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Ergodic properties of discrete groups; inheritance to normal subgroups and invariance under quasiconformal deformations

By

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§0. Introduction

There are many studies on the ergodic properties of discrete isometry groups Γ acting on the n-dimensional hyperbolic space $B^n = \{x \in \mathbf{R}^m; |x| < 1\}$ and on the sphere at infinity $S^{n-1} = \{x \in \mathbf{R}^n; |x| = 1\}$ (cf. [N]). Among them, Lyons and Sullivan's work [LS] is remarkable. They obtained the conditions concerning covering transformation groups, under which normal (regular) covers of a compact hyperbolic manifold are recurrent or Liouville. In other words, we may say that they showed what normal subgroups inherit the ergodicity of the action on S^{n-1} with respect to the Lebesgue measure from a cocompact discrete group. In connection with this problem, in the present paper, we consider in what degree any normal subgroup Γ' of Γ inherits ergodicity on $S^{n-1} \times S^{n-1}$ ($= B^n/\Gamma$ is recurrent) and ergodicity on S^{n-1} ($= B^n/\Gamma$ is Liouville). Particularly, in the case where $n = 2$, we can characterize $S^{n-1} \times S^{n-1}$-ergodicity of Γ by conservativity of the action on S^{n-1} of Γ':

Theorem. *A Riemann surface B^2/Γ is recurrent if and only if any non-trivial normal subgroup of Γ is conservative.*

We develop those arguments in the first part "inheritance to normal subgroups" (§4 and §5) after the sections of several preliminaries and preparations. The first part also contains some investigations on the following two conjectures which seem interesting in the course of our arguments:

(C1) If B^n/Γ is recurrent and Γ' is a normal subgroup of Γ such that any subgroup of Γ/Γ' is a finitely generated solvable group, then B^n/Γ' is Liouville (cf. [LS]).

(C2) Γ acts on S^{n-1} ergodically if and only if any normal subgroup Γ' of Γ acts on S^{n-1} either conservatively or totally dissipatively.

In the second part "invariance under quasiconformal deformations" (§6 and §7), we study whether the ergodic properties on S^{n-1} are preserved or not, by

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deformations of Γ under quasi-isometric automorphisms of B^n. For $n \geq 3$, most of these problems turn out to be trivial by Mostow-Sullivan rigidity. Hence we restrict ourselves to the case $n = 2$. The above conjecture (C2) suggests that if conservativity and totally dissipativity of Fuchsian groups were preserved under quasiconformal mappings, ergodicity would be also preserved. But in §7, we exhibit an example of a quasiconformal automorphism compatible with Γ which maps the conservative part of positive measure to a null set. Actually, Lyons [L] has already constructed by stochastic methods a pair of quasiconformally equivalent Riemann surfaces, one has the Liouville property, but the other has not. In §6, as an extension of this example, we prove by using geometric normal covers that all the ergodic components of a Fuchsian group may disappear by quasiconformal deformation.

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§ 1. Preliminaries, harmonic functions and ergodic components

Let B^n be the n-dimensional unit ball with the hyperbolic metric $d\rho = 2|dx|/1 - |x|^2$. We denote by $\mathrm{Isom}^+(\mathrm{B}^n)$ the group of orientation preserving isometric automorphisms. It acts not only on B^n but also on the sphere at infinity $S^{n-1} = \{|x| = 1\}$. The group $\mathrm{Isom}^+(B^n)$ has the canonical matrix representation and under this topology we consider discrete subgroups of $\mathrm{Isom}^+(\mathrm{B}^n)$. It is well known that a subgroup $\Gamma \subset \mathrm{Isom}^+(\mathrm{B}^n)$ is *discrete* if and only if it acts on B^n properly discontinuously. A complete n-dimensional hyperbolic manifold R is a quotient of B^n by a torsion-free discrete group Γ. Throughout this paper we use the same notation for a class of complete hyperbolic manifolds as that of the corresponding class of hyperbolic discrete groups. We classify discrete groups by ergodic properties of the action on S^{n-1} which carries the normalized Lebesgue measure dm with $m(S^{n-1}) = 1$. Such properties may be translated into the existance of Green's or harmonic functions on the hyperbolic manifolds by the Poisson integral, where the above terms are in the hyperbolic sense, namely, with respect to the Laplace-Beltrami operator

$$\Delta_h = \frac{(1-r^2)^2}{4}\left[\Delta + \frac{2(n-2)r}{1-r^2}\frac{\partial}{\partial r}\right],$$

where $r = |x|$ (cf. [N, Chap. 5]). When $n = 2$, we say a hyperbolic discrete group acting on $D = B^2$ or the upper half plane model H to be *Fuchsian.* In this case, the hyperbolic harmonicity is the same as the usual one.

The limit set $\Lambda(\Gamma)$ of the orbit of some reference point by a discrete group Γ is important for our study. We say Γ to be *of the first kind* if $\Lambda(\Gamma) = S^{n-1}$ and otherwise *of the second kind.* More precisely, we define several types of the limit points of Γ.

Definition 1.1. Let Q be a set of countable points of B^n not accumulating to any point of B^n. Then we define

$L(Q) = \{x \in S^{n-1} |$ there are points of Q which accumulate to $x\}$

$\overline{H}(Q) = \{x \in S^{n-1} |$ there are points of Q which accumulate to x in some horosphere tangential to S^{n-1} at $x\}$

$H(Q) = \{x \in S^{n-1} |$ in every horosphere tangential to S^{n-1} at x, there are points of $Q\}$

$C(Q) = \{x \in S^{n-1} |$ there are points of Q which accumulate to x in some cone region with vertex $x\}$

$\underline{C}(Q) = \{x \in S^{n-1} |$ in every cone region with vertex x, there are points of $Q\}$

Let Γ be a discrete group acting on B^n. For $Q = \{\gamma(0); \gamma \in \Gamma\}$, we define $\Lambda(\Gamma) = L(Q)$, $\Lambda_h(\Gamma) = H(Q)$, $\Lambda_{\bar{h}}(\Gamma) = \overline{H}(Q)$ and $\Lambda_c(\Gamma) = C(Q)$.

Remark. (1) $\Lambda(\Gamma)$, $\Lambda_h(\Gamma)$, $\Lambda_{\bar{h}}(\Gamma)$ and $\Lambda_c(\Gamma)$ do not depend on the choice of orbits, hence we may take the orbit of the origin in the above definition.

(2) Note that $H(Q) = \overline{H}(Q)$ a.e. and $C(Q) = \underline{C}(Q)$ a.e. A proof of these facts is, for example in [N, Th.2.1.2], assuming that Q is an orbit by a discrete group. But this assumption is not essential. $\overline{H} - H$ contains so-called Garnett points. Sullivan first remarked that their measure is zero [Su, §IV].

Class $\mathrm{O_G}$. A hyperbolic manifold $R = B^n/\Gamma$ belongs to the class $\mathrm{O_G}$ if it admits no Green's function. In other words, the harmonic measure of the ideal boundary of R is identically zero. We can characterize this class by the following properties of the discrete group Γ. See for example [N, p. 109, Th. 6.3.6]. The equivalence (3)$\Leftrightarrow$(4) can be found in [Su, § 3].

Proposition 1.1. *The followings are equivalent*:

(1) $\Gamma \in \mathrm{O}_G$.

(2) Γ *is of divergence type, i.e.,* $\sum_{\gamma \in \Gamma} (1 - |\gamma(0)|)^{n-1} = \infty$.

(3) $\Lambda_c(\Gamma) = S^{n-1}$ (a.e.).

(4) $m(\Lambda_c(\Gamma)) > 0$.

(5) Γ acts ergodically on $S^{n-1} \times S^{n-1}$.

Classes $\mathrm{O^m_{HB}}$ $(1 \le m \le \infty)$, $\mathrm{U_{HB}}$. We denote by $\mathrm{O^m_{HB}}$ $(1 \le m < \infty)$ a class of hyperbolic manifolds such that the vector space

$\mathrm{HB(R)} = \{$(real) bounded (hyperbolically) harmonic functions on $R\}$

has at most m dimension. Let $u \in \mathrm{HB(R)}$ be positive. Suppose that for every $v \in \mathrm{HB(R)}$, $u \geq v \geq 0$ implies the existence of a constant c with $cu = v$. Then the function u is called *HB-minimal* on R. The dimension of HB(R) is the same as the number of linearly independent HB-minimal functions on R. In the case where $n = 2$, we can characterize the class $\mathrm{O}_{\mathrm{HB}}^{\mathrm{m}}$ by the *harmonic boundary* $\Delta_w(R)$ of *Wiener's compactification* R_w^* of R (cf. [SN, Chap. IV]). That is, $R \in \mathrm{O}_{\mathrm{HB}}^{\mathrm{m}}$ $(1 \leq m < \infty)$ if and only if $\Delta_w(R)$ consists of at most m points. As a limiting class of $\mathrm{O}_{\mathrm{HB}}^{\mathrm{m}}$, we define $\mathrm{O}_{\mathrm{HB}}^{\infty}$ as a class of hyperbolic manifolds R such that HB(R) has at most a countable number of HB-minimal functions as basis. Further, by U_{HB} we denote a class of R on which there exists at least one HB-minimal function. Remark that we assume if $R \in \mathrm{O}_{\mathrm{G}}$, R has a HB-minimal, i.e., $\mathrm{O}_{\mathrm{G}} \subset \mathrm{U}_{\mathrm{HB}}$, which is different from the usual notation as in e.g. [SN]. We see $R \in \mathrm{U}_{\mathrm{HB}} - \mathrm{O}_{\mathrm{G}}$ if and only if $\Delta_w(R)$ contains an isolated point.

