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Local times and related sample Path properties of certain self-similar processes

By

Norio KÔNO and Narn-Rueih SHIEH*

1. Introduction

Let $X(t)$, $t \in R_+$, be a real-valued measurable stochastic process. We say that X is H-self-similar (abbrev. H-*ss*), $H > 0$, if for any $c > 0$ $\{X(ct)\}$ and $\{c^H X(t)\}$ have the same finite dimensional distributions. We say that X is of stationary increments (abbrev. *si*) if for any $b \geq 0$ $\{X(t+b) - X(b)\}$ and $X(t) - X(0)\}$ have the same finite dimensional distributions. We say that X is symmetric α-stable (abbrev. $S\alpha S$) if all finite linear combinations $\sum_{i=1}^n a_i X(t_i)$ are symmetric α-stable random variables. A bibliographical guide to the development of self-similar processes can be found in Taqqu [18]. We also mention that there are two important classes of H-*ss si* $S\alpha S$ processes, namely linear fractional stable and (real) harmonizable fractional stable processes, which are defined respectively by

$$\begin{aligned}\Delta_{H,\alpha}(a, b; t) = \int_{-\infty}^{\infty} &\{a[(t-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha}] \\ &+ b[(t-u)_-^{H-1/\alpha} - (-u)_-^{H-1/\alpha}]\} Z_\alpha(\mathrm{d}u)\,, \quad \text{and}\end{aligned}$$

$$\Psi_{H,\alpha}(a, b; t) = Re \int_{-\infty}^{\infty} \frac{e^{itu} - 1}{iu} (a u_+^{1-H-1/\alpha} + b u_-^{1-H-1/\alpha}) \tilde{Z}_\alpha(\mathrm{d}u)\,,$$

where $0 < H < 1$, $0 < \alpha < 2$, $H \neq 1/\alpha$, a and $b \in R$ such that $a^2 + b^2 > 0$, and Z_α and $\tilde{Z}_\alpha$ are respectively real and complex symmetric Lévy α-stable motions. See Cambanis–Maejima [6] for detailed discussions on the distributional properties and the limiting theorems of these two processes.

The investigation on the "fine" sample path properties of *ss* processes has been stimulated a lot by the intensive works of Vervaat [19, 20]. Regarded as a contribution to this expanding topic, it is the purpose of this paper to study the local times and the related path properties of certain *ss* processes. In this aspect, we mention that Kôno [10] and Kôno-Maejima [11] proved the existence of square-integrable local times and Nolan [14, 15] discussed the joint continuity of local times for certain stable processes including $\Psi_{H,\alpha}(1, 1)$ [14, Proposition 4.9].

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In § 2 of this paper, we shall prove (Theorem 2.1 (ii)) that, for certain *ss si* processes with bivariate densities, almost every path has local times $\varphi(x, t)$ which are continuous in t for a.e. x; in Berman's [5] terminology, it is the temporal continuity of φ. We shall also find (Theorem 2.1 (iii)) an estimate on the modulus of such a continuity. In § 3, we shall prove (Theorem 3.1) certain infinite local oscillations of the sample paths and shall prove (Theorems 3.3 and 3.4) some results on the Hausdorff measures and dimensions of level sets. Then in § 4, we shall prove (Theorem 4.3) the joint continuity of local times of linear fractional stable processes in case the stability parameter $\alpha \geq 1$ and (Theorem 4.4) a related uniform dimension result for level sets; these results are derived from the local nondeterminism of the processes (Proposition 4.1). In the final § 5, we shall prove (Theorems 5.1 and 5.2) that the small value of the scaling parameter H can imply the differentiability of $\varphi(x, t)$ in x. For example, let $H < 1/3$. Our results assert that for certain *H-ss si* processes the derivative $\varphi'(x, t)$ in x exists and is Hölder continuous; this corresponds to Geman-Horowitz [8, Theorem (28.5a)] for Gaussian processes.

As a summary, we may say that what we have done in this paper shows that "Berman's principle", i.e. the connections between the smoothness of local times and the irregularity of sample paths, is also valid for certain *ss* non-Gaussian processes.

2. Local times of certain *ss* processes with bivariate densities

Let $X(t)$, $t \in R_+$, be a real-valued measurable stochastic process. We start with following assumption on the bivariates $(X(s), X(t))$, $0 < s < t \leq T < \infty$.

Assumption (A). *The characteristic function* $(z_1, z_2) \to E \exp[i(z_1 X(s) + z_2 X(t))]$ *is nonnegative and integrable over* $(z_1, z_2) \in R^2$.

Consequently, so are the characteristic functions $z \to E \exp[izX(t)]$ *and* $z \to E \exp[iz(X(t) - X(s))]$, $z \in R$. Then, the joint density $p(s, t; x, y)$ of $(X(s), X(t))$ exists and is continuous in (x, y), moreover $p(s, t; x, y) \leq p(s, t; 0, 0)$; so are true for the density $p(t; x)$ of $X(t)$ and the density $\sigma(s, t; x)$ of $X(t) - X(s)$. The conditional density $\rho(s, t; x, y)$ of $X(t) - X(s) = y$ given $X(s) = x$ then also exists. Now, we impose on the second assumption.

