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Singular Cauchy Problems of Higher Order with Characteristic Initial Surface

By

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Introduction

The present article is concerned with the Cauchy problem of linear partial differential equation with holomorphic coefficients in complex domain. The purpose is to give an explicit representation of the singularity of the solution for meromorphic Cauchy data.

In the case where the initial surface is non-characteristic, this problem has been studied by several authors: see Y. Hamada [2] in case of simple characteristics, see Y. Hamada-J. Leray-C. Wagschal [3] in case of constant multiple characteristics, see Y. Hamada-G. Nakamura [4], D. Shiltz-J. Vaillant-C. Wagschal [10] and T. Kobayashi [8] in case of involutive characteristics, and see, for instance, J. Urabe [12] and C. Wagschal [14] and so on in other cases.

On the other hand, we can consider this problem even in the case where the initial surface is characteristic. Indeed, the Cauchy problem for Fuchsian partial differential operator (in the sense of M. S. Baouendi-C. Goulaouic [1]) has a unique holomorphic local solution under some conditions (see Y. Hasegawa [5], M. S. Baouendi-C. Goulaouic [1]). J. Urabe [13] treated a special class of operaters in $\mathbf{C}^2$ whose principal parts are $t\partial_t^2 - \partial_x^2$ and whose characteristic exponents are constant. He gave an explicit representation of the singularity of the solutions by means of hypergeometric functions. S. Ouchi [8] treated second order operators whose principal parts are of simple characteristics multiplied by t^2. He used the multi-phase functions and showed that the solutions are holomorphic except on the characteristic sets.

In this paper, we treat a class of operators $L(x; D_x)$, $x = (x_0, x_1, \dots, x_n) = (x_0, x')$ of order $2m$ ($m \in \mathbf{N}$), which are, roughly speaking, transformed to operators with simple characteristics by change of variables $x_0 = y_0^2$, $x' = y'$ (see (A.1) and (A.2)). By (A.1), these are of Fuchs type with weight m. So we consider the Cauchy problem

$$\begin{cases} L(x; D_x)u(x) = 0, \\ D_0^k u(0; x') = v_k(x') \qquad (k = 0, \dots, m-1), \end{cases}$$

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where $v_k(x')$ has poles along $x_1 = 0$. This problem is a generalization of J. Urabe [13].

What is difficult in studying Fuchsian operators is that not only the principal part but also the lower order terms affect very much the singularity of solutions. As is well-known, in the Cauchy problem for the second order operator in two independent variables

$$L = tD_t^2 - D_x^2 + (1 - c)D_t$$

with initial surface $t = 0$, the values of c is of dicisive importance in discussing the singularity of solutions. If L is of order $2m$ with $m \geqq 2$, then we have m values playing the role of c above. And hence we need a strong condition (A.3) in order that the representation of the solution be the same as that of J. Urabe [13] (see Remark 1.5 in section 1).

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§ 1. Assumptions and results

Let Ω be an open neighborhood of the origin in $\mathbf{C}^{n+1} = \mathbf{C} \times \mathbf{C}^n$ with standard coordinates $x = (x_0, x_1, \dots, x_n) = (x_0, x')$ and $L = L(x; D_x)$ a linear partial differential operator of order $2m$ ($m \in \mathbf{N}$) whose coefficients are holomorphic in Ω. We use the notation $D_x = (D_0, D_1, \dots, D_n) = (D_0, D')$, $D_j = \dfrac{\partial}{\partial x_j}$ ($j = 0, 1, \dots, n$), and $D'^{\beta} = D_1^{\beta_1} \dots D_n^{\beta_n}$ for $\beta = (\beta_1, \dots, \beta_n)$ and $|\beta| = \beta_1 + \cdots + \beta_n$.

We shall impose three conditions (A.1), (A.2) and (A.3) on the symbol $L(x, \xi)$, $\xi = (\xi_0, \xi_1, \dots, \xi_n) = (\xi_1, \xi')$.