We characterize those classes $\mathrm{O}_{\mathrm{HB}}^{\mathrm{m}}$ $(1 \leq m \leq \infty)$ and U_{HB} by indivisible invariant measurable sets on S^{n-1} under discrete groups. Originally they were due to Constantinescu-Cornea [CC] for Fuchsian groups.

Definition 1.2. Let Γ be a discrete group acting on B^n. We call a Lebesgue measurable set E on S^{n-1} with $m(E) > 0$ *ergodic component* for Γ if

(1) [invariant] $E = \gamma(E)$ (a.e.) for every $\gamma \in \Gamma$, and
(2) [indivisible] if a subset E' of E with $m(E') > 0$ has the property (1), then $E = E'$ (a.e.).

Proposition 1.2. *If $E \subset S^{n-1}$ is an ergodic component for a discrete group Γ, then the projection to $R = B^n/\Gamma$ of the harmonic measure*

$$w_E(z) = \int_E \left(\frac{1 - |z|^2}{|x - z|^2}\right)^{n-1} dm(x)$$

is HB-minimal on R. Conversely, if u is HB-minimal on R, then the lift $\tilde{u}(z)$ to B^n has non-tangential limits which take a positive constant on some ergodic component E for Γ and zero on $S^{n-1} - E$ a.e.

Proof. We denote the canonical projection $B^n \to R$ by π. Let u be the projection of $w_E(z)$, v a harmonic function on R satisfying $u \geq v \geq 0$. Since $\tilde{v} = v \circ \pi$ is a bounded harmonic function on B^n, it has a non-tangential limit at almost every x on S^{n-1} (cf. [N, pp. 91–92]), which we denote by the same letter $\tilde{v}(x)$. By the inequality $w_E \geq \tilde{v} \geq 0$ on B^n, we have $1_E(x) \geq \tilde{v}(x) \geq 0$ a.e. on S^{n-1}, hence $\tilde{v}(x) = 0$ a.e. on $S^{n-1} - E$. If $\tilde{v}(x)$ is not constant a.e. on E, using the Γ-automorphic value $\tilde{v}(x)$, we can take a measurable subset E' of E with $0 < m(E') < m(E)$ which is Γ-invariant in a.e. sense. But this violates that E is an ergodic component for Γ. Therefore there is a constant c $(0 \leq c \leq 1)$ such that $\tilde{v}(x) = c1_E(x)$ a.e. Since the bounded harmonic function $\tilde{v}(z)$ is reproduced by the Poisson integral of $\tilde{v}(x)$, we get $\tilde{v}(z) = cw_E(z)$, hence $v = cu$ on R. This means that u is HB-minimal.

Conversely, for a HB-minimal function u on R, let $\tilde{u}(z) = u \circ \pi(z)$ and $E = \{x \in S^{n-1} |$ the non-tangential limit $\tilde{u}(x)$ is positive$\}$. Note that E is Γ-invariant and $m(E) > 0$. We will show that E is indivisible. If there is a Γ-invariant measurable subset E' of E satisfying $0 < m(E') < m(E)$, we can define a Γ-automorphic boundary value $\tilde{v}(x)$ so that $\tilde{v}(x) = \tilde{u}(x)$ on E' and $\tilde{v}(x) = 0$ on $S^{n-1} - E'$. It is clear that $0 \leq \tilde{v}(x) \leq \tilde{u}(x)$ a.e. and $\tilde{v}(x) \not\equiv$ const. $\times\, \tilde{u}(x)$. Then the Poisson integrals $\tilde{v}(z)$ and $\tilde{u}(z)$ satisfy the same condition, hence so are the projections v and u on R. But this contradicts that u is HB-minimal. We have seen that E is an ergodic component for Γ. It is obvious that $\tilde{u}(x) =$ const. on E, for otherwise, E would be divisible.

This shows one-to-one correspondence between ergodic components for Γ and HB-minimal functions on B^n/Γ up to constant multiples. Hence we have;

Proposition 1.3. (1) $\Gamma \in \mathrm{O}_{\mathrm{HB}}^{\mathrm{m}}$ $(1 \leq m < \infty)$ *if and only if* S^{n-1} *is a disjoint union of at most m ergodic components for* Γ *up to null sets.*

(2) $\Gamma \in \mathrm{O}_{\mathrm{HB}}^{\infty}$ *if and only if* S^{n-1} *is a disjoint union of at most a countable number of ergodic components for* Γ *up to null sets.*

(3) $\Gamma \in \mathrm{U}_{\mathrm{HB}}$ *if and only if* S^{n-1} *has an ergodic component for* Γ.

§ 2. Conservative part and dissipative part

Classes C, $\mathrm{U_c}$ *and* $\mathrm{W_c}$. The action of a discrete group Γ divide S^{n-1} into two measurable sets, *conservative* part $\mathscr{R}(\Gamma)$ and *dissipative* part $\mathscr{D}(\Gamma)$. These are determined up to null sets by the following requirement: $\mathscr{D}(\Gamma)$ has a measurable fundamental set, and for each subset $A \subset \mathscr{R}(\Gamma)$ with positive measure, $\#\{\gamma \in \Gamma | m(\gamma(A) \cap A) > 0\} = \infty$ (cf. [Ag, appendix]). We define

$$\mathrm{C} = \{\Gamma;\ \mathscr{R}(\Gamma) = S^{n-1} \text{ a.e., that is, } \Gamma \text{ is a conservative group}\}$$

$$\mathrm{U_c} = \{\Gamma;\ m(\mathscr{R}(\Gamma)) > 0, \text{ that is, } \Gamma \text{ is not a totally dissipative group}\}.$$

When $n = 2$, Pommerenke [P$_1$] introduced "Fuchsian groups of accessible type". For Fuchsian groups Γ of convergence type, i.e., $\sum_{\gamma \in \Gamma} (1 - |\gamma(0)|) < \infty$, we call a Blaschke product $g(z) = \prod_{\gamma \in \Gamma} \gamma(z) \exp(-i \arg \gamma(0))$ the Green's function of Γ with respect to 0.

(1) Γ is of *accessible type* if $g'(z)$ has a finite non-tangential limit on a set of positive measure of ∂D.

(2) Γ is of *fully accessible type* if $g'(z)$ has a finite non-tangential limit a.e. on ∂D.

(3) Γ is of *Parreau-Widom type* if $g'(z)$ belongs to the Nevanlinna class, i.e., $\int_0^{2\pi} \log^+ |g'(re^{i\theta})| d\theta$ is bounded for $0 < r < 1$.

$$W_c = \{\Gamma;\ \Gamma \text{ is not of Pareau-Widom type}\}.$$

What is the set where $g'(z)$ has a finite non-tangential limit?

Proposition 2.1 (Pommerenke [P_1, Th. 1]). *Let Γ be a Fuchsian group of convergence type acting on D. Then the following measurable sets on ∂D are coincident up to null sets*:

(a) *A set where $g'(z)$ has a finite non-tangential limit*
(b) $\partial D - \Lambda_{\tilde{h}}(\Gamma)$
(c) *The dissipative part $\mathscr{D}(\Gamma)$.*

Therefore for Fuchsian groups, we know $\mathrm{C} = \{\text{not accessible type}\} \cup \{\text{divergence type}\}$ and $\mathrm{U_c} = \{\text{not fully accessible type}\} \cup \{\text{divergence type}\}$.

Later, Sullivan [Su] generalized Proposition 2.1 in higher dimensional cases by a similar argument to the original proof of Proposition 2.1. Further, as has been mentioned in Remark (2) after Definition 1.1, he showed that $\Lambda_{\tilde{h}}(\Gamma) = \Lambda_h(\Gamma)$ a.e. In sum;

Proposition 2.1′. *For any dimensional hyperbolic discrete group Γ, $\mathscr{R}(\Gamma) = \Lambda_h(\Gamma)$ a.e.*

Note that a discrete group of the second kind does not belong to C, and whose limit set is a null set does not belong to $\mathrm{U_c}$.

A characterization of conservativity for Riemann surfaces. Even in the case where $n = 2$, it is not easy to distinguish what Riemann surfaces belong to C, $\mathrm{U_c}$ or $\mathrm{W_c}$ in general. So we restrict ourselves to planar Riemann surfaces such as $D - Q$, where $Q = \{q_n\}$ is a countable number of points of D not accumulating to any point of D. A necessary and sufficient condition for $D - Q$ to belong to each one of C, $\mathrm{U_c}$ and $\mathrm{W_c}$ is as follows. These results are essentially due to Pommerenke [P_3, Ex. 1] & [P_1, Th. 8].