Assumption (B). *For every* $\varepsilon : 0 < \varepsilon < T$ *and* $a : a > 0$, *there exist positive numbers* $C = C_{\varepsilon, a}$ *and* $\delta = \delta_{\varepsilon, a}$ *such that*

$$\sup_{|x| \leq a} \rho(s, t; x, 0) \leq C\sigma(s, t; 0)$$

for all $s, t : \varepsilon < s < t \leq T$ *and* $t - s < \delta$.

When $X(t)$ is a process of independent increments, (B) holds trivially. When $X(t)$ is a centered Gaussian process of which the correlation function is uniformly smaller than 1, then, by the expression of conditional Gaussian densities, we see that (B) also holds with $C = 1$. In §4, we shall prove that (B) holds for a large class of *ss* stable processes.

Remark. Observe that in general, for each a, $C_{\varepsilon,a} \uparrow \infty$ and $\delta_{\varepsilon,a} \downarrow 0$ as $\varepsilon \downarrow 0$, and in case they are bounded away from ∞ and 0 respectively we may just let $\varepsilon = 0$ in (*B*). However, as we shall see in §4, in many cases we have to work on $[\varepsilon, T]$ rather than $[0, T]$; cf. Pitt [16, Comment, p 324] for the Gaussian case. With the same reason, we consider the supremum in (B) being over $[-a, a]$ rather than the whole R.

We recall the following definition of local times as occupation densities. Let $f(t)$ be a real measurable function. The local time of $f(\cdot)$ over a finite interval $I \subset R_+$, denoted by $\varphi(x, I)$, is defined to be the Radon-Nikodym derivative of the occupation measure $\mu(\Lambda, I) = \text{Leb}\,\{t \in I \mid f(t) \in \Lambda\}$, $\Lambda \in \mathscr{B}(R)$ with respect to Lebesgue measure on R, whenever the former is absolutely continuous with respect to the latter. Note that, in general, $\varphi(x, I)$ is defined only a.e. x. When $I = [0, t]$, we also use the notation $\varphi(x, t)$.

Theorem 2.1. *Suppose that* $X(t)$ *is* H-*ss*, $0 < H < 1$, *and satisfies* (A) *&* (B). *Moreover, assume that* $p(1; 0) > 0$. *Then,* $X(t)$ *a.s. has local times* $\varphi(x, t)$, $x \in R$ *and* $0 < t \le T$ *such that* (*more precisely, a version of* $\varphi(x, t)$ *can be chosen such that*)

(i) $\varphi(x, t, \omega)$ *is jointly measurable in* (x, t, ω);

(ii) $\varphi(x, t)$ *is continuous in* t *for a.e.* x, *and*

(iii) *For each* $a > 0$, $E\displaystyle\int_{-a}^{a} \varphi^2(x, I)dx \le \text{Const.}\,|I|^{2-H}$

for all intervals $I \subset [\varepsilon, T]$ *with the length* $|I| < \delta$, *where* δ *is the positive number in* (B). *In* (ii), *the statement holds a.s., and the constant in* (iii) *does not depend on* I (*however it may depends on* a).

Proof. Firstly, by the definitions of p, ρ and σ and (B), we have that, whenever $0 < t - s < \delta$ for all x and y

$$\begin{aligned} p(s, t; x, y) &\le p(s, t; 0, 0) \\ &= p(s; 0)\rho(s, t; 0, 0) \\ &\le Cp(s; 0)\sigma(s, t; 0) \end{aligned}$$

here, it is sufficient to let $a = 1$ in (B). Since X is H-*ss* and *si*, we then have

$$\begin{aligned} p(s, t; x, y) &\le Cp(s; 0)p(t - s; 0) \\ &= Cp^2(1; 0)s^{-H}(t - s)^{-H}, \end{aligned}$$

where we have used the following identity for the densities of an H-*ss* process.

$$p(ct; c^H x) = c^{-H}p(t; x) \qquad \text{for all } c > 0 \text{ and } x \in R.$$

Set $g(s, t) = Cp^2(1; 0)s^{-H}(t - s)^{-H}$, since $g(s, t)$ is integrable over $0 < s < t \le T$, by Geman [7, Theorem B] we see that the first and the second assertion hold for any interval in $[\varepsilon, T]$ of which length is smaller than δ. Since $[\varepsilon, T]$ is a finite interval, we can obtain local times over the whole interval $[\varepsilon, T]$ by a standard patch-up procedure, i.e. we partition $[\varepsilon, T]$ into $\bigcup_{i-1}^{n}[T_{i-1}, T_i]$ and define $\varphi(x, [\varepsilon, T]) = \sum_{i=1}^{n}\varphi(x, [T_{i-1}, T_i])$, where $T_0 = \varepsilon$ and $T_n = T$. Finally, we can obtain the local time $\varphi(x, T)$ by defining it to be $\lim_{n\to\infty}\varphi(x, [1/n, T])$.