First, we write $L(x, \xi)$ as

$$L(x, \xi) = \sum_{k+|\beta| \leqq 2m} a_{k,\beta}(x) \xi_0^k \xi'^{\beta}. \tag{1.1}$$

We define

$$d(k, \beta) = \max\left(k + \left[\frac{|\beta| + 1}{2}\right] - m, 0\right)$$

for all (k, β) satisfying $k + |\beta| \leqq 2m$, where $[a]$ denotes the largest integer not exceeding a.

(A.1) *$a_{k,\beta}(x)$ is divisible by $x_0^{d(k,\beta)}$, that is, $a_{k,\beta}(x)$ is written in the form:*

$$a_{k,\beta}(x) = x_0^{d(k,\beta)} \tilde{a}_{k,\beta}(x) \qquad (k + |\beta| \leqq 2m)$$

with a holomorphic function $\tilde{a}_{k,\beta}$ in Ω. Especially, $\tilde{a}_{2m,0} \equiv 1$.

Next, let $L^H(x;\xi)$ be the principal symbol of $L(x;\xi)$, and $\widehat{L^H}(y;\eta)$ the symbol obtained from $L^H(x;\xi)$ by the change of variables $x_0 = y_0{}^2$, $x' = y'$ in the following way:

$$\widehat{L^H}(y;\eta) = L^H\left(y_0{}^2, y'; \frac{1}{2y_0}\eta_0, \eta'\right),$$

where $y = (y_0, y_1, \ldots, y_n) = (y_0, y')$ and $\eta = (\eta_0, \eta_1, \ldots, \eta_n) = (\eta_0, \eta')$. By (A.1), the coefficients in $\widehat{L^H}$ are holomorphic in Ω.

(A.2) *Equation* $\widehat{L^H}(0;\eta_0, 1, 0, \ldots, 0) = 0$ *has* $2m$ *distinct roots.*

($\widehat{L^H}(0;\eta_0, 1, 0, \ldots, 0)$ is in fact a polynomial with respect to η_0^2 of degree m by (A.1) and we denote the roots by $\{\eta_{0r}^{\pm}\}_{r=1}^{m}$. See the proof of Proposition 1.1.)

Finally, the condition (A.1) allows us to write $L(x;D_x)$ as

$$L(x, D_x) = \sum_{|\beta|\leqq 2m}\ \sum_{0<\kappa\leqq s(\beta)} x_0^{-\kappa} L_\beta^\kappa(x'; x_0D_0)D'^\beta + L^-(x; x_0D_0, D'), \tag{1.2}$$

where $s(\beta) = m - \left[\dfrac{|\beta|+1}{2}\right]$ and each of $L_\beta^\kappa(x';\lambda)$'s [resp. $L^-(x;\lambda,\xi')$] is a polynomial with respect to λ [resp. (λ, ξ')] whose coefficients are holomorphic at $x' = 0$ [resp. $x = 0$].

(A.3) *There exists a constant* $c \in \mathbf{C}\backslash\{1, 2, \ldots,\}$ *such that* $L_\beta^\kappa(x';\lambda)$ *is divisible by* $\prod_{p=0}^{\kappa-1}(\lambda - p)(\lambda - c - p)$ *for all* β *and* κ ($|\beta| \leqq 2m$, $0 < \kappa \leqq s(\beta)$).

Note that $L_\beta^\kappa(x';\lambda)$ is always divisible by $\prod_{p=0}^{\kappa-1}(\lambda - p)$. Condition (A.3) is clearly equivalent to the following:

(A.3)$'$. *There exists a constant* $c \in \mathbf{C}\backslash\{1, 2, \ldots\}$ *such that*

$$\Delta_\lambda^j\left\{L_\beta^\kappa(x';\lambda)\Big/\prod_{p=0}^{\kappa-1}(\lambda - p)\right\}\Bigg|_{\lambda=c} = 0 \qquad (j = 0, 1, \ldots, \kappa - 1)$$

for all β *and* κ ($|\beta| \leqq 2m$, $0 < \kappa \leqq s(\beta)$) *in a neighborhood of the origin* $x' = 0$, *where* Δ_λ *is an operator acting on functions of* λ:

$$(\Delta_\lambda f)(\lambda) = f(\lambda+1) - f(\lambda), \qquad \Delta_\lambda^j f = \Delta_\lambda(\Delta_\lambda^{j-1} f), \qquad \Delta_\lambda^0 f(\lambda) = f(\lambda).$$

We shall sometimes write (A.3)$_c$ [resp. (A.3)$'_c$] instead of (A.3) [resp. (A.3)$'$] in order to specify the constant c.

