chern class and c_k^* is its complex conjugate. In case of $BSU(n)$, an easy calculation leads to

$$\begin{aligned} MU^*BSU(n) &\cong MU^*BU(n)/(F_1) \\ &\cong MU^*[[c_2, \cdots, c_n]], \end{aligned}$$

where $F_1=\sum_{MU} t_i$ (cf. the Conner-Floyd first Chern class $c_1=\sum t_i$). Then the next problem is the case of $BSO(n)$. We must concentrate on a calculation of a 2-primary part of $MU^*BSO(n)$. For an odd prime p,

$$MU^*_{(p)}BSO(n)\cong MU^*_{(p)}BO(n)\cong MU^*_{(p)}[[c_2, c_4, \cdots, c_{2k}, \cdots : 2k\leqq n]].$$

We get the answer immediately when n is odd. Since $BO(n)\cong BZ/2\times BSO(n)$ in this case, the diagram

$$\begin{array}{ccc} MU^*BU(n) & \longrightarrow & MU^*BSU(n) \\ \downarrow & & \downarrow \\ MU^*BO(n) & \longrightarrow & MU^*BSO(n) \end{array}$$

becomes a pushout diagram. But we still don't have a simple description of $MU^*BSO(2n)$.

P. Landweber [2] studied the cobordism of the classifying spaces of abelian groups. His result claims that BP^*BA is even-concentrated, i.e., $BP^*BA\cong BP^{even}BA$, where A is any abelian group. N. Tezuka and N. Yagita [3] and A. Kono and N. Yagita [1] investigated more general cases. The authers of the former paper studied the ordinary cohomology of the classifying space of the finite p-group G with p^3 elements through the Brown-Peterson cohomolgy, and obtained the result that BP^*BG is also even-concentrated. The authers of the latter paper conjectured that this property holds

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for all compact Lie groups, and showed that their conjecture is true for $G=SO(4)$, G_2, F_4, E_6, $PU(3)$, $PSU(4k+3)$ and $Spin(n)(n\leqq 10)$.

The main purpose of this paper is to show that the conjecture is true for $SO(6)$.

Theorem 3.11. (1) *The Atiyah-Hirzeburch spectral sequence for $BP^*BSO(6)$ collapses at the E_{32}-term.*

(2) *$BP^*BSO(6)$ is concentrated in even degrees:*

$$BP^*BSO(6)\cong BP^*[[c_2, c_4, c_6]]\{1, \widetilde{4\chi}\}$$
$$\oplus BP^*[[c_2, \cdots c_6]]/(c_2-c_2^*, \cdots, c_6-c_6^*)\{c_3, c_5\},$$

where the isomorphism is just a module isomorphim.

Brown-Peterson cohomology of $SU(6)/SO(6)$ is studied in Section 2 in order to know a certain differential to be non-trivial in the Atiyah-Hirzeburch spectral sequence for $BSO(6)$. We carry out an actual computation of the spectral sequence in Section 3. Section 4 contains a table of $Q_i w_k$ which is used for computing differentials of spectral sequences.

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2. The Brown-Peterson cohomology of $SU(6)/SO(6)$

The ordinary cohomology of $SU(6)/SO(6)$ with the coefficient $Z/2$ is well-known:

$$H^*SU(6)/SO(6)\cong \Lambda(e_2, e_3, e_4, e_5, e_6), \tag{2.1}$$

where e_k is the image of the Stiefel-Whitney class w_k under the induced map of $SU(6)/SO(6)\to BSO(6)$. The action of the Steenrod operation is completely determined by Wu formula

$$Sq^i w_j=\sum_{k=0}^{i}\binom{j-k-1}{i-k}w_{i+j-k}w_k \qquad \text{for } i\leqq j, \tag{2.2}$$

so that we can compute the Bockstein spectral sequence in order to know the ordinary cohomology of $SU(6)/SO(6)$ with the coefficient $Z_{(2)}$.

degree	$H^*(SU(6)/SO(6); Z_{(2)})$		degree	$H^*(SU(6)/SO(6); Z_{(2)})$	
20	$Z_{(2)}\{e_{23456}\}$		10		$Z/2\{e_{235}\}$
18		$Z/2\{e_{3456}\}$	9	$Z_{(2)}\{e_{45+36}\}$	$Z/2\{e_{36}\}$
16		$Z/2\{e_{2356}\}$	8		$Z/2\{e_{35}\}$
15	$Z_{(2)}\{e_{456}\}$		7		$Z/2\{e_{34+25}\}$
14	$Z_{(2)}\{e_{2345+356}\}$	$Z/2\{e_{356}\}$	6	$Z_{(2)}\{e_6\}$	
13		$Z/2\{e_{346+256}\}$	5	$Z_{(2)}\{e_{23+5}\}$	$Z/2\{e_5\}$
12		$Z/2\{e_{345}\}$	3		$Z/2\{e_3\}$
11	$Z_{(2)}\{e_{236+56}\}$	$Z/2\{e_{56}\}$	0	$Z_{(2)}\{1\}$	

In the above table, e_{23+5} means $e_2e_3+e_5$ and so on.

Now we consider the Atiyah-Hirzeburch spectral sequence converging to $\boldsymbol{BP}^*\boldsymbol{SU}(6)/\boldsymbol{SO}(6)$:

(2.3) $$E_2 \cong BP^* \otimes H^*(SU(6)/SO(6); Z_{(2)}) \Longrightarrow BP^*SU(6)/SO(6).$$

Recall that $BP^*=Z_{(2)}[v_1, v_2, \cdots]$, where $|v_n|=-2(2^n-1)$. The differentials of this spectral sequece are quite complicated, yet the following lemma gives a guide line for computing.

Lemma 2.4 ([4]). *The first differential of the Atiyah-Hirzeburch spectral sequence converging to* $P(m)^*X$ *is* $d_{2pm-1}=v_m \otimes Q_m$, *where* Q_m *is the Milnor operation.*

$P(m)^*(\ )$ is a multiplicative cohomology theory whose coefficient ring is $P(m)^*=Z/2[v_m, v_{m+1}, \cdots]$. The Milnor operation mentioned above is a cohomology operation inductively defined as follows.

