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Remarks on torus principal bundles

By

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In this paper we study principal bundles $X \xrightarrow{\pi} M$ over a compact complex manifold M whose structure group is a compact complex torus $T = V/\Lambda$. The total space X of such a principal bundle is usually not a Kähler space even if the base manifold M is.

Typical examples are Hopf manifolds, or the Calabi-Eckmann manifolds diffeomorphic to a product of spheres. These are principal bundles over a product of projective spaces, the fibre is an elliptic curve. Those and other special examples have been studied in detail, see [Cal-Eck], [Maeda], [Nakamura], [Akao].

We develop the theory starting from the base manifold M, often assuming that it (i.e. $\mathrm{H}^2(M)$) has a Hodge decomposition. For a T-principal bundle $X \xrightarrow{\pi} M$ we define a characteristic class $c^{\mathbf{Z}} \in \mathrm{H}^2(M, \Lambda)$ (1.3) and invariants $\varepsilon\colon \mathrm{H}_T^{0,1} \to \mathrm{H}_M^{0,2}$, $\gamma\colon \mathrm{H}_T^{1,0} \to \mathrm{H}_M^{1,1}$ (1.5). It will turn out that these can be computed from $c^{\mathbf{Z}}$ and determine the d_2 differentials of the Leray spectral sequence converging to $\mathrm{H}^{\cdot}(X, \mathbf{C})$ and of a spectral sequence converging to $\mathrm{H}_X^{\cdot\,\cdot}$ (with a variant computing $\mathrm{H}^{\cdot}(\Theta_X)$). This spectral sequence was constructed by Borel in his appendix to [Hirzebruch] and was used there to compute the Hodge ring of Calabi-Eckmann manifolds. Since in our case all those spectral sequences degenerate on E_3-level, Betti numbers, Hodge numbers, and the space of infinitesimal deformations of X can be computed in general (Theorem 1.6).

In bundles with $\varepsilon = 0$ the torus T can be replaced by any other torus of the same dimension (e.g. Calabi-Eckmann manifolds), whereas for $\varepsilon \neq 0$ (e.g. Iwasawa manifold) the periods of T must be related to intrinsic data of M (Chapter 7, Chapter 8).

If M is simply-connected, then it is fairly easy to construct simply-connected bundles, even with first Chern class $c_1(X) = 0$. They do not carry a Kähler metric by Blanchard's theorem (1.7), in fact they cannot carry a complex Kähler structure for purely topological reasons (11.4).

If moreover M is a complex surface and T an elliptic curve, then we get a lot of interesting simply-connected complex threefolds with $c_1 = 0$. According to Wall's classification of real six-dimensional manifolds, the only diffeomor-

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phism invariant is the Betti number $b_2(M)$. So we find complex structures with different Kodaira dimension on the same C^∞ manifold (Chapter 13).

Small deformations of Calabi-Eckmann manifolds have been described in [Akao], those of the Iwasawa manifold in [Nakamura]. Suwa has studied infinitesimal deformations of holomorphic Seifert fibre spaces in general [Suwa_1], [Suwa_2]. In our special situation things are fairly easy, and we can describe the infinitesimal deformations using the invariants ε and γ.

This paper is mainly a result of my stay in Japan. I had many interesting discussions with Japanese mathematicians there but in the first place I would like to thank Kenji Ueno who invited me to Kyoto University and helped me in many ways to understand both mathematics and the way of life of that fascinating country.

1. Notation, basic facts, and main theorem

1.1. Notation. $T = V/\Lambda$ always denotes an n-dimensional compact complex torus, defined by a lattice $\Lambda \subset V$ in the n-dimensional complex vector space V. M is a compact complex manifold of dimension m, and $\pi: X \to M$ denotes a T-principal bundle.

Canonical identifications concerning the torus will be made frequently. In particular we use $T_0(T) = \mathrm{H}^0(\Theta_T) = V$, $\mathrm{H}^i(\Theta_T) = \mathrm{H}_T^{0,i} \otimes V$, $\mathrm{H}_T^{1,0} = \mathrm{H}^0(\Omega_T^1) = \mathrm{H}^0(\Theta_T)^\vee = V^\vee$, $\mathrm{H}_T^{p,q} = \mathrm{H}_T^{p,0} \otimes \mathrm{H}_T^{0,q}$, $\Lambda = \mathrm{H}_1(T, \mathbf{Z})$, $\mathrm{H}_1(T, \mathbf{Z})^\vee = \mathrm{H}^1(T, \mathbf{Z})$.

Whenever there is a Hodge decomposition for the cohomology, pr_{pq}: $\mathrm{H}^{p+q}(Y, \mathbf{C}) \to \mathrm{H}_Y^{pq}$ denotes the projection onto the (p, q)-component, $\mathrm{pr}_{pq}(\omega) =: \omega^{pq}$.

Hodge numbers and Betti numbers of X will be written in the form

$$\begin{array}{ccccccc} & & \cdots & & & & \cdots \\ & \cdots & & \cdots & & & \cdots \\ \cdots & & h_X^{1,1} & & \cdots & & b_2(X) \\ & h_X^{1,0} & & h_X^{0,1} & & & b_1(X) \\ & & h_X^{0,0} & & & & b_0(X) \end{array}$$

1.2. Cocycles. Such principal bundles are described by elements of $\mathrm{H}^1(M, \mathcal{O}_M(T))$. For a Čech 1-cocycle (ϕ_{ij}) the function ϕ_{ij}: $U_i \cap U_j \to T$ identifies $(z, t) \in U_i \times T$ with $(z, t') = (z, \phi_{ij}(z) + t) \in U_j \times T$ in different trivializations.

1.3. The characteristic class. Taking local sections of the constant sheaves $0 \to \Lambda \to V \to T \to 0$ we get an exact sequence of sheaves on M

$$0 \to \Lambda \to \mathcal{O}_M \otimes V \to \mathcal{O}_M(T) \to 0$$

and from this the exact cohomology sequence

$$\cdots \to \mathrm{H}^1(M, \Lambda) \to \mathrm{H}_M^{0,1} \otimes V \to \mathrm{H}^1(\mathcal{O}_M(T)) \xrightarrow{c^{\mathbf{Z}}} \mathrm{H}^2(M, \Lambda) \to \mathrm{H}_M^{0,2} \otimes V \to \cdots$$

So the defining cocycle of the bundle in $\mathrm{H}^1(\mathcal{O}_M(T))$ determines a characteristic class

$$\boxed{c^{\mathbf{Z}} = c^{\mathbf{Z}}(X) = c^{\mathbf{Z}}(X \xrightarrow{\pi} M) \in \mathrm{H}^2(M, \Lambda) = \mathrm{H}^2(M, \mathbf{Z}) \otimes \Lambda\,.}$$

The inclusion $\Lambda \subset V \cong \mathbf{C}^n$ induces a map from the $\mathbf{Z}$-module $\mathrm{H}^2(M, \Lambda)$ (of rank $b_2(M)\cdot 2n$) to the $(b_2(M)\cdot n)$-dimensional vector space $\mathrm{H}^2(M, V) = \mathrm{H}^2(M, \mathbf{C}) \otimes_{\mathbf{C}} V$. The image of $c^{\mathbf{Z}}$ defines a characteristic class

$$\boxed{c = c(X) = c(X \xrightarrow{\pi} M) \in \mathrm{H}^2(M, \mathbf{C}) \otimes V}$$

1.4. Basic facts. (a) *The translation invariant vector fields* $\mathrm{H}^0(\Theta_T) = V$ *on* T *induce an n-dimensional space of everywhere linear independent vector fields on* X. *This gives exact sequences*

$$(*) \qquad 0 \to \mathcal{O}_X \otimes_{\mathbf{C}} V \to \Theta_X \to \pi^*\Theta_M \to 0$$

$$(*)^\vee \quad 0 \to \pi^*\Omega^1_M \to \Omega^1_X \to \mathcal{O}_X \otimes_{\mathbf{C}} \mathrm{H}^{1,0}_T \to 0\,.$$

(b) *For the sheaves of relative vector fields and of relative differentials and for the canonical bundle this means*

$$\Theta_{X/M} = \mathcal{O}_X \otimes_{\mathbf{C}} V \cong \mathcal{O}_X^{\oplus n}$$

$$\Omega^1_{X/M} = \mathcal{O}_X \otimes_{\mathbf{C}} \mathrm{H}^{1,0}_T \cong \mathcal{O}_X^{\oplus n}$$

$$\mathcal{K}_X = \pi^*\mathcal{K}_M$$

(c)
$$e(X) = 0 \qquad c_1(X) = \pi^*c_1(M)$$

$$\chi(\Omega^p_X) = 0 \qquad c_2(X) = \pi^*c_2(M)$$

$$c_3(X) = 0$$

(d) *Furthermore*

$$\mathrm{R}^i\pi_*\mathcal{O}_X = \mathcal{O}_M \otimes_{\mathbf{C}} \mathrm{H}^{0,i}_T$$

$$\mathrm{R}^i\pi_*\Theta_{X/M} = \mathcal{O}_M \otimes_{\mathbf{C}} \mathrm{H}^{0,i}_T \otimes_{\mathbf{C}} V = \mathcal{O}_M \otimes_{\mathbf{C}} \mathrm{H}^i(\Theta_T)$$

$$\mathrm{R}^i\pi_*\Theta_X = \pi_*\Theta_X \otimes_{\mathbf{C}} \mathrm{H}^{0,i}_T$$

$$\mathrm{R}^i\pi_*\Omega^1_{X/M} = \mathcal{O}_M \otimes_{\mathbf{C}} \mathrm{H}^{0,i}_T \otimes_{\mathbf{C}} \mathrm{H}^{1,0}_T = \mathcal{O}_M \otimes_{\mathbf{C}} \mathrm{H}^{1,i}_T$$

$$\mathrm{R}^i\pi_*\Omega^1_X = \pi_*\Omega^1_X \otimes_{\mathbf{C}} {}^{0,i}_T$$

(e) *The long exact sequences obtained by pushing forward* $(*)$ *split up and all the extensions coincide*:

$$\begin{array}{ccccccccc}
0 \to & R^i\pi_*\Theta_{X/M} & \to & R^i\pi_*\Theta_X & \to & \Theta_M \otimes H_T^{0,i} & \to 0 \\
& \| & & \| & & \| & \\
0 \to & \mathcal{O}_M \otimes V \otimes H_T^{0,i} & \to & \pi_*\Theta_X \otimes H_T^{0,i} & \to & \Theta_M \otimes H_T^{0,i} & \to 0 \\
0 \to & \Omega_M^1 \otimes H_T^{0,i} & \to & R^i\pi_*\Omega_X^1 & \to & R^i\pi_*\Omega_{X/M}^1 & \to 0 \\
& \| & & \| & & \| & \\
0 \to & \Omega_M^1 \otimes H_T^{0,i} & \to & \pi_*\Omega_X^1 \otimes H_T^{0,i} & \to & \mathcal{O}_M \otimes H_T^{1,0} \otimes H_T^{0,i} & \to 0
\end{array}$$

Proof. The invariant vector fields on T are also invariant under changes of bundle coordinates, so they define global vector fields on X. This gives $(*)$, and the second sequence in (a) is just the dual. (b) follows directly from the exact sequences defining $\Theta_{X/M}$ and $\Omega_{X/M}^1$

$$0 \to \Theta_{X/M} \to \Theta_X \to \pi^*\Theta_M \to 0$$

$$0 \to \pi^*\Omega_M^1 \to \Omega_X^1 \to \Omega_{X/M}^1 \to 0$$

and $\Lambda^{\max}\Omega_X^1 = \pi^*\Lambda^{\max}\Omega_M^1 \otimes \Lambda^{\max}\Omega_{X/M}^1$. (c) is a direct consequence of the multiplicative property of the Euler characteristic (resp. of the χ_y-genus) (cf. [Borel] = [Hirzebruch] App. II.8) and $\chi_y(T) = 0$. The Chern classes come from downstairs because Θ_X is an extension of a trivial sheaf by $\pi^*\Theta_M$. The first equation in (d) holds because transition functions act trivially on the cohomology of a fibre, the rest is an easy consequence. For (e) observe that since the bundle is locally trivial, locally (on M) $\Theta_M(U) \otimes H_T^{0,i}$ is a direct summand of $H^i(\pi^{-1}(U), \Theta_X)$ in a canonical way. Therefore $R^i\pi_*\Theta_X \to \Theta_M \otimes H_T^{0,i}$ is surjective.

1.5. Invariants. *The relevant information on the bundle $X \xrightarrow{\pi} M$ is contained in the following invariants of a T-principal bundle:*

(a) *The extension class of the sequence $0 \to \Omega_M^1 \to \pi_*\Omega_X^1 \to \mathcal{O}_M \otimes H_T^{1,0} \to 0$, that is $\gamma \in \mathrm{Ext}^1(\mathcal{O}_M \otimes H_T^{1,0}, \Omega_M^1) = H^1(\Omega_M^1) \otimes (H_T^{1,0})^\vee$ or equivalently*

$$\boxed{\gamma\colon H_T^{1,0} \to H_M^{1,1}}$$

(b) *The transgression of the fibre bundle, i.e. the first possibly nontrivial d_2-homomorphism $E_2^{0,1} \to E_2^{2,0}$ in the Leray spectral sequence of the constant sheaf $\mathbf{C}_X$*

$$0 \to H^1(M, \mathbf{C}) \to H^1(X, \mathbf{C}) \to H^1(T, \mathbf{C}) \xrightarrow{\delta} H^2(M, \mathbf{C}).$$

Together with the transgressions in integral cohomology and homology there are

$$\boxed{\begin{array}{c} \delta\colon H^1(T, \mathbf{C}) \to H^2(M, \mathbf{C}) \\ \delta^{\mathbf{Z}}\colon H^1(T, \mathbf{Z}) \to H^2(M, \mathbf{Z}) \\ \delta_{\mathbf{Z}}\colon H_2(M, \mathbf{Z}) \to H_1(T, \mathbf{Z})\,. \end{array}}$$

(c) *The first possibly nontrivial d_2-homomorphism* $\mathrm{H}^0(\mathrm{R}^1\pi_*\mathcal{O}_X) \to \mathrm{H}^2(\pi_*\mathcal{O}_X)$ *in the Leray spectral sequence of* $\mathcal{O}_X$

$$\boxed{\varepsilon\colon \mathrm{H}_T^{0,1} \to \mathrm{H}_M^{0,2}\,.}$$

(d) *The characteristic classes* $c^{\mathbf{Z}} \in \mathrm{H}^2(M, \Lambda)$ *and* $c \in \mathrm{H}^2(M, \mathbf{C}) \otimes V$ *of the bundle as defined in* 1.3.

All these invariants are related to each other, they determine Hodge and Betti numbers and also the space of infinitesimal deformations of X. The main general results that we will prove in this paper are:

1.6. Theorem. *Let* $X \xrightarrow{\pi} M$ *be a T-principal bundle as described above. Then*:

(a) *Borel's spectral sequence* ${}^{p,q}E_2^{s,t} = \sum \mathrm{H}_M^{i,s-i} \otimes \mathrm{H}_T^{p-i,t-p+i}$ *which computes the Hodge numbers of X degenerates on E_3-level, and the d_2-differential is determined by ε and γ* (4.3).

(b) *The same holds for Borel's spectral sequence computing the cohomology* $h^{\cdot}(\Theta_X)$ (14.7).

(c) *Leray spectral sequence* $E_2^{s,t} = \mathrm{H}^s(M, \mathbf{C}) \otimes \mathrm{H}^t(T, \mathbf{C})$ *which computes the Betti numbers of X degenerates on E_3-level, and the d_2-differential is determined by δ* (5.1).