Before stating our main theorem, let us define the auxiliary functions $U_p^{(c)}(\theta, \rho)$, $X_p^{(c)}(\theta, \rho)$ and $Y_p^{(c)}(\theta, \rho)$, which are fundamental to describe the singularity of the solution of our Cauchy problem (C.P.) below.

First, we introduce the so-called wave forms $f_p(s)$, $k_p(s)$ with a complex parameter p:

$$f_p(s) = \frac{1}{\Gamma(p+1)} s^p ,$$

$$k_p(s) = \frac{\partial}{\partial p} f_p(s) = \frac{1}{\Gamma(p+1)} (\log s + \psi(p+1)) ,$$

where $\psi(p) = \dfrac{\Gamma'(p)}{\Gamma(p)}$. Especially $k_p(s) = |p+1|!(-1)^{p-1} s^p$ for $p = -1, -2, \dots$.

Next, we introduce the multi-valued functions $U_p^{(c)}(\theta, \rho)$ as the solution of the Cauchy preblem;

$$\begin{cases} P_c U_p^{(c)} = \{\theta D_\theta^2 - D_\rho^2 + (1-c) D_\theta\} U_p^{(c)} = 0 , \\ U_p^{(c)}(0, \rho) = f_p(\rho) , \end{cases}$$

where $D_\theta = \dfrac{\partial}{\partial \theta}$, $D_\rho = \dfrac{\partial}{\partial \rho}$ and c is a constant ($c \in \mathbf{C} \setminus \{1, 2, \dots\}$). We can write down $U_p^{(c)}(\theta, \rho)$ explicitly as:

$$U_p^{(c)}(\theta, \rho) = \frac{1}{\Gamma(p+1)} \rho^p F\left(\frac{-p}{2}, \frac{-p+1}{2}, 1-c; z\right)$$

where $z = \dfrac{4\theta}{\rho^2}$. Now we define $X_p^{(c)}(\theta, \rho)$ and $Y_p^{(c)}(\theta, \rho)$ as follows:

$$X_p^{(c)}(\theta, \rho) = \frac{\partial}{\partial p} U_p^{(c)}(\theta, \rho) ,$$

$$Y_p^{(c)}(\theta, \rho) = \theta D_\theta X_{p+1}^{(c)}(\theta, \rho) .$$

Note that $X_p^{(c)}$ is the solution of the Cauchy problem:

$$\begin{cases} P_c X_p^{(c)} = 0 , \\ X_p^{(c)}(0, \rho) = k_p(\rho) . \end{cases}$$

For the property of $U_p^{(c)}(\theta, \rho)$, $X_p^{(c)}(\theta, \rho)$ and $Y_p^{(c)}(\theta, \rho)$, see [13].

Now let us consider the Cauchy problem with initial surface $S = \{x_0 = 0\}$, whose Cauchy data have poles along $x_1 = 0$. By the principle of superposition, we have only to consider the following problems:

$$\begin{cases} L(x; D_x) u(x) = 0 \\ D_0^k u(0, x') = \delta_{l,k} w(x') k_{-\alpha}(x_1) , \qquad k = 0, \dots, m-1, \qquad (l = 0, \dots, m-1) , \end{cases} \tag{C.P.}$$

where α is an integer, $\delta_{l,k}$ is Kronecker's delta and $w(x')$ is a holomorphic function in a neighborhood of the origin $x' = 0$.