(2.5) $$Q_0=Sq^1, \qquad Q_n=[Q_{n-1}, Sq^{2^n}]=Q_{n-1}Sq^{2^n}+Sq^{2^n}Q_{n-1}.$$

Using this lemma and Wu formula, especially the action of Q_i $(0 \leqq i \leqq 4)$ on w_k $(2 \leqq k \leqq 6)$, we can determine the E_4-term of the spectral sequence (2.3). The formulae of Q_iw_k are arranged in Section 4.

degree	E_4-term		
20	$BP^*\{e_{23456}\}$		
18			$BP^*/(2, v_1)\{e_{3456}\}$
16			$BP^*/(2, v_1)\{e_{2356}\}$
15	$BP^*\{2e_{456}\}$		
14	$BP^*\{e_{2345+356}\}$		$BP^*/(2, v_1)\{e_{356}\}$
12		$BP^*/2\{e_{345}\}$	
11	$BP^*\{2e_{236+56}\}$	$BP^*/2\{e_{56}\}$	
10		$BP^*/2\{e_{235}\}$	
9	$BP^*\{e_{45+36}\}$		$BP^*/(2, v_1)\{e_{36}\}$
8			$BP^*/(2, v_1)\{e_{35}\}$
7		$BP^*2\{e_{34+26}\}$	
6	$BP^*\{2e_6\}$		
5	$BP^*\{e_{23+5}\}$		
3		$BP^*/2\{e_3\}$	
0	$BP^*\{1\}$		

We need more information to determine further differentials. For that, we use the Serre spectral sequences for the fibration $SU(6)/SO(6) \rightarrow BSO(6) \rightarrow BSU(6)$:

(2.6) $$E_2 \cong BP^*SU(6)/SO(6) \otimes H^*(BSU(6); Z_{(2)}) \Longrightarrow BP^*BSO(6)$$

and for the fibration $SU/SO \rightarrow BSO \rightarrow BSU$:

(2.7) $$E_2 \cong BP^*SU/SO \otimes H^*(BSU\,;\,Z_{(2)}) \Longrightarrow BP^*BSO\,.$$

There are some elements in these spectral sequences that should equal to zero in the E_∞-term, i.e., $c_k - c_k^* = 0$ in $BP^*BSO(6)$. c_k^* is a complex conjugate of c_k and $c_k = \sum t_1 \cdots t_k$, so we can describe c_k^* as follows.

Lemma 2.8. *These formulae hold in* $BP^*BSU(6)$ *(and slightly different for* BP^*BSU*):*

$$c_2^* - c_2 = -3v_1c_3 + 6v_1^2c_4 + 6v_1^3c_2c_3 - 15v_1^3c_5 + 5v_2c_2c_3 - 5v_2c_5 + \textit{higher terms}$$

$$c_3^* - c_3 = -2c_3 + \textit{higher terms}$$

$$c_4^* - c_4 = -5v_1c_5 + \textit{higher terms}$$

$$c_5^* - c_5 = -2c_5 + \textit{higher terms}$$

$$c_6^* - c_6 = c_6(4v_1^3c_3 + 3v_2c_3 - 8v_1^4c_4 - 6v_1v_2c_4 - 18v_1^5c_2c_3 + 26v_1^5c_5 - 24v_1^2v_2c_2c_3 + 30v_1^2v_2c_5 + \textit{higher terms}).$$

Hence the leading terms of the above formulae are boundary elements in the Serre spectral sequence (2.6).

Proof. Since complex conjugation is a ring homomorphism, we have

(2.9) $$c_k^* = \sum \iota(t_1) \cdots \iota(t_k)\,,$$

where

(2.10) $$\begin{aligned}\iota(t) &= \exp_{BP}(-\log_{BP}(t)) \\ &= -t - v_1t^2 - v_1^2t^3 - (2v_1^3 + v_2)t^4 - (4v_1^4 + 3v_1v_2)t^5 \\ &\quad -9(v_1^5 + v_1^2v_2)t^6 + \text{higher } t \text{ terms.}\end{aligned}$$

We have a commutative diagram:

(2.11) $$\begin{array}{ccc} BP^*BU(6) & \longrightarrow & BP^*BSU(6) \cong BP^*[[c_2, \cdots, c_6]]/(F_1) \\ \downarrow & & \downarrow \\ BP^*BO(6) & \longrightarrow & BP^*BSO(6)\,, \end{array}$$

where

(2.12) $$\begin{aligned} BP^*BO(6) &\cong BP^*[[c_1, \cdots, c_6]]/(c_1 - c_1^*, \cdots, c_6 - c_6^*) \\ &\cong BP^*[[F_1, c_2, \cdots, c_6]]/(F_1 - F_1^*, c_2 - c_2^*, \cdots, c_6 - c_6^*)\,. \end{aligned}$$

F_1 is the image of c_1 under the map of $B\det^*: BP^*BU(1) \to BP^*BU(6)$. Therefore we have

(2.13) $$\begin{aligned} F_1 &= \sum_{i=1}^{6}{}_{BP} t_i \\ &= -v_1c_2 - v_1^3c_4 - v_1^5c_2c_4 + v_2c_2^2 - 2v_2c_4 + v_1^2v_2c_2^3 - 2v_1^2v_2c_2c_4 + \cdots \\ &\quad + c_1(1 + v_1^2c_2 + v_1^3c_3 + v_1^4c_4 + v_1^5c_2c_3 + 2v_2c_3 - v_1v_2c_2^2 + 2v_1v_2c_4 + 2v_1^2v_2c_2c_3 + \cdots) \\ &\quad + c_1^2(-2v_1^3c_2 - 2v_2c_2 + \cdots) + \text{higher } c_1 \text{ terms.} \end{aligned}$$

Then the image of c_1 under the map $\boldsymbol{BP^*BU}(6)\to \boldsymbol{BP^*BSU}(6)$ is

$$(2.14)\qquad \begin{aligned} &v_1c_2-v_1^3c_2^2+v_1^3c_4-v_2c_2^2+2v_2c_4-v_1^4c_2c_3+2v_1^5c_2^3\\ &\quad -v_1^5c_2c_4-2v_1v_2c_2c_3+3v_1^2v_2c_2^3-2v_1^2v_2c_2c_4+\cdots. \end{aligned}$$

The pushout of the above diagram (2.11) is

$$(2.15)\qquad \begin{aligned} &\boldsymbol{BP}^*[[c_1, \cdots, c_6]]/(\boldsymbol{F}_1, c_2-c_2^*, \cdots, c_6-c_6^*)\\ &\quad \cong \boldsymbol{BP}^*[[c_2, \cdots, c_6]]/(c_2-c_2^*, \cdots, c_6-c_6^*). \end{aligned}$$

We can then prove this lemma through simple calculations. A leading term of any element x in the ideal $(c_2-c_2^*, \cdots, c_6-c_6^*)$ must become a boundary element because x should be trivial in $\boldsymbol{BP^*BSO}(6)$.