(d) *Under the identification* $\mathrm{H}^1(T, \mathbf{Z}) = \mathrm{Hom}\,(\Lambda, \mathbf{Z})$ *the characteristic class* $c^{\mathbf{Z}} \in \mathrm{H}^2(M, \mathbf{Z}) \otimes \Lambda$ *and the map* $\delta^{\mathbf{Z}}\colon \mathrm{H}^1(T, \mathbf{Z}) \to \mathrm{H}^2(M, \mathbf{Z})$ *coincide* (6.1).

(e) *δ is obtained from $\delta^{\mathbf{Z}}$ by scalar extension* (6.2).

(f) *Assume that* $\mathrm{H}^2(M)$ *is has a Hodge decomposition. Then δ determines ε and γ, and vice versa* (*Chapter* 6).

The invariant δ somehow measures the twisting of the bundle modulo torison, and it also appears in

1.7. Blanchard's Theorem [Blanchard]. *Assume that the base space M is a Kähler manifold. Then the total space X is a Kähler manifold if and only if* $\delta = 0$.

According to (c) and (d) of the previous theorem, $\delta = 0$ if and only if the characteristic class $c^{\mathbf{Z}}$ is torsion, and then all the invariants behave like for a trivial bundle. So from our point of view, this is the less interesting case. In contrary, we will construct simply-connected spaces (which requires a simply-connected base and δ injective, see Chapter 11), mainly as elliptic principal bundles over algebraic surfaces (Chapter 12, Chapter 13), where the topological structure of X is determined by simple invariants.

2. Example: Calabi-Eckmann manifolds

These are (non-Kähler) principal bundles with fibre $T = \mathbf{C}/(\mathbf{Z} \oplus \tau\mathbf{Z})$ over a product $M = \mathbf{P}^{m_1} \times \mathbf{P}^{m_2}$ of complex projective spaces whose total space is diffeomorphic to a product of spheres $S^{2m_1+1} \times S^{2m_2+1}$. If $(x_0\colon\ldots\colon x_{m_1}; y_0\colon\ldots\colon y_{m_2})$ are homogenous coordinates of M, the bundle is trivial over the standard affine coordinate patches $U_{ij} = \{x_i \neq 0,\, y_j \neq 0\}$ and the transition functions are given by

$$\phi_{ij,kl}([x],[y]) = \frac{1}{2\pi i}\left(\log\frac{x_k}{x_i} + \tau\cdot\log\frac{y_l}{y_j}\right).$$

Since $\tau \notin \mathbf{R}$ it is easy to show that the bundle is diffeomorphic to $S^{2m_1+1} \times S^{2m_2+1} \subset \mathbf{C}^{m_1+1} \times \mathbf{C}^{m_2+1}$ with the standard projection to $\mathbf{P}^{m_1} \times \mathbf{P}^{m_2}$ via

$$S^{2m_1+1} \times S^{2m_2+1} \ni (x,y) \mapsto \left([x],[y],\frac{1}{2\pi i}(\log x_i + \tau\cdot\log y_j) \bmod \mathbf{Z} \oplus \tau\mathbf{Z}\right) \in U_{ij} \times T.$$

The Hodge algebra has been investigated in [Borel], II.9. The Hodge numbers are $(0 \leq m_1 \leq m_2)$

$$h_X^{p,q} = \begin{cases} 1 & \text{if } p \leq m_1 \text{ and } q = p,\ p+1 \\ 1 & \text{if } p > m_2 \text{ and } q = p,\ p-1 \\ 0 & \text{otherwise} \end{cases}$$

(see 9.4). In 4.3 we investigate the spectral sequence that Borel used for his computation. $m_1 = 0$ defines a Hopf manifold. In the easiest simply connected case $m_1 = m_2 = 1$ we get a complex threefold diffeomorphic to $S^3 \times S^3$ with Hodge numbers

$$\begin{array}{ccccccccc}
 & & & 1 & & & & & 1 \\
 & & 0 & & 1 & & & & 0 \\
 & 0 & & 1 & & 0 & & & 0 \\
0 & & 1 & & 1 & & 0 & & 2 \\
 & 0 & & 1 & & 0 & & & 0 \\
 & & 1 & & 0 & & & & 0 \\
 & & & 1 & & & & & 1
\end{array}$$

Akao has studied the small deformations in [Akao]. He starts from the description of Calabi-Eckmann manifolds as a quotient of $(\mathbf{C}^{m_1+1} - 0) \times (\mathbf{C}^{m_2+1} - 0)$ by an action of the additive group $\mathbf{C}$ via diagonal matrices $(e^{t\cdot I}, e^{t\cdot\tau\cdot I})$. Deforming the identity matrices to pairs (A, B) and dividing out scalar multiples $(\mu A, \mu B)$ (defining biholomorphically isomorphic manifolds) one gets all small deformations (see 15.4).

3. Example: Iwasawa manifold (cf. [Nakamura])

Let G be the complex Lie group biholomorphic to $\mathbf{C}^3$ but with multiplication defined by

$$\begin{bmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a_2 & a_3 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & z_2 + a_2 & z_3 + a_3 + z_2 a_1 \\ 0 & 1 & z_1 + a_1 \\ 0 & 0 & 1 \end{bmatrix}$$

Λ denotes the lattice of Gaussian integers $\mathbf{Z} \oplus i \cdot \mathbf{Z}$, and Γ is the discrete subgroup of G consisting of those matrices with all entries in Λ. Then via $(z_1, z_2, z_3) \mapsto (z_1, z_2)$ we get a map

$$X := G/\Gamma \xrightarrow{\pi} M := \mathbf{C}/\Lambda \times \mathbf{C}/\Lambda = T \times T .$$

This is an analytic T-principal bundle, $T = \mathbf{C}/\Lambda$ acting by the matrices of the form $a_1 = a_2 = 0$, $a_3 \in T$. Fixing a local lifting $\tilde{z}_2$ for the coordinate z_2 on T local trivializations are

$$\begin{array}{ll} \pi^{-1}(U) \ni (z_1, z_2, z_3) \mapsto (z_1, z_2, t) = & (z_1, z_2, z_3 - z_1 \tilde{z}_2) \in U \times U \times T \\ \bmod \Gamma & \bmod \Lambda \times \Lambda \times \Lambda \end{array}$$

The inverse mapping is given by $z_3 = t + z_1 \tilde{z}_2$. So the transition functions are $\phi_{ij} = z_1 \cdot \lambda_{ij}$ for some $\lambda_{ij} \in \Lambda$ representing the difference between two liftings of z_2. The G-invariant holomorphic 1-form $-z_1 dz_2 + dz_3$ on G descends to a form ω on X which in local bundle coordinates is

$$\omega = dt + \tilde{z}_2 \cdot dz_1 .$$

The invariants are ([Nakamura] or Chapter 10)

$$\begin{array}{ccccccccc} & & & 1 & & & & & 1 \\ & & 3 & & 2 & & & & 4 \\ & 3 & & 6 & & 2 & & & 8 \\ 1 & & 6 & & 6 & & 1 & & 10 \\ & 2 & & 6 & & 3 & & & 8 \\ & & 2 & & 3 & & & & 4 \\ & & & 1 & & & & & 1 \end{array}$$

Another example of Nakamura shows that our results hold only for principal bundles. He constructs a parallelizable manifold with $h_X^{0,1} = 1$ which is a non-principal 2-torus bundle over an elliptic curve. But by 7.4 we know $h_X^{0,1} = 3$ in the principal bundle case.

4. Spectral sequences of Ω_X^p

4.1. Bundle coordinates. Under a change of bundle coordinates as described in 1.2, the leading term of a differential form $\omega = dz_I \wedge d\bar{z}_{\bar{I}} \wedge dt'_J \wedge d\bar{t}'_{\bar{J}}$ remains unchanged but there are additional components coming from the derivatives of ϕ_{ij}:

$$\phi_{ij}^*\omega = dz_I \wedge d\bar{z}_{\bar{I}} \wedge dt_J \wedge d\bar{t}_{\bar{J}} + \sum_{K\bar{K}L\bar{L}} f_{K\bar{K}L\bar{L}} dz_K \wedge d\bar{z}_{\bar{K}} \wedge dt_L \wedge d\bar{t}_{\bar{L}}$$

with of course $|K| + |L| = |I| + |J|$, $|\bar{K}| + |\bar{L}| = |\bar{I}| + |\bar{J}|$, but only summands with higher base degree occur, i.e. only those with $|K| + |\bar{K}| > |I| + |\bar{I}|$ and $|K| \geq |I|$, $|\bar{K}| \geq |\bar{J}|$.

4.2. Borel spectral sequence. The sheaf Ω_X^p can be resolved by the Dolbeault complex $(\mathscr{A}_X^{p,\cdot}, \bar{\partial})$ of C^∞-$(p, .)$-forms. The direct image complex $\pi_*\mathscr{A}_X^{p,\cdot}$ can be filtered by the base degree of the forms: $F^s\pi_*\mathscr{A}_X^{p,q}$ consists of those forms that in local bundle coordinates (z, t) can be written as a linear combination of $dz_I \wedge d\bar{z}_{\bar{I}} \wedge dt_J \wedge d\bar{t}_{\bar{J}}$ with $|I| + |\bar{I}| \geq s$.

Taking global sections $A_X^{p,q} = \Gamma\mathscr{A}_X^{p,q}$ we get a filtered complex $(A_X^{p,\cdot}, \bar{\partial})$ of modules and from this the spectral sequence as usual. This is the filtration introduced in [Borel] 4.1, and we keep the notation from there. The usual spectral sequence graduation is given by (s, t), corresponding to the filtration, i.e. to the total base and fibre degree of differential forms. We also include (p, q) denoting the $(\partial, \bar{\partial})$-type but p is constant in each of the sequences and we always have $p + q = s + t$.

$${}^{p,q}E_r^{s,t} = \frac{\{\omega \in F^s A_X^{p,q} | \bar{\partial}\omega \in F^{s+r} A_X^{p,q+1}\}}{F^{s+1}A_X^{p,q} + \bar{\partial}(F^{s-r+1}A_X^{p,q-1})}$$

$${}^{p,q}E_r^{s,t} \xrightarrow{d_r} {}^{p,q+1}E_r^{s+r,t-r+1}$$

$$Gr\mathrm{H}_X^{p,q} = \bigoplus_{s+t=p+q} {}^{p,q}E_\infty^{s,t}$$

Note that p is not changed by the differential, and q is determined by $p + q = s + t$, so in fact there is a single spectral sequence for each p, computing the cohomology of Ω_X^p.

Let $\mathscr{A}_{X/M}^{i,j}$ denote the bundle of global C^∞-(i, j)-forms on the fibres. Then the first levels can be interpreted as follows [Borel]:

$${}^{p,q}E_0^{s,t} = \bigoplus_i \Gamma(\mathscr{A}_M^{i,s-i} \otimes \mathscr{A}_{X/M}^{p-i,t-p+i})$$

$${}^{p,q}E_1^{s,t} = \bigoplus_i A_M^{i,s-i} \otimes \mathrm{H}_T^{p-i,t-p+i}$$

$${}^{p,q}E_2^{s,t} = \bigoplus_i \mathrm{H}_M^{i,s-i} \otimes \mathrm{H}_T^{p-i,t-p+i}$$

$$q = 0: \qquad {}^{p,0}E_2^{s,t} = \mathrm{H}_M^{s,0} \otimes \mathrm{H}_T^{t,0}$$

$$p = 0: \qquad {}^{0,q}E_2^{s,t} = \mathrm{H}_M^{0,s} \otimes \mathrm{H}_T^{0,t}$$

The map from $A_X^{p,q}$ to ${}^{p,q}E_0^{\cdot,\cdot}$ is given by locally taking only the well-defined terms with lowest base degree in each fibre; these are glued together to a section of $\mathscr{A}_M^{i,j} \otimes \mathscr{A}_{X/M}^{p-i,q-j}$. The differential d_0 is then $\bar{\partial}$ in fibre direction, so the map to

${}^{p,q}E_1^{s,t}$ is taking Dolbeault cohomology on the torus. The bundle consisting of the Hodge spaces $\mathrm{H}^{i,j}_{\pi^{-1}(z)}$ is trivial, and we are left with a form on M times a cohomology class of T. The next differential d_1 equals $\bar\partial$ of these forms in base direction. These facts are described in [Borel]. We will now investigate d_2 in our special situation. The basic maps are

$$\varepsilon\colon {}^{0,1}E_2^{0,1} = \mathrm{H}_T^{0,1} \xrightarrow{d_2} {}^{0,2}E_2^{2,0} = \mathrm{H}_M^{0,2}$$

$$\gamma\colon {}^{1,0}E_2^{0,1} = \mathrm{H}_T^{1,0} \xrightarrow{d_2} {}^{1,1}E_2^{2,0} = \mathrm{H}_M^{1,1}\,.$$

4.3. Proposition. (a) *d_2 is a derivation on the product of Hodge algebras $\sum_{p+q=s+t}\sum_i \mathrm{H}_M^{i,s-i}\otimes \mathrm{H}_T^{p-i,q-s+i}$:*

$$d_2(\omega\wedge\eta) = d_2\omega\wedge\eta + \omega\wedge d_2\eta\,.$$

(b) $d_2(\mathrm{H}_M^{i,j}) = 0$.

(c) $\gamma\colon \mathrm{H}_T^{1,0}\to \mathrm{H}_M^{1,1}$ *is the invariant introduced in* 1.5(a).

(d) $\varepsilon\colon \mathrm{H}_T^{0,1}\to \mathrm{H}_M^{0,2}$ *is the invariant introduced in* 1.5(c), *i.e. the d_2 map from Leray spectral sequence for $\mathcal{O}_X$. It vanishes if and only if the spectral sequence for $p=0$ degenerates at E_2-level, i.e. if $h_X^{0,q} = \sum h_M^{0,i}\cdot h_T^{0,q-i}$.*

(e) $d_r = 0$ *for* $r > 2$.

Proof. $d_2(\omega)$ is computed by lifting the cohomology class to a global C^∞ form on X, then taking $\bar\partial$ and projecting back to E_2. Since the projection respects wedge products, d_2 behaves like a differential. This proves (a). Because $\omega\in \mathrm{H}_M^{i,j}$ can be lifted to the $\bar\partial$-closed form $\pi^*\omega$, all $d_r(\omega)$ vanish for $r\geq 2$. So (b) holds. (e) follows because statements (a) and (b) hold also for $r>2$, and the generators dt_i and $d\bar t_i$ of $\mathrm{H}_T^{\cdot,\cdot}$ are not affected by higher d_r because of their degrees.