Put $T = \{x_0 = x_1 = 0\}$. From Remark 1.1 below, we see that if $z \in S - T$ is sufficiently close to the origin, the Cauchy problem (C.P.) has a unique holomorphic solution in a neighborhood of z (see [1]). The solution is expected to have singularities along the characteristic surfaces of L issuing from T. The characteristic surfaces are S and $V_r = \{\phi_r^+(x) = 0\} \cup \{\phi_r^-(x) = 0\}$ $(r = 1, \dots, m)$, where $\phi_r^\pm(x)$ is expressed as $\phi_r^\pm(x) = \rho_r(x) \pm 2\theta_r(x)^{1/2}$ for some holomorphic functions $\theta_r(x)$ and $\rho_r(x)$, which will be constructed in Proposition 1.1 at the end of this section.

Theorem. *Under the assumptions* (A.1) (A.2) *and* $(\mathrm{A.3})_c$, *there exists one and only one solution of the Cauchy problem* (C.P.) *and it is extended holomorphically to the universal covering space* $\mathscr{R}(\omega - (\bigcup_{r=1}^m V_r) \cup S)$, *where* ω *is a connected neighborhood of the origin of* $\mathbf{C}^{n+1}$. *More precisely, the solution is expressed as*:

$$u(x) = \sum_{r=1}^{m} \sum_{p=2l-\alpha}^{\infty} (g_p^r(x) X_p^{(c)}(\theta_r(x), \rho_r(x)) + h_p^r(x) Y_p^{(c)}(\theta_r(x), \rho_r(x))) , \tag{1.3}$$

where $g_p^r(x)$ *and* $h_p^r(x)$ *are holomorphic functions in* ω *and the sum on the right-hand side is uniformly convergent on every compact subset of* $\mathscr{R}(\omega - (\bigcup_{r=1}^m V_r) \cup S)$.

Remark 1.1. Note that $d(k, 0) = \max(k - m, 0)$, and $d(k, \beta) \geqq k - m + 1$ if $|\beta| \geqq 1$. It follows that any symbol $L(x; \xi)$ which satisfies (A.1) is Fuchsian in the sense of Baouendi-Goulaouic [1] with respect to the hyperplane $x_0 = 0$ of order $2m$ and with weight m.

Remark 1.2. If L is an operator satisfying (A.1) [resp. $(\mathrm{A.3})_c$], any holomorphic change of variables, which preserves the hyperplane $x_0 = 0$, transforms L into another operator L' which also satisfies (A.1) [resp. $(\mathrm{A.3})_c$].

Remark 1.3. In the case $m = 1$, the conditions (A.1), (A.2) and (A.3) are equivalent to the following: $L(x; \xi)$ is a second order Fuchsian symbol with respect to the hyperplane $x_0 = 0$ with weight 1. The characteristic exponent, which is equal to the constant c in $(\mathrm{A.3})_c$, is not positive integer. Moreover, the coefficient of ξ_1^2 does not vanish at the origin.

Remark 1.4. If L_1 and L_2 satisfy (A.1), then the composition $L_1 \circ L_2$ also satisfies (A.1), and if both of L_1 and L_2 satisfy $(\mathrm{A.3})_c$, $L_1 \circ L_2$ also satisfies $(\mathrm{A.3})_c$. We can verify the latter assertion by lemma E in the Appendix.

Remark 1.5. We give an example of a 4th order operator to justify the assumption (A.3). Let L be the composition of two operators L_1 and L_2; $L = L_1 \circ L_2$, and each L_r $(r = 1, 2)$ a second order operator in $\mathbf{C}^2 = \mathbf{C}_t \times \mathbf{C}_x$ as follows:

$$L_r = tD_t^2 - a_r D_x^2 + (1 - c_r) D_t$$

where c_r and a_r are constants, $a_r \neq 0$ and $c_r \in \mathbf{C} \backslash \{1, 2, \dots\}$. Note that each L_r satisfies the conditions (A.1), (A.2) and $(\mathrm{A.3})_{c_r}$ from remark 1.3, while L satisfies (A.1) but (A.2) if and only if $a_1 \neq a_2$; and $(\mathrm{A.3})_c$ if and only if $c_1 = c_2 = c$.