As for the third assertion, we observe that for any $[\alpha, \beta] \subset [\varepsilon, T]$ with $\beta - \alpha < \delta$, again by *H-ss*, *si* and (B) we have, for each $a > 0$,

$$\int_{-a}^{a}\int_{\alpha}^{\beta}\int_{s}^{\beta} p(s, t; x, x)dtdsdx \leq C\int_{-a}^{a}\int_{\alpha}^{\beta}\int_{s}^{\beta} p(s; x)p(t-s; 0)dtdsdx$$

$$= Cp(1; 0)\int_{-a}^{a}\int_{\alpha}^{\beta}\int_{s}^{\beta} p(s; x)(t-s)^{-H}dtdsdx\,.$$

Since

$$\int_{-\infty}^{\infty} p(s; x)dx = 1 \qquad \text{for all } s\,,$$

the above integral is

$$\leq Cp(1; 0)(1-H)^{-1}(\beta-\alpha)^{2-H}\,.$$

Thus, in view of Berman [4, Theorem 3.1] we have

$$E\int_{-a}^{a}\varphi^2(x, [a, b])dx \leq 2\int_{-a}^{a}\int_{\alpha}^{\beta}\int_{s}^{\beta} p(s, t; x, x)dtdsdx$$

$$\leq 2Cp(1; 0)(1-H)^{-1}(\beta-\alpha)^{2-H}\,.$$

3. Local oscillations and level sets of the sample paths

Firstly, we recall that the approximate lim sup (resp. the approximate limit) of a nonnegative measurable function $f(\cdot)$ at t is $+\infty$, if and only if, for all $K > 0$ t is not a point of dispersion for $\{s : f(s) > K\}$ (resp. for all $K > 0$ t is a point of dispersion for $\{s : f(s) < K\}$), see Geman-Horowitz [8, Appendix] for the more detailed definition and discussion.

Theorem 3.1. *Let $X(t)$ be the ss process in Theorem 2.1. With probability one,*

$$\text{(i)}\quad \text{approx.}\lim_{s\to t}\frac{|X(s)-X(t)|}{|s-t|} = +\infty \ \textit{for a.e. } t \in [\varepsilon, T]\,, \textit{ and}$$

$$\text{(ii)}\quad \text{approx.}\limsup_{s\to t}\frac{|X(s)-X(t)|}{|s-t|^{1+H}\psi(|s-t|)} = +\infty \ \textit{for all } t \in [\varepsilon, T]\,,$$

where $\psi(r)$, $r \geq 0$ is any right-continuous function decreasing to 0 as $r \downarrow 0$.

Corollary 3.2. *With probability one, $X(\cdot, \omega)$ is nowhere Hölder continuous of any order $>1 + H$.*

Remark. Kôno [10] proved that for certain general *ss* processes, the path is nowhere Hölder continuous of any order >2. The Corollary 3.2 above improves his result and actually relates to the scaling parameter H.

Proof. The first assertion follows directly from Theorem 2.1 (ii), see Geman [7, Theorem A]. To prove the second assertion, we shall work on [0, 1] for the notational convenience (with a linear change of variable, we can transform to $[\varepsilon, T]$). Fix an $a > 0$. Define a process $M(t)$, $t \in [0, 1]$, by

$$M(t) = \left(2 \sum_{j=1}^{2^n} \int_{-a}^{a} \varphi^2\left(x, \left[\frac{j-1}{2^n}, \frac{j}{2^n}\right]\right) \mathrm{d}x\right)^{1/2},$$

whenever $2^{-n-1} < t \leq 2^{-n}$, which is Berman's [5] modulator of the local time. Here we consider the case $m = 2$ in his paper. By Theorem 2.1 (iii), we have

$$\begin{aligned}
&\liminf_{n\to\infty} 2^n [EM^2(2^{-n})]^{1/2} [(2^{-n})^{1+H} \psi(2^{-n})]^{1/2} \\
&\leq \text{Const.} \liminf_{n\to\infty} (2^n)^{1+(1-(2-H))/2-(1+H)/2} \psi^{1/2}(2^{-n}) \\
&= 0 .
\end{aligned}$$

Then, by Berman [5, (4.4) and the statement after it] we see that the second assertion holds whenever $|X(t)| < a$. Since a is arbitrary, the latter restriction can be removed.

Next, we recall that the Hausdorff measure $H_\psi(\Lambda)$ of a Borel subset Λ of R is defined to be

$$H_\psi(\Lambda) = \liminf_{\varepsilon\downarrow 0} \left\{ \sum_{n=1}^{\infty} \psi(|I_n|) : \{I_n\} \text{ is a countable cover of } \Lambda \text{ by compact intervals and } |I_n| \leq \varepsilon \right\}.$$

Here, the measure function $\psi(r)$ is any nondecreaing right-continuous function of $r \geq 0$ with $\psi(0) = 0$.

The "progressive level set" Z_t is defined by $Z_t = \{s \in [0, T] | X(s) = X(t)\}$.

Theorem 3.3. *Let $X(t)$ be the ss process in Theorem 2.1. Let the measure function be*

$$\psi(r) = r^{(1-H)/2} |\log r|^\theta, \qquad \theta > 1/2 .$$

With probability one, $H_\psi(Z_t) = +\infty$ for a.e. t.