(c): We resolve

$$0\to \Omega_M^1 \to \pi_*\Omega_X^1 \to \mathcal{O}_M\otimes \mathrm{H}_T^{1,0}\to 0$$

by

$$0\to \mathscr{A}_M^{1,\cdot}\to \mathscr{B}^\cdot \to \mathscr{A}_M^{0,\cdot}\otimes \mathrm{H}_T^{1,0}\to 0$$

where $\mathscr{B}^q(U)$ consists of those $(1,q)$-forms on $\pi^*(U)$ that are harmonic in fibre direction, i.e. which in local bundle coordinates can be written as $\sum\omega_i\wedge dt_i + \eta$ with ω_i and η being forms on M. The map to $\mathscr{A}_M^{0,q}\otimes \mathrm{H}_T^{1,0}$ is the (well-defined) projection to $\sum\omega_i\wedge dt_i$. Then γ as defined in 1.5 is the connecting homomorphism $\mathrm{H}^0(\mathscr{A}_M^{0,\cdot}\otimes \mathrm{H}_T^{1,0})\to \mathrm{H}^1(\mathscr{A}_M^{1,\cdot})$, it is the obstruction to lifting dt_i to a global section of $\pi_*\Omega_X^1$: Locally on a system of trivializing neighbourhoods (U_α) for the bundle on M, it can be lifted to dt_i, and the difference of two liftings on $U_{\alpha\beta}$ is $dt_i - \phi_{\alpha\beta}^* dt_i$ and defines a 1-cocycle in $\mathscr{A}_M^{1,0}$ which must be a boundary since the sheaf is acyclic. Thus there is a 0-chain $(\rho_\alpha)\in \mathrm{H}^0(\mathscr{A}_M^{1,\cdot})$ such that $(dt_i - \rho_\alpha)$ is a global form, the lifting to $\Gamma\mathscr{B}^0$, and its differential $\bar\partial\rho_\alpha$ is $\gamma(dt_i)$. But this is exactly how the differential in the spectral sequence works.

So it remains to show (d): The d_2 differential $\mathrm{H}^0(M, \mathrm{R}^1\pi_*\mathcal{O}_X) \to \mathrm{H}^2(\pi_*\mathcal{O}_X)$ of Leray spectral sequence may be computed by using the same resolution of $\mathcal{O}_X$ as above: $\mathrm{R}\pi_*\mathcal{O}_X = (\pi_*\mathscr{A}_X^{0,\bullet}, \bar\partial)$. An element of $\mathrm{H}^0(M, \mathrm{R}^1\pi_*\mathcal{O}_X)$ is represented by a cocycle $(\bar\partial\psi_\alpha)$ where $\psi_\alpha \in \Gamma(\pi^{-1}U_\alpha, \mathscr{A}_X^{0,0})$. Its image in $\mathrm{H}^2(\pi_*\mathcal{O}_X)$ is represented by the cocycle $(\delta\psi) = (\chi_{\alpha\beta})$, $\chi_{\alpha\beta} \in \Gamma(\pi^{-1}(U_\alpha \cap U_\beta), \mathscr{A}_X^{0,0})$. The same argumentation as in (c) but with $d\bar t_i$ instead of dt_i proves the assertion.

4.4. Corollary. *The spectral sequence degenerates at E_3-level and the d_2-differential is wholly determined by the two maps $\mathrm{H}_T^{1,0} \xrightarrow{\gamma} \mathrm{H}_M^{1,1}$ and $\mathrm{H}_T^{0,1} \xrightarrow{\varepsilon} \mathrm{H}_M^{0,2}$.*

4.5. Remark. The Leray spectral sequence for Ω_X^p also converges to $\mathrm{H}_X^{p,q}$ but has E_2-term $E_2^{i,j} = \mathrm{H}^i(\mathrm{R}^j\pi_*\Omega_X^p)$. Except for $p = 0$ the higher direct image sheaves are non-trial, the twisting being measured by γ. The d_2-differential, on the other hand, is determined by ε, and the Leray spectral sequence should degenerate if $\varepsilon = 0$.

5. Leray spectral sequence of $\mathbf{C}_X$

Taking de Rham cohomology (with complex valued forms) instead of Dolbeault cohomology, we get the usual Leray spectral sequence converging to $\mathrm{H}^\bullet(X, \mathbf{C})$. Here the constant sheaf $\mathbf{C}_X$ is resolved by the de Rham complex $(\mathscr{A}_X^\bullet, d)$ of C^∞-forms, the filtration is again defined by base degree. Everything works like described above, now defining a spectral sequence with

$$E_2^{s,t} = \mathrm{H}^s(M, \mathbf{C}) \otimes \mathrm{H}^t(T, \mathbf{C})\,.$$

The basic map is $\delta\colon E_2^{0,1} = \mathrm{H}^1(T, \mathbf{C}) \xrightarrow{d_2} E_2^{2,0} = \mathrm{H}^2(M, \mathbf{C})$. With the same arguments as in 4.3 we get:

5.1. Proposition. (a) *d_2 is a derivation on the product of cohomology rings $\mathrm{H}^\bullet(M, \mathbf{C}) \otimes \mathrm{H}^\bullet(T, \mathbf{C})$, i.e. $d_2(\omega \cup \eta) = d_2\omega \cup \eta + \omega \cup d_2\eta$.*

(b) *$d_2(\mathrm{H}^s(M, \mathbf{C})) = 0$.*

(c) *$d_r = 0$ for $r > 2$.*

5.2. Proposition. *The following statements are equivalent*:

(i) *The Leray spectral sequence for $\mathbf{C}_X$ degenerates at E_2-level*

(ii) *$\delta\colon \mathrm{H}^1(T, \mathbf{C}) \to \mathrm{H}^2(M, \mathbf{C})$ is the zero map*

(iii) *The restriction map $\mathrm{H}^2(X, \mathbf{C}) \to \mathrm{H}^2(T, \mathbf{C})$ takes a non-zero value in $\mathrm{H}_T^{1,1}$*

By Blanchard's Theorem (1.7), *for a Kähler base space these statements are equivalent to X being a Kähler manifold.*

Proof. All the $\mathrm{R}^i\pi_*\mathbf{C}_X$ are constant, and any element of $\mathrm{H}_T^{1,1} \subset \mathrm{H}^2(T, \mathbf{C}) = \mathrm{H}^0(M, \mathrm{R}^2\pi_*\mathbf{C}_X)$ is a Kähler class and therefore induces isomorphisms in the cohomology of the fibres. So (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) is the statement of [Deligne], (2.11).

5.3. Proposition. *Let $\iota\colon T \to X$ denote the inclusion of a fibre.*

(a) *For each p there is an exact sequence*

$$\mathrm{H}^{p-2}(M, \mathbf{C}) \otimes \mathrm{H}^1(T, \mathbf{C}) \xrightarrow{d_2} \mathrm{H}^p(M, \mathbf{C}) \xrightarrow{\pi^*} \mathrm{H}^p(X, \mathbf{C})$$

where $d_2(\omega \otimes \theta) = \omega \cup \delta(\theta)$.

(b) *There is an exact sequence*

$$0 \to \mathrm{H}^1(M, \mathbf{C}) \xrightarrow{\pi^*} \mathrm{H}^1(X, \mathbf{C}) \xrightarrow{\iota^*} \mathrm{H}^1(T, \mathbf{C}) \xrightarrow{\delta} \mathrm{H}^2(M, \mathbf{C}) \xrightarrow{\pi^*} \mathrm{H}^2(X, \mathbf{C}).$$

(c) *If δ is non-zero, then the pull-back $\pi^*\colon \mathrm{H}^{2m}(M, \mathbf{C}) \to \mathrm{H}^{2m}(X, \mathbf{C})$ from top dimensional cohomology of the base space ($m = \dim_{\mathbf{C}} M$) is the zero map.*

Proof. The pull back map occurs in Leray spectral sequence as the composite $\mathrm{H}^p(M) = E_2^{p,0} \to E_\infty^{p,0} \hookrightarrow \mathrm{H}^p(X)$ ([Whitehead], XIII.7.2*). Since the spectral sequence degenerates on E_3-level, the first map only divides out the image of $f\colon E_2^{p-2,1} \xrightarrow{d_2} E_2^{p,0}$. For $p = 2$ we can extend the sequence to the left by the standard spectral sequence argument. In top dimension, since the target space is one-dimensional, it suffices to show that f is non-zero. But if $b \in \mathrm{H}^1(T)$ is any element with $\delta(b) \neq 0$, and $a \in \mathrm{H}^{2m-2}(M)$ is the Poincaré dual of $\delta(b)$ ([Dold] VIII.8.13), then $f(a \otimes b) = a \cup b$ is non-zero in $\mathrm{H}^{2m}(M)$.

5.4. Leray-Serre spectral sequence in (integral) homology. (see [Whitehead], XIII.4.9, XIII.7) This is a first quadrant spectral sequence with $E^2_{p,q} = \mathrm{H}_p(X, \mathrm{H}_q(\mathscr{F}))$ ($\mathrm{H}_q(\mathscr{F})$ the local coefficient system of the fibration) converging to the homology of X with differentials $d_r\colon E^r_{p,q} \to E^r_{p-2,q+1}$. Always $E^2_{0,q} = \mathrm{H}_q(T)$ and, since $\mathrm{H}_0(\mathscr{F})$ is trivial, $E^2_{p,0} = \mathrm{H}_p(M)$. There is a commutative diagram ([Whitehead] XIII.7.8, 7.9)

$$\begin{tikzcd}
& \mathrm{H}_2(X, T) \arrow[dl, "\pi_*"'] \arrow[dr, "\partial_*"] & \\
\mathrm{H}_2(X) \arrow[d, equal] & & \mathrm{H}_1(T) \arrow[d, equal] \\
E^2_{2,0} \arrow[rr, "d_2"] & & E^2_{0,1}
\end{tikzcd}$$

where π_* is the surjective ([Whitehead] XIII.7.3) projection map to $H_\bullet(X, pt) = \mathrm{H}_\bullet(X)$ in relative homology and ∂_* is the connecting homomorphism from the long exact homology sequence of the pair (X, T). The transgression is by definition the (well-defined) map $\partial_* \circ \pi_*^{-1}$.

5.5. Leray-Serre spectral sequence in integral cohomology. This is dual to 5.4. The transgression is now the composition of the connecting homomorphism $\mathrm{H}^1(T, \mathbf{Z}) \to \mathrm{H}^2(X, T; \mathbf{Z})$ and the inverse of the injective map $\mathrm{H}^2(M, \mathbf{Z}) = \mathrm{H}^2(M, pt; \mathbf{Z}) \xrightarrow{\pi^*} \mathrm{H}^2(X, T; \mathbf{Z})$.

Since there are no higher differentials or possibly nontrivial local coefficient systems involved, statement 5.3(b) holds also in integral cohomology and homology:

5.6. Proposition. *There are exact sequences*

(a) $0 \to \mathrm{H}^1(M, \mathbf{Z}) \xrightarrow{\pi^*} \mathrm{H}^1(X, \mathbf{Z}) \xrightarrow{\iota^*} \mathrm{H}^1(T, \mathbf{Z}) \xrightarrow{\delta^{\mathbf{Z}}} \mathrm{H}^2(M, \mathbf{Z}) \xrightarrow{\pi^*} \mathrm{H}^2(X, \mathbf{Z})$.

(b) $H_2(X, \mathbf{Z}) \xrightarrow{\pi_*} \mathrm{H}_2(M, \mathbf{Z}) \xrightarrow{\delta_{\mathbf{Z}}} \mathrm{H}_1(T, \mathbf{Z}) \xrightarrow{\iota_*} \mathrm{H}_1(X, \mathbf{Z}) \xrightarrow{\pi_*} \mathrm{H}_1(M, \mathbf{Z}) \to 0$

5.7. Corollary. *$c_1(X) = \pi^* c_1(M)$ is zero if and only if $c_1(M) \in \operatorname{im} \delta^{\mathbf{Z}}$.*

6. Relations between the invariants

6.1. Theorem. *Under the identification* $\mathrm{H}^1(T, \mathbf{Z}) = \mathrm{Hom}(\Lambda, \mathbf{Z})$ *the characteristic class* $c^{\mathbf{Z}} \in \mathrm{H}^2(M, \mathbf{Z}) \otimes \Lambda$ *and the map* $\delta^{\mathbf{Z}}: \mathrm{H}^1(T, \mathbf{Z}) \to \mathrm{H}^2(M, \mathbf{Z})$ *coincide.*

Proof. We resolve $\mathbf{C}_X$ by the de Rham complex and compute $\delta^{\mathbf{Z}}$ analogous to the proof of 4.3(d). It is easy to see that this corresponds to a Čech cocycle representing $c^{\mathbf{Z}}$.

6.2. Proposition. *δ is obtained from $\delta^{\mathbf{Z}}$ by scalar extension:*

$$\delta = \delta^{\mathbf{Z}} \otimes \mathrm{id}_{\mathbf{C}}: \mathrm{H}^1(T, \mathbf{C}) = \mathrm{H}^1(T, \mathbf{Z}) \otimes \mathbf{C} \to \mathrm{H}^2(M, \mathbf{Z}) \otimes \mathbf{C} = \mathrm{H}^2(M, \mathbf{C}).$$

In particular, δ commutes with complex conjugation.

Proof. This follows from the universal coefficient theorem (e.g. [Dold], VI.7.8) because δ and $\delta^{\mathbf{Z}}$ are the transgressions in cohomology with complex and integral coefficients (Chapter 5).

6.3. Theorem. *Assume that* $\mathrm{H}^2(M)$ *has a Hodge decomposition. Let* $\bar{\ }$ *denote complex conjugation. Then identifying* $\mathrm{H}^1(T, \mathbf{C})$ *with* $\mathrm{H}_T^{1,0} \oplus \mathrm{H}_T^{0,1}$ *we can write*

$$\varepsilon = \mathrm{pr}_{02} \circ \delta|_{H_T^{0,1}}, \qquad \varepsilon \circ \mathrm{pr}_{01} = \mathrm{pr}_{02} \circ \delta,$$

$$\gamma = \mathrm{pr}_{11} \circ \delta|_{H_T^{1,0}},$$

$$\delta(a^{10} + a^{01}) = \overline{\varepsilon(\overline{a^{10}})} + \gamma(a^{10}) + \overline{\gamma(\overline{a^{01}})} + \varepsilon(a^{01}) \qquad (a^{10} \in \mathrm{H}_T^{1,0}, a^{01} \in \mathrm{H}_T^{0,1})$$

Proof. All the maps follow the same pattern: Take a closed 1-form ω on T, lift it to a global 1-form $\tilde{\omega}$ on X that locally can be written as $\tilde{\omega}|_{\pi^{-1}U_\alpha} = \omega + \eta_\alpha$, where η_α is a 1-form on U_α. Then the exterior derivatives of the η_α define a closed global 2-form on the base which represents the image of ω. For δ we have to take de Rham cohomology while ε and γ are defined by Dolbeault cohomology. The claims follow from

$$\delta(dt_i) = d(\eta_\alpha^i) = \overline{\bar{\partial}(\overline{\eta_\alpha^i})} + \bar{\partial}(\eta_\alpha^i) = \overline{\varepsilon(d\bar{t}_i)} + \gamma(dt_i)$$

$$\delta(d\bar{t}_i) = d(\overline{\eta_\alpha^i}) = \overline{\bar{\partial}(\eta_\alpha^i)} + \bar{\partial}(\overline{\eta_\alpha^i}) = \overline{\gamma(dt_i)} + \varepsilon(d\bar{t}_i)$$

6.4. Corollary. (a) $\delta = 0 \iff \gamma = 0$ *and* $\varepsilon = 0$.

(b) ε *injective* $\iff$ δ *injective.*

(c) δ *injective* $\iff$ $\delta^{\mathbf{Z}}$ *injective.*

Proof. (a) and (c) are obvious. (b): $0 = \delta(a) = \delta(a^{10} + a^{01}) \Rightarrow \overline{\varepsilon(\overline{a^{10}})} = 0$, $\varepsilon(a^{01}) = 0 \Rightarrow a^{10} = a^{01} = 0 \Rightarrow a = 0$. So δ is injective.