Lemma 2.8 claims that $2c_3$, $v_1\otimes c_3$, $2c_5$, $v_1\otimes c_5$, and $v_2\otimes c_3c_6$ in the Serre spectral sequence (2.6) are boundary elements. On the one hand, the spectral sequence (2.7) shows that there must be some $\boldsymbol{BP}^*$-generators in $\boldsymbol{BP^*SU}(6)/\boldsymbol{SO}(6)$ which project to the above elements because $\boldsymbol{BPQ^*SU/SO}\cong \Lambda_{\boldsymbol{BPQ}^*}(f_5, f_9, f_{13}, \cdots)$ and $\boldsymbol{BP^*SU/SO}$ must contains $2f_5$, v_1f_5, $2f_9$, v_1f_9, $2f_{13}$, v_1f_{13}, $\cdots$. Since their degrees are 3, 5, 7, 9, and 11, e_3, e_{23+5}, e_{34+25}, one of e_{45+36} and e_{36}, and one of e_{56} and $2e_{236+56}$ are infinite cycles in the spectral sequence (2.3). In fact, the remaining non-trivial differentials are only $d_5(2e_6)$ and $d_7(e_{36})$. We need the following lemma to prove this fact.

Lemma 2.16 ([4]). *Suppose there is a relation* $\sum v_nx_n\equiv 0 \bmod I^2$ *in* $\boldsymbol{BP}^*X$, *where* $I=(2, v_1, v_2, \cdots)$, *then there exists* $y\in \boldsymbol{H}^*(X; \boldsymbol{Z}/p)$ *such that* $\boldsymbol{Q}_i(y)=\rho(x_i)$, *where* $\rho: \boldsymbol{BP}^*X\to \boldsymbol{H}^*(X; \boldsymbol{Z}/p)$.

The paper [4] gave consideration to odd prime case, but its proof is still valid for the prime 2.

Lemma 2.17. $d_5(2e_6)=v_1^2\otimes e_{56}$ *and* $d_7(e_{36})=v_2\otimes e_{2356}$.

Proof. It is easy to prove $d_7(e_{36})=v_2\otimes e_{2356}$ (Lemma 2.4 with $m=2$). If we suppose $d_5(2e_6)=0$ in the spectral sequence (2.3) then $\boldsymbol{BP^*SU}(6)/\boldsymbol{SO}(6)$ has $\boldsymbol{BP}^*$-generators 1, $\tilde{e}_3$, $\tilde{e}_{23+5}$, $\widetilde{2e}_6$, $\tilde{e}_{34+25}$, $\tilde{e}_{35}$, $\tilde{e}_{36+45}$, $\tilde{e}_{235}$, $\tilde{e}_{56}$, $\widetilde{2e}_{236+56}$, $\tilde{e}_{345}$, $\tilde{e}_{356}$, $\tilde{e}_{2345+356}$, $\widetilde{2e}_{456}$, $\tilde{e}_{2356}$, $\tilde{e}_{3456}$, and $\tilde{e}_{23456}$ because this spectral sequence collapses at the $\boldsymbol{E}_8$-term. There are of course some relations among these elements. We need Lemma 2.16 in order to detect their relations. One of them is $v_1\tilde{e}_{356}+v_2\tilde{e}_{3456}=\sum a_ix_i$, $a_i\in I^2$, because the following equations hold in $\boldsymbol{H}^*(\boldsymbol{SU}(6)/\boldsymbol{SO}(6); \boldsymbol{Z}_{(2)})$

$$(2.18)\qquad \begin{aligned} &\boldsymbol{Q}_0e_{236}=\boldsymbol{Q}_3e_{236}=\boldsymbol{Q}_4e_{236}=0\\ &\boldsymbol{Q}_1e_{236}=e_{356}\\ &\boldsymbol{Q}_2e_{236}=e_{3456}. \end{aligned}$$

But this relation leads to a contradiction. Consider the Serre spectral sequence (2.6) again. We have Lemma 2.8, which says

(2.19)
$$d_6(\tilde{e}_3)=v_1\otimes c_3,\quad d_6(\tilde{e}_{23+5})=2c_3,\quad d_8(\tilde{e}_{34+25})=v_1\otimes c_5,$$
$$d_8(\tilde{e}_{36+45})=2c_5,\quad d_{12}(\tilde{e}_{56})=v_2\otimes c_3c_6.$$

Then $d_6(v_1\tilde{e}_{356}+v_2\tilde{e}_{3456})\equiv v_1^2\tilde{e}_{56}\otimes c_3 \mod I^3$, on one hand, $d_6(\sum a_ix_i)\equiv v_1v_2\widetilde{2e}_{456}\otimes c_3$ or 0 $\mod I^3$. But we have supposed $d_5(2e_6)=0$ in the Atiyah-Hirzeburch spectral sequence for $SU(6)/SO(6)$. This means $v_1^2e_{56}\neq 0$, so there is no relation between $v_1^2\tilde{e}_{56}$ and $v_1v_2\widetilde{2e}_{456}$.

The differentials in the Atiyah-Hirzeburch spectral sequence for $SU(6)/SO(6)$ that should be considered are exhausted. Hence we have the following theorem.

Theorem 2.20. *$BP^*SU(6)/SO(6)$ has a BP^*-filtration whose associated graded module is*

$$\begin{aligned}&BP^*\{1\}\oplus BP^*/2\{e_3\}\oplus BP^*\{e_{23+5}\}\oplus BP^*\{4e_6\}\\ &\oplus BP^*/2\{e_{34+25}\}\oplus BP^*/(2,v_1)\{e_{35}\}\oplus BP^*\{e_{36+45}\}\\ &\oplus BP^*/2\{e_{235}\}\oplus BP^*\{2e_{236}\}\oplus BP^*/(2,v_1^2)\{e_{56}\}\oplus BP^*/2\{e_{345}\}\\ &\oplus BP^*\{e_{2345+356}\}\oplus BP^*/(2,v_1)\{e_{356}\}\oplus BP^*\{2e_{456}\}\\ &\oplus BP^*/(2,v_1,v_2)\{e_{2356}\}\oplus BP^*/(2,v_1)\{e_{3456}\}\oplus BP^*\{e_{23456}\}.\end{aligned}$$

Proof. This is a direct consequence of Lemma 2.4 and Lemma 2.17.