Proof. We consider again the case [0, 1] and the modulator $M(t)$ defined in the proof of Theorem 3.1. Since

$$EM^2(2^{-n}) \leq \text{Const.} (2^{-n})^{1-H} ,$$

we have

$$\sum_{n=1}^{\infty} \frac{EM^2(2^{-n})}{[\psi(2^{-n})]^2} \leq \text{Const.} \sum_{n=0}^{\infty} \frac{(2^{-n})^{1-H}}{(2^{-n})^{1-H} n^{2\theta}} < \infty .$$

Then, by Berman [5, (4.6) and the statement after it] we see that the assertion holds.

The Hausdorff dimension of a Borel subset Λ of R is defined to be

$$\dim \Lambda = \inf \{\beta | H_{\psi}(\Lambda) = 0, \psi(r) = r^{\beta}\}$$
$$= \sup \{\alpha | H_{\psi}(\Lambda) = +\infty, \psi(r) = r^{\alpha}\} .$$

We mention that the following dimension result for the zero set $Z_0 = \{s \in [0, T] | X(s) = 0\}$ cannot be derived directly from local times. Reason: it is now at a fixed level and local times in general are defined up to a set of Lebesgue measure zero.

Theorem 3.4. *Let $X(t)$ be the ss process in Theorem 2.1. Assume moreover that $X(\cdot)$ has at most countably many discontinuies (for example, $X(\cdot)$ is "cadlag"). Then $\dim Z_0 \geq 1 - H$ with positive probability.*

Remark. Takashima [17] introduced the concept of the "ergodic" *ss* processes. For such a process, in Theorem 3.4 then we have $\dim Z_0 \geq 1 - H$ a.s.; this gives a lower bound estimate for Takashima [17, Proposition 5.2].

Proof. Fix an $\varepsilon : 0 < \varepsilon < T$, and let $Z_0^{\varepsilon} = \{t \in [\varepsilon, T] | X(t) = 0\}$. Set $\psi_{\theta}(z) = (1/2\theta)\chi_{[-\theta,\theta]}(z)$, $z \in R$ and $0 < \theta \leq 1$. Following the arguments in the proof of Marcus [13, Theorem 1] or Pitt [16, Proposition 3.1], we can find a.s. a sequence $\theta_n \downarrow 0$ such that for all rationals $t \in [\varepsilon, T]$

$$\lim_{n \to \infty} \int_{\varepsilon}^{t} \psi_{\theta_n}(X(s)) ds \triangleq L_0^{\varepsilon}(t)$$

exists. We mention that (B) enforces that the conditions set up in [13, 16] for the convergence above are actually satisfied. In [16], it is the case $k = 2$.

Then, we have a measure $L_0^{\varepsilon}(dt)$ constructed from the "right-continuous modification" of $L_0(t)$. It is easy to check that whenever $X(\cdot)$ is continuous at some $t_0 \in [\varepsilon, T]$ and $X(t_0) \neq 0$ then $L_0^{\varepsilon}(J) = 0$ for all small neighborhood of t_0. By Pitt [16, Proposition 3.2], the measure $L_0^{\varepsilon}(dt)$ has no atoms, and hence $L_0^{\varepsilon}(dt)$ is supported on Z_0^{ε} whenever $X(\cdot)$ has at most countably many discontinuities. When $[a, b] \subset [\varepsilon, T]$ and $b - a < \delta$, δ: the positive number in (B), letting $k(r) = r^{-\tau}$, $0 < \tau < 1 - H$ and $r \geq 0$, and using the same arguments as those in the proof of Theorem 2.1, we see that

$$\int_a^b \int_s^b p(s, t; 0, 0) k(t - s) dt ds < \infty .$$

Moreover, we have

$$0 < \int_a^b p(t;0)\mathrm{d}t = p(1;0)\int_a^b \frac{\mathrm{d}t}{t^H} < \infty .$$

Therefore, by the proof of Marcus [13, Theorem 1] we see that the assertion holds for Z_0^ε. Then, let $\varepsilon \downarrow 0$.

4. Joint continuity of local times of linear fractional stable processes

We set

$$f(u) = a\{(1-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha}\} + b\{(1-u)_-^{H-1/\alpha} - (-u)_-^{H-1/\alpha}\}, \qquad u \in R,$$

and set

$$f(t,u) = \begin{cases} t^{H-1/\alpha} f(u/t), & t > 0 \\ 0 & t = 0 . \end{cases} \tag{4.1}$$

Then, the linear fractional stable processes $\Delta_{H,\alpha}(a, b; t)$ defined in §1 is also expressed by

$$X(t) = \int_{-\infty}^{\infty} f(t,u) Z_\alpha(\mathrm{d}u) . \tag{4.2}$$

Note that, for all $s, t: 0 < s < t$ and all $u \in R$,

$$f(t,u) - f(s,u) = f(t-s, u-s) . \tag{4.3}$$

Takashima [17] considered a class of *ss si SαS* processes determined by (4.1), (4.2) and (4.3). In the below, we prove our results for linear fractional stable processes; yet most of our arguments can be extended to Takashima's *ss* processes.