Lemma 2.1. (1) *The Riemann surface $R = D - Q$ belongs to the class* C *if and only if $m(C(Q)) = 1$.*

(2) *$R \in \mathrm{U_c}$ if and only if $m(C(Q)) > 0$.*

(3) *$R \in \mathrm{W_c}$ if and only if $\{q_n\}$ does not satisfy the Blaschke condition, i.e., $\sum (1 - |q_n|) = \infty$.*

Proof. (1) and (2): We may assume that the origin 0 is not in Q. Let π be a holomorphic universal covering map $D = \{|z| < 1\} \to R = \{|w| < 1\} - Q$ such that $\pi(0) = 0$, Γ the corresponding Fuchsian group, and $g(z)$ the Green's function of Γ with respect to 0. It is easy to see that the projection of $-\log|g(z)|$ to R is the usual Green's function of R with a pole at 0. Since Q has zero capacity, it coincides with $-\log|w|$. Therefore $-\log|g(z)| = -\log|\pi(z)|$, which implies that $g(z) = e^{i\theta}\pi(z)$. We consider the image under g of the Green's fundamental domain G of Γ with respect to 0. It is a starlike domain which consists of radial segments, the images of Green's lines. Since $g(z) = e^{i\theta}\pi(z)$ is unbranched, no Green's line ends at a branched point. Hence $g(G) = e^{i\theta}\pi(G)$ is a rotation of $D - \bigcup_{n=1}^{\infty} \left[q_n, \frac{q_n}{|q_n|}\right)$. Let T be a set of $x \in \partial D$ where $g(G)$ contains a small Stolz angular region of vertex x with any large opening. Then it is clear that $T = \partial D - C(Q)$ (a.e.). Pommerenke proved that $m(T) = m(\mathscr{D}(\Gamma))$, from which follow the equivalences $\Gamma \notin \mathrm{U_c} \Leftrightarrow m(T) = 1$ and $\Gamma \in \mathrm{C} \Leftrightarrow m(T) = 0$ [P_2, Cor.]. They yield our assertions.

(3): $R \notin \mathrm{W_c}$ if and only if R satisfies the Widom condition for some (hence every) $p \in R$, that is, letting $g(w, p)$ be the Green's function on R with a pole at p and $B(t, p)$ the first Betti number of $\{w \in R;\ g(w, p) > t\}$, it holds that $\int_0^\infty B(t, p)dt < \infty$. For $R = D - Q$ $(0 \notin Q)$, $g(w, 0) = -\log|w|$ and $B(t, 0) = \#\{q \in Q;\ -\log|q| > t\}$. Then by $e^{-t} = r$,

$$\int_0^\infty B(t, 0)dt = \int_0^1 \#\{q \in Q; |q| < r\}\frac{dr}{r}.$$

This integral converges if and only if $\int_0^1 \#\{q \in Q; |q| < r\}dr$ converges, which is equal to $\sum(1 - |q_n|)$.

§ 3. Strict inclusion relations between the classes

Theorem 3.1. *The following system of strict inclusion relations holds.*

$$\begin{array}{ccccccc} \mathrm{O_G} & \longrightarrow & \mathrm{O^m_{HB}} & \longrightarrow & \mathrm{O^\infty_{HB}} & \longrightarrow & \mathrm{U_{HB}} \\ & & & & \downarrow & & \downarrow \\ & & & & \mathrm{C} & \longrightarrow & \mathrm{U_c} \end{array}$$

Proof. The inclusion relations are evident if we remark that ergodic components are in the conservative part. The strictness $\mathrm{O_G} \to \mathrm{O^m_{HB}} \to \mathrm{O^\infty_{HB}}$ can be found in [T_2], and the others are seen if we construct the following two examples of arbitrary dimensional hyperbolic manifolds R:

(1) $R \in \mathrm{C}$ but $R \notin \mathrm{U_{HB}}$

(2) $R \in \mathrm{U_{HB}}$ but $R \notin \mathrm{C}$

(1) has appeared in [VM]. To construct (2), we prepare $M \in \mathrm{O_{HB}} - \mathrm{O_G}$ which contains a compact totally geodesic submanifold N with codimension 1 such that $M - N$ is connected (cf. [T_2]). Then by Lemma 3.1 (II) in the following subsection, we obtain a discrete group G whose limit set $\Lambda(G)$ is the ergodic component for G. We know $G \notin \mathrm{C}$, for G is of the second kind, but $G \in \mathrm{U_{HB}}$.

Geometric covers obtained by cutting along totally geodesic submanifolds. Let M be an n-dimensional complete hyperbolic manifold which contains a compact totally geodesic submanifold N with codimension 1 such that $M-N$ is connected. Let R be the hyperbolic manifold with boundary obtained from $M-N$ be attaching two borders $\{N^+, N^-\}$ corresponding to N. Then R is represented as K/G, where K is a closed region in B^n with totally geodesic boundary and G is a discrete subgroup of Isom^+ (B^n) under which K is invariant. Since $\bar{K}\cap S^{n-1}$ is closed, G-invariant and contained in $\Lambda(G)$, K is actually the Nielsen convex hull $K(\Lambda(G))$ of the limit set $\Lambda(G)$.

Conversely, if we start from a given hyperbolic manifold $R=K(\Lambda(G))/G$ with borders N^+ and N^- which are totally geodesic and mutually isometric, then by glueing N^+ to N^-, we obtain the complete hyperbolic manifold M. From a viewpoint of covering transformation groups, the corresponding discrete group to M is constructed as follows: Let $S^+(S^-)$ be an orthogonal hemisphere in $\partial K(\Lambda(G))$ such that $S^+/\mathrm{stab}_G(S^+)=N^+$ (resp. $S^-/\mathrm{stab}_G(S^-)=N^-$). Let τ be an element of $\mathrm{Isom}^+(\mathrm{B}^n)$ which maps the interior of S^+ to the exterior of S^- and induces the isomorphism between the stabilizers $\mathrm{stab}_G(S^+)\to\mathrm{stab}_G(S^-)$. Then the combination theorem concerning discrete groups asserts that the group F generated by G and τ is discrete, represented by an HNN-extension $G*_\tau$ and $M=B^n/F$ (cf. [Mg, p. 78]).

Lemma 3.1. *Using the same notations under the same circumstances as above,* (I) $M=B^n/F\in O_G \Leftrightarrow m(\Lambda(G))=0$ *and* (II) $M\in O_{HB}-O_G\Leftrightarrow \Lambda(G)$ *is the ergodic component for* G (*with positive measure*).

Proof. We have the following diagram of covering maps

$$\begin{tikzcd} B^n \arrow[r, "\pi_G"] \arrow[dr, "\pi_F"'] & R^* \arrow[d, "\pi"] & \supset & \mathring{R} \arrow[d, hook] \\ & M & \supset & M-N \end{tikzcd}$$

where $R^*=B^n/G$, and assume $\pi_G(0)\in\mathring{R}=R-(N^+\cup N^-)$. Let l_θ be the geodesic ray in B^n from 0 towards $\theta\in S^{n-1}$. Then $\theta\in\Lambda(G)$ if and only if $\pi_G(l_\theta)$ hits neither N^+ nor N^-, which is equivalent to the condition that $\pi_F(l_\theta)$ does not hit $N(\subset M)$.

(I) Assume that $M\in O_G$. Let T be a ball in B^n with center 0 and $\pi_G(T)\subset\mathring{R}$. By the recurrence of the geodesic rays, the set $\{\theta\colon \pi_F(l_\theta)$ recurs infinitely often to $\pi_F(T)\}$ is of full measure. Therefore,

$$\begin{aligned} m(\Lambda(G)) &= m(\{\theta\colon \pi_F(l_\theta) \text{ does not hit } N\}) \\ &= m(\{\theta\colon \pi_G(l_\theta) \text{ recurs infinitely often to } \pi_G(T)\}) \\ &= m(\Lambda_c(G)). \end{aligned}$$

Since G is of the second kind and $m(\Lambda_c(G))$ is either zero or one (Proposition 1.1), we have $m(\Lambda(G))=0$.

Conversely, if $M \notin \mathrm{O_G}$, the measure of the geodesic rays $\{\pi_F(l_\theta)\}$ which hit N (compact) infinitely often is zero. This implies that for almost every θ, l_θ does not pass $\{f(K(\Lambda(G)))\}_{f\in F}$ infinitely many times. Then we easily see that $\bigcup_{f\in \Gamma} f(\Lambda(G)) = S^{n-1}$ a.e., especially we know $m(\Lambda(G)) > 0$.

(II) Note that for any $f \in F - G$, $m(f(\Lambda(G)) \cap \Lambda(G)) = 0$. By (I), we may assume that $m(\Lambda(G)) > 0$. If $\Lambda(G)$ is not an ergodic component for G, then there are G-invariant sets E_1 and E_2 with positive measure such that $E_1 \amalg E_2 = \Lambda(G)$. So the action of F on S^{n-1} is not ergodic because $F(E_1)$ and $F(E_2)$ are both F-invariant and $m(F(E_1) \cap F(E_2)) = 0$. This shows that part "$\Rightarrow$".