3. The even-concentratedness of $BP^*BSO(6)$

We consider the Atiyah-Hirzeburch spectral sequence for $BSO(6)$ in this section:

(3.1)
$$E_2\cong BP^*\otimes H^*(BSO(6);Z_{(2)})\Longrightarrow BP^*BSO(6).$$

It is well-known that

(3.2)
$$H^*(BSO(6);Z_{(2)})\cong Z_{(2)}[p_1,p_2,\chi]\oplus Z/2[p_1,p_2,\chi,w_3,w_5]\{w_3,w_5,\alpha\},$$

where p_1, p_2, χ, w_3, w_5, and α are the elements which respectively project to w_2^2, w_4^2, w_6, w_3, w_5, and $w_2w_5+w_3w_4$ under the reduction map $\rho: H^*(BSO(6);Z_{(2)})\to H^*(BSO(6);Z/2)$. Notice that Chern class elements c_2, c_3, c_4, c_5, and c_6 are respectively project to p_1, w_3^2, p_2, w_5^2, and χ^2 under the map $H^*(BSU(6);Z_{(2)})\to H^*(BSO(6);Z_{(2)})$. Hence its E_2-term is as follows.

(3.3)
$$\begin{aligned}E_2&\cong BP^*[p_1,p_2,\chi]\oplus BP^*/2[p_1,p_2,\chi,w_3]\{w_3\}\\ &\quad\oplus BP^*/2[p_1,p_2,\chi,w_3,w_5]\{w_5,\alpha\}\\ &\cong BP^*[p_1,p_2,\chi^2]\{1,\chi\}\oplus BP^*/2[p_1,p_2,\chi^2,w_3^2]\{w_3,w_3^2,\chi w_3,\chi w_3^2\}\\ &\quad\oplus BP^*/2[p_1,p_2,\chi^2,w_3^2,w_5^2]\\ &\quad\cdot\{w_5,w_3w_5,\chi w_5,\chi w_3w_5,w_5^2,w_3w_5^2,\chi w_5^2,\chi w_3w_5^2,\\ &\qquad\alpha,w_3\alpha,\chi\alpha,\chi w_3\alpha,w_5\alpha,w_3w_5\alpha,\chi w_5\alpha,\chi w_3w_5\alpha\}\\ &\cong BP^*[p_1,p_2,\chi^2]\{1,\chi\}\oplus BP^*/2[p_1,p_2,\chi^2,w_3^2]\{w_3,w_3^2,\chi w_3,\chi w_3^2\}\\ &\quad\oplus BP^*/2[p_1,p_2,\chi^2,w_3^2,w_5^2]\end{aligned}$$

$$\cdot\{w_5,\ w_3w_5,\ \chi w_5,\ \chi w_3w_5,\ w_5^2,\ w_3^2\alpha+w_3w_5^2,\ \chi w_5^2,\ \chi w_2w_5^2,$$
$$\alpha,\ w_3\alpha,\ \chi\alpha,\ \chi w_3\alpha+\chi w_5^2,\ w_5\alpha,\ w_3w_5\alpha+w_5^3,\ \chi w_5\alpha,\ \chi w_3w_5\alpha\}\ .$$

In the first place, p_1, p_2, χ, w_3^2, and w_5^2 are infinite cycles because they are in the image of $\boldsymbol{H}^*(\boldsymbol{BSU}(6);\boldsymbol{Z}_{(2)})$. We can compute the differential d_3 on the other generators from Lemma 2.4 and Table 4.1.

degree	3	5	6	7	10
x	w_3	w_5	χ	α	$w_3\alpha$
$d_3(x)$	w_3^2	w_3w_5	χw_3	w_5^2	$w_3^2\alpha+w_3w_5^2$

11	12	12	13	14
χw_5	χw_3^2	$w_5\alpha$	$\chi\alpha$	χw_3w_5
0	χw_3^3	$w_3w_5\alpha+w_5^3$	$\chi w_3\alpha+\chi w_5^2$	$\chi w_3^2w_5$

16	18	21
χw_5^2	$\chi w_5\alpha$	$\chi w_3w\alpha$
$\chi w_3w_5^2$	χw_5^3	$\chi w_5(w_3^2\alpha+w_3w_5^2)$

Consider the entries in the columns for $d_3(x)$ to be v_1-tensored. Consequently, we get the $\boldsymbol{E}_4$-term.

(3.4)
$$\begin{aligned}
\boldsymbol{E}_4\cong&\boldsymbol{BP}^*[p_1,\ p_2,\ \chi^2]\{1,\ 2\chi\}\oplus\boldsymbol{BP}^*/2[p_1,\ p_2,\ \chi^2]\{\chi w_5\}\\
&\oplus\boldsymbol{BP}^*/(2,\ v_1)[p_1,\ p_2,\ \chi^2,\ w_3^2]\{w_3^2,\ \chi w_3,\ \chi w_3^2w_5\}\\
&\oplus\boldsymbol{BP}^*/(2,\ v_1)[p_1,\ p_2,\ \chi^2,\ w_3^2,\ w_5^2]\\
&\quad\cdot\{w_3w_5,\ w_5^2,\ w_3^2\alpha+w_3w_5^2,\ w_3w_5\alpha+w_5^3,\ \chi w_3\alpha+\chi w_5^2,\\
&\qquad\chi w_3w_5^2,\ \chi w_5^3,\ \chi w_5(w_5^2\alpha+w_3w_5^2)\}\\
\cong&\boldsymbol{BP}^*[p_1,\ p_2,\ \chi^2]\{1,\ 2\chi\}\oplus\boldsymbol{BP}^*/2[p_1,\ p_2,\ \chi^2]\{\chi w_5\}\\
&\oplus\boldsymbol{BP}^*/(2,\ v_1)[p_1,\ p_2,\ \chi^2,\ w_3^2]\{w_3^2,\ \chi w_3^2w_5\}\\
&\oplus\boldsymbol{BP}^*/(2,\ v_1)[p_1,\ p_2,\ \chi^2,\ w_3^2,\ w_5^2]\\
&\quad\cdot\{w_3w_5,\ \chi w_3,\ w_5^2,\ w_3^2\alpha+w_3w_5^2,\ w_3w_5\alpha+w_5^3,\ \chi w_3\alpha+\chi w_5^2,\\
&\qquad\chi w_5^3,\ \chi w_5(w_3^2\alpha+w_3w_5^2)\}\ .
\end{aligned}$$

Lemma 2.17 implies $d_5(2\chi)=v_1^2\otimes\chi w_5$ in this spectral sequence since there is a map of spectral sequences

$$\begin{array}{ccc}
\boldsymbol{BP}^*\otimes\boldsymbol{H}^*(\boldsymbol{BSO}(6);\boldsymbol{Z}_{(2)}) & \Longrightarrow & \boldsymbol{BP}^*\boldsymbol{BSO}(6)\\
\downarrow & & \downarrow\\
\boldsymbol{BP}^*\otimes\boldsymbol{H}^*(\boldsymbol{SU}(6)/\boldsymbol{SO}(6);\boldsymbol{Z}_{(2)}) & \Longrightarrow & \boldsymbol{BP}^*\boldsymbol{SU}(6)/\boldsymbol{SO}(6)\ ,
\end{array}$$

and there is no element except χw_5 whose degree is (0, 11). It is easy to see that the differential d_5 is trivial for other generators.