Proposition 4.1. *Let $1 \le \alpha < 2$ and $0 < \varepsilon < T < \infty$. For all $m \ge 2$ and $t_1 < t_2 < \cdots < t_m$, $t_j \in [\varepsilon, T]$, and $v_1, \ldots, v_m \in R$, the kernel of $\Delta_{H,\alpha}(a, b)$ has the following positive* lim inf:

$$\liminf_{(t_m - t_1)\downarrow 0} \inf_{v_j \in R} \frac{\|[f(t_m, \cdot) - f(t_{m-1}, \cdot)] - \sum_{j=1}^{m-1} v_j [f(t_j, \cdot) - f(t_{j-1}, \cdot)]\|_\alpha^\alpha}{\|f(t_m, \cdot) - f(t_{m-1}, \cdot)\|_\alpha^\alpha} = C_m > 0,$$

where $t_0 = 0$ and $\|f(t, \cdot)\|_\alpha$ denotes $L^\alpha(R)$ norm with respect to the variable $\cdot$.

Proof. Suppose on the contrary that $C_m = 0$ for some m. Then there exists sequences t_j^n and a_j^n, $j = 1, 2, \ldots, m$ and $n = 1, 2, 3, \ldots$, $\varepsilon \le t_1^n < t_2^n < \cdots < t_m^n \le T$, $(t_m^n - t_1^n) \downarrow 0$ as $n \uparrow \infty$ and $a_j^n \in R$ such that

$$\lim_{n\to\infty} \frac{\|[f(t_m^n, \cdot) - f(t_{m-1}^n, \cdot)] - \sum_{j=1}^{m-1} a_j^n f(t_j^n, \cdot)\|_\alpha^\alpha}{\|f(t_m^n, \cdot) - f(t_{m-1}^n, \cdot)\|_\alpha^\alpha} = 0 . \tag{4.4}$$

Using (4.1) and (4.3), we see that

$$f(t_m^n, u) - f(t_{m-1}^n, u) = (t_m^n - t_{m-1}^n)^{H-1/\alpha} f\left(\frac{u - t_{m-1}^n}{t_m^n - t_{m-1}^n}\right) .$$

By a linear change of variable:

$$u = t_{m-1}^n + (t_m^n - t_{m-1}^n)u_1 ,$$

the numerator of (4.4) becomes

$$(4.5)\quad (t_m^n - t_{m-1}^n)^{H\alpha}\left\| f(u_1) - \sum_{j=1}^{m-1} a_j^n \left(\frac{t_j^n}{t_m^n - t_{m-1}^n}\right)^{H-1/\alpha} f\left(\frac{t_{m-1}^n + (t_m^n - t_{m-1}^n)u_1}{t_j^n}\right)\right\|_{L^\alpha(du_1)}^\alpha ,$$

and the denominator of (4.4) becomes

$$(t_m^n - t_{m-1}^n)^{H\alpha}\|f(u_1)\|_{L^\alpha(du_1)}^\alpha .$$

Therefore, the limit in (4.4) is 0 means that the $L^\alpha(du_1)$ norm in (4.5) tends to 0 as $n \uparrow \infty$. Since $\alpha \geq 1$, there must exist a subsequence $t_j^{n'}$ and $a_j^{n'}$ such that

$$(4.6)\qquad f(u_1) = \lim_{n' \to \infty} \sum_{j=1}^{m-1} a_j^{n'} \left(\frac{t_j^{n'}}{t_m^{n'} - t_{m-1}^{n'}}\right)^{H-1/\alpha} f\left(\frac{t_{m-1}^{n'} + (t_m^{n'} - t_{m-1}^{n'})u_1}{t_j^{n'}}\right)$$

for a.e. u_1. We argue that (4.6) is impossible. We write $f(\cdot)$ in the above summund as

$$f(1 + v_j^{n'}) = a\{(-v_j^{n'})_+^{H-1/\alpha} - (-1 - v_j^{n'})_+^{H-1/\alpha}\}$$
$$+ b\{(-v_j^{n'})_-^{H-1/\alpha} - (-1 - v_j^{n'})_-^{H-1/\alpha}\} ,$$

where

$$v_j^n = v_j^n(u_1) = \frac{(t_{m-1}^n - t_j^n) + (t_m^n - t_{m-1}^n)u_1}{t_j^n} .$$

Since, for each $j = 1, 2, \ldots, m-1$, $\varepsilon \leq t_j^n$ and $0 < (t_{m-1}^n - t_j^n) < (t_m^n - t_1^n) \downarrow 0$ as $n \uparrow \infty$, $v_j^n \to 0+$ as $n \uparrow \infty$ for each $u_1 > 0$. Thus, $f(1 + v_j^n) \to b\{(0 \text{ or } \infty) - 1\}$ according to $H > 1/\alpha$ or $H < 1/\alpha$. The limit is in any case independent on u_1. This means that $f(u_1)$ given by (4.6) is constant a.e. on $u_1 > 0$, which is impossible from the definition of $f(\cdot)$.

Now, we mention some literature background concerning Proposition 4.1. Berman [3] introduced the concept of local nondeterminism for Gaussian processes. Nolan [14] later extended this concept to stable processes. In view of [14, Theorem 3.2 (b)], what we have proved in Proposition 4.1 then can be stated as that $\Delta_{H,\alpha}(a, b)$, $1 \leq \alpha < 2$ and $0 < H < 1$ is locally nondeterministic over any $[\varepsilon, T]$ with $\varepsilon > 0$. As a corollary to the case $m = 2$ in Proposition 4.1, we have

Corollary 4.2. *When* $1 \leq \alpha < 2$ *and* $0 < H < 1$, $\Delta_{H,\alpha}(a, b)$ *satisfies the assumption* (B) *in* §2.