The first Hodge numbers are

$$h_X^{0,1} = h_M^{0,1} + \dim \ker \varepsilon$$

$$h_X^{1,0} = h_M^{1,0} + \dim \ker \gamma$$

$$b_1(X) = b_1(M) + \dim \ker \delta$$

6.5. Relation between integral and complex structure on the torus. (cf. [GH] p. 300 ff or [Wells] VI.1.6) We have to connect integral structure and Hodge decomposition of the cohomology of the torus.

Let $(\lambda_1 \dots \lambda_{2n})$ be a basis of $\Lambda = \mathrm{H}_1(T, \mathbf{Z})$, embedded in the complex vector space V with basis $(e_1 \dots e_n)$ and corresponding complex coordinates $(t_1 \dots t_n)$. Let $\Omega = (\omega_{i\nu})$ be the $(n \times 2n)$ period matrix, i.e. its ν-th column contains the t-coordinates of λ_ν. So Ω is the matrix of the $\mathbf{C}$-linear map

$$\psi\colon \mathrm{H}_1(T, \mathbf{C}) = \Lambda \otimes \mathbf{C} \to V \qquad \psi(\lambda_\nu) = \sum \omega_{i\nu} e_i$$

induced by the inclusion $\Lambda \subset V$ with respect to the bases $(\lambda_1 \dots \lambda_{2n})$ of $\Lambda \otimes \mathbf{C}$ and $(e_1 \dots e_n)$ of V. If $(x_1 \dots x_{2n})$ is the real coordinate system of V whose unit vectors are the λ_ν, then the coordinate change is

$$t_i = \sum_\nu \omega_{i\nu} x_\nu \, .$$

Coordinates of V descend to coordinates of $T = V/\Lambda$. So we get two bases of the de Rham cohomology $\mathrm{H}^1(T, \mathbf{C})$. The first one consists of $(dx_1 \dots dx_{2n})$ and reflects the integral structure $\mathrm{H}^1(T, \mathbf{Z}) \subset \mathrm{H}^1(T, \mathbf{C})$, it is dual to the basis $(\lambda_1 \dots \lambda_{2n})$ of $\Lambda \otimes \mathbf{C} = \mathrm{H}_1(T, \mathbf{C})$. The second basis is formed by $(dt_1 \dots dt_n, d\bar{t}_1 \dots d\bar{t}_n)$, where the first vectors span $\mathrm{H}_T^{1,0}$ and the last ones span $\mathrm{H}_T^{0,1}$. The differential forms are transformed as

$$dt_i = \sum \omega_{i\nu} dx_\nu$$

$$d\bar{t}_i = \sum \bar{\omega}_{i\nu} dx_\nu$$

So the change of basis $\mathrm{H}_T^{1,0} \oplus \mathrm{H}_T^{0,1} \xrightarrow{=} \mathrm{H}^1(T, \mathbf{C})$ is described by ${}^t\tilde{\Omega} = ({}^t\Omega \, {}^t\bar{\Omega})$, where ${}^t\Omega$ and ${}^t\bar{\Omega}$ correspond to the injections of $\mathrm{H}_T^{1,0}$ and $\mathrm{H}_T^{0,1}$, respectively. The inverse of $\tilde{\Omega}$ is usually denoted by $\tilde{\Pi} = (\Pi | \bar{\Pi})$, $\Pi = (\pi_{\nu i})$. So $\Omega \cdot \Pi = 1_n$, $\Omega \cdot \bar{\Pi} = 0$, $\Pi \cdot \Omega + \bar{\Pi} \cdot \bar{\Omega} = 1_{2n}$, and

$$dx_\nu = \sum \pi_{\nu i} dt_i + \sum \bar{\pi}_{\nu i} d\bar{t}_i \, .$$

Let $c^{\mathbf{Z}} = \sum \xi_{k\nu} \alpha_k \otimes \lambda_\nu \in \mathrm{H}^2(M, \mathbf{Z}) \otimes \Lambda$ be the characteristic class, i.e. $\delta^{\mathbf{Z}}\colon \mathrm{H}^1(T, \mathbf{Z}) \to \mathrm{H}^2(M, \mathbf{Z})$ is given by the integral $(b_2(M) \times 2n)$-matrix $D := (\xi_{k\nu})$ with respect to the dual basis of $(\lambda_1 \dots \lambda_{2n})$ and some basis $(\alpha_1 \dots \alpha_{b_2})$ of $\mathrm{H}^2(M, \mathbf{Z})$ (ignoring torsion) (6.1).

The invariant δ is described by D with respect to the (dx_ν) (6.3) and by $D \cdot {}^t\tilde{\Omega}$ with respect to the $(dt_i, d\bar{t}_i)$:

$$\delta(dt_i) = \sum_k \left(\sum_\nu \xi_{k\nu} \omega_{i\nu} \right) \alpha_k = \sum_k \zeta_{ki} \alpha_k$$

where (ζ_{ki}) is the $(b_2(M) \times n)$-matrix $D \cdot {}^t\Omega$.

The characteristic class $c^{\mathbf{Z}} \in \mathrm{H}^2(M, \mathbf{Z}) \otimes \Lambda$ can be considered as an element of $\mathrm{H}^2(M, \mathbf{C}) \otimes_{\mathbf{C}} (\Lambda \otimes \mathbf{C})$. The invariant c defined in 1.3 is then $c = \mathrm{id} \otimes \psi(c^{\mathbf{Z}}) \in \mathrm{H}^2(M, \mathbf{C}) \otimes V$, i.e.

$$c = \sum \xi_{k\nu}\alpha_k \otimes \psi(\lambda_\nu) = \sum \xi_{k\nu}\omega_{i\nu}\alpha_k \otimes e_i = \sum \zeta_{ki}\alpha_k \otimes e_i \,.$$

Now assume that $\mathrm{H}^2(M)$ has a Hodge decomposition, the projections to the different components being described by matrices P_{20}, P_{11}, P_{02} with respect to the (α_k) and some bases of the Hodge components such that $P_{02} = \overline{P_{20}}$. Then the composite matrix has the form

$$P \cdot D \cdot {}^t\tilde{\Omega} = \begin{bmatrix} P_{20} \\ P_{11} \\ P_{02} \end{bmatrix} \cdot D \cdot ({}^t\Omega\; {}^t\bar{\Omega}) = \begin{bmatrix} \bar{E} & 0 \\ C & \bar{C} \\ 0 & E \end{bmatrix}$$

where E and C describe ε and γ, respectively (6.3). The symmetry in the matrix comes from the fact that δ commutes with complex conjugation.

The characteristic class $c^{\mathbf{Z}}$ had been defined by the cohomology sequence of $0 \to \Lambda \to \mathcal{O}_M \otimes V \to \mathcal{O}_M(T) \to 0$. Since the first inclusion of sheaves factorizes over $\Lambda \subset V \subset \mathcal{O}_M \otimes V$, in cohomology we have

$$\begin{tikzcd} {} \arrow[r] & \mathrm{H}^1(\mathcal{O}_M(T)) \arrow[r, "c^{\mathbf{Z}}(\cdot)"] & \mathrm{H}^2(M, \Lambda) \arrow[r, "\phi"] \arrow[d] & \mathrm{H}_M^{0,2} \otimes V \arrow[r] & {} \\ & & H^2(M, \mathbf{C}) \otimes V \arrow[ur] & & \end{tikzcd}$$

Thus the obstruction map for a given $c^{\mathbf{Z}}$ being the characteristic class of some bundle sees only $c = \sum \zeta_{ki}\alpha_k \otimes e_i$ and projects the $\mathrm{H}^2(M, \mathbf{C})$-part to its $(0, 2)$-component. In matrix notation this is $P_{02} \cdot D \cdot {}^t\Omega$, i.e. the 0-block in the matrix for $P \cdot D \cdot {}^t\tilde{\Omega}$ above. This proves

6.6. Proposition. *Consider any element $\tilde{c}^{\mathbf{Z}} \in \mathrm{H}^2(M, \mathbf{Z}) \otimes \Lambda$ or, equivalently, $\tilde{\delta}^{\mathbf{Z}}: \mathrm{H}^1(T, \mathbf{Z}) \to \mathrm{H}^2(M, \mathbf{Z})$. This is the characteristic class of some T-principal bundle on M if and only if the obstruction*

$$\boxed{\Delta = \mathrm{pr}_{02} \circ \tilde{\delta}|_{H_T^{1,0}}: \mathrm{H}_T^{1,0} \to \mathrm{H}_M^{0,2}}$$

is zero, where $\tilde{\delta} = \tilde{\delta}^{\mathbf{Z}} \otimes \mathrm{id}_{\mathbf{C}}: \mathrm{H}^1(T, \mathbf{C}) \to \mathrm{H}^2(M, \mathbf{C})$.

6.7. On the other hand, if we start with two maps $\tilde{\gamma}: \mathrm{H}_T^{1,0} \to \mathrm{H}_M^{1,1}$ and $\tilde{\varepsilon}: \mathrm{H}_T^{0,1} \to \mathrm{H}_M^{0,2}$, we define $\tilde{\delta}: \mathrm{H}^1(T, \mathbf{C}) \to \mathrm{H}^2(M, \mathbf{C})$ by the formula in 6.3. Then these invariants come from a bundle iff $\tilde{\delta} = \tilde{\delta}^{\mathbf{Z}} \otimes \mathrm{id}_{\mathbf{C}}$ for some $\tilde{\delta}^{\mathbf{Z}}: \mathrm{H}^1(T, \mathbf{Z}) \to \mathrm{H}^2(M, \mathbf{Z})$.

7. Bundles with $\varepsilon = 0$

7.1. Proposition. *There is an injective map*

$$\Phi\colon \mathrm{Pic}(M) \otimes_{\mathbf{Z}} \Lambda = \mathrm{H}^1(\mathcal{O}_M^*) \otimes_{\mathbf{Z}} \Lambda \to \mathrm{H}^1(\mathcal{O}_M(T))$$

compatible with taking characteristic classes, i.e. if $\sum \mathcal{L}_i \otimes \lambda_i$ *is a combination of line bundles in* $\mathrm{Pic}(M) \otimes \Lambda$ *then the characteristic class* $c^{\mathbf{Z}}$ *of* $\Phi(\sum \mathcal{L}_i \otimes \lambda_i)$ *equals* $\sum c_1(\mathcal{L}_i) \otimes \lambda_i \in \mathrm{H}^2(M, \Lambda)$.

$$\begin{tikzcd}
\mathrm{Pic}(M) \otimes \Lambda \arrow[r, hook, "\Phi"] \arrow[d, "c_1 \otimes \mathrm{id}"] & \mathrm{H}^1(\mathcal{O}_M(T)) \arrow[d, "c^{\mathbf{Z}}"] \\
\mathrm{H}^2(M, \mathbf{Z}) \otimes \Lambda \arrow[r, "="] & \mathrm{H}^2(M, \Lambda)
\end{tikzcd}$$

Proof. Consider the following diagram of $\mathbf{Z}$-modules (with exact rows) obtained by tensoring the exponential sequence by Λ and applying the inclusion $\Lambda \hookrightarrow V$. The rightmost vertical homomorphism maps $\sum \zeta_j \otimes \lambda_j \in \mathbf{C}^* \otimes_{\mathbf{Z}} \Lambda$ to $\sum \log \zeta_j \cdot \lambda_j \in V \bmod \Lambda$.

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r] \arrow[d, equal] & \mathbf{C} \otimes_{\mathbf{Z}} \Lambda \arrow[r, "\exp"] \arrow[d] & \mathbf{C}^* \otimes_{\mathbf{Z}} \Lambda \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Lambda \arrow[r] & V \arrow[r] & T \arrow[r] & 0
\end{tikzcd}$$

Sheafifying over M and taking cohomology yield the following diagram. Short diagram chasing shows that Φ is injective.

$$\begin{tikzcd}
\mathrm{H}^1(\Lambda) \arrow[r] \arrow[d, equal] & \mathrm{H}_M^{0,1} \otimes_{\mathbf{Z}} \Lambda \arrow[r] \arrow[d] & \mathrm{H}^1(\mathcal{O}_M^*) \otimes_{\mathbf{Z}} \Lambda \arrow[r] \arrow[d, "\Phi"] & \mathrm{H}^2(M, \Lambda) \arrow[r] \arrow[d, equal] & \mathrm{H}_M^{0,2} \otimes_{\mathbf{Z}} \Lambda \arrow[d] \\
\mathrm{H}^1(\Lambda) \arrow[r] & \mathrm{H}_M^{0,1} \otimes_{\mathbf{C}} V \arrow[r] & \mathrm{H}^1(\mathcal{O}_M(T)) \arrow[r] & \mathrm{H}^2(M, \Lambda) \arrow[r] & \mathrm{H}_M^{0,2} \otimes_{\mathbf{C}} V
\end{tikzcd}$$

7.2. Corollary. *If* $\mathrm{H}^2(M)$ *has a Hodge decomposition, then the image of* Φ, *i.e. the set of isomorphism classes of principal bundles constructed in the previous proposition, equals*

$$\begin{aligned}
\mathrm{im}\, \Phi &= \{\textit{Isomorphism classes of } T\textit{-principal bundles with } c^{\mathbf{Z}} \in (\mathrm{H}_M^{1,1} \cap \mathrm{H}^2(M, \mathbf{Z})) \otimes \Lambda\} \\
&= \{\textit{Isomorphism classes of } T\textit{-principal bundles with } \varepsilon = 0\}
\end{aligned}$$

Moreover, any $\tilde{c}^{\mathbf{Z}} \in (\mathrm{H}_M^{1,1} \cap \mathrm{H}^2(M, \mathbf{Z})) \otimes \Lambda$ *is the characteristic class of such a bundle.*

Proof. This follows from $c_1(\mathrm{H}^1(\mathcal{O}_M^*)) = \mathrm{H}_M^{1,1} \cap \mathrm{H}^2(M, \mathbf{Z})$, 6.2, 6.3, and 6.6.

7.3. Remark. The torus itself does not play any particular role here.

7.4. Fibre bundles over curves. If $\dim M = 1$, then ε vanishes for dimension reasons. So the Hodge numbers $h_X^{0,q}$ behave like for a product. The Betti numbers, however, can be smaller: Consider for example the primary Kodaira surfaces (cf. [BPV] p. 147) which are bundles over an elliptic curve with $h_X^{0,1} = 2$ and $b_1(X) = 3$.

7.5. Elliptic fibre bundles over $\mathbf{P}^m$. Here of course $\varepsilon = 0$, so in the non-trivial case γ must be a surjective map onto $\mathrm{H}_M^{1,1} \cong \mathbf{C}$. But then multiplication by $\gamma(dt)$ is an isomorphism except in first and top cohomology of $\mathbf{P}^m$. This means $h_X^{0,0} = h_X^{0,1} = h_X^{m,m+1} = h_X^{m+1,m+1} = 1$, the other Hodge numbers are zero.