(3.6)
$$\begin{aligned} \boldsymbol{E}_6 \cong & \boldsymbol{BP}^*[p_1, p_2, \chi^2]\{1, 4\chi\} \oplus \boldsymbol{BP}^*/(2, v_1^2)[p_1, p_2, \chi^2]\{\chi w_5\} \\ & \oplus \boldsymbol{BP}^*/(2, v_1)[p_1, p_2, \chi^2, w_3^2]\{w_3^2, \chi w_3^2 w_5\} \\ & \oplus \boldsymbol{BP}^*/(2, v_1)[p_1, p_2, \chi^2, w_3^2, w_5^2] \\ & \cdot \{w_3 w_5, \chi w_3, w_5^2, w_3^2\alpha + w_3 w_5^2, w_3 w_5 \alpha + w_5^3, \chi w_3 \alpha + \chi w_5^2, \\ & \quad \chi w_5^3, \chi w_5(w_3^2\alpha + w_3 w_5^2)\} . \\ \cong & \boldsymbol{BP}^*[p_1, p_2, \chi^2]\{1, 4\chi\} \oplus \boldsymbol{BP}^*/(2, v_1^2)[p_1, p_2, \chi^2]\{\chi w_5\} \\ & \oplus \boldsymbol{BP}^*/(2, v_1)[p_1, p_2, \chi^2, w_3^2]\{w_3^2, w_5^2, \chi w_3^2 w_5, \chi w_3^3\} \\ & \oplus \boldsymbol{BP}^*/(2, v_1)[p_1, p_2, \chi^2, w_3^2, w_5^2] \\ & \cdot \{w_3 w_5, \chi w_3, w_3^2\alpha + w_3 w_5^2, w_3 w_5 \alpha + w_5^3 + \chi w_3^3, \chi w_3 \alpha + \chi w_5^2, w_5^4 + p_1 w_3^2 w_5^2 \\ & + p_2 w_3^4, \chi w_5(w_3^2\alpha + w_3 w_5^2), \chi w_5(w_5^4 + p_1 w_3^2 w_5^2 + p_2 w_3^4) + \chi^2 w_3^2(w_3^2\alpha + w_3 w_5^2)\} . \end{aligned}$$

Since the $\boldsymbol{E}_6(\boldsymbol{E}_7)$-term is almost a $\boldsymbol{P}(2)^*$-free module, there is no indeterminancy owing to computing d_7 from Lemma 2.4 and Table 4.1.

degree	6	8	9	11
x	4χ	$w_3 w_5$	χw_3	χw_5
$d_7(x)$	0	$w_3 w_5 \alpha + w_5^3 + \chi w_3^3$	$\chi w_3 \alpha + \chi w_5^2$	$\chi^2 w_3^2$

13	17	21
$w_3^2\alpha + w_3 w_5^2$	$\chi w_3^2 w_5$	χw_5^3
$w_5^3 + p_1 w_3^2 w_5^2 + p_2 w_3^4$	$\chi^2 w_3^4$	$\chi^2 w_3^2 w_5^2$

31
$\chi w_5(w_3^2\alpha + w_3 w_5^2)$
$\chi^2 w_3^2(w_3^2\alpha + w_3 w_5^2) + \chi w_5(w_5^4 + p_1 w_3^2 w_5^2 + p_2 w_3^4)$

In the above table, we omitted tensoring v_2 to the entries in the columns for d_7. The fect that $d_7(4\chi)=0$ cannot be determined in the same manner. But it is actually true because

(3.7)
$$\begin{aligned} & d_7(4\chi) \in \boldsymbol{E}_7^{-6,13} \cong \boldsymbol{Z}/2\{v_2 \otimes p_1 \chi w_3, v_2 \otimes (w_3^2\alpha + w_3 w_5^2)\} \\ & \underset{\cong}{\overset{d_7}{\longrightarrow}} \boldsymbol{E}_7^{-12,20} \cong \boldsymbol{Z}/2\{v_2^2 \otimes (p_1 \chi w_3 \alpha + p_1 \chi w_5^2), v_2^2 \otimes (w_5^4 + p_1 w_3^2 w_5^2 + p_2 w_3^4)\} \end{aligned}$$

and $d_7 d_7 = 0$.

Hence the $\boldsymbol{E}_8$-term of the spectral sequence is as follows.

(3.8)
$$\begin{aligned} E_8 \cong\ & BP^*[p_1, p_2, \chi^2]\{1, 4\chi\} \oplus BP^*/(2, v_1)[p_1, p_2, \chi^2]\{v_1 \otimes \chi w_5\} \\ & \oplus BP^*/(2, v_1)[p_1, p_2, w_3^2]\{w_3^2, w_5^2\} \\ & \oplus BP^*/(2, v_1)[p_1, p_2, \chi^2]\{\chi^2 w_5^2\} \\ & \oplus BP^*/(2, v_1, v_2)[p_1, p_2, \chi^2, w_3^2]\{\chi^2 w_3^2, \chi^2 w_3^2 w_5^2\} \\ & \oplus BP^*/(2, v_1, v_2)[p_1, p_2, \chi^2, w_3^2, w_5^2] \\ & \cdot \{w_3 w_5 \alpha + w_5^3 + \chi w_3^3, \chi w_3 \alpha + \chi w_5^2, w_5^4 + p_1 w_3^2 w_5^2 + p_2 w_3^4, \\ & \chi w_5 (w_5^4 + p_1 w_3^2 w_5^2 + p_2 w_3^4) + \chi^2 w_3^2 (w_3^2 \alpha + w_3 w_5^2)\}\,. \end{aligned}$$

Now we we must take it into consideration whether d_9 and d_{11} are trivial or not. Recall again that there are some elements which should be zero in the E_∞-term, i.e., be boundary elements of some differentials. Those are the lowest degree terms of the eiements in the ideal $(e_2 - e_2^*, \cdots, e_6 - e_6^*)$. Thus we can compute them to a certain extent.

Lemma 3.9. *The differential d_9 is trivial and $d_{11}(v_1 \otimes \chi w_5) = v_2^2 \otimes \chi^2 w_5^2$.*

Proof. For the following equation, $v_2^2 \otimes \chi^2 w_5^2$ turns out to be a boundary element.