The solution of the second order Cauchy problem

$$\begin{cases} L_r u_r = 0 \\ u_r(0, x) = w_r(x) k_{-\alpha_r}(x)\,, \end{cases}$$

is of the form

$$u_r(t, x) = \sum_{p=-\alpha_r}^{\infty} (g_p^r(t, x) X_p^{(c_r)}(a_r t, x) + h_p^r(t, x) Y_p^{(c_r)}(a_r t, x))$$

for each $r = 1, 2$, as is shown in the theorem. Then, it is natural to consider whether the solution for the 4th order Cauchy problem, say,

$$\begin{cases} Lu = 0 \\ u(0, x) = w(x) k_{-\alpha}(x) \\ D_t u(0, x) = 0\,, \end{cases} \tag{1.4}$$

is of the form

$$u(t, x) = \sum_{r=1,2} \sum_{p=-\alpha}^{\infty} \{g_p^r(t, x) X_p^{(c_r)}(a_r t, x) + h_p^r(t, x) Y_p^{(c_r)}(a_r t, x)\}\,, \tag{1.5}$$

with holomorphic functions $g_p^r(t, x)$ and $h_p^r(t, x)$ in a neighborhood of $(t, x) = (0, 0)$. The answer is "no".

Counterexample. If $a_1 \neq a_2$ and if $c_1 \neq c_2$, the solution of the Cauchy problem (1.4) is not of the form (1.5).

To see this, we investigate the transport systems. The transport system of, say, (g_p^1, h_p^1) is as follows:

$$\begin{cases} a_1(a_2 - a_1)\{(2tD_t + 1)h_p^1 - 2D_x g_p^1\} \\ \quad = (L_1 + E_1 A_2 + B_1 B_2) g_{p-1}^1 + (E_1 B_2 + B_1 E_2) h_{p-1}^1 \\ \quad\quad + (L_1 B_2 + B_1 L_2) g_{p-2}^1 + (L_1 E_2 + E_1 F_2) h_{p-2}^1 \\ \quad\quad + L_1 L_2 g_{p-3}^1\,, \\ 2(a_2 - a_1)\{D_t g_p^1 - a_1 D_x h_p^1\} \\ \quad = (B_1 A_2 + A_1 B_2) g_{p-1}^1 + (F_1 + B_1 B_2 + A_1 E_2) h_{p-1}^1 \\ \quad\quad + (F_1 A_2 + A_1 L_2) g_{p-2}^1 + (F_1 B_2 + B_1 F_2) h_{p-2}^1 \\ \quad\quad + F_1 F_2 h_{p-3}^1\,, \end{cases} \tag{1.6}$$

where $A_r = 2D_t + \dfrac{c_1 - c_r}{t}$, $B_r = -2a_r D_x$, $E_r = a_1(2tD_t + 1 + c_1 - c_r)$ and $F_r = L_r + 2c_1 D_t + \dfrac{c_1(c_1 - c_r)}{t}$ $(r = 1, 2)$.

The Cauchy problem (1.6) with Cauchy data $g_p^1(0, x)$ has a unique holomorphic solution (g_p^1, h_p^1) if the Cauchy data and the right-hand side of (1.6) are holomorphic. However, on the right-hand side of (1.6), the coefficients of the operators A_2 and F_2 are not holomorphic if $c_1 \neq c_2$. Therefore we cannot construct in general the solution of the form (1.5).

We shall conclude this section with the following proposition which enables us to construct the characteristic hypersurfaces V_r $(r = 1, \dots, m)$.