Conversely, suppose that $M \notin \mathrm{O_{HB}}$. Then there are F-invariant sets A_1 and A_2 with positive measure such that $A_1 \amalg A_2 = S^{n-1}$. Set $E_i = A_i \cap \Lambda(G)$ $(i = 1, 2)$. E_1 and E_2 are G-invariant and $E_1 \cap E_2 = \phi$. We now show that $m(E_i) > 0$, which implies that $\Lambda(G)$ is not an ergodic component for G. If, say $m(E_1) = 0$, then $\Lambda(G)$ is contained in A_2 a.e., hence $F(\Lambda(G))$ is contained in $F(A_2) = A_2$ a.e. In (I) we have seen that $F(\Lambda(G)) = S^{n-1}$ a.e. under the assumption $M \notin \mathrm{O_G}$. Thus $S^{n-1} = A_2$, which contradicts $m(A_1) > 0$.

Remark. For Fuchsian groups, we have the following system of strict inclusion relations. Most of them were shown in [KT].

$$\begin{tikzcd}
\mathrm{O_G} \arrow[r] & \mathrm{O^m_{HB}} \arrow[r] & \mathrm{O^\infty_{HB}} \arrow[r] \arrow[d] & \mathrm{U_{HB}} \arrow[r] \arrow[d] & \mathrm{O_{AB}} \arrow[d] \\
 & & \mathrm{C} \arrow[r] & \mathrm{U_c} \arrow[r] & \mathrm{W_c}
\end{tikzcd}$$

Here, $\mathrm{O_{AB}}$ is a class of Riemann surfaces which admit no bounded analytic functions except for constants. The famous Myrberg's example is an example of a Riemann surface R such that $R \in \mathrm{O_{AB}}$ but $R \notin \mathrm{U_c}$. Indeed, the 2-sheeted covering surface of $\mathbf{C}$ with an infinite number of branched points above $\mathbf{N} = \{1, 2, 3, \dots\}$ is belongs to $\mathrm{O_G}$ (see Theorem 4.2 in the next section).

§4. Finite covers

Before we consider normal covers in general, we will see in this section that the ergodicity is almost preserved when it is passed to a subgroup of finite index (cf. [$\mathrm{T_2}$] and [Mo]). We begin by showing two fundamental lemmas which will be used later.

Remark. The symbols $=$, $\subset$, $\varnothing$, etc. are used in "almost every" sense unless we mention specifically, in §4. If there is no danger of confusion, we sometimes omit "almost every" or "up to null sets" in the remainder of this paper.

Lemma 4.1. *Let G be a discrete group, H a subgroup of G, and A an ergodic component for H. Then $E = \bigcup_{g\in G} g(A)$ is an ergodic component for G. In particular, $H \in \mathrm{U_{HB}} \Rightarrow G \in \mathrm{U_{HB}}$ and $H \in \mathrm{O^m_{HB}} \Rightarrow G \in \mathrm{O^m_{HB}}$.*

Proof. It is clear that E is invariant under G. If there is a G-invariant subset E_1 of E satisfying $m(E) > m(E_1) > 0$. Then $E_2 = E - E_1$ is G-invariant and $m(E) > m(E_2) > 0$. Set $A_i = A \cap E_i$ $(i = 1, 2)$. They are H-invariant and $A_1 \amalg A_2 = A$. Hence, we may assume, say $A_1 = A$. This implies $A \subset E_1$, from which follows $E = G(A) \subset G(E_1) = E_1$. It contradicts to $m(E) > m(E_1)$.

Lemma 4.2. *For each $h \in \mathrm{Isom}(\mathrm{B}^n)$, there are positive constants $t = t(h)$ and $T = T(h)$ such that*

$$tm(A) \leq m(h(A)) \leq Tm(A) \qquad \text{for every measurable set } A \text{ on } S^{n-1}.$$

Proof. It is known that $m(h(A)) = \int_A \left(\frac{1 - |h^{-1}(0)|^2}{|x - h^{-1}(0)|^2}\right)^{n-1} dm(x)$. Estimating the kernel from below and above, we have

$$\left(\frac{1 - |h^{-1}(0)|^2}{(1 + |h^{-1}(0)|)^2}\right)^{n-1} m(A) \leq m(h(A)) \leq \left(\frac{1 - |h^{-1}(0)|^2}{(1 - |h^{-1}(0)|)^2}\right)^{n-1} m(A).$$

The next result is the first step to consider the problem touched upon in the introduction.

Lemma 4.3. *If H is a subgroup of Γ of finite index and $\Gamma \in \mathrm{U_{HB}}$, then $H \in \mathrm{U_{HB}}$.*

Proof. Fix a left coset decomposition $\Gamma = \gamma_1 H \amalg \ldots \amalg \gamma_n H$. Let A be an ergodic component of Γ. It is H-invariant. Suppose that there is no ergodic component for H in A. Then for any $\varepsilon > 0$, there exists $E \subset A$ such that $0 < m(E) < \varepsilon$ and E is H-invariant. Set $\varepsilon = m(A) \Big/ \sum_{k=1}^{n} T(\gamma_k)$, where $T(\gamma_k)$ is the constant as in Lemma 4.2. For this ε, we take E as above. Let $B = \bigcup_{k=1}^{n} \gamma_k(E)$. It is Γ-invariant and contained in A. But, this contradicts to the fact that A is an ergodic component for Γ, because

$$0 < m(B) \leq \sum_{k=1}^{n} m(\gamma_k(E)) \leq \sum T(\gamma_k) m(E) < m(A).$$

Therefore there is an ergodic component for H in A.

Further, we conjecture that $\Gamma \in \mathrm{U_{HB}} \Rightarrow H \in \mathrm{U_{HB}}$ when H is normal in Γ and Γ/H is cyclic. This conjecture combined with the next Proposition 4.1 would be related to the problem (C1) mentioned in the introduction, which was raised in [LS, p. 304].

Proposition 4.1. *Let Γ be a discrete isometry group acting on B^n which is in $\mathrm{O_G}$, Γ' a normal subgroup of Γ. If Γ' belongs to $\mathrm{U_{HB}}$, then Γ' actually belongs to $\mathrm{O_{HB}}$.*

A proof is given in [VM]. It is based on the following lemma:

Lemma 4.4. *Let H be a normal subgroup of Γ and $A \subset S^{n-1}$ an ergodic component for H. Then each $\gamma(A)$ $(\gamma \in \Gamma)$ is an ergodic component for H.*

Proof. We show $\gamma(A)$ is H-invariant and indivisible. $\gamma(A)$ is H-invariant: For every $h \in H$, there is some $h_1 \in H$ such that $h \circ \gamma = \gamma \circ h_1$, because H is a normal subgroup of Γ. Hence $h \circ \gamma(A) = \gamma \circ h_1(A) = \gamma(A)$. $\gamma(A)$ is indivisible: Let $\gamma(A')$ be a H-invariant measurable subset of $\gamma(A)$. For any $h \in H$, $\gamma \circ h(A') = h_1 \circ \gamma(A') = \gamma(A')$, hence $h(A') = A'$. Since $A' \subset A$, and A is an ergodic component, we have $\gamma(A') = \varnothing$ or $\gamma(A') = \gamma(A)$.

Corollary 4.1. *Let Γ be a discrete isometry group acting on B^n which is in $\mathrm{O_{HB}}$, Γ' be a normal subgroup of Γ. If Γ' belongs to $\mathrm{U_{HB}}$, then Γ' actually belongs to $\mathrm{O^{\infty}_{HB}}$.*

This follows from Lemmas 4.1 and 4.4. It is a parallel result to Proposition 4.1.

We return to the position after Lemma 4.3. We can see that each ergodic component is divided at most n pieces by an n-cover. That is;

Theorem 4.1. *Let Γ' be a subgroup of Γ with $[\Gamma : \Gamma'] = n < \infty$. If $\Gamma \in \mathrm{O_{HB}}$, then $\Gamma' \in \mathrm{O^{n}_{HB}}$. More generally, if $\Gamma \in \mathrm{O^{N}_{HB}}$ $(1 \leq N \leq \infty)$, then $\Gamma' \in \mathrm{O^{nN}_{HB}}$.*

Proof. We show only the former statement. The latter is carried out by the same proof. Let $H = \bigcap_{\gamma \in \Gamma} \gamma \Gamma' \gamma^{-1}$. It is easy to see that H is normal in Γ and is of finite index. We define $m = [\Gamma' : H] < \infty$. Fix left coset decompositions

$$\Gamma = f_1 \Gamma \amalg \ldots \amalg f_n \Gamma' \qquad (f_1 = 1) \text{ and}$$

$$\Gamma' = e_1 H \amalg \ldots \amalg e_m H \qquad (e_1 = 1)\,.$$

We may take $\{g_{ij}\} = \{f_i e_j\}_{j=1,\ldots,m}^{i=1,\ldots,n}$ as a system of representatives for a left coset decomposition of Γ by H.

By Lemma 4.3, H has an ergodic component A. By Lemma 4.4, we see each $g_{ij}(A)$ is an ergodic component for H. Thus $\{g_{ij}(A)\}$ are mutually coincident or disjoint. Their union is S^{n-1}, for $\bigcup_{i,j} g_{ij}(A)$ is Γ-invariant and $\Gamma \in \mathrm{O_{HB}}$. Let $\dim \mathrm{HB(H)}$ be the number of ergodic components for H and k the number of elements of $\{g_{ij}\}$ which fix A. Then, with each ergodic component for H, k sets of $\{g_{ij}(A)\}$ coincident. Hence $\dim \mathrm{HB(H)} = mn/k$.