(3.10)
$$\begin{aligned} & \left(v_2 c_6 + \frac{4}{3} v_1^3 c_6\right)(c_2^* - c_2) - (5 v_1^6 c_2 c_6 + 9 v_1^3 v_2 c_2 c_6)(c_3^* - c_3) - \frac{13}{3} v_1^2 v_2 c_6 (c_4^* - c_4) \\ & + (3 v_1^6 c_6 + 15 v_1^3 v_2 c_6)(c_5^* - c_5) + \left(v_1 - \frac{5}{3} v_2 c_2\right)(c_6^* - c_6) = -5 v_2^2 w_5^2 w_6^2 + \text{higher terms}. \end{aligned}$$

Hence $v_2^2 \otimes \chi^2 w_5^2 \in E_8^{-12,22}$ requires some element in $E_9^{-4,13}$, $E_{11}^{-2,11}$, or $E_{13}^{0,9}$ which maps to $v_2^2 \otimes \chi^2 w_5^2$ by the differential d_9, d_{11}, or d_{13}. Since $E_8^{0,9} \cong E_8^{-4,13} \cong 0$ and $E_8^{-2,11} \cong Z/2\{v_1 \otimes \chi w_5\}$, we can conclude that $d_9(v_1 \otimes \chi w_5) = 0$ and $d_{11}(v_1 \otimes \chi w_5) = v_2^2 \otimes \chi^2 w_5^2$. Moreover, if we assume $d_9(4\chi) \in E_9^{-8,16} \cong Z/2\{v_1 v_2 \otimes p_1 \chi w_5\}$ to be non-trivial, $v_2^3 \otimes p_1 \chi^2 w_5^2$ $(= d_{11}(v_1 v_2 \otimes p_1 \chi w_5))$ must be zero, which is, in fact, a non-trivial element in the E_{11}-term. It is obvious that d_9 and d_{11} are trivial for other generators.

The diferential d_{13} is also trivial for degree reasons, and the remaining differentials to be considered are completely determined by Lemma 2.4 and Table 4.1.

Theorem 3.11. (1) *The Atiyah-Hirzeburch spectral sequence for $BP^*BSO(6)$ collapses at the E_{34}-term.*

(2) *$BP^*BSO(6)$ is concentrated in even degree:*

$$BP^*BSO(6) \cong BP^*[[c_2, c_4, c_6]]\{1, \widetilde{4\chi}\} \oplus BP^*[[c_2, \cdots, c_6]]/(c_2 - c_2^*, \cdots, c_6 - c_6^*)\{c_3, c_5\}\,,$$

where the isomorphism is just a module isomorphism.

Proof.

(3.12)
$$\begin{aligned} E_{15} \cong E_{14} \cong E_{13} \cong E_{12} & \\ \cong\ & BP^*[p_1, p_2, \chi^2]\{1, 4\chi\} \\ & \oplus BP^*/(2, v_1)[p_1, p_2, w_3^2]\{w_3^2, w_5^2\} \end{aligned}$$

$$\oplus BP^*/(2, v_1, v_2^2)[p_1, p_2, \chi^2]\{\chi^2 w_5^2\}$$
$$\oplus BP^*/(2, v_1, v_2)[p_1, p_2, \chi^2, w_3^2]\{\chi^2 w_3^2, \chi^2 w_3^2 w_5^2\}$$
$$\oplus BP^*/(2, v_1, v_2)[p_1, p_2, \chi^2, w_3^2, w_5^2]$$
$$\cdot\{w_3w_5\alpha+w_5^3+\chi w_3^3,\ \chi w_3\alpha+\chi w_5^2,\ w_5^4+p_1w_3^2w_5^2+p_2w_3^4,$$
$$\chi w_5(w_5^4+p_1w_3^2w_5^2+p_2w_3^4)+\chi^2 w_3^2(w_3^2\alpha+w_3w_5^2)+p_1\chi^2(w_3w_5\alpha+w_5^3+\chi w_3^3)\}\,.$$

The differential d_{15} can be computed from Lemma 2.4 and Table 4.1 as follows.

(3.13)
$$d_{15}(4\chi)=0$$
$$d_{15}(w_3w_5\alpha+w_5^3+\chi w_3^3)=v_3\otimes(\chi^2 w_3^6+w_5^2(w_5^4+p_1w_3^2w_5^2+p_2w_3^4))$$
$$d_{15}(\chi w_3\alpha+\chi w_5^2)=v_3\otimes(\chi w_5(w_5^4+p_1w_3^2w_5^2+p_2w_3^4)+\chi^2 w_3^2(w_3^2\alpha+w_3w_5^2)$$
$$+p_1\chi^2(w_3w_5\alpha+w_5^3+\chi w_3^3))\,.$$

The fact that $d_{15}(4\chi)=0$ follows from $d_{15}d_{15}=0$. Hence the E_{16}-term is

(3.14)
$$E_{16}\cong BP^*[p_1, p_2, \chi^2]\{1, 4\chi\}$$
$$\oplus BP^*/(2, v_1)[p_1, p_2, w_3^2]\{w_3^2, w_5^2\}$$
$$\oplus BP^*/(2, v_1, v_2^2)[p_1, p_2, \chi^2]\{\chi^2 w_5^2\}$$
$$\oplus BP^*/(2, v_1, v_2)[p_1, p_2, \chi^2, w_3^2]\{\chi^2 w_3^2, \chi^2 w_3^2 w_5^2, w_5^4+p_1w_3^2w_5^2+p_2w_3^4\}$$
$$\oplus BP^*/(2, v_1, v_2, v_3)[p_1, p_2, \chi^2, w_3^2, w_5^2]$$
$$\cdot\{\chi^2 w_3^6+w_5^2(w_5^4+p_1w_3^2w_5^2+p_2w_3^4),\ \chi w_5(w_5^4+p_1w_3^2w_5^2+p_2w_3^4)$$
$$+\chi^2 w_3^2(w_3^2\alpha+w_3w_5^2)+p_1\chi^2(w_3w_5\alpha+w_5^3+\chi w_3^3)\}\,.$$

By the same manner, we get $d_{31}(\beta)=p_1\chi^2(\chi^2 w_3^6+w_5^2(w_5^4+p_1w_3^2w_5^2+p_2w_3^4))$, where β is the last generator in the E_{16}-term above. $d_r(4\chi)=0$ $(r\geqq 16)$ for degree reasons. Thus the E_{32}-term, which is also the E_∞-term, is expressed as follows.