8. Bundles with $\varepsilon \neq 0$

8.1. The image of the map $\mathrm{H}^2(M, \mathbf{Z}) \to \mathrm{H}_M^{0,2}$ induced by the inclusion $\mathbf{Z} \hookrightarrow \mathcal{O}_M$ is an additive subgroup $A = (\mathrm{H}^2(M, \mathbf{Z}))^{02} \subset \mathrm{H}_M^{0,2}$. Usually A is dense in $\mathrm{H}_M^{0,2}$.

$$\begin{tikzcd}
\mathrm{H}^1(T, \mathbf{C}) \arrow[rrr, "\mathrm{pr}_{02}"] \arrow[ddd, "\delta"'] & & & \mathrm{H}_T^{0,1} \arrow[ddd, "\varepsilon"] \\
& \mathrm{H}^1(T, \mathbf{Z}) \arrow[r] \arrow[ul] \arrow[d, "\delta^{\mathbf{Z}}"] & \Lambda' \arrow[ur] \arrow[d] & \\
& \mathrm{H}^2(M, \mathbf{Z}) \arrow[r] \arrow[dl] & A \arrow[dr] & \\
\mathrm{H}^2(M, \mathbf{C}) \arrow[rrr, "\mathrm{pr}_{02}"] & & & \mathrm{H}_M^{0,2}
\end{tikzcd}$$

The above diagram implies

8.2. Proposition. *Let $\Lambda' = \mathrm{pr}_{01}\,(\mathrm{H}^1(T, \mathbf{Z})) \subset \mathrm{H}_T^{0,1}$ be the dual lattice of Λ. Then $\varepsilon(\Lambda') \subset A \cap \varepsilon(\mathrm{H}_T^{0,1})$.*

So the cohomology classes connected to bundles with $\varepsilon \neq 0$ are those not coming from line bundles on M.

For fixed M, there is a restriction on the periods of a torus T which is the fibre of a principal bundle over M with $\varepsilon \neq 0$: The dual lattice must be mapped to the (countable) set A. This means that in contrast to γ for a given M there are only few possible tori T for which a T-principal bundle with, say, injective ε exists.

9. Fibrations by elliptic curves

9.1. Now suppose the fibres are 1-dimensional. Then after choosing a generator dt for $\mathrm{H}_T^{1,0}$ the d_2 differentials become (up to sign) multiplication by $\gamma(dt)$ and $\varepsilon(d\bar{t})$ in the Hodge algebra of M (4.3). The only possibly non-zero terms and differentials are

$$\begin{tikzcd}
{}^{p,q-1}E_2^{p+q-3,2} = & & \mathrm{H}_M^{p-1,q-2}\otimes \mathrm{H}_T^{1,1} \arrow[dl, "\gamma"'] \arrow[dr, "-\varepsilon"] & \\
{}^{p,q}E_2^{p+q-1,1} = & \mathrm{H}_M^{p,q-1}\otimes \mathrm{H}_T^{0,1} \arrow[dr, "\varepsilon"'] & \oplus & \mathrm{H}_M^{p-1,q}\otimes \mathrm{H}_T^{1,0} \arrow[dl, "\gamma"] \\
{}^{p,q+1}E_2^{p+q+1,0} = & & \mathrm{H}_M^{p,q+1}\otimes \mathrm{H}_T^{0,0} &
\end{tikzcd}$$

The $\mathrm{H}_T^{\bullet,\bullet}$ are all 1-dimensional but they help to remember the effect of d_2 on the $\mathrm{H}_M^{\bullet,\bullet}$: The map starting at $\mathrm{H}_M^{\bullet,\bullet}\otimes \mathrm{H}_T^{0,1}$ is multiplication by $\varepsilon(d\bar{t})$, the one starting at $\mathrm{H}_M^{\bullet,\bullet}\otimes \mathrm{H}_T^{1,0}$ multiplies by $\gamma(dt)$ and the last one starting at $\mathrm{H}_M^{\bullet,\bullet}\otimes \mathrm{H}_T^{1,1}$ by $\gamma(dt) - \varepsilon(d\bar{t})$.

9.2. We take $(1, \tau)$ as basis of Λ, 1 as a generator of V and use the notation from 6.5. The change of bases is now

$$dt = dx_1 + \tau\cdot dx_2 \qquad dx_1 = \frac{1}{\bar{\tau}-\tau}(\bar{\tau}\cdot dt - \tau\cdot d\bar{t})$$

$$d\bar{t} = dx_1 + \bar{\tau}\cdot dx_2 \qquad dx_2 = \frac{1}{\bar{\tau}-\tau}(-dt + d\bar{t})\,.$$

Thus any $\lambda \in \Lambda$ can be written as

$$\lambda = \frac{\bar{\tau}\lambda - \tau\bar{\lambda}}{\bar{\tau}-\tau}\cdot 1 + \frac{\bar{\lambda}-\lambda}{\bar{\tau}-\tau}\cdot\tau\,.$$

With $c^{\mathbf{Z}} = a\otimes 1 + b\otimes\tau \in \mathrm{H}^2(M,\mathbf{Z})\otimes\Lambda$ and $c = \eta\otimes 1 \in \mathrm{H}^2(M,\mathbf{C})\otimes V$ we can write the relations between the invariants as follows. Δ is the obstruction for $c^{\mathbf{Z}}$ being the characteristic class of a bundle (6.6).

$$c^{\mathbf{Z}} = a\otimes 1 + b\otimes\tau = \frac{\bar{\tau}\cdot\eta - \tau\cdot\bar{\eta}}{\bar{\tau}-\tau}\otimes 1 + \frac{\bar{\eta}-\eta}{\bar{\tau}-\tau}\otimes\tau$$

$$c = (a + \tau\cdot b)\otimes 1 = \eta\otimes 1$$

$$\Delta = (a+\tau\cdot b)^{02} = \eta^{02}$$

$$\delta\colon\qquad dx_1 \mapsto a = \frac{1}{\bar{\tau}-\tau}(\bar{\tau}\cdot\eta - \tau\cdot\bar{\eta})$$

$$dx_2 \mapsto b = \frac{1}{\bar{\tau}-\tau}(-\eta + \bar{\eta})$$

$$dt \mapsto a + \tau\cdot b = \eta$$

$$d\bar{t} \mapsto a + \bar{\tau}\cdot b = \bar{\eta}$$

$$\varepsilon\colon\qquad d\bar{t} \mapsto (a + \bar{\tau}\cdot b)^{02} = \bar{\eta}^{02}$$

$$\gamma\colon\qquad dt \mapsto (a + \tau\cdot b)^{11} = \eta^{11}$$

9.3. In order to construct an elliptic principal bundle with $\varepsilon \neq 0$, we have to find a, $b \in \mathrm{H}^2(M, \mathbf{Z})$ and $\tau \in \mathbf{C} - \mathbf{R}$ such that $\Delta = (a + \tau \cdot b)^{02} = 0$ and $\varepsilon(d\bar{t}) = (a + \bar{\tau} \cdot b)^{02} \neq 0$, i.e. $a^{02} = -\tau \cdot b^{02}$ but $b^{02} \neq 0$. This is equivalent to finding a, $b \in \mathrm{H}^2(M, \mathbf{Z})$ such that a^{02} and b^{02} are linear dependent over $\mathbf{C}$ but independent over $\mathbf{R}$, and τ is the ratio between them.

9.4. Elliptic bundles over $M = \mathbf{P}^{m_1} \times \mathbf{P}^{m_2}$. Since $h_M^{0,1} = h_M^{0,2} = 0$, the characteristic class map is bijective in this case. If we vary the transition functions defining the Calabi-Eckmann manifolds by parameters $\lambda_1 = l_1 + k_1\tau$, $\lambda_2 = l_2 + k_2\tau \in \Lambda$

$$\phi_{ij,kl}([x],[y]) = \frac{1}{2\pi i}\left(\lambda_1 \cdot \log\frac{x_k}{x_i} + \lambda_2 \cdot \log\frac{y_l}{y_j}\right)$$

we get a family of fibre bundles with characteristic classes

$$c^{\mathbf{Z}} = H_1 \otimes \lambda_1 + H_2 \otimes \lambda_2 = (l_1 H_1 + l_2 H_2) \otimes 1 + (k_1 H_1 + k_2 H_2) \otimes \tau \in \mathrm{H}^2(M, \Lambda)$$

where H_1 and H_2 are generators for the integral cohomology of the two factors (the Chern classes of the hyperplane bundles). These are all elliptic principal bundles over M. Now $\varepsilon = 0$ and $\gamma(dt) = \lambda_1 H_1 + \lambda_2 H_2 \in \mathrm{H}_M^{1,1}$.

Since the Hodge numbers of M are concentrated in the diagonal

$$h_M^{r,r} = \begin{cases} r+1 & 0 \leq r \leq m_1 \\ m_1 + 1 & m_1 < r < m_2 \\ m_1 + m_2 - r + 1 & m_1 \leq r \leq m_1 + m_2 \end{cases}$$

(assuming $m_1 \leq m_2$), for a given p the only contributions to the spectral sequence are

$$^{p,p}E_2^{2p-2,2} = \mathrm{H}_M^{p-1,p-1} \otimes \mathrm{H}_T^{1,1} \to \mathrm{H}_M^{p,p}\mathrm{H}_T^{0,1} = {}^{p,p+1}E_2^{2p,1}$$

$$^{p,p-1}E_2^{2p-2,1} = \mathrm{H}_M^{p-1,p-1} \otimes \mathrm{H}_T^{1,0} \to \mathrm{H}_M^{p,p} = {}^{p,p}E_2^{2p,0}$$

Both are multiplication by $\gamma(dt)$ in the first factor. So if λ_1 and λ_2 are both non-zero, the maps are injective for $p \leq m_2$ and surjective for $p \geq m_1 + 1$ which implies the result stated in Chapter 2 (even if $m_1 = 0$).

10. Iwasawa manifold

10.1. We are now able to compute the invariants of the bundle introduced in Chapter 3. The global holomorphic form ω is a lifting of dt with $\bar{\partial}\omega = 0$ which means $\gamma = 0$ in the spectral sequence. On the other hand, if superscripts distinguish between the two factors of $M = T \times T$, for the complex conjugate $\bar{\partial}\bar{\omega} = \overline{\partial\omega} = -d\bar{t}^1 \wedge d\bar{t}^2 \in \mathrm{H}_M^{0,2}$, so $\delta(d\bar{t}) = \varepsilon(d\bar{t}) = -d\bar{t}^1 \wedge d\bar{t}^2 \in \mathrm{H}_M^{0,2}$ and $\delta(dt) = \overline{\varepsilon(d\bar{t})} = -dt^1 \wedge dt^2 \in \mathrm{H}_M^{2,0}$ in this example. We can compute the characteristic class:

$$\begin{aligned}
\delta(dx_1) &= \frac{1}{2}\delta(dt + d\bar{t}) \\
&= -\frac{1}{2}(dt^1 \wedge dt^2 + d\bar{t}^1 \wedge d\bar{t}^2) \\
&= -\frac{1}{2}((dx_1^1 + i\cdot dx_2^1) \wedge (dx_1^2 + i\cdot dx_2^2) + (dx_1^1 - i\cdot dx_2^1) \wedge (dx_1^2 - i\cdot dx_2^2)) \\
&= -dx_1^1 \wedge dx_1^2 + dx_2^1 \wedge dx_2^2 \\
\delta(dx_2) &= \frac{-i}{2}\delta(dt - d\bar{t}) \\
&= \frac{-i}{2}(-dt^1 \wedge dt^2 + d\bar{t}^1 \wedge d\bar{t}^2) \\
&= \frac{-i}{2}(-(dx_1^1 + i\cdot dx_2^1) \wedge (dx_1^2 + i\cdot dx_2^2) + (dx_1^1 - i\cdot dx_2^1) \wedge (dx_1^2 - i\cdot dx_2^2)) \\
&= -dx_1^1 \wedge dx_2^2 - dx_2^1 \wedge dx_1^2
\end{aligned}$$

Thus

$$\begin{aligned}
c^{\mathbf{Z}} &= (-dx_1^1 \wedge dx_1^2 + dx_2^1 \wedge dx_2^2) \otimes 1 + (-dx_1^1 \wedge dx_2^2 - dx_2^1 \wedge dx_1^2) \otimes i \\
&\in \mathrm{H}^2(M, \mathbf{Z}) \otimes \Lambda
\end{aligned}$$

10.2. ***T*-bundles over** $T \times T$. Let us investigate to which extent the Gauß lattice can be replaced by a different one in the above construction. So we start with an elliptic curve T, set $M = T \times T$ and ask if there is a T-principal bundle on M with $\gamma = 0$ and $\varepsilon \neq 0$. Thus $c^{\mathbf{Z}} = a \otimes 1 + b \otimes \tau$ with a, $b \in (\mathrm{H}_M^{2,0} \oplus \mathrm{H}_M^{0,2}) \cap \mathrm{H}^2(M, \mathbf{Z})$ such that $a^{02} + \tau b^{02} = 0$ but $b \neq 0$ (see 9.2). If we write

$$a = \alpha \cdot dt^1 \wedge dt^2 + \bar{\alpha} \cdot d\bar{t}^1 \wedge d\bar{t}^2\,, \qquad b = \beta \cdot dt^1 \wedge dt^2 + \bar{\beta} \cdot d\bar{t}^1 \wedge d\bar{t}^2$$

then by $dt^i = dx_1^i + \tau dx_2^i$ and $\bar{\alpha} + \tau\bar{\beta} = 0$ the integrality condition is equivalent to

$$\begin{aligned}
\bar{\tau}\beta + \tau\bar{\beta} &\in \mathbf{Z}\,, & \beta + \bar{\beta} &\in \mathbf{Z}\,, \\
\tau\bar{\tau}(\beta + \bar{\beta}) &\in \mathbf{Z}\,, & \tau\beta + \bar{\tau}\bar{\beta} &\in \mathbf{Z}\,, \\
\tau\bar{\tau}(\tau\beta + \bar{\tau}\bar{\beta}) &\in \mathbf{Z}\,, & \tau^2\beta + \bar{\tau}^2\bar{\beta} &\in \mathbf{Z}\,.
\end{aligned}$$

Since the equations are homogeneous, it suffices to find $\beta \in \mathbf{C}$ such that all the expressions are rational. One of $\beta + \bar{\beta}$, $\tau\beta + \bar{\tau}\bar{\beta}$ must be non-zero, so the lattice must satisfy the conditions

$$\tau + \bar{\tau} \in \mathbf{Q}\,, \qquad \tau\bar{\tau} \in \mathbf{Q}\,.$$

These are also sufficient because $\tau^2\beta + \bar{\tau}^2\beta^2 = (\tau + \bar{\tau})(\tau\beta + \bar{\tau}\bar{\beta}) - 2\tau\bar{\tau}(\beta + \bar{\beta})$, any $0 \neq \beta \in \mathbf{Z}$ with $\tau\bar{\tau}\beta$ and $(\tau + \bar{\tau})\beta$ integral will do.

The invariants can be computed from the spectral sequence (9.1), see 13.6. They are the same as for the Iwasawa manifold (with $\tau = i$ and $\beta = -1$) given in Chapter 3.

11. Topology of the total space

We will now investigate homotopy and homology properties of the bundle. Since $\pi_1(T)$ is the only nontrivial homotopy group of a torus, the long exact homotopy sequence of the bundle yields

11.1. Proposition. *$\pi_i(X) \cong \pi_i(M)$ for $i \geq 3$. The first homotopy groups fit in the exact sequence*

$$0 \to \pi_2(X) \to \pi_2(M) \to \pi_1(T) \to \pi_1(X) \to \pi_1(M) \to 0\,.$$

11.2. Proposition. (a) *If $b_1(M) = 0$, then there is an exact sequence*

$$0 \to \mathrm{H}^1(X, \mathbf{C}) \xrightarrow{\iota^*} \mathrm{H}^1(T, \mathbf{C}) \xrightarrow{\delta} \mathrm{H}^2(M, \mathbf{C}) \xrightarrow{\pi^*} \mathrm{H}^2(X, \mathbf{C}) \xrightarrow{\iota^*} \mathrm{H}^2(T, \mathbf{C})\,.$$

(b) *If $\pi_1(M) = 0$, then there are exact sequences*

$$0 \to \mathrm{H}^1(X, \mathbf{Z}) \xrightarrow{\iota^*} \mathrm{H}^1(T, \mathbf{Z}) \xrightarrow{\delta^{\mathbf{Z}}} \mathrm{H}^2(M, \mathbf{Z}) \xrightarrow{\pi^*} \mathrm{H}^2(X, \mathbf{Z}) \xrightarrow{\iota^*} \mathrm{H}^2(T, \mathbf{Z})$$

$$\mathrm{H}_2(T, \mathbf{Z}) \xrightarrow{\iota_*} \mathrm{H}_2(X, \mathbf{Z}) \xrightarrow{\pi_*} \mathrm{H}_2(M, \mathbf{Z}) \xrightarrow{\delta_{\mathbf{Z}}} \mathrm{H}_1(T, \mathbf{Z}) \xrightarrow{\iota_*} \mathrm{H}_1(X, \mathbf{Z}) \to 0\,.$$

Proof. If $b_1(M) = 0$, then we can extend the sequence from 5.3(b) one step further to the right because $E_2^{1,1} = 0$ (Serre spectral sequence, [Whitehead] XIII 7.10). But for $\mathbf{Z}$-coefficients (5.6) we must assume that the base space is simply connected in order to conclude that the local coefficient system is trivial.