Let $\{A_p\}$ $(p = 1, \ldots, mn/k)$ be the ergodic components for H. By Lemma 4.1, $\bigcup_{j=1}^{m} e_j(A_p) = \bigcup_{e \in \Gamma'} e(A_p)$ is an ergodic component for Γ'. We classify $\{A_p\}$ by the following relation:

$A_1 \sim A_2 \quad \Leftrightarrow \quad A_1$ and A_2 are contained in the same ergodic component for Γ'

We denote the equivalent class by $[A_q]$. Then $d = \dim \mathrm{HB}(\Gamma') = \#\{[A_q]\}$. Let r_p be the number of elements of $\{e_j\}$ which fix A_p. Since precisely k elements of $\{g_{ij}\}$ fix A_p and $\{e_j\} \subset \{g_{ij}\}$, r_p is not larger than k for every p. The equivalent class $[A_q]$ contains m/r_q sets. Therefore,

$$\frac{mn}{k} = \sum_{q=1}^{d} \frac{m}{r_q} \geq \sum_{q=1}^{d} \frac{m}{k} = \frac{md}{k},$$

that is, $n \geq \dim \mathrm{HB}(\Gamma')$.

Finally we remark that the classes $\mathrm{O_G}$ and C are preserved by finite covers.

Theorem 4.2. *Let H be a subgroup of Γ of finite index. Then $\Lambda_c(H)$ and $\Lambda_h(H)$ coincide precisely with $\Lambda_c(\Gamma)$ and $\Lambda_h(\Gamma)$ respectively. Especially, $H \in \mathrm{O_G} \Leftrightarrow \Gamma \in \mathrm{O_G}$ and $H \in \mathrm{C} \Leftrightarrow \Gamma \in \mathrm{C}$.*

Proof. It is clear that $\Lambda_c(H) \subset \Lambda_c(\Gamma)$ and $\Lambda_h(H) \subset \Lambda_h(\Gamma)$. We will show the inverse inclusions. Let $\Gamma = He_1 \amalg \ldots \amalg He_n$ be a right coset decomposition of Γ by H. Let $\{\gamma_m(0)\}_{\gamma_m \in \Gamma}$ be a sequence of conical (horospherical) approach to $x \in \Lambda_c(\Gamma)$ $(\Lambda_h(\Gamma))$. Then at least one coset He_i contains infinite number of elements $\{\gamma_{m_k}\}$ of $\{\gamma_m\}$. Since $\gamma_{m_k} e_i^{-1} \in H$, the orbit of $e_i(0)$ by H contains a sequence of conical (horospherical) approach to x.

§5. Inheritance to normal covers

Roughly speaking, by normal covers ideal boundary of a hyperbolic manifold is divided evenly. So we expect that normal subgroups of a discrete group G should inherit some ergodic property from G. The following theorem explains such an aspect well.

Theorem 5.1 *Let Γ be a discrete subgroup of* $\mathrm{Isom}^+(\mathrm{B}^n)$ *which belong to* $\mathrm{O_G}$. *Then its non-trivial normal subgroup H belongs to* C.

Proof. Assume that H is not conservative. Then H has a fundamental set A of positive measure in the dissipative part $\mathscr{D}(H)$. On the other hand, the assumption $\Gamma \in \mathrm{O_G}$ implies that $\Lambda_c(\Gamma)$ has full measure (Proposition 1.1). Let $w_A(z)$ be the harmonic measure of A with respect to z, that is, $w_A(z) = \int_A \left(\frac{1-|z|^2}{|x-z|^2}\right)^{n-1} dm(x)$. Since $w_A(z)$ has the non-tangential limit $1_A(x)$ at almost every $x \in S^{n-1}$, we can choose a point $x \in A \cap \Lambda_c(\Gamma)$ such that $w_A(z) \to 1$ as z approaches to x non-tangentially. Take a subsequence $\{\gamma_n\}$ of Γ such that $\gamma_n^{-1}(0) \to x$ in a cone. Then $w_A(\gamma_n^{-1}(0)) \to 1$ as $n \to \infty$, where $w_A(\gamma_n^{-1}(0)) = w_{\gamma_n(A)}(0) = m(\gamma_n(A))$. We define $A_n = \gamma_n(A)$.

For any $h \in H - \{1\}$, there is some $h' \in H - \{1\}$ such that

$$h(A_n) \cap A_n = \gamma_n\{\gamma_n^{-1} h \gamma_n(A) \cap A\} = \gamma_n\{h'(A) \cap A\}$$

by normality of H. Since A is a fundamental set of $\mathscr{D}(H)$, we have $m(h(A_n)\cap A_n)=0$. For an arbitrary $h\in H$ which is not identity, we take a constant $t>0$ as in Lemma 4.2. Then $m(h(A_n))+m(A_n)\geq(1+t)m(A_n)$. This value exceeds 1 when n is sufficiently large, because $m(A_n)\to 1$. Therefore $h(A_n)$ and A_n must intersects with positive measure, which contradicts the result $m(h(A_n)\cap A_n)=0$.

An application of Theorem 5.1 is given in [VM]. In the case where $n=2$, the converse is also true. Namely, we can characterize Riemann surfaces of $\mathrm{O_G}$ by their normal covers (cf. Velling [V]).

Theorem 5.2. *A hyperbolic Riemann surface R belongs to the class $\mathrm{O_G}$ if and only if its any non-universal normal cover belongs to the class* C.

Proof. We have only to show "if part". Assume that $R\notin\mathrm{O_G}$. Further we may assume that R does not have border. We take a simple closed curve α in R which is homotopically non-trivial and divide R into two subregions. Since the Wiener's harmonic boundary $\Delta_w(R)$ is not empty and $\Delta_w(R)\cap\alpha=\varnothing$, at least one of the subregions does not belong to

$$\mathrm{SO_{HB}}=\{\text{bordered Riemann surfaces whose ideal boundary has null harmonic measure}\}$$

by its characterization theorem (cf. [T_1, §1].) Let Y be the other subregion. We cut off Y from R and instead of Y paste a conformal disk U to make a new Riemann surface $\hat{R}$. Since $\hat{R}$ has the subregion not belonging to $\mathrm{SO_{HB}}$, we see $\hat{R}\notin\mathrm{O_G}$.

Let $\hat{\pi}$ be a holomorphic universal covering map $D\to\hat{R}$ satisfying $\hat{\pi}(0)\in U$, and H the corresponding Fuchsian group. Set $\hat{\pi}^{-1}(U)=\{U_n\}$. They are neighborhoods of the orbit points of the origin by H, which have the same finite hyperbolic diameter. Since $\hat{R}\notin\mathrm{O_G}$, we know $m(\Lambda_c(H))=0$ by Proposition 1.1. Hence, almost every x on ∂D satisfies the condition (P): there is some Stolz angular region in $D-\{U_n\}$ with vertex x on ∂D. We prepare the copies $\{Y_n\}$ of Y, cut off $\{U_n\}$ from $\hat{R}$, and instead of each of $\{U_n\}$ paste each of $\{Y_n\}$. The resulting Riemann surface we denote by R'. It is a holomorphic normal cover of R, because there is the covering map π as follows:

$$\pi|_{Y_n}\colon Y_n\to Y \text{ is the identity map for each } n\,,$$

$$\pi|_{R'-\{Y_n\}}=\hat{\pi}|_{D-\{U_n\}}\,.$$

This normal covering surface R' does not belong to C, because R' has the subregion $D-\{U_n\}$ which satisfies the condition (P) [P_1, Th. 3].

By analogy to $\mathrm{O_G}$, one may feel that the conjecture (C2) concerning $\mathrm{O_{HB}}$ mentioned in the introduction is natural. To prove a relating result to this conjecture for $n=2$ (Theorem 5.3), we need a lemma:

Lemma 5.1. *Let $S = \{s_n\}$ be a set of a countable number of points in D which has no accumulated points in D. Suppose that there exists a positive constant δ such that for each point z of D, the hyperbolic distance from z to S is less than δ. For any measurable set A on ∂D, we denote the harmonic measure of A by $w_A(z)$, and define $Q = \{z \in S | w_A(z) \geq 1/2\}$. Then $C(Q) = A$ (a.e.).*

Proof. $w_A(z)$ has the non-tangential limit at almost every $x \in \partial D$, denoted also by $w_A(x)$. Below all the statements of this proof is in "a.e." sense. Since $w_A(x) = 1_A(x)$, we have the inclusion $C(Q) \subset \{x \in \partial D | w_A(x) \geq 1/2\} = \{x \in \partial D | w_A(x) = 1\} = A$. Similarly, $C(S - Q) \subset \{x \in \partial D | w_A(x) \leq 1/2\} = \partial D - A$. But by the distribution of S, we know $C(Q) \cup C(S - Q) = \partial D$. Therefore $C(Q) = A$.