(3.15)
$$E_{32}\cong BP^*[p_1, p_2, \chi^2]\{1, 4\chi\}$$
$$\oplus BP^*/(2, v_1)[p_1, p_2, w_3^2]\{w_3^2, w_5^2\}$$
$$\oplus BP^*/(2, v_1, v_2^2)[p_1, p_2, \chi^2]\{\chi^2 w_5^2\}$$
$$\oplus BP^*/(2, v_1, v_2)[p_1, p_2, \chi^2, w_3^2]\{\chi^2 w_3^2, \chi^2 w_3^2 w_5^2, w_5^4+p_1w_3^2w_5^2+p_2w_3^4\}$$
$$\oplus BP^*/(2, v_1, v_2, v_3)[p_1, p_2, \chi^2, w_3^2, w_5^2]/(p_1\chi^2)$$
$$\cdot\{\chi^2 w_3^6+w_5^2(w_5^4+p_1w_3^2w_5^2+p_2w_3^4)\}$$
$$\oplus BP^*/(2, v_1, v_2, v_3, v_4)[p_1, p_2, \chi^2, w_3^2, w_5^2]$$
$$\cdot\{p_1\chi^2(\chi^2 w_3^6+w_5^2(w_5^4+p_1w_3^2w_5^2+p_2w_3^4))\}\,.$$

The E_{32}-term is generated by elements in even degrees, hence this spectral sequence collapses at the E_{32}-term. The differentials which are not concerned with χ are all derived from the ideal $(c_2-c_2^*, \cdots, c_6-c_6^*)$. Therefore there is no further relation among $c_2, \cdots, c_6$. This fact affirms that the map from the pushout of the diagram (2.11) to $BP^*BSO(6)$ is monic. Hence we obtain the theorem.

4. The table for $Q_m w_k$ in $H^*(BSO(6); Z_{(2)})$

Table 4.1. *We can compute $Q_i w_k$ as follows.*

$Q_0 w_2 = w_3 \quad Q_0 w_3 = 0 \quad Q_0 w_4 = w_5 \quad Q_0 w_5 = 0 \quad Q_0 w_6 = 0$

$Q_1 w_2 = w_2 w_3 + w_5 \quad Q_1 w_3 = w_3^2 \quad Q_1 w_2 = w_3 w_4 \quad Q_1 w_5 = w_3 w_5 \quad Q_1 w_6 = w_3 w_6$

$Q_2 w_2 = w_2^2 w_3 + w_3^3 + w_2^2 w_5 + w_4 w_5 + w_3 w_6$

$Q_2 w_3 = w_2^2 w_3^2 + w_5^2$

$Q_2 w_4 = w_2^2 w_3 w_4 + w_3 w_4^2 + w_3^2 w_5 + w_2 w_4 w_5 + w_2 w_3 w_6 + w_5 w_6$

$Q_2 w_5 = w_2^2 w_3 w_5 + w_3 w_4 w_5 + w_2 w_5^2 + w_3^2 w_6$

$Q_2 w_6 = w_2^2 w_3 w_6 + w_3 w_4 w_6 + w_2 w_5 w_6$

$Q_3 w_2 = w_2^7 w_3 + w_2^4 w_3^3 + w_2 w_3^5 + w_2^6 w_5 + w_3^4 w_5 + w_2^4 w_4 w_5 + w_4^3 w_5 + w_3 w_4 w_5^2 + w_2 w_5^3 + w_2^4 w_3 w_6$
$\quad + w_3 w_4^2 w_6 + w_3^2 w_5 w_6 + w_2 w_3 w_6^2 + w_5 w_6^2$

$Q_3 w_3 = w_2^6 w_3^2 + w_3^6 + w_2^4 w_5^2 + w_4^2 w_5^2 + w_3^2 w_6^2$

$Q_3 w_4 = w_2^2 w_3 w_4 + w_3^5 w_4 + w_2^4 w_3 w_4^2 + w_3 w_4^4 + w_2^4 w_3^2 w_5 + w_2^5 w_4 w_5 + w_2 w_4^3 w_5 + w_2 w_3 w_4 w_5^2$
$\quad + w_2^2 w_5^3 + w_4 w_5^3 + w_2^5 w_3 w_6 + w_2 w_3 w_1^2 w_6 + w_2^4 w_5 w_6 + w_2 w_3^2 w_5 w_6 + w_4^2 w_5 w_6 + w_3 w_5^2 w_6$
$\quad + w_2^2 w_3 w_6^2 + w_3 w_4 w_6^2$

$Q_3 w_5 = w_2^6 w_3 w_5 + w_3^5 w_5 + w_2^4 w_3 w_4 w_5 + w_3 w_4^3 w_5 + w_2^5 w_5^2 + w_3^2 w_4 w_5^2 + w_2 w_4^2 w_5^2 + w_2 w_3 w_5^3 + w_5^4$
$\quad + w_2^4 w_3^2 w_6 + w_3^2 w_4^2 w_6 + w_3^3 w_5 w_6 + w_3 w_5 w_6^2$

$Q_3 w_6 = w_2^6 w_3 w_6 + w_3^5 w_6 + w_2^4 w_3 w_4 w_6 + w_3 w_4^3 w_6 + w_2^5 w_5 w_6 + w_3^2 w_4 w_5 w_6 + w_2 w_4^2 w_5 w_6$
$\quad + w_2 w_3 w_5^2 w_6 + w_5^3 w_6 + w_3^3 w_6^2 + w_2^2 w_5 w_6^2 + w_3 w_6^3$

$Q_4 w_2 = w_2^{15} w_3 + w_2^{12} w_3^3 + w_2^9 w_3^5 + w_2^3 w_3^9 + w_3^{11} + w_2^{14} w_5 + w_2^8 w_3^4 w_5 + w_2^2 w_3^8 w_5 + w_2^{12} w_4 w_5$
$\quad + w_3^8 w_4 w_5 + w_2^8 w_4^3 w_5 + w_4^7 w_5 + w_2^8 w_3 w_4 w_5^2 + w_3 w_4^5 w_5^2 + w_2^9 w_5^3 + w_2 w_4^4 w_5^3 + w_3^3 w_4 w_5^4$
$\quad + w_2 w_3 w_4^2 w_5^4 + w_2 w_3^2 w_5^5 + w_2^2 w_4 w_5^5 + w_4^2 w_5^5 + w_3 w_5^6 + w_2^{12} w_3 w_6 + w_3^9 w_6 + w_2^8 w_3 w_4^2 w_6$
$\quad + w_3 w_4^6 w_6 + w_2^8 w_3^2 w_5 w_6 + w_3^2 w_4^4 w_5 w_6 + w_3^4 w_5^3 w_6 + w_2^2 w_3 w_5^4 w_6 + w_2^9 w_3 w_6^2 + w_2 w_3 w_4^4 w_6^2$
$\quad + w_2^8 w_5 w_6^2 + w_3^4 w_4 w_5 w_6^2 + w_4^4 w_5 w_6^2 + w_3^5 w_6^3 + w_2^3 w_3 w_6^4 + w_3^3 w_6^4 + w_2^2 w_5 w_6^4 + w_4 w_5 w_6^4$
$\quad + w_3 w_6^5$