Proposition 1.1. *Under the assumptions* (A.1) *and* (A.2), *the first order Cauchy problem*

$$\begin{cases} L^H(x; \phi_x) = 0 \\ \phi(0; x) = x_1 \end{cases} \tag{1.7}$$

has $2m$ *distinct solutions* $\{\phi_r^\pm\}_{r=1,\dots,m}$, *each of which can be written in the form*

$$\phi_r^\pm(x) = \rho_r(x) \pm 2\theta_r(x)^{1/2}, \qquad (r = 1, \dots, m)$$

where $\rho_r(x)$ *and* $\theta_r(x)$ *are holomorphic in a neighborhood of the origin, and furthermore,* θ_r *is expressed as*

$$\theta_r(x) = x_0 \sigma_r(x) \qquad (r = 1, \dots, m) \tag{1.8}$$

where $\sigma_r(x)$ *is holomorphic and* $\sigma_r(0) \neq 0$.

Proof. Put $\tilde{\phi}(y) = \phi(y_0^2, y')$. We consider the new Cauchy problem

$$\begin{cases} \widehat{L^H}(y; \tilde{\phi}_y) = 0 \\ \tilde{\phi}(0, y') = y_1 \end{cases}$$

Taking account of (A.2), implicit function theorem and Cauchy-Kovalevskaya theorem guarantee that this problem has $2m$ local solutions which are holomorphic in y in a neighborhood of the origin. Moreover, we can easily verify by the definition of $\widehat{L^H}$ that if $\tilde{\phi}(y)$ is a solution of (1.9), then $\tilde{\phi}^\vee(y)$ is also a solution of it, where $\tilde{\phi}^\vee(y) = \tilde{\phi}(-y_0, y')$. On the other hand, $\widehat{L^H}(0; \eta_0, 1, 0, \dots, 0)$ is a polynomial of η_0^2, since

$$d(k, \beta) = \left[\frac{k+1}{2}\right] \qquad \text{if} \quad k + |\beta| = 2m\,. \tag{1.9}$$

And hence, taking account of (A.2), $\eta_{0_r}^\pm \neq 0$ $(r = 1, \dots, m)$. Therefore $\dfrac{\partial}{\partial y_0}\tilde{\phi}(0) \neq 0$, and since $\dfrac{\partial}{\partial y_0}\tilde{\phi}^\vee(0) = -\dfrac{\partial}{\partial y_0}\tilde{\phi}(0)$, we have $\dfrac{\partial}{\partial y_0}\tilde{\phi}^\vee(0) \neq \dfrac{\partial}{\partial y_0}\tilde{\phi}(0)$. Thus we can classify the $2m$ solutions of (1.9) into m couples $\{\tilde{\phi}_r^\pm\}_{r=1,\dots,m}$ by means of the relation

$$\tilde{\phi}_r^+(-y_0, y') = \tilde{\phi}_r^-(y_0, y') \qquad (r = 1, \dots, m)\,.$$

Put

$$\theta_r = \left(\frac{\tilde{\phi}_r^+ - \tilde{\phi}_r^-}{4}\right)^2, \qquad \rho_r = \frac{\tilde{\phi}_r^+ + \tilde{\phi}_r^-}{2} \qquad (r = 1, \dots, m)\,.$$

Then θ_r and ρ_r are holomorphic in x in a neighborhood of the origin since they are even in y_0. Moreover, noting $(\tilde{\phi}_r^+ - \tilde{\phi}_r^-)(0) = 0$ and $\frac{\partial}{\partial y_0}(\tilde{\phi}_r^+ - \tilde{\phi}_r^-)(0) = 2\frac{\partial}{\partial y_0}\tilde{\phi}_r^+(0) \neq 0$, we can express θ_r as (1.8). Q.E.D.

§2. Preliminary calculations

We shall prove the theorem in the following way. Suppose that the solution of (C.P.) be equal to a series u of type (1.3). To determine the coefficients $\{g_p^r\}$ and $\{h_p^r\}$, we operate L term by term to the series. It turns out that anyone of derivatives of $X_p^{(c)}$'s and $Y_p^{(c)}$'s is again a linear combination of $X_p^{(c)}$'s and $Y_p^{(c)}$'s (see Lemma 2.1). Putting $Lu = 0$, we get a system of partial differential equations which $\{g_p^r\}$ and $\{h_p^r\}$ solve. We may call it *transport system.* Coefficients are not holomorphic in general (Remark 1.5). However in our case, the condition (A.3) guarantees that they are holomorphic (Proposition 3.1). If we prescribe holomorphic Cauchy data, the transport system has one and only one holomorphic solution $\{g_p^r, h_p^r\}$ (Proposition 3.4). Thus we obtain a formal solution of type (1.3) of (C.P.). In section 4, we shall prove that the formal solution is in fact convergent in a neighborhood of the origin. In section 5, we shall prove the fundamental formulas used mainly in the proof of Proposition 3.1.