Proof. Observe that, by Proposition 4.1 with $m = 2$, for $\varepsilon < s < t < T$,

$$2\|z_1 f(s, \cdot) + z_2[f(t, \cdot) - f(s, \cdot)]\|_\alpha^\alpha \geq |z_1|^\alpha \inf_{\tau \in R} \|f(s, \cdot) - \tau[f(t, \cdot) - f(s, \cdot)]\|_\alpha^\alpha$$
$$+ |z_2|^\alpha \inf_{\tau \in R} \|f(t, \cdot) - \tau f(s, \cdot)\|_\alpha^\alpha$$
$$\geq C_2\{|z_1|^\alpha \|f(s, \cdot)\|_\alpha^\alpha + |z_2|^\alpha \|f(t, \cdot) - f(s, \cdot)\|_\alpha^\alpha\} ,$$

whenever $0 < t - s < \delta$, for some δ. Thus

$$e^{-\|z_1 f(s,\cdot)+z_2(f(t,\cdot)-f(s,\cdot))\|_\alpha^\alpha} \le e^{-C_2/2\{|z_1|^\alpha \|f(s,\cdot)_\alpha^\alpha + |z_2|^\alpha \|f(t,\cdot)-f(s,\cdot)\|_\alpha^\alpha\}},$$

which is equivalent to

$$Ee^{i(z_1 X(s)+z_2(X(t)-X(s)))} \le Ee^{i(C_2/2)^{1/\alpha} z_1 X(s)} Ee^{i(C_2/2)^{1/\alpha} z_2(X(t)-X(s))}.$$

By Fourier inversion formula and the symmetry of $(X(s), X(t))$, we have

$$\sup_{x,y} p(s, t; x, y) \le p(s, t; 0, 0) \le \left(\frac{4}{C_2^2}\right)^{1/\alpha} p(s; 0)p(t-s; 0).$$

Letting $x = y$, $|x| \le a$ in the above display and using the notations in § 2, we have

$$\sup_{|x|\le a} \rho(s, t; x, 0) \le \left[\left(\frac{4}{C_2^2}\right)^{1/\alpha} \sup_{|x|\le a} \frac{p(s; 0)}{p(s; x)}\right] \sigma(s, t; 0).$$

The quantity in the above bracket is a positive finite number whenever $0 < \varepsilon \le s \le T$, since $p(s; x) = s^{-H} p(1; s^{-H} x)$ and $p(1; \cdot)$ is continuous and everywhere positive.

From those general ideas and arguments firstly appeared in Berman [3] and later elaborated by Pitt [16], Geman-Horowitz [8 §§ 24–26] and Nolan [15, Lemma 2.2], it is now well-known that local nondeterminism is the essential technique leading to the existence of jointly continuous local times. Thus, using Proposition 4.1 and the arguments in the proofs of [3, Theorem 8.1], [16, Theorem 2] and [15, Theorem 4.2], we can prove that

Theorem 4.3. *When $1 \le \alpha < 2$ and $0 < H < 1$, the linear fractional stable process $\Delta_{H,\alpha}(a, b)$ has jointly continuous local times $\varphi(x, t)$. Moreover, φ a.s. has the following Hölder continuities. Let $0 < \varepsilon < T < \infty$, $K \subset R$ be compact and $I \subset [\varepsilon, T]$ be any interval with rational endpoints, then*

(i) $|\varphi(x, I) - \varphi(y, I)| \le C_1 |x - y|^\gamma$ *for all $x, y \in K$, where $\gamma : 0 < \gamma < \min(1, (1-H)/2H)$, and*

(ii) $\sup_{x\in K} \varphi(x, I) \le C_2 |I|^\delta$,

where $\delta : 0 < \delta < 1 - H$. In the above $C_i = C_i(\omega, K, \varepsilon, T)$ are a.s. finite positive random variables.

We mention that it would be interesting to improve Theorem 4.3 (ii) so that the conclusions also relate to the stability parameter α.

Applying Theorem 4.3 (ii) and Berman [2, pp 76–78], we have the following uniform dimension result for level sets.

Theorem 4.4. *When $1 \le \alpha < 2$ and $0 < H < 1$, the following holds with probability one: for any interval $I \subset [0, T]$,*

$$\dim \{t \in I | \Delta_{H,\alpha}(a, b; t) = x\} \ge 1 - H,$$

for all x such that $\varphi(x, I) > 0$.

We should also mention that, in case $H > 1/\alpha$ it is known that the paths $X(\cdot, \omega)$ are continuous, while in case $H < 1/\alpha$ Maejima [12] has shown that the paths $X(\cdot, \omega)$ are nowhere bounded and hence everywhere discontinuous. However, Theorem 4.4 is applicable in both cases. Although in [2, p 77] Berman required the continuity of paths, yet his argument can be modified to hold for measurable paths, cf. Geman-Horowitz [8, Theorem (13, 4)].

5. Differentiability of local times in the space variable for certain *ss* processes

Let $X(t)$ again be a real-valued measurable H-*ss si* process. In this section, we shall prove that, under certain conditions, the small value of H can imply the smoothness of local time $\varphi(x, T)$ in x, for each fixed T, up to some differentiability order. We begin with an elementary result.