$Q_4 w_3 = w_2^{14} w_3^2 + w_2^8 w_3^6 + w_2^2 w_3^{10} + w_2^{12} w_5^2 + w_3^8 w_5^2 + w_2^8 w_4^2 w_5^2 + w_4^6 w_5^2 + w_3^2 w_4^2 w_5^4 + w_2^2 w_5^6$
$\quad + w_2^8 w_3^2 w_6^2 + w_3^2 w_4^4 w_6^2 + w_3^4 w_5^2 w_6^2 + w_2^2 w_3^2 w_6^4 + w_5^2 w_6^4$

$Q_4 w_4 = w_2^{14} w_3 w_4 + w_2^8 w_3^5 w_4 + w_2^2 w_3^9 w_4 + w_2^{12} w_3 w_4^2 + w_3^9 w_4^2 + w_2^8 w_3 w_4^4 + w_3 w_4^8 + w_2^{12} w_3^2 w_5$
$\quad + w_3^{10} w_5 + w_2^{13} w_4 w_5 + w_2 w_3^8 w_4 w_5 + w_2^9 w_4^3 w_5 + w_2 w_4^7 w_5 + w_2^9 w_3 w_4 w_5^2 + w_2 w_3 w_4^5 w_5^2$
$\quad + w_2^{10} w_5^3 + w_2^8 w_4 w_5^3 + w_2^2 w_4^4 w_5^3 + w_4^5 w_5^3 + w_2 w_3^3 w_4 w_5^4 + w_2^2 w_3 w_4^2 w_5^4 + w_3 w_4^3 w_5^4$
$\quad + w_2^2 w_3^2 w_5^5 + w_2^3 w_4 w_5^5 + w_3^2 w_4 w_5^5 + w_5^7 + w_2^{13} w_3 w_6 + w_2 w_3^9 w_6 + w_2^9 w_3 w_4^2 w_6$
$\quad + w_2 w_3 w_4^6 w_6 + w_2^{12} w_5 w_6 + w_2^9 w_3^2 w_5 w_6 + w_3^8 w_5 w_6 + w_2^8 w_4^2 w_5 w_6 + w_2 w_3^2 w_4^4 w_5 w_6$
$\quad + w_4^6 w_5 w_6 + w_2^8 w_3 w_5^2 w_6 + w_3 w_4^4 w_5^2 w_6 + w_2 w_3^4 w_5^3 w_6 + w_2^3 w_3 w_5^4 w_6 + w_3^3 w_5^4 w_6 + w_2^2 w_5^5 w_6$

$$+w_2^{10}w_3w_6^2+w_2^8w_3w_4w_6^2+w_2^2w_3w_4^4w_6^2+w_3w_4^5w_6^2+w_2w_3^4w_4w_5w_6^2+w_3w_5^4w_6^2$$
$$+w_2w_3^5w_6^3+w_3^4w_5w_6^3+w_3^4w_3w_6^4+w_2^2w_3w_4w_6^4+w_3w_4^2w_6^4+w_3^2w_5w_6^4+w_2w_4w_5w_6^4$$
$$+w_2w_3w_6^5+w_5w_6^5$$

$$Q_4w_5=w_2^{14}w_3w_5+w_2^8w_3^5w_5+w_2^2w_3^9w_5+w_2^{12}w_3w_4w_5+w_3^9w_4w_5+w_2^8w_3w_4^3w_5+w_3w_4^7w_5+w_2^{13}w_5^2$$
$$+w_2w_3^8w_5^2+w_3^8w_3^2w_4w_5^2+w_2^9w_4^2w_5^2+w_3^2w_4^5w_5^2+w_2w_4^6w_5^2+w_2^9w_3w_5^3+w_2w_3w_4^4w_5^3$$
$$+w_2^8w_5^4+w_3^4w_4w_5^4+w_4^4w_5^4+w_2w_3^3w_5^5+w_2^2w_3w_4w_5^5+w_3w_4^2w_5^5+w_2^3w_5^6+w_3^2w_5^6$$
$$+w_2^{12}w_3^2w_6+w_3^{10}w_6+w_2^8w_3^2w_4^2w_6+w_3^2w_4^6w_6+w_2^8w_3^3w_5w_6+w_3^3w_4^4w_5w_6+w_3^5w_5^3w_6$$
$$+w_2^2w_3^2w_5^4w_6+w_2^8w_3w_5w_6^2+w_3^5w_4w_5w_6^2+w_3w_4^4w_5w_6^2+w_2w_3^4w_5^2w_6^2+w_3^6w_6^3$$
$$+w_2^2w_3w_5w_6^4+w_3w_4w_5w_6^4+w_2w_5^2w_6^4+w_3^2w_6^5$$

$$Q_4w_6=w_2^{14}w_3w_6+w_2^8w_3^5w_6+w_2^2w_3^9w_6+w_2^{12}w_3w_4w_6+w_3^9w_4w_6+w_2^8w_3w_4^3w_6+w_3w_4^7w_6$$
$$+w_2^{13}w_5w_6+w_2w_3^8w_5w_6+w_2^8w_3^2w_4w_5w_6+w_2^9w_4^2w_5w_6+w_3^2w_4^5w_5w_6+w_2w_4^6w_5w_6$$
$$+w_2^9w_3w_5^2w_6+w_2w_3w_4^4w_5^2w_6+w_2^8w_5^3w_6+w_3^4w_4w_5^3w_6+w_4^4w_5^3w_6+w_2w_3^3w_5^4w_6$$
$$+w_2^2w_3w_4w_5^4w_6+w_3w_4^2w_5^4w_6+w_2^3w_5^5w_6+w_3^2w_5^5w_6+w_2^8w_3^3w_6^2+w_3^3w_4^4w_6^2+w_2^{10}w_5w_6^2$$
$$+w_3^4w_4^2w_5w_6^2+w_2^2w_4^4w_5w_6^2+w_3^5w_5^2w_6^2+w_5^5w_6^2+w_2^8w_3w_6^3+w_3^5w_4w_6^3+w_3w_4^4w_6^3$$
$$+w_2w_3^4w_5w_6^3+w_2^4w_5w_6^4+w_2^2w_3w_6^5+w_3w_4w_6^5+w_2w_5w_6^5$$

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