Theorem 5.3. *A Riemann surface R belongs to the class $\mathrm{O_{HB}}$ if and only if for any closed set E of countable points in R, any holomorphic normal cover of the Riemann surface $R - E$ is either conservative or totally dissipative.*

Proof. Assume that $R \notin \mathrm{O_{HB}}$. Then there exists a non-constant harmonic measure w on R. We choose a set of a countable number of points S in R so that it has no accumulated points in R and the hyperbolic distance from any point of R to S is less than some positive constant. Let $\pi: D \to R$ be a holomorphic universal covering map, $\tilde{w} = w \circ \pi$ and $\tilde{S} = \pi^{-1}(S)$. Let A be a set of points on ∂D where $\tilde{w}(z)$ has a non-tangential limit 1. Since $\tilde{w}(z)$ is non-constant, $0 < m(A) < 1$. We define $\tilde{E} = \{z \in \tilde{S}: \tilde{w}(z) \geq 1/2\}$. Then by Lemma 5.1, $C(\tilde{E}) = A$ (a.e.). Set $E = \pi(\tilde{E})$. It is closed and countable. The restriction π to $D - \tilde{E}$ is a holomorphic normal covering map of $R - E$. The Riemann surface $D - \tilde{E}$ belongs to U_c, but not belongs to C, by Lemma 2.1.

Coversely assume that $R \in \mathrm{O_{HB}}$. For any closed set E of countable points in R, the Riemann surface $R - E$ also belongs to $\mathrm{O_{HB}}$. Then the assertion follows directly from the general theorem:

Theorem 5.4. *Let Γ be a discrete subgroup of* $\mathrm{Isom}^+(\mathrm{B}^n)$ *which belongs to O_{HB}. Then its normal subgroup Γ' either belongs to* C *or not belongs to* U_c.

Proof. We will show that $\Lambda_h(\Gamma')$ is invariant under Γ. Then the ergodicity of the action of Γ forces $m(\Lambda_h(\Gamma'))$ to be one or zero. The former means that $\Gamma' \in \mathrm{C}$ and the latter $\Gamma' \notin \mathrm{U_c}$ by Proposition 2.1′. If $x \in \Lambda_h(\Gamma')$, then there is a sequence $\{\gamma_n'\}$ of Γ' such that $\{\gamma_n'(0)\}$ approaches to x horospherically. Let γ be any element of Γ. A sequence $\{\gamma \circ \gamma_n' \circ \gamma^{-1}\}$ belongs to Γ' by normality, and the images of a reference point $\gamma(0)$ under $\{\gamma \circ \gamma_n' \circ \gamma^{-1}\}$ approach to $\gamma(x)$ horocyclically. Hence $\gamma(x) \in \Lambda_h(\Gamma')$. This shows that $\Lambda_h(\Gamma')$ is invariant under Γ.

§6. QC non-invariance of $\mathrm{O}_{\mathrm{HB}}^{\infty}$ and U_{HB}

In the remainder of this paper, we treat only Fuchsian groups because our problems turn out to be trivial by Mostow-Sullivan rigidity in the higher dimensional cases. See Remark in the end of §7.

In the classification theory of Riemann surfaces, we often pose a question whether a given class $\mathcal{O}$ of Riemann surfaces has a property of quasiconformal invariance, or not. Here, we say $\mathcal{O}$ is quasiconformally (QC) invariant if it satisfies the following property:

"For any Riemann surface $R \in \mathcal{O}$, if there is a QC homeomorphism of R onto another Riemann surface R', then R' belongs to $\mathcal{O}$."

Riemann surfaces of finite conformal type are of course QC invariant. Further it is known that the class $\mathrm{O_G}$ is QC invariant (Pfluger). Moreover, the classes defined by degeneration for harmonic functions with finite Dirichret integrals, $\mathrm{O^n_{HD}}$ $(1 \leq n \leq \infty)$ and $\mathrm{U_{HB}}$ are QC invariant (cf. [SN, Chap. III]). These follow from a general result that a QC homeomorphism between two Riemann surfaces has a homeomorphic extension to their Royden's compactifications which preserves harmonic boundary points. (Lyons' example below implies that a QC homeomorphism between Riemann surfaces need not have homeomorphic extension to their Wiener's compactifications because if it has, the harmonic boundary is preserved [SN, p. 282]. Recently, Segawa [Se] constructed a concrete example of a pair of Denjoy domains whose QC homeomorphism cannot be extended to their Martin's compactifications). We can see both $\mathrm{O_{AB}}$ and $\mathrm{O_{AD}}$ are not QC invariant by an application of Myrberg's well-known example [SN, Chap. II]. For Denjoy domains, $\hat{C} - E$ belongs to $\mathrm{O_{AB}}$ where E is a compact set on R if and only if the linear measure of E is zero. Thus we can also construct a counterexample for QC invariance of $\mathrm{O_{AB}}$ by using the Beurling-Ahlfors extension [BA].

A class $\mathcal{O}$ of Fuchsian groups is QC invariant if it satisfies;

"For any Fuchsian group $\Gamma \in \mathcal{O}$, if there is a QC automorphism f of D such that $\Gamma' = f\Gamma f^{-1}$ is Fuchsian, then Γ' belongs to $\mathcal{O}$."

For example, the class of Fuchsian groups whose exponent of convergence is equal to 1, which includes $\mathrm{O_G}$, is QC invariant [FR]; the class of Fuchsian groups whose limit set has null linear measure is not QC invariant [$\mathrm{T_1}$].

Note that QC homeomorphisms which appear in the above examples are defined essentially on the whole extended complex plane $\hat{\mathbf{C}}$. It had been difficult to construct QC homeomorphisms between non-planar surfaces which show non-invariance of $\mathrm{O^n_{HB}}$ $(1 \leq n \leq \infty)$ and $\mathrm{U_{HB}}$. In this section, we prove that the classes $\mathrm{O^n_{HB}}$ and $\mathrm{U_{HB}}$ are not QC invariant. For $\mathrm{O_{HB}} = \mathrm{O^1_{HB}}$, Lyons [L] constructed an example such that R and R' are QC equivalent, while $R \in \mathrm{O_{HB}}$ and $R' \in \mathrm{O^2_{HB}} - \mathrm{O_{HB}}$. Our examples in this section are based on his example.

In the beginning, we remark for a finite integer n, it easily follows from Lyons' that $\mathrm{O^n_{HB}}$ is not QC invariant. Let R be as above, α a simple closed curve which runs transversely around a handle of R. We cut open R along α, and the resulting two border curves we denote by α_+ and α_-. Then paste cyclically n pieces with α_- on α_+ to form an n-covering surface S of R. Since R and R' are QC equivalent, there is a QC map of S onto the n-cover S' of

R'. By the number of Wiener's harmonic boundary points (or, by Theorem 4.1 and the two- region test (cf. [SN, p. 242])), we see $S \in O^{n}_{HB}$ but $S' \notin O^{2n-1}_{HB}$ $(\in O^{2n}_{HB})$. Hence for $1 \leq n < \infty$, O^{n}_{HB} is not QC invariant.

For O^{∞}_{HB} and U_{HB}, we need more complicated construction. In the remainder of this section, we show that a Riemann surface in O^{∞}_{HB} may be mapped by a QC homeomorphism onto a Riemann surface not in U_{HB}. Especially, we obtain QC non-invariance of O^{∞}_{HB} and U_{HB}.

Lemma 6.1. *There are Riemann surfaces M and M', both belonging to the class O_{HB}, and a QC homeomorphism $f: M \to M'$, having the following properties*:

(1) *A countable number of disjoint simple closed curves $\{c_n\}$ not accumulating to any point of M, divide M into two subregions A and B.*

(2) $A \notin SO_{HB}$, $B \in SO_{HB}$, $A' = f(A) \in SO_{HB}$ *and* $B' = f(B) \notin SO_{HB}$.

Proof. By Lemma 8.1 and Proposition 8.3 of Lyons [L], we obtain O_{HB}-Riemann surfaces M and M' homeomorphic by a QC map $f: M \to M'$, satisfying that; there are open sets $\mathscr{A}$ and $\mathscr{B}$ in M such that (I) $M - (\mathscr{A} \cup \mathscr{B})$ consists of a countable number of disjoint simple closed curves not accumulating to any point of M, and (II) a Brownian motion $\{X_t(\omega)\}_{t \geq 0}$ on M eventually stays in $\mathscr{A}$, whereas a Brownian motion on M' eventually stays in $\mathscr{B}' = f(\mathscr{B})$. Here, we assume Brownian motions are defined on a probability space $(\Omega, \mathscr{F}, P)$ with a filtration $\{\mathscr{F}_t\}_{t \geq 0}$.