11.3. Proposition. (a) *$b_1(X) = 0$ if and only if $b_1(M) = 0$ and δ: $\mathrm{H}^1(T, \mathbf{C}) \to \mathrm{H}^2(M, \mathbf{C})$ is injective.*

(b) *In that case, the restriction to the fibre $\mathrm{H}^2(X, \mathbf{C}) \to \mathrm{H}^2(T, \mathbf{C})$ is zero, the pull-back $\mathrm{H}^2(M, \mathbf{C}) \to \mathrm{H}^2(X, \mathbf{C})$ is surjective, and*

$$b_2(X) = b_2(M) - b_1(T) = b_2(M) - 2n\,.$$

Proof. (a) follows directly from the preceeding theorem. For (b) note that with $\delta = d_2$: $E_2^{0,1} \to E_2^{2,0}$ also d_2: $E_2^{0,2} \to E_2^{2,1}$ is injective. Thus $b_2(X) = b_2(M) - b_1(T)$, and $\mathrm{H}^2(M, \mathbf{C}) \to \mathrm{H}^2(X, \mathbf{C})$ is surjective.

11.4. Corollary. *If $b_1(X) = 0$, then m-fold products $\mathrm{H}^2(X, \mathbf{C}) \otimes \cdots \otimes \mathrm{H}^2(X, \mathbf{C}) \to \mathrm{H}^{2m}(X, \mathbf{C})$ are zero. In particular, there is no Kähler structure on the topological manifold underlying X.*

Proof. By the proposition, all those products come from downstairs. So $(m+1)$-fold products vanish for dimension reasons. By 5.3 this holds already for m.

11.5. Proposition. (a) *X is simply connected if and only if $\pi_1(M)$ is zero and $\delta_{\mathbf{Z}}$ is surjective.*

(b) *In that case the inclusion of a fibre induces zero in homology*: $\mathrm{H}_2(T, \mathbf{Z}) \xrightarrow{0} \mathrm{H}_2(X, \mathbf{Z})$, *thus* $\mathrm{H}_2(X, \mathbf{Z}) = \ker \delta_{\mathbf{Z}}$.

Proof. If M is 1-connected, then $\delta_{\mathbf{Z}}$ coincides with $\pi_2(M) \to \pi_1(T)$ in the homotopy sequence. So (a) follows from 11.1. The proof of (b) is dual to the proof of 11.3.

Analogously:

11.6. Proposition. *Assume that M is simply-connected. Then* $\mathrm{H}^1(X, \mathbf{Z}) = 0$ *if and only if $\delta^{\mathbf{Z}}$ is injective, and in that case* $\mathrm{H}^2(X, \mathbf{Z}) \xrightarrow{\iota^*} \mathrm{H}^2(T, \mathbf{Z})$ *is zero.*

Note that by the universal coefficient theorem (e.g. [Dold], VI.7.8) if M is simply-connected then $\delta^{\mathbf{Z}}$ is the dual of $\delta_{\mathbf{Z}}$. So if $\delta_{\mathbf{Z}}$ is surjective, then $\delta^{\mathbf{Z}}$ is injective. The converse is by no means true, however. But if $\delta^{\mathbf{Z}}$ is an injection onto a direct summand of $\mathrm{H}^2(M, \mathbf{Z})$, then $\delta_{\mathbf{Z}}$ is surjective.

11.7. In general, if we only assume that M is simply connected, then the universal covering $\tilde{X}$ of X is also a fibre bundle over M, with connected fibres since $\pi_1(T)$ generates $\pi_1(X)$. The fibre $\tilde{T}$ is a covering space of T, it is compact exactly if this covering is finite, i.e. if X has finite fundamental group. In fact, $\tilde{T}$ is an Abelian complex Lie group, and $\tilde{X} \to M$ is a $\tilde{T}$-principal bundle. If, for example, T is an elliptic curve, then X is the quotient of a $\mathbf{C}^*$-bundle (total space of a line bundle with zero section removed) by a linear $\mathbf{Z}$-action exactly if $\pi_1(X)$ is not finite. The most extreme case is that $b_1(X)$ equals the fibre dimension n. Then $\tilde{T}$ is $\mathbf{C}^n$ and the bundle is the quotient of an affine principal bundle by the lattice Λ.

If, on the other hand, $\pi_1(X)$ is finite, then we can replace the torus T by a finite covering $\tilde{T}$ which is a compact complex torus again and get a principal bundle with simply-connected total space.

12. Elliptic fibrations over surfaces

This might be the easiest interesting case. Since the total space is a complex 3-manifold, we can use C.T.C. Wall's results on the topology of real 6-manifolds:

12.1. Theorem (C.T.C. Wall's classification of 6-manifolds). *Let $X \xrightarrow{\pi} M$ and $X' \xrightarrow{\pi'} M'$ be two elliptic principal bundles with structure groups T and T' over compct complex surfaces M and M'. Assume that X and X' are simply-connected with torsion-free homology and that the second Stiefel-Whitney classes of the underlying real 6-manifolds* ($w_2(X) = \pi^* c_1(M) \bmod 2$, $w_2(X') = \pi'^* c_1(M') \bmod 2$) *are zero. Then the following statements are equivalent*:

(a) *X and X' are diffeomorphic.*
(b) *X and X' are (orientend) homotopy equivalent.*
(c) *$b_2(X) = b_2(X')$ and $b_3(X) = b_3(X')$.*

Proof. This is [Wall], Thm. 5 and 6, applied to our situation. Here triple products in $\mathrm{H}^2(X)$ are always zero (11.4). Since the characteristic classes of X are pull-backs from M and the pull-back morphism $\mathrm{H}^4(M) \to \mathrm{H}^4(X)$ is zero (5.3), the first Pontrjagin class always vanishes. Therefore the Betti numbers b_2 and b_3 are the only remaining parameters in the classification.

12.2. Theorem (Almost complex structures on 6-manifolds) ([Wall, Thm. 9). *The homotopy classes of almost-complex structures on the 6-manifold underlying a compact complex 3-manifold X are in 1-1 correspondence to elements in $\mathrm{H}^2(X, \mathbf{Z})$ that reduce to the second Stiefel-Whitney class of X.*

12.3. Proposition. *Let X be an elliptic principal bundle over a simply-connected compact complex surface M and assume that the transgression $\delta_{\mathbf{Z}}$: $\mathrm{H}_2(M, \mathbf{Z}) \to \mathrm{H}_1(T, \mathbf{Z})$ is surjective with torsion-free kernel (or that $\delta^{\mathbf{Z}}$: $\mathrm{H}^1(T, \mathbf{Z}) \to \mathrm{H}^2(M, \mathbf{Z})$ is an injection onto a direct summand). Then:*

(a) $\pi_1(X) = 0$.

(b) $\mathrm{H}_2(X, \mathbf{Z})$ *is a free abelian group of rank* $b_2(X) = b_2(M) - 2$.

(c) $\mathrm{H}_3(X, \mathbf{Z})$ *is a free abelian group of rank* $b_3(X) = 2 \cdot b_2(M) - 2$.

(d) X is diffeomorphic to a connected sum $(S^3 \times S^3) \# \cdots \# (S^3 \times S^3) \# Y$ *where Y is obtaned from S^6 by disjoint surgery operations $S^3 \times D^3 \to S^6$.*

(e) *There is no Kähler manifold diffeomorphic to X.*

Proof. (a) and (b) follow from 11.5. Poincaré duality and Universal Coefficient Theorem ([Dold], VIII.8.1, VI.7.10) imply $\mathrm{H}_3(X, \mathbf{Z}) \cong \mathrm{H}^3(X, \mathbf{Z}) \cong (\mathrm{H}_3(X, \mathbf{Z}))^\vee \oplus \mathrm{Ext}\,(\mathrm{H}_2(X, \mathbf{Z}), \mathbf{Z})$, so from (b) we deduce that $\mathrm{H}_3(X, \mathbf{Z})$ is torsion-free. The rank is determinded by $e(X) = 0$. (d) is contained in [Wall], (e) is 11.4.

12.4. Remark. So in order to construct interesting bundles on a simply-connected surface M, we start with a candidate for a characteristic class $c^{\mathbf{Z}} = a_1 \otimes \lambda_1 + a_2 \otimes \lambda_2 \in \mathrm{H}^2(M, \mathbf{Z}) \otimes \Lambda$. A corresponding principal bundle exists iff the obstruction $\Delta = \lambda_1 \cdot a_1^{02} + \lambda_2 \cdot a_2^{02} \in \mathrm{H}_M^{0,2}$ vanishes (6.6). This is always fulfilled if the a_i are Chern classes of line bundles.

By the preceeding proposition, X will be simply-connected if a_1, a_2 form a basis of a direct summand of $\mathrm{H}^2(M, \mathbf{Z})$. Moreover, if $c_1(M)$ is in the span of the a_i then $c_1(X)$ will be zero (5.7). Then by 12.1 the diffeomorphism class of the total space is determined only by $b_2(M)$. If $\mathrm{H}^2(M)$ has a Hodge decomposition, then ε and γ are determined by $c^{\mathbf{Z}}$ (9.2).

12.5. Computation of the Hodge numbers. According to 9.1 the only non-trivial terms in the spectral sequence of $\mathcal{O}_X$ are ${}^{0,q}E_2^{q-1,1} \to {}^{0,q+1}E_2^{q,0}$. Contributions come from

$$
\begin{aligned}
{}^{0,3}E_2^{2,1} &= \mathrm{H}_M^{0,2} \otimes \mathrm{H}_T^{0,1} \\
{}^{0,2}E_2^{1,1} &= \mathrm{H}_M^{0,1} \otimes \mathrm{H}_T^{0,1} \\
{}^{0,1}E_2^{1,0} &= \mathrm{H}_M^{0,1} \\
{}^{0,0}E_2^{0,0} &= \mathbf{C} \\
{}^{0,1}E_2^{0,1} &= \mathrm{H}_T^{0,1} \xrightarrow{\varepsilon} \mathrm{H}_M^{0,2} = {}^{0,2}E_2^{2,0}
\end{aligned}
$$

Thus

$$h_X^{0,0} = 1 \quad h_X^{0,1} = h_M^{0,1} + \dim\ker\varepsilon \quad h_X^{0,2} = h_M^{0,1} + h_M^{0,2} - \dim\operatorname{im}\varepsilon \quad h_X^{0,3} = h_M^{0,2}$$

The other Hodge numbers are not so easy to compute, because multiplication in the Hodge ring of M is involved. For $p = 1$ the information is contained in (see 9.1):

$$\begin{array}{rcl} {}^{1,q-1}E_2^{q-2,2} = & & \mathrm{H}_M^{0,q-2} \otimes \mathrm{H}_T^{1,1} \\ & & \swarrow^{\gamma} \qquad \searrow^{-\varepsilon} \\ {}^{1,q}E_2^{q,1} = & & \mathrm{H}_M^{1,q-1} \otimes \mathrm{H}_T^{0,1} \oplus \mathrm{H}_M^{0,q} \otimes \mathrm{H}_T^{1,0} \\ & & \searrow^{\varepsilon} \qquad \swarrow^{\gamma} \\ {}^{1,q+1}E_2^{q+2,0} = & & \mathrm{H}_M^{1,q+1} \otimes \mathrm{H}_T^{0,0} \end{array}$$

12.6. Surfaces with $h_M^{1,0} = h_M^{0,1} = 0$. In this case, the only nontrivial terms in the spectral sequence of Ω_X^1 appear in

$$\begin{aligned} {}^{1,3}E_2^{0,2} &= \mathrm{H}_M^{0,2} \otimes \mathrm{H}_T^{1,1} \\ {}^{1,1}E_2^{0,2} &= \mathrm{H}_T^{1,1} \xrightarrow{\gamma\oplus-\varepsilon} \mathrm{H}_M^{1,1} \otimes \mathrm{H}_T^{0,1} \oplus \mathrm{H}_M^{0,2} \otimes \mathrm{H}_T^{1,0} = {}^{1,2}E_2^{2,1} \\ {}^{1,0}E_2^{0,1} &= \mathrm{H}_T^{1,0} \xrightarrow{\gamma} \mathrm{H}_M^{1,1} = {}^{1,1}E_2^{2,0} \end{aligned}$$

The morphism $\gamma \oplus -\varepsilon$ is injective iff one of the maps is nonzero, i.e. iff $\delta \neq 0$. Assuming this (otherwise all Hodge numbers equal those of $M \times T$) we get

$$h_X^{1,0} = \dim\ker\gamma \quad h_X^{1,1} = h_M^{1,1} - \dim\operatorname{im}\gamma \quad h_X^{1,2} = h_M^{1,1} + h_M^{0,2} - 1 \quad h_X^{1,3} = h_M^{0,2}\,.$$

If moreover $\delta\colon \mathrm{H}^1(T, \mathbf{C}) \to \mathrm{H}^2(M, \mathbf{C})$ is injective, the Betti numbers are determined by 11.3 (and $e(X) = 0$). Then the invariants are (with $e := \dim\operatorname{im}\varepsilon$, $g := \dim\operatorname{im}\gamma$)

$$\begin{array}{ccc} \begin{array}{c} 1 \\ 1-g \quad 1-e \\ h_M^{0,2} \quad h_M^{1,1}-g \quad h_M^{0,2}-e \\ h_M^{0,2} \quad h_M^{1,1}+h_M^{0,2}-1 \quad h_M^{1,1}+h_M^{0,2}-1 \quad h_M^{0,2} \\ h_M^{0,2}-e \quad h_M^{1,1}-g \quad h_M^{0,2} \\ 1-e \quad 1-g \\ 1 \end{array} & \qquad & \begin{array}{c} 1 \\ 0 \\ b_2(M)-2 \\ 2b_2(M)-2 \\ b_2(M)-2 \\ 0 \\ 1 \end{array} \end{array}$$

13. Examples of elliptic fibrations over surfaces

13.1 Surfaces with $b_1 = 0$ and $b_2 = 2$. Let M be a compact complex surface with $b_1 = 0$ and $b_2 = 2$, consequently with Euler characteristic $e = c_2 = 4$. A look on the classification table ([BPV], Chapter VI) shows us that Miyaoka-Yau inequality $c_1^2 \leq 3 \cdot c_2$ holds, and that $\chi(\mathcal{O}_M) = 1 - h_M^{0,1} + h_M^{0,2}$ must be positive (in the algebraic case this is clear because $2h_M^{1,0} = b_1(M) = 0$, and for the possibly non-algebraic elliptic surfaces we know $\chi > 0$ unless alle fibres are (possibly multiple) non-singular elliptic curves, which would imply $e(M) = 0$ ([BPV], III.11.4, V.12.2 and the remark preceeding it)). Together with Noether's formula $\frac{1}{12}(c_1^2 + c_2) = \chi(\mathcal{O}_M) \in \mathbf{Z}$ this only leaves the invariants

$$c_1^2 = 8\,, \qquad c_2 = 4\,, \qquad \chi(\mathcal{O}_M) = 1\,, \qquad h_M^{1,0} = h_M^{2,0} = 0\,.$$

So M is either rational, i.e. a (simply-connected) Hirzebruch surface Σ_r (for $r = 1$ a blown-up $\mathbf{P}^2$, otherwise M is minimal), or it is surface of general type with these special invariants. In the latter case M can either be the blow-up of a ball-quotient surface with $c_1^2 = 9$, $c_2 = 3$ (the only known example being Mumford's fake $\mathbf{P}^2$) which must have infinite fundamental group, or it is minimal. For minimal surfaces of general type with those invariants two constructions due to Beauville and Kuga (cf. [BPV], VII.11) are known, but both lead to infinite fundamental groups. In any case for a bundle with $\gamma \neq 0$ the invariants are those of Calabi-Eckmann manifolds:

$$\begin{array}{ccccccccc}
 & & & 1 & & & & \qquad & 1 \\
 & & 0 & & 1 & & & & 0 \\
 & 0 & & 1 & & 0 & & & 0 \\
0 & & 1 & & 1 & & 0 & & 2 \\
 & 0 & & 1 & & 0 & & & 0 \\
 & & 1 & & 0 & & & & 0 \\
 & & & 1 & & & & & 1
\end{array}$$

13.2. Bundles over Hirzebruch surfaces. Let M be one of the Hirzebruch surfaces Σ_r. Then $\pi_1(M) = 0$, $h_M^{2,0} = 0$, and $b_2(M) = 2$. So for any given characteristic class in $\mathrm{H}^2(M, \mathbf{Z}) \otimes \Lambda$ there is a unique bundle. If $\delta^{\mathbf{Z}}$ is an isomorphism, then the total space of this bundle is diffeomorphic to $S^3 \times S^3$ and the Hodge numbers are the same as in the Calabi-Eckmann case.