Theorem 5.1. *Let $X(t)$ be H-ss, si. Let $\psi(\theta)$ be the characteristic function of $X(1)$, i.e. $\psi(\theta) = E[\exp i\theta X(1)]$, $\theta \in R$. Suppose that for some nonnegative integer r*

$$\int_{-\infty}^{\infty} |\theta|^{2r}|\psi(\theta)|d\theta < \infty\,, \qquad \textit{and} \tag{5.1}$$

$$H < 1/(2r+1)\,. \tag{5.2}$$

Then, the local time $\varphi(x, T)$ of $X(t)$ exists and $\varphi^{(k)}(x, T)$, the k-th derivatives of $\varphi(x, T)$, also exist up to $k = r$. Moreover $\varphi^{(k)}(x, T, \omega) \in L^2(R \times \Omega)$.

Proof. Consider the integral

$$E\int_{-\infty}^{\infty}\iint_{0<s<t\leq T} |\theta|^{2r}[e^{i\theta\cdot(X(t)-X(s))}]dsdtd\theta\,.$$

By H-*ss* and *si*, it is dominated by

$$\iint_{0<s<t\leq T}\int_{-\infty}^{\infty} |\theta|^{2r}|\psi(\theta\cdot(t-s)^H)|d\theta$$

$$= \iint_{0<s<t\leq T}\frac{dsdt}{(t-s)^{(2r+1)H}}\cdot\int_{-\infty}^{\infty}|\theta|^{2r}|\psi(\theta)|d\theta\,,$$

which is finite by (5.1) and (5.2). Then, the conclusions follow from Fourier inversion formula, cf. Berman [1, Lemma 5.1] with p there is now $2r$.

Remark. Theorem 5.1 extends Kôno-Maejima [11, Theorem 6.1] from $r = 0$ to $r \geq 0$. Note that, in case $r = 0$ it only asserts the existence of square-integrable local times, while for $r > 0$ it only asserts the existence of continuous $\varphi^{(k)}(x, T)$ up to $k = r - 1$.

Now, we impose on the following assumption concerning the "approximately independent increments" property of characteristic functions. For symmetric stable processes, it is essentically the local nondeterminism of the process, see Nolan [14, Theorem 3.2].

Assumption (B_p). *For some $p \geq 2$, there exists A_p and C_j, $j = 1, 2, \ldots, p$, such that*

$$|E[e^{i\sum_{j=1}^{p}\theta_j(X(t_j)-X(t_{j-1}))}]| \leq A_p \prod_{j=1}^{p} |E[e^{iC_j\theta_j(X(t_j)-X(t_{j-1}))}]|$$

for all $0 = t_0 < t_1 < \cdots < t_m \leq T$ and all $\theta_j \in R$.

As we have seen in §4, it may happen the occasion that we need to consider time interval $[\varepsilon, T]$ with $\varepsilon > 0$. Also note that, in case $X(t)$ are symmetric (B_2) is essentially equivalent to the assumption (B) in §2 with $\varepsilon = 0$ there.

Theorem 5.2. *Let $X(t)$ be H-ss, si and satisfy (B_p) for some even $p \geq 2$. Suppose that for some nonnegative integer r,*

$$\int_{-\infty}^{\infty} |\theta|^{2r+2/p+\varepsilon}|\psi(\theta)|d\theta < \infty, \qquad \text{for some } \varepsilon > 0\,, \qquad \text{and} \tag{5.3}$$

$$H < 1/(2r + 2/p + 1)\,, \tag{5.4}$$

where $\psi(\theta)$ denotes again the characteristic function of $X(1)$. Then the local time $\varphi(x, T)$ is of class C^r in x and in fact $\varphi^{(r)}(x, T)$ is Hölder continuous of certain order.

Remark 1. From the proof, we see that the assumptions of Theorem 5.2 can be weaken to

(i) $Ee^{i\theta\cdot(X(t)-X(s))} \sim Ee^{i\theta\cdot(t-s)^H X(1)}$, *and*

(ii) (B_p) *holds when* $(t_m - t_1)$ *is small enough.*

Remark 2. When (B_p) holds for all even $p \geq 2$, for example the linear fractional stable processes in §4, then certainly (5.4) can be replaced by

$$H < 1/(2r + 1)\,. \tag{5.5}$$

The cases $r = 0$ and 1 are deserved to be mentioned explicitely. (i) $r = 0$. It asserts that $\varphi(x, T)$ is Hölder continuous when $0 < H < 1$. This is consistent with Theorem 4.3 (i) for linear fractional stable processes. (ii) $r = 1$. It asserts that $\varphi'(x, T)$ is Hölder continuous when $0 < H < 1/3$. This corresponds to the "unproved" Gaussian case in Geman-Horowitz [8, Theorem (28.5.a)].