Let $S(\omega)$ be a Borel measurable function on Ω defined by $S(\omega) = \sup_t \{X_t(\omega) \notin \mathscr{A}\}$. Then the condition that the Brownian motion $\{X_t(\omega)\}$ starting at $z \in M$ eventually stays in $\mathscr{A}$, is formulated by $P_z(S(\omega) < \infty) = 1$. Note that this doesn't depend on z. Let $\{A_m\}_{m=0,1,\ldots}$ be connected components of $\mathscr{A}$. Since the events $\{\omega: X_t(\omega) \in A_m$ for ${}^{\forall}t > S(\omega)\}$ cannot occur simultaneously,

$$\sum_m P_z(X_t(\omega) \in A_m \text{ for } {}^{\forall}t > S(\omega)) = 1 .$$

Hence we may assume that there is a component A_0 of $\mathscr{A}$ such that

$$P_z(X_t(\omega) \in A_0 \text{ for } {}^{\forall}t > S(\omega)) = \delta > 0 .$$

For a positive time r, set $\varepsilon(r) = P_z(X_t(\omega) \in A_0$ for ${}^{\forall}t \geq r)$. Since $P_z(S(\omega) < \infty) = 1$, $\varepsilon(r)$ converges increasingly to δ as r goes to infinity. So we have a sufficiently large number T such that

$$P_z(X_t(\omega) \in A_0 \text{ for } {}^{\forall}t \geq T\} = \varepsilon > 0 \ \ldots \ (1) .$$

We now prove that $A_0 \notin SO_{HB}$. Assume that $A_0 \in SO_{HB}$. Since $P_z(X_s(\omega)$ cannot stay in A_0 for all $s \geq 0) = P_z(X_s(\omega) \notin A_0$ for some $s \geq 0)$ is the harmonic measure of the relative boundary ∂A_0 with the variable $z \in A_0$, the assumption $A_0 \in SO_{HB}$ is equivalent to the condition

$$P_z(X_s(\omega) \notin A_0 \text{ for } {}^{\exists}s \geq 0) = 1 \qquad \text{for any } z \text{ in } M \ \ldots \ (2) .$$

We show

$$P_z(X_t(\omega) \notin A_0 \text{ for } {}^\exists t \geq T) = 1 \ \ldots\ (3)$$

in order to deduce a contradiction to (1). Using conditional probability,

$$P_z(X_t(\omega) \notin A_0 \text{ for } {}^\exists t \geq T) = P_z P_z(X_t(\omega) \notin A_0 \text{ for } {}^\exists t \geq T | \mathscr{F}_T)\,.$$

By the Markov property of $\{X_t\}$,

$$P_z(X_t \notin A_0 \text{ for } {}^\exists t \geq T | \mathscr{F}_T) = P_{X_{T(\omega)}}(X_s(\omega) \notin A_0 \text{ for } {}^\exists s \geq 0)\,,$$

and it is equal to 1 a.s. ω by the condition (2). We have got (3), hence proved $A_0 \notin \mathrm{SO_{HB}}$.

By the same way, we can choose a component B'_0 of $\mathscr{B}'$ such that $B'_0 \notin \mathrm{SO_{HB}}$. Put $B_0 = f^{-1}(B'_0)$. Then we denote by A the component of $M - \bar{B}_0$ which contains A_0. Take $B = M - \bar{A}$. It is obvious that A and B are connected, and they have the common relative boundary of simple closed curves $\{c_n\} \subset M - (\mathscr{A} \cup \mathscr{B})$. Since A contains $A_0 \notin \mathrm{SO_{HB}}$, A does not belong to $\mathrm{SO_{HB}}$. Then it follows from $M \in \mathrm{O_{HB}}$ and the two region test that $B \in \mathrm{SO_{HB}}$. By the same reason, $A' = f(A) \in \mathrm{SO_{HB}}$ and $B' = f(B) \notin \mathrm{SO_{HB}}$.

Theorem 6.1. *There are Riemann surfaces $R \in \mathrm{O}^\infty_{\mathrm{HB}}$ and $R' \notin \mathrm{U_{HB}}$ which are QC equivalent.*

Proof. Let M and M' be Riemann surfaces in $\mathrm{O_{HB}}$, f a QC homeomorphism of M onto M', and A and B subregions of M obtained in Lemma 6.1. Since the relative boundary of B consists of a countable number of simple closed curves $\{c_n\}$, we can construct a hyperbolic Riemann surface $\hat{B}$ without border by pasting a conformal disk U_n to each curve c_n of B as in the proof of Theorem 5.2. Let $\hat{\pi}\colon D \to \hat{B}$ be a holomorphic universal cover and H be the Fuchsian group of $\hat{\pi}$. We may take a fundamental region ω of H in D such that one component of $\hat{\pi}^{-1}(U_n)$ completely contained in ω for every n. For each $h \in H$, we prepare a copy A_h of A. From $h(\omega)$, we remove the preimages of $\{U_n\}$, then attach A_h so that each boundary of A_h is on the corresponding preimage of $\{c_n\}$. After such surgeries for all $h \in H$, we obtain the Riemann surface which we denote by R. We define a covering map $\pi\colon R \to M$ as follows:

$$\pi|_{A_h}\colon A_h \to A \text{ is the identity map for every } h \in H$$

$$\pi = \hat{\pi} \text{ on } R - \bigcup_{h \in H} A_h$$

Then, π is a holomorphic normal covering map of R onto M.

We will show that R belongs to the class $\mathrm{O}^\infty_{\mathrm{HB}}$. Since $M \in \mathrm{O_{HB}}$, by Corollary 4.1, we have only to show that $R \in \mathrm{U_{HB}}$. The Riemann surface M has the subregion $A \notin \mathrm{SO_{HB}}$. Hence the characterization theorem asserts that $\Delta_w(M) \cap (\bar{A}^w - \overline{\partial A}^w) \neq \varnothing$, where $-_w$ means the closure in the Wiener's compactification M^*_w of M, and $\Delta_w(M)$ the harmonic boundary. Since $M \in \mathrm{O_{HB}} - \mathrm{O_G}$, $\Delta_w(M)$ consists of a single point p, therefore $\Delta_w(M) \cap (\bar{A}^w - \overline{\partial A}^w) = \{p\}$. We choose a copy A_h of A in R. Then the holomorphic mapping $\pi: R \to M$ is univalent on A_h and satisfies the condition that $\pi(\partial A_h) = \partial A$. By [$T_2$, Prop. A], we have the continuous extension of π which gives a homeomorphism from $\Delta_w(R) \cap (\bar{A}_h^w - \overline{\partial A}_h^w)$ onto $\Delta_w(M) \cap (\bar{A}^w - \overline{\partial A}^w) = \{p\}$. Therefore, $\Delta_w(R) \cap (\bar{A}_h^w - \overline{\partial A}_h^w)$ also consists of a single point. Since $\bar{A}_h^w - \overline{\partial A}_h^w$ is open in R_w^*, this implies that $\Delta_w(R)$ contains an isolated point, hence $R \in \mathrm{U_{HB}}$. We have seen that $R \in \mathrm{O_{HB}^{\infty}}$.

By the same way as for R, we construct the normal cover R' of M' and the covering map $\pi': R' \to M'$. Then, there is a QC homeomorphism F of R onto R' such that $f \circ \pi = \pi' \circ F$. We now prove that the Riemann surface R' does not belong to $\mathrm{U_{HB}}$: Assume that $R' \in \mathrm{U_{HB}}$. Again by Corollary 4.1, R' must be in $\mathrm{O_{HB}^{\infty}}$, for it covers $M' \in \mathrm{O_{HB}}$. Let G' be the preimage $\pi'^{-1}(B')$ of B' in R'. It is connected. Since $B' \notin \mathrm{SO_{HB}}$, there is a non-constant bounded harmonic function u on B' such that $u = 0$ on $\partial B'$. Then $u \circ \pi'$ is a non-constant bounded harmonic function on G' such that $u \circ \pi' = 0$ on $\partial G'$. This means that $G' \notin \mathrm{SO_{HB}}$. If G' is not in $\mathrm{U_{HB}}$ as a Riemann surface, then by the same reason as the first part of the proof, $\Delta_w(R') \cap (\overline{G'}^w - \overline{\partial G'}^w)$ does not have isolated points, which violates $R' \in \mathrm{O_{HB}^{\infty}}$. Hence G' is forced to belong to $\mathrm{U_{HB}} \subset \mathrm{O_{AB}}$. But by the construction of R', G' is a planar domain which admits non-constant bounded analytic functions. It is a contradiction. Therefore $R' \notin \mathrm{U_{HB}}$.

Remark. (1) By Theorem 5.4, we know $R' \in \mathrm{C}$ or $R' \notin \mathrm{U_c}$. But, we do not know whether R' belongs to $\mathrm{U_c}$ or not.