In this section, we prepare three preliminary lemmas of which we shall make use in the next section. We omit the proofs because they are obtained by simple calculations.

From now on, we fix the constant c in (A.3) and write X_p, Y_p instead of $X_p^{(c)}$, $Y_p^{(c)}$ respectively. First, we express the derivatives of X_p and Y_p with respect to θ and ρ as linear combinations of X_q's and Y_q's using the definition of them. Here we introduce operators E_k acting on functions of λ:

$$(E_k f)(\lambda) = \begin{cases} \dfrac{1}{k!}\Delta_\lambda^k \dfrac{f(\lambda)}{\Lambda_{k+1}(\lambda)} & (k \geqq 0) \\ 0 \quad (k = -1) \end{cases}$$

where $\Lambda_l(\lambda) = \lambda(\lambda - 1)\dots(\lambda - l + 1)$.

Lemma 2.1.

$$\theta^l D_\theta^l X_p = \sum_{k=0}^{[l/2]} b_{l,2k}(c)\theta^k X_{p-2k} + \sum_{k=0}^{[(l-1)/2]} b_{l,2k+1}(c)\theta^k Y_{p-2k-1},$$

$$\theta^l D_\theta^l Y_p = \sum_{k=1}^{[(l+1)/2]} \tilde{b}_{l,2k-1}(c)\theta^k X_{p-2k+1} + \sum_{k=0}^{[l/2]} \tilde{b}_{l,2k}(c)\theta^k Y_{p-2k},$$

$$D_\rho^l X_p = X_{p-l},$$

$$D_\rho^l Y_p = Y_{p-l},$$

and especially for $k < l$,

$$b_{l,2k} = \left\{E_{k-1}\left(\frac{1}{\lambda - k}A_l\right)\right\}\bigg|_{\lambda=c}, \qquad b_{l,2k+1} = (E_k A_l)|_{\lambda=c},$$

$$\tilde{b}_{l,2k-1} = \left\{E_{k-1}\left(\frac{\lambda}{\lambda - k}A_l\right)\right\}\bigg|_{\lambda=c}, \qquad \tilde{b}_{l,2k} = \{E_k(\lambda A_l)\}|_{\lambda=c}.$$

Lemma 2.1′. *It follows from the above lemma that if* $K(\lambda)$ *is a polynomial with respect to* λ *of order* l *divisible by* $\lambda(\lambda-1)\dots(\lambda-l'+1)$ $(0 \leqq l' \leqq l)$, *then* $K(\theta D_\theta)X_p$ *and* $K(\theta D_\theta)Y_p$ *can be expressed as*

$$K(\theta D_\theta)X_p = \sum_{k=0}^{[l/2]} B_{2k}(c)\theta^k X_{p-2k} + \sum_{k=0}^{[(l-1)/2]} B_{2k+1}(c)\theta^k Y_{p-2k-1},$$

$$K(\theta D_\theta)Y_p = \sum_{k=1}^{[(l+1)/2]} \tilde{B}_{2k-1}(c)\theta^k X_{p-2k+1} + \sum_{k=0}^{[l/2]} \tilde{B}_{2k}(c)\theta^k Y_{p-2k},$$

and especially for $k < l'$,

$$B_{2k} = \left\{E_{k-1}\left(\frac{1}{\lambda - k}K(\lambda)\right)\right\}\bigg|_{\lambda=c}, \qquad B_{2k+1} = (E_k K(\lambda))|_{\lambda=c},$$

$$\tilde{B}_{2k-1} = \left\{E_{k-1}\left(\frac{\lambda}{\lambda - k}K(\lambda)\right)\right\}\bigg|_{\lambda=c}, \qquad \tilde{B}_{2k} = \{E_k(\lambda K(\lambda))\}|_{\lambda=c}.$$

Note that $B_j(c)$ *and* $\tilde{B}_j(c)$ *are polynomials of* c.