To prove the theorem, we need the following

Lemma 5.3 (Kôno [9, Theorem 1]). *Let $X(u)$ be a real separable stochastic process. If there exist $p \geq 1$, nonnegative integer r, and a continuous function $\sigma(h)$ such that*

$$E[|\Delta_h^{(r+1)}X(u)|^p]^{1/p} \leq \sigma(h)\,, \qquad \text{and}$$

$$\int_{0+} h^{-(1+r+1/p)}\sigma(h)h^{-\gamma}dh < \infty \qquad \text{for some } \gamma: 0 < \gamma < 1\,.$$

Then $X(u)$ is of class C^r in u and $X^{(r)}(u)$ is Hölder continuous of order γ. In the above, $\Delta_h^{(r+1)}$ denotes the $r+1$ iterates of $(\Delta_h f)(x) \triangleq f(x+h) - f(x)$, $h > 0$ and $x \in R$.

The proof of Theorem 5.2 is based on an application of Lemma 5.3 to "the process" $x \to \varphi(x, T)$. To estimate

$$E[|\Delta_h^{(r+1)}\varphi(x, T)|^p], \tag{5.7}$$

we find that the arguments in Berman [3, p 92] can be adapted to our need. Note that

$$\Delta_h^{(r+1)} e^{i\theta x} = e^{i\theta x}\{e^{i\theta h} - 1\}^{r+1}.$$

Thus, in view of [3, (8.2) and (8.7)], (5.7) is dominated by

$$(\text{Const.})_p \int\cdots\int_{0<t_1<\dots<t_p\le T} \int\cdots\int_{R^p} \left\{ |Ee^{i\sum_{j=1}^p \theta_j X(t_j)}| \prod_{j=1}^p |e^{i\theta_j h} - 1|^{r+1} \right\} \prod_{j=1}^p d\theta_j \prod_{j=1}^p dt_j. \tag{5.8}$$

Using an elementary inequality: $|e^{iu} - 1| \le |u|^\delta$ for all $u \in R$ and all δ, $0 < \delta < 1$, we see that (5.8) is dominated by

(5.9)
$$(\text{Const.})_p h^{p\delta(r+1)} \int\cdots\int_{0<t_1<\dots<t_p\le T} \int\cdots\int_{R^p} \left\{ |Ee^{i\sum_{j=1}^p \theta_j X(t_j)}| \prod_{j=1}^p |\theta_j|^{\delta(r+1)} \right\} \prod_{j=1}^p d\theta_j \prod_{j=1}^p dt_j.$$

Using the transformation: $\theta_j = v_j - v_{j+1}$, $j = 1, \dots, p-1$ and $\theta_p = v_p$, we see that

$$\begin{aligned}
|Ee^{i\sum_{j=1}^p \theta_j X(t_j)}| &= |Ee^{i\sum_{j=1}^p v_j(X(t_j) - X(t_{j-1}))}|, \qquad t_0 = 0, \\
&\le A_p \prod_{j=1}^p |Ee^{iC_j v_j(X(t_j) - X(t_{j-1}))}| \\
&\le A_p \prod_{j=1}^p |\psi(C_j(t_j - t_{j-1})^H v_j)|,
\end{aligned}$$

where $\psi(\cdot)$ again denotes the characteristic function of $X(1)$ as that in Theorem 5.1. In the above, we have used (B_p) and the H-*ss si* for the last two inequalities. Berman has argued that $\Pi_{j=1}^p |\theta_j|^{\delta(r+1)}$ can be dominated by a sum of 2^{p-1} terms, each term is of the form $\Pi_{j=1}^p |v_j|^{k_j\delta(r+1)}$ with $k_j = 0$, 1 or 2. Using the transformation $C_j(t_j - t_{j-1})^H v_j = w_j$ and (5.3), we have

$$\begin{aligned}
E[|\Delta_h^{(r+1)}\varphi(x, T)|^p] \le (\text{Const.})_p \cdot h^{p\delta(r+1)} \cdot \left(\int_{-\infty}^{\infty} |\theta|^{2\delta(r+1)} |\psi(\theta)| d\theta \right)^p \\
\cdot \int\cdots\int_{0<t_1<\dots<t_p\le T} \left\{ \frac{1}{\prod_{j=1}^p (t_j - t_{j-1})^{2\delta H(r+1)+H}} \right\} \prod_{j=1}^p dt_j.
\end{aligned}$$

The last integral is finite whenever

$$(5.10)\qquad 2\delta H(r+1)+H<1\,, \qquad \text{and}$$

$$(5.11)\qquad 2\delta(r+1)<2r+2/p+\varepsilon\,.$$

On the other hand, if we require that

$$\int_{0+} h^{-(1+r+1/p)}h^{\delta(r+1)}h^{-\gamma}dh<\infty$$

for some $\gamma: 0<\gamma<1$, then it is necessary and sufficient that

$$\left(1+r+\frac{1}{p}\right)-\delta(r+1)<1\,,$$

or equivalently

$$(5.12)\qquad \frac{r+1/p}{r+1}<\delta\,.$$

When (5.4) holds, we can always find $\delta: 0<\delta<1$ so that all the (5.10) (5.11) and (5.12) hold. Thus, Lemma 5.3 is actually applicable with $\sigma(h)=h^{\delta(r+1)}$; this completes the proof of Theorem 5.2.

Division of Mathematics
Dept. of Fundamental Sciences
Fac. of Integrated Human Studies
Kyoto University
Department of Mathematics
National Taiwan University
Taipei, Taiwan

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