(2) There exists a continuous mapping ρ of Wiener's compactification M_w^* of M onto Royden's compactification M_R^* such that ρ is the identity on M and $\rho(\Delta_w(M)) = \Delta_R(M)$ [SN, p. 230]. Since $\Delta_w(M)$ consists of a single point p, $\Delta_R(M)$ also consists of a single point $q = \rho(p)$. We now show that q is in $\overline{\bigcup c_n}^R$. If not, for the subregion $B \in \mathrm{SO_{HB}} \subset \mathrm{SO_{HD}}$ [SN, p. 241], we have $\bar{B}^R \cap \Delta_R(M) = \emptyset$. Then by Kusunoki-Mori's theorem [SN, p. 159], the double $\hat{B}$ of B about $\partial B = \bigcup c_n$ belongs to $\mathrm{O_G}$. The QC homeomorphism $f: B \to B'$ is extended to a QC homeomorphism $\hat{f}$ of $\hat{B}$ onto $\hat{B}'$, the double of B'. But $\hat{B}' \notin \mathrm{O_G}$, because $B' \notin \mathrm{SO_{HB}}$. It contradicts to the fact that the class $\mathrm{O_G}$ is invariant under QC homeomorphisms. Hence we know $q \in \overline{\bigcup c_n}^R$. But since p is in $\bar{A}^w - \overline{\partial A}^w$, we also know $p \notin \overline{\bigcup c_n}^w$. This argument shows that even if a subregion B of a Riemann surface M satisfies $\bar{B}^w \cap \Gamma_w(M) = \emptyset$, the double $\hat{B}$ need not belong to $\mathrm{O_G}$, namely, Kusunoki-Mori's type theorem fails in Wiener's compactification.

§7. QC non-invariance of conservativity

In this section, we show that each of C, $\mathrm{U_c}$ and $\mathrm{W_c}$ is not QC invariant when $n = 2$. Shiga [Sh] showed the fact for $\mathrm{W_c}$, however once again, we explain it by using Lemma 2.1. We have only to construct a QC automorphism f of D and $Q = \{q_n\}$ so that $\{q_n\}$ does not satisfy the Blaschke condition, while $\{f(q_n)\}$ satisfies it. In the upper half plane $H = \{\operatorname{Im} z > 0\}$, set $q_n = i/n$, and $f(z) = |z|^{\alpha-1} z$ $(\alpha > 1)$. Then $\sum |q_n| = \infty$, but $\sum |f(q_n)| < \infty$. It is easy to see that the conjugation by a Möbius transformation $H \to D$ yields our example.

Concerning C and U_c, it is rather easy to show merely that they are not QC invariant. Indeed, a characterization of C for Denjoy domains [P_3, Ex. 4] combined with the Beurling-Ahlfors' well-known example, exhibits QC non-invariance of C. So we will construct a stronger example: There are Riemann surfaces R and R' such that $R \in \mathrm{C}$ and $R' \notin \mathrm{U_c}$, but they are QC equivalent. We prepare the following two lemmas.

Lemma 7.1. *Let φ be a quasisymmetric (strictly increasing) function from* $\mathbf{R}$ *onto* $\mathbf{R}$ *which is singular w.r.t. the Lebesgue measure* $|\ |$ *on* $\mathbf{R}$. *Put* $\psi(x) = \varphi(x) + x$, $f = \psi^{-1}$ *and* $g = \varphi \circ f$. *Then we have;*

(1) *f and g are quasisymmetric.*

(2) *f and g are absolutely continuous w.r.t.* $|\ |$.

(3) $\mathbf{R}$ *is a disjoint union of two sets A and B both of positive measure such that* $|f(A)| = 0$ *and* $|g(B)| = 0$.

Proof. (1) Since φ is quasisymmetric, there is a constant $k \geq 1$ such that for every $x \in \mathbf{R}$ and $t > 0$,

$$\frac{1}{k} \leq \frac{\varphi(x+t) - \varphi(x)}{\varphi(x) - \varphi(x-t)} \leq k\,.$$

Hence

$$\begin{aligned}\psi(x+t) - \psi(x) &= \varphi(x+t) - \varphi(x) + t \\ &\leq k(\varphi(x) - \varphi(x-t) + t) = k(\psi(x) - \psi(x-t))\end{aligned}$$

and

$$\begin{aligned}\psi(x) - \psi(x-t) &= \varphi(x) - \varphi(x-t) + t \\ &\leq k(\varphi(x+t) - \varphi(x) + t) = k(\psi(x+t) - \psi(x))\,.\end{aligned}$$

This means that ψ is quasisymmetric. Then the inverse $f = \psi^{-1}$ and the composition $g = \varphi \circ f$ are also quasisymmetric.

(2) f is absolutely continuous because the inverse ψ maps any set of positive measure on $\mathbf{R}$ to a set of positive measure. To see g is absolutely continuous, we consider the inverse g^{-1} and take any set $E' = \varphi(E)$ of positive measure on $\mathbf{R}$. Since $g^{-1}(\varphi(E)) = \psi(E)$ and $|\psi(E)| \geq |\varphi(E)|$, $|E'| > 0$ implies $|g^{-1}(E')| > 0$.

(3) Since φ is singular, there is a set A' of null measure on $\mathbf{R}$ whose image by φ is full on $\mathbf{R}$. Take $A = \psi(A')$ and $B = \mathbf{R} - A$. Then A is clearly of positive measure and so is B because $B = \psi(\mathbf{R} - A')$. But, $|f(A)| = 0$ and $|g(B)| = 0$.

Lemma 7.2 *Let f be a QC automorphism of D, and Q a set of countable points of D. Then $f(C(Q))$ coincides precisely with $C(f(Q))$.*

Proof. We have only to prove that $f(C(Q)) \subset C(f(Q))$, because the inverse inclusion follows from $f^{-1}(C(f(Q)) \subset C(f^{-1}(f(Q)))$. Let x be an arbitrary point of $C(Q)$. Then there is a Stolz angular region Δ with a vertex x in which a subsequence $\{q_n\}$ of Q converges to x. Note that f has the QC extension to the whole complex plane. By the three point property of quasicircles, we can see $f(\Delta)$ is contained in some Stolz angular region in D with the vertex $f(x)$ (cf. [BR, Lemma I]). Hence $f(q_n)$ converges to $f(x)$ conically.

Theorem 7.1. *There are a conservative Fuchsian group acting on D and a QC (even quasi-isometric) automorphism f such that $\Gamma' = f\Gamma f^{-1}$ is a totally dissipative Fuchsian group. In other words, the horocyclic limit set of Γ is of full measure but that of Γ' is of null measure.*

Proof. We translate Lemma 7.1 into the unit disk D by the conjugation of the Möbius transformation $H \to D$. Then we get two disjoint sets A, B of positive measure whose union is ∂D, and absolutely continuous quasisymmetric functions f, g on ∂D such that $m(f(A)) = 0$, $m(g(B)) = 0$. We denote the Beurling-Ahlfors extension [BA] of f and g by the same notations f and g respectively.

Let S be a set of points in D which satisfies the assumption of Lemma 5.1. By this lemma, we choose a subset Q of S so that $C(Q) = A$ (a.e.). Set $R' = f(D - Q)$ and $R = g(D - Q)$. By Lemma 7.2, $C(f(Q)) = f(C(Q))$. Since $C(Q) = A$ (a.e.) and f is absolutely continuous, $C(f(Q)) = f(A)$ (a.e.), that is, $m(C(f(Q)) = 0$. This implies the Riemann surface R' is not in $\mathrm{U_c}$ by Lemma 2.1. Similarly, we have $C(g(Q)) = g(C(Q))$, $\partial D - B = C(Q)$ (a.e.). Hence $m(C(g(Q)) = 1$, which shows that $R \in \mathrm{C}$. But, $f \circ g^{-1}$ is a QC mapping of R onto R'.

Let Γ and Γ' be the Fuchsian groups corresponding to the Riemann surfaces R and R' respectively. Then by Proposition 2.1, $m(\Lambda_h(\Gamma)) = 1$ and $m(\Lambda_h(\Gamma')) = 0$. In other words, the action of Γ on ∂D is conservative, while that of Γ' is totally dissipative. However, there is a QC automorphism Φ of D, even quasi-isometry w.r.t. the hyperbolic metric on D by Douady-Earle's theorem [DE], such that $\Phi\Gamma\Phi^{-1} = \Gamma'$.

Remark. (1) Let Γ be a discrete isometry group acting on the hyperbolic space B^n of dimension n, where $n \geq 3$. Sullivan proved that if the action of Γ on the sphere at infinity is conservative with respect to the canonical measure, then Γ is Mostow-rigid. Here we say Γ is *Mostow-rigid* if any discrete isometry group Γ' acting on B^n which is conjugate to Γ by a quasi-isometric automorphism f of B^n, is actually conjugate to Γ by an isometric automorphism. Particularly we know conservative groups for $n \geq 3$ are quasi-isometrically invariant. While in the case of Fuchsian groups, i.e., $n = 2$, conservative groups need not be Mostow-rigid. In addition, it has been shown by our theorem that even the class C is not preserved under quasi-isometric automorphisms. (Recently, Astala-Zinsmeister [AZ] proved that for a Fuchsian group Γ, it is of divergence type ($\in \mathrm{O_G}$) if and only if Γ has a "weakened" rigidity property.)

(2) Sullivan [Su, Th. IV] also showed the following equivalent condition $(*)$ to conservativity of the action on the sphere at infinity of a group Γ of hyperbolic motions: For some/any point p on $R = B^n/\Gamma$, we denote by $V(r)$ the hyperbolic volume of the points of R within a distance r from p, and by $H(r)$

the hyperbolic volume of the ball of radius r in B^n. Then,

$$(*) \qquad V(r)/H(r) \to 0 \quad \text{as } r \to \infty .$$

This condition is represented by the hyperbolic metric on R. But, our example shows that the condition $(*)$ is not quasi-isometric invariant in the case of dimension 2.

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