Next, let $K(x, \xi)$ be a homogeneous polynomial with respect to ξ of degree l. Denote by $K_k(x; \xi, \eta)$ the sum of terms of degree k in ξ and of degree $l-k$ in η of $K(x; \xi + \eta)$, that is to say,

$$K(x, r\xi + s\eta) = \sum_{k=0}^{l} K_k(x; r\xi, s\eta) = \sum_{k=0}^{l} r^k s^{l-k} K_k(x; \xi, \eta),$$

where $r, s \in \mathbf{C}$. We shall write

$$K^{(j)}(x; \xi) = \frac{\partial}{\partial \xi_j} K(x; \xi),$$

$$K^{(j,k)}(x; \xi) = \frac{\partial^2}{\partial \xi_j \partial \xi_k} K(x; \xi).$$

And we define $K(x; \theta_x D_\theta + \rho_x D_\rho)$ to be

$$K(x; \theta_x D_\theta + \rho_x D_\rho) = \sum_{k=0}^{l} K_k(x; \theta_x, \rho_x) D_\theta^k D_\rho^{l-k},$$

where θ and ρ are functions of x and

$$\theta_x = \left(\frac{\partial \theta}{\partial x_0}, \dots, \frac{\partial \theta}{\partial x_n}\right), \qquad \rho_x = \left(\frac{\partial \rho}{\partial x_0}, \dots, \frac{\partial \rho}{\partial x_n}\right).$$

Lemma 2.2. *Let us put*

$$\theta^l K(x; \theta_x D_\theta + \rho_x D_\rho) X_p(\theta, \rho) = K^1 X_{p-l} + K^2 Y_{p-l} + K^3 X_{p-l+1} + K^4 Y_{p-l+1} + \cdots,$$

$$\theta^l K(x; \theta_x D_\theta + \rho_x D_\rho) Y_p(\theta, \rho) = \tilde{K}^1 X_{p-l} + \tilde{K}^2 Y_{p-l} + \tilde{K}^3 X_{p-l+1} + \tilde{K}^4 Y_{p-l+1} + \cdots,$$

where ... *is a linear combination of* $\{X_q, Y_q\}_{p-l+2 \leq q \leq p}$ *whose coefficients are functions of* x. *Then we have*

$$K^1 = \sum_{k=0}^{[l-2]} \theta^{l-k} K_{2k}(x; \theta_x, \rho_x),$$

$$K^2 = \sum_{k=0}^{[(l-1)/2]} \theta^{l-k} K_{2k+1}(x; \theta_x, \rho_x),$$

$$K^3 = \sum_{k=0}^{[(l-1)/2]} k(c-1-k)\theta^{l-k-1} K_{2k+1}(x; \theta_x, \rho_x),$$

$$K^4 = \sum_{k=0}^{[l/2]} k(c-k)\theta^{l-k-1} K_{2k}(x; \theta_x, \rho_x),$$

$$\tilde{K}^1 = \theta K^2,$$

$$\tilde{K}^2 = K^1,$$

$$\tilde{K}^3 = \sum_{k=0}^{[l/2]} k(c+1-k)\theta^{l-k} K_{2k}(x; \theta_x, \rho_x),$$

$$\tilde{K}^4 = \sum_{k=0}^{[(l-1)/2]} \{(k+1)c - k^2\}\theta^{l-k-1} K_{2k+1}(x; \theta_x, \rho_x).$$

Finally, let $P(x; \xi)$ be a polynomial in ξ of degree l, $P^H(x; \xi)$ the principal part of P and $Q^H(x, \xi)$ the principal part of $Q(x, \xi) = P(x, \xi) - P^H(x, \xi)$.

Lemma 2.3.

$$\begin{aligned} &P(x