Such bundles have been constructed by Maeda, also in higher dimensions over base spaces which are $\mathbf{P}^{m_2}$-bundles over $\mathbf{P}^{m_1}$ ([Maeda]).

13.3. Bundles over other rational surfaces. Every blow-up adds a direct summand $\mathbf{Z}$ to $\mathrm{H}^2(M, \mathbf{Z})$. Let $\sigma: M \to \Sigma_r$ be a k-fold blow-up of Σ_r. If we take the pull-back of the characteristic class of a bundle on Σ_r and add all the classes of exceptional divisors (in order to kill $c_1(X)$), we can define a lot of simply-connected bundles with torsion-free homology and $c_1(X) = 0$ on M. The invariants are

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 1 & & \\
 & 0 & & b_2(M)-1 & & 0 & \\
0 & & b_2(M)-1 & & b_2(M)-1 & & 0 \\
 & 0 & & b_2(M)-1 & & 0 & \\
 & & 1 & & 0 & & \\
 & & & 1 & & &
\end{array}
\qquad
\begin{array}{c}
1 \\ 0 \\ b_2(M)-2 \\ 2\cdot b_2(M)-2 \\ b_2(M)-2 \\ 0 \\ 1
\end{array}
$$

with $b_2(M) = k + 2$, the diffeomorphism type is determined by this invariant.

13.4. Simply connected surfaces. Since the minimal model $\hat{M}$ of a simply connected compact complex surface M is again simply connected, the Enriques-Kodaira classification ([BPV], Chapter VI) tells us that the minimal model must be either rational or K3 or proper elliptic (i.e. of Kodaira dimension $\kappa = 1$) or of general type. In any case $b_2(M) \geq b_2(\hat{M}) \geq 1$. Since there is always a rational surface with isomorphic second cohomology (the intersection form does not play any role here), the total space of any simply-connected elliptic principal bundle (with torsion-free homology and $w_2 = 0$) over any simply-connected surface is diffeomorphic to a bundle over a rational surface, which means that there are complex structures of different Kodaira dimension on the same differentiable manifold. If the second Stiefel-Whitney classes coincide, they are even homotopic as almost-complex structures.

13.5. Remark. If X is any complex 3-fold diffeomorphic to $S^3 \times S^3$ with Kodaira dimension $\kappa(X) = 2$, then its algebraic dimension is also 2 and by [Akao] Part I, Theorem 1 and (the proof of) Corollary 3, it admits a torus action with possibly singular quotient space M of general type whose minimal resolution is a (then simply-connected) surface $\tilde{M}$ with $h_{\tilde{M}}^{0,2} = 0$. By [Akao], Corollary 4, the rational cohomology ring of M equals the one of $\mathbf{P}^1 \times \mathbf{P}^1$.

13.6. Bundles over an abelian surface. Now we consider an elliptic fibre bundle over an abelian surface, assuming $\delta \neq 0$. As before, we set $e := \operatorname{rank} \varepsilon$, $g := \operatorname{rank} \gamma$. But now we also have to consider the map $\mathrm{H}_M^{0,1} \otimes \mathrm{H}_T^{1,0} \to \mathrm{H}_M^{1,2}$, multiplication by $\gamma(dt)$. Its rank h can take the values 0 ($\Leftrightarrow g = 0$), 1 (e.g. $\gamma(dt) = dt^1 \wedge d\bar{t}^2$), or 2 (e.g. $\gamma(dt)$ a Kähler form). Furthermore we need $f := \operatorname{rank}(\mathrm{H}_M^{1,0} \oplus \mathrm{H}_M^{0,1} \to \mathrm{H}_M^{1,2})$ (induced by multiplication by $\varepsilon(d\bar{t})$ on the first and by $\gamma(dt)$ on the second summand). But $f = 2$ if $e = 1$ and $f = h$ if $e = 0$, and g is determined by h, so the parameters for the spectral sequence are only $e \in \{0, 1\}$ and $h \in \{0, 1, 2\}$, not both zero. Then the invariants are:

$$
\begin{array}{ccccccc|c}
 & & & 1 & & & & 1 \\
 & & 5-f-g & & 3-e & & & 4 \\
 & 3-h & & 8-f-g & & 3-e & & 8 \\
1 & & 6-h & & 6-h & & 1 & 10 \\
 & 3-e & & 8-f-g & & 3-h & & 8 \\
 & & 3-e & & 5-f-g & & & 4 \\
 & & & 1 & & & & 1
\end{array}
$$

For $\varepsilon \neq 0$, i.e. $e = 1$, there are three possible sets of Hodge numbers, and they all occur for small deformations of the Iwasawa manifold, see 14.6.

13.7. Fibrations over a K3 surface. According to 12.6, the invariants of a bundle with δ injective can take three different sets of values depending on the ranks g and e of γ and ε. In any case X is simply-connected with trivial canonical bundle.

$$
\begin{array}{ccccccc|c}
 & & & 1 & & & & 1 \\
 & & 1-g & & 1-e & & & 0 \\
 & 1 & & 20-g & & 1-e & & 20 \\
1 & & 20 & & 20 & & 1 & 42 \\
 & 1-e & & 20-g & & 1 & & 20 \\
 & & 1-e & & 1-g & & & 0 \\
 & & & 1 & & & & 1
\end{array}
$$

For example, one can take the Calabi-Eckmann fibration over $\mathbf{P}^1 \times \mathbf{P}^1$ and pull it back to a K3-surface which is a 2-sheeted cover ramified along a smooth curve of bidegree $(4, 4)$.

But this is only one example. The most interesting ones may be those with $\varepsilon \neq 0$, which should be quite numerous if the Picard number is small.

14. Infinitesimal deformations

14.1. We will now study the space $\mathrm{H}^1(\Theta_X)$ of infinitesimal deformations of X. Combining the exact sequence from Leray spectral sequence for $\Theta_{X/M}$, Θ_X and $\pi^*\Theta_M$ (horizontal) and cohomology sequences from 1.4(e), we obtain the diagram

$$
\begin{array}{cccccccc}
0\to & \mathrm{H}^0(\Theta_M) & \to & \mathrm{H}^0(\pi^*\Theta_M) & \to & 0 & & \\
 & \downarrow & & \downarrow & & \downarrow & & \\
0\to & \mathrm{H}^1(\pi_*\Theta_{X/M}) & \to & \mathrm{H}^1(\Theta_{X/M}) & \to & \mathrm{H}^0(\mathrm{R}^1\pi_*\Theta_{X/M}) & \to & \mathrm{H}^2(\pi_*\Theta_{X/M}) \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0\to & \mathrm{H}^1(\pi_*\Theta_X) & \to & \mathrm{H}^1(\Theta_X) & \to & \mathrm{H}^0(\mathrm{R}^1\pi_*\Theta_X) & \to & \mathrm{H}^2(\pi_*\Theta_X) \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0\to & \mathrm{H}^1(\Theta_M) & \to & \mathrm{H}^1(\pi^*\Theta_M) & \to & \mathrm{H}^0(\mathrm{R}^1\pi_*\pi^*\Theta_M) & \to & \mathrm{H}^2(\Theta_M) \\
 & \downarrow & & \downarrow & & \downarrow & & \\
\to & \mathrm{H}^2(\pi_*\Theta_{X/M}) & \to & \mathrm{H}^2(\Theta_{X/M}) & & \mathrm{H}^1(\mathrm{R}^1\pi_*\Theta_{X/M}) & & \\
 & \downarrow & & & & & & \\
 & \mathrm{H}^2(\pi_*\Theta_X) & & & & & &
\end{array}
$$

14.2. With all the isomorphisms from above this becomes

$$
\begin{array}{cccccccc}
0\to & \mathrm{H}^0(\Theta_M) & \to & \mathrm{H}^0(\Theta_M) & \to & 0 & & \\
 & \downarrow{\scriptstyle\gamma^5} & & \downarrow & & \downarrow & & \\
0\to & \mathrm{H}^{0,1}_M\otimes V & \to & \mathrm{H}^{0,1}_X\otimes V & \to & \mathrm{H}^{0,1}_T\otimes V & \xrightarrow{\varepsilon^3} & \mathrm{H}^{0,2}_M\otimes V \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0\to & \mathrm{H}^1(\pi_*\Theta_X) & \to & \mathrm{H}^1(\Theta_X) & \to & \mathrm{H}^{0,1}_T\otimes \mathrm{H}^0(\Theta_X) & \to & \mathrm{H}^2(\pi_*\Theta_X) \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0\to & \mathrm{H}^1(\Theta_M) & \to & \mathrm{H}^1(\pi^*\Theta_M) & \to & \mathrm{H}^{0,1}_T\otimes \mathrm{H}^0(\Theta_M) & \xrightarrow{\varepsilon^2} & \mathrm{H}^2(\Theta_M) \\
 & \downarrow{\scriptstyle\gamma^4} & & \downarrow & & \downarrow{\scriptstyle\gamma^1} & & \\
\xrightarrow{\varepsilon^3} & \mathrm{H}^{0,2}_M\otimes V & \to & \mathrm{H}^{0,2}_X\otimes V & & \mathrm{H}^{0,1}_T\otimes V\otimes \mathrm{H}^{0,1}_M & & \\
 & \downarrow & & & & & & \\
 & \mathrm{H}^2(\pi_*\Theta_X) & & & & & &
\end{array}
$$

14.3. Remark. The vertical connecting homomorphisms γ^1, γ^4, γ^5 are induced by (the dual of) γ (1.5(a), 1.4(e)), γ^5 is γ^1, tensored by the identity of $\mathrm{H}^{0,1}_T$. The horizontal d_2-map ε^3 is $\varepsilon\otimes\mathrm{id}_V$. The maps with a superscript occur in Borel spectral sequence which we will investigate in 14.7.

14.4. In the much more general situation of a holomorphic Seifert fibre space Suwa considers the following decomposition of $\mathrm{H}^1(\Theta_X)$ derived from the above diagram [Suwa$_1$]:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_T & \longrightarrow & D & \longrightarrow & A_F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \longrightarrow & \mathrm{H}^1(\Theta_X) & \longrightarrow & J & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_P & \longrightarrow & G & \longrightarrow & A_D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here

$A_T := \operatorname{coker} \gamma^5$	deformations preserving the T-action with quotient space M
$A_F := \ker \varepsilon^3$	deformations of T
$A_P := \ker \gamma^4$	deformations of M preserving the fibration
$A_D := \ker \gamma^1 \cap \ker \varepsilon^2$	deformations destroying the fibre structure

A_T is the space of infinitesimal 'twist deformations', i.e. deformations which are still T-principal bundles with the same structure group over the fixed base space M. All of them are unobstructed ([Suwa$_1$], Thm. 3.3). ε^3 is the obstruction map for a deformation of T inducing a global infinitesimal deformation. The deformations in $A_T \oplus A_F \oplus A_P$ are still torus principal bundles.

14.5. Invariants of the deformations. Under a deformation in $A := A_T \oplus A_F \oplus A_P$, the characteristic class $c^{\mathbf{Z}}$ (and therefore also δ) remains unchanged if the cohomology of the deformed manifolds M' and T' is identified with that of M and T, respectively. For A_T-deformations M and T are not changed, so ε and γ also remain the same.

14.6. Deformations of the Iwasawa manifold. For the Iwasawa manifold the computations are very easy because the tangent sheaf is trivial and γ is 0. The connecting homomorphisms in the vertical sequences are also 0, and we compute $\dim \mathrm{H}^1(\Theta_X) = \dim \mathrm{H}^1(\pi_* \Theta_X) = 6$, $\dim A_T = 2$, $\dim A_P = 4$ and $A_F = A_D = 0$. Therefore each infinitesimal deformation of M induces an infinitesimal deformation of X which is still a T-principal bundle. But deformations of T cannot be globalized—anyway Λ is a very special lattice. In fact, the small deformations have been computed by Kodaira and Nakamura, see [Nakamura], Sect. 3. The Hodge numbers of the deformations are also given there. ε remains nonzero in all cases while γ can take different values, see 13.6. Depending on γ (i.e. $h = 0, 1, 2$ in 13.6), three different sets of values for the Hodge numbers occur (A_T, $A_P - 0$, and the complement):

1	1	1	1
3 2	2 2	2 2	4
3 6 2	2 5 2	1 5 2	8
1 6 6 1	1 5 5 1	1 4 4 1	10
2 6 3	2 5 2	2 5 1	8
2 3	2 2	2 2	4
1	1	1	1

(Recall that while Hodge numbers are constant in complex-analytic families of Kähler manifolds, they are only upper-semicontinuous in the non-Kähler case, see [Wells], V.6.5, V.6.6.)

14.7. Spectral sequence converging to $\mathrm{H}^r(\Theta_X)$. In general we can compute $\mathrm{H}^i(\Theta_X) = \mathrm{H}^{m-n+i}(\Omega_X^1 \otimes \pi^*\mathscr{K}_M)^\vee$ using Borel spectral sequence for $p = 1$ but with a twist by $\mathscr{K}_M$. Such twists with vector bundles on the base space have been included in [Borel]. Writing the twisted Hodge space $\mathrm{H}^q(\Omega_M^p \otimes \mathscr{K}_M)$ as $\mathrm{H}_M^{p,q}(\mathscr{K}_M)$ the spectral sequence is (again $p + q = s + t$ but only considering the case $p = 1$)

$$
\begin{aligned}
{}^{1,q}\tilde{E}_2^{s,t} &= \mathrm{H}_M^{0,s}(\mathscr{K}_M) \otimes \mathrm{H}_T^{1,t-1} \oplus \mathrm{H}_M^{1,s-1}(\mathscr{K}_M) \otimes \mathrm{H}_T^{0,t} \\
&= (\mathrm{H}_M^{0,m-s} \otimes \mathrm{H}^{n-t+1}(\Theta_T) \oplus \mathrm{H}^{m-s+1}(\Theta_M) \otimes \mathrm{H}_T^{0,n-t})^\vee \\
&= (\mathrm{H}_M^{0,m-s} \otimes \mathrm{H}_T^{0,n-t+1} \otimes V \oplus \mathrm{H}^{m-s+1}(\Theta_M) \otimes \mathrm{H}_T^{0,n-t})^\vee
\end{aligned}
$$

$$
\mathrm{H}^{n+m-q}(\Theta_X)^\vee \cong Gr\,\mathrm{H}_X^{1,q}(\mathscr{K}_X) = \bigoplus_{s+t=1+q} {}^{1,q}\tilde{E}_\infty^{s,t}
$$

($\cdot^\vee$ here means Serre duality.) The spectral sequence has no ring structure any more but still a $\mathrm{H}_M^{\cdot,\cdot}$-module structure $\mathrm{H}_M^{i,j}(\mathscr{K}_M) \otimes \mathrm{H}_M^{r,s} \to \mathrm{H}_M^{i+r,j+s}(\mathscr{K}_M)$. With the same arguments like in 4.3 one can show

14.8. Proposition. *Let $\tilde{\varepsilon}$: $\mathrm{H}_T^{i,j} \to \mathrm{H}_M^{0,2} \otimes \mathrm{H}_T^{i,j-1}$ and $\tilde{\gamma}$: $\mathrm{H}_T^{i,j} \to \mathrm{H}_M^{1,1} \otimes \mathrm{H}_T^{i-1,j}$ be the iterates derived from ε and γ by Leibniz' rule. Then $\tilde{d}_2(\omega \otimes \vartheta) = \omega \cdot (\tilde{\varepsilon}(\vartheta) + \tilde{\gamma}(\vartheta))$. The higher differentials are zero.*

The differential is thus

$$\begin{array}{ccccc}
{}^{1,q}\tilde{E}_2^{s,t} & = & \mathrm{H}_M^{0,s}(\mathcal{K}_M)\otimes \mathrm{H}_T^{1,t-1} & \oplus & \mathrm{H}_M^{1,s-1}(\mathcal{K}_M)\otimes \mathrm{H}_T^{0,t} \\
\downarrow{\scriptstyle \tilde{d}_2} & & \downarrow{\scriptstyle \dot{\varepsilon}} & \searrow{\scriptstyle \dot{\gamma}} & \downarrow{\scriptstyle \ddot{\varepsilon}} \\
{}^{1,q+1}\tilde{E}_2^{s+2,t-1} & = & \mathrm{H}_M^{0,s+2}(\mathcal{K}_M)\otimes \mathrm{H}_T^{1,t-2} & \oplus & \mathrm{H}_M^{1,s+1}(\mathcal{K}_M)\otimes \mathrm{H}_T^{0,t-1} \\
({}^{1,q+1}\tilde{E}_2^{s+2,t-1})^\vee & = & \mathrm{H}_M^{0,m-s-2}\otimes \mathrm{H}_T^{0,n-t+2}\otimes V & \oplus & \mathrm{H}^{m-s-1}(\Theta_M)\otimes \mathrm{H}_T^{0,n-t+1} \\
\downarrow{\scriptstyle \tilde{d}_2^\vee} & & \downarrow{\scriptstyle \dot{\varepsilon}^\vee} & \swarrow{\scriptstyle \dot{\gamma}^\vee} & \downarrow{\scriptstyle \ddot{\varepsilon}^\vee} \\
({}^{1,q}\tilde{E}_2^{s,t})^\vee & = & \mathrm{H}_M^{0,m-s}\otimes \mathrm{H}_T^{0,n-t+1}\otimes V & \oplus & \mathrm{H}^{m-s+1}(\Theta_M)\otimes \mathrm{H}_T^{0,n-t}
\end{array}$$

A careful consideration of those maps shows

14.9. Proposition. (a) $\dot{\gamma}^\vee$ *is the connecting homomorphism in the cohomology sequence of* 1.4(e).

(b) $\dot{\varepsilon}^\vee$ *is the* d_2*-morphism* $\mathrm{H}_M^{0,m-s-2}\otimes \mathrm{H}_T^{0,n-t+2}\to \mathrm{H}_M^{0,m-s}\otimes \mathrm{H}_T^{0,n-t+1}$ *from the spectral sequence of* $\mathcal{O}_X$*, tensored by* id_V.

(c) $\ddot{\varepsilon}^\vee$ *is not so easy to describe but it vanishes if* $\varepsilon = 0$.

In order to compute $\mathrm{H}^1(\Theta_X)$ we have to take $s+t-1=q=n+m-1$. Since only $s\le m+1$ and $t\le n+1$ can give contributions, we need to consider only $s=m-1$, m, $m+1$ and get the following spaces and d_2-differentials (first in spectral sequence notation, then their duals):

$$ {}^{1,n+m-2}\tilde{E}_2^{m-1,n} \longrightarrow {}^{1,n+m-1}\tilde{E}_2^{m+1,n-1} \longrightarrow 0$$

$$ {}^{1,n+m-2}\tilde{E}_2^{m-2,n+1} \longrightarrow {}^{1,n+m-1}\tilde{E}_2^{m,n} \longrightarrow 0$$

$$ 0 \longrightarrow {}^{1,n+m-1}\tilde{E}_2^{m-1,n+1} \longrightarrow {}^{1,n+m}\tilde{E}_2^{m+1,n}$$

$$\mathrm{H}_M^{0,m-1}(\mathcal{K}_M)\otimes \mathrm{H}_T^{1,n-1}\oplus \mathrm{H}_M^{1,m-2}(\mathcal{K}_M)\otimes \mathrm{H}_T^{0,n} \longrightarrow \mathrm{H}_M^{1,m}(\mathcal{K}_M)\otimes \mathrm{H}_T^{0,n-1}\longrightarrow 0$$

$$\mathrm{H}_M^{0,m-2}(\mathcal{K}_M)\otimes \mathrm{H}_T^{1,n} \longrightarrow \mathrm{H}_M^{0,m}(\mathcal{K}_M)\otimes \mathrm{H}_T^{1,n-1}\oplus \mathrm{H}_M^{1,m-1}(\mathcal{K}_M)\otimes \mathrm{H}_T^{0,n}\longrightarrow 0$$

$$0\longrightarrow \mathrm{H}_M^{0,m-1}(\mathcal{K}_M)\otimes \mathrm{H}_T^{1,n}\longrightarrow \mathrm{H}_M^{1,m}(\mathcal{K}_M)\otimes \mathrm{H}_T^{0,n}$$

$$\mathrm{H}_M^{0,1}\otimes \mathrm{H}_T^{0,1}\otimes V\oplus \mathrm{H}^2(\Theta_M)\xleftarrow{\gamma^1\oplus\varepsilon^2}\mathrm{H}^0(\Theta_M)\otimes \mathrm{H}_T^{0,1}\longleftarrow 0$$

$$\mathrm{H}_M^{0,2}\otimes V\xleftarrow{\varepsilon^3+\gamma^4}\mathrm{H}_T^{0,1}\otimes V\oplus \mathrm{H}^1(\Theta_M)\longleftarrow 0$$

$$0\longleftarrow \mathrm{H}_M^{0,1}\otimes V\xleftarrow{\gamma^5}\mathrm{H}^0(\Theta_M)$$

15. Special cases

15.1. If the spectral sequence degenerates, the invariants are

$$h^r(\Theta_X) = \sum_{i+j=r-1}\left(n\cdot\binom{n}{j+1}\cdot h_M^{0,i} + \binom{n}{j}\cdot h^{i+1}(\Theta_M)\right)$$

$$= \sum_{i+j=r}\binom{n}{j}(n\cdot h_M^{0,i} + h^i(\Theta_M))$$

15.2. Remark. In the case $h_M^{0,q} = 0$ for $q > 0$, all the γ-differentials are zero. All remaining differentials come from ε, but this is zero because $h_M^{0,2} = 0$. Therefore the spectral sequence degenerates, and $h^r(\Theta_X) = \binom{n}{r} + \sum \binom{n}{r-i} h^i(\Theta_M)$.

15.3. Bundles over curves. In this case always $\varepsilon = 0$, and

$$^{1,q}\tilde{E}_2^{s,t} = \mathrm{H}_M^{0,s}(\mathscr{K}_M) \otimes \mathrm{H}_T^{1,t-1} \oplus \mathrm{H}_M^{1,s-1}(\mathscr{K}_M) \otimes \mathrm{H}_T^{0,t}$$

is non-zero only for $s = 0$, 1, 2. The only possibly non-zero maps are $\dot{\gamma}^t$: $\mathrm{H}_M^{0,0}(\mathscr{K}_M) \otimes \mathrm{H}_T^{1,t-1} \to \mathrm{H}_M^{1,1}(\mathscr{K}_M) \otimes \mathrm{H}_T^{0,t-1}$. But $\mathrm{H}_M^{0,s}(\mathscr{K}_M) = \mathrm{H}^s(\mathscr{K}_M)$ and $\mathrm{H}_M^{1,s+1}(\mathscr{K}_M) = \mathrm{H}^{s+1}(\mathscr{K}_M{}^2)$, so the only case where both are non-zero is M an elliptic curve, $s = 0$. Therefore the spectral sequence degenerates for non-elliptic curves, and the cohomology is given by 15.1.

If M is an elliptic curve, then $\mathrm{H}_M^{i,j}(\mathscr{K}_M) = \mathrm{H}_M^{i,j}$. The first summand of $^{1,q}\tilde{E}_2^{s,t}$ is the starting point of $\dot{\gamma}^t$ (for $s = 0$) and the second one receives $\dot{\gamma}^{t-1}$ (for $s = 2$), and $\dot{\gamma}^t$ is nothing but $\gamma \otimes \mathrm{id}_{H_T^{0,t-1}}$: $\mathrm{H}_T^{1,0} \otimes \mathrm{H}_T^{0,t-1} \to \mathrm{H}_M^{1,1} \otimes \mathrm{H}_T^{0,t-1}$. So in the non-trivial case this map is surjective with a kernel of dimension $(n-1)\binom{n}{t-1}$. In the spectral sequence we still have

$$\begin{array}{ll} ^{1,q}\tilde{E}_\infty^{0,t}: & \ker \dot{\gamma}^t \\ ^{1,q}\tilde{E}_\infty^{1,t}: & \mathrm{H}_M^{0,1} \otimes \mathrm{H}_T^{1,t-1} \oplus \mathrm{H}_M^{1,0} \otimes \mathrm{H}_T^{0,t} \\ ^{1,q}\tilde{E}_\infty^{2,t}: & \operatorname{coker} \dot{\gamma}^{0,t+1} \end{array}$$

Assuming $\gamma \neq 0$ we get $h^{n+1-q}(\Theta_X) = (n-1)\binom{n}{q} + n\binom{n}{q-1} + \binom{n}{q} = n\binom{n+1}{q}$. Depending on the genus g of the curve, the result is thus

$$h^r(\Theta_X) = (n+3)\binom{n}{r} \qquad (g = 0)$$

$$h^r(\Theta_X) = (n+1)\binom{n+1}{r} \qquad (g = 1, \gamma = 0)$$

$$h^r(\Theta_X) = n\binom{n+1}{r} \qquad (g = 1, \gamma \neq 0)$$

$$h^r(\Theta_X) = n\binom{n}{r} + \binom{n}{r-1}((n+3)g - 3) \qquad (g \geq 2)$$

15.4. Calabi-Eckmann manifolds. Here $h^0(\Theta_M) = m_1^2 + 2m_1 + m_2^2 + 2m_2$, and $h^i(\Theta_M) = 0$ if $i > 0$, and the spectral sequence degenerates by 15.2. Thus $\mathrm{H}^1(\Theta_X) \cong \mathrm{H}^1(\Theta_T) \oplus \mathrm{H}^0(\Theta_M)$. Only 'fibre deformations' ($\dim A_F = 1$) and 'fibre destroying deformations' ($A_D \cong \mathrm{H}^0(\Theta_M)$, $\dim A_D = (m_1 + 2)m_1 + (m_2 + 2)m_2$) occur. While all small deformations have Kodaira dimension $-\infty$, the algebraic dimension drops for the 'fibre destroying deformations' ([Akao], Part II, Prop. 2 and 3).

15.5. Elliptic fibrations over manifolds with $h_M^{0,1} = 0$. Here the diagram reduces to

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \mathrm{H}_X^{0,1} \arrow[r] \arrow[d] & \mathbf{C} \arrow[r, "\varepsilon^3"] \arrow[d] & \mathrm{H}_M^{0,2} \arrow[d] \\
0 \arrow[r] & \mathrm{H}^1(\pi_*\Theta_X) \arrow[r] \arrow[d] & \mathrm{H}^1(\Theta_X) \arrow[r] \arrow[d] & \mathrm{H}^0(\Theta_X) \arrow[r] \arrow[d] & \mathrm{H}^2(\pi_*\Theta_X) \arrow[d] \\
0 \arrow[r] & \mathrm{H}^1(\Theta_M) \arrow[r] \arrow[d, "\gamma^4"] & \mathrm{H}^1(\pi^*\Theta_M) \arrow[r] \arrow[d] & \mathrm{H}^0(\Theta_M) \arrow[r, "\varepsilon^2"] \arrow[d] & \mathrm{H}^2(\Theta_M) \\
0 \arrow[r] & \mathrm{H}_M^{0,2} \arrow[r] \arrow[d] & \mathrm{H}_X^{0,2} \arrow[r] & 0 & \\
 & \mathrm{H}^2(\pi_*\Theta_X) & & &
\end{tikzcd}
$$

If we assume that in addition $h_M^{0,2} = 0$, then $\mathrm{H}^1(\Theta_X) \cong \mathrm{H}^1(\Theta_M) \oplus \mathrm{H}^1(\Theta_T) \oplus \mathrm{H}^0(\Theta_M)$. If X is a nontrivial elliptic fibration with $\varepsilon = 0$ over a K3-surface, $h^0(\Theta_X) = 1$, $h^1(\Theta_X) = 20$, $h^2(\Theta_X) = 19$, $h^3(\Theta_X) = 0$. Besides the 1-dimensional A_F only A_P-deformations coming from the base space exist, but γ^4 gives an obstruction for lifting those deformations to X.

15.6. If M is a surface of general type with $h_M^{0,1} = h_M^{0,2} = 0$, then $\mathrm{H}^0(\Theta_M) = 0$ and $\chi(\Theta_M) = 6$ by Hirzebruch-Riemann-Roch.

15.7. Rigid spaces. In order to construct a rigid total space, we must get $\gamma^1 \oplus \varepsilon^2$ and $\varepsilon^3 + \gamma^4$ injective and γ^5 surjective. Any bundle on a rigid surface of general type with $h_M^{0,1} = 0$ such that $\varepsilon\colon \mathrm{H}_T^{0,1} \to \mathrm{H}_M^{0,2}$ is non-zero will do, e.g. any ball quotient surface with $0 \neq p_g = \chi(\mathcal{O}_M) - 1 = \frac{1}{3}c_2(M) - 1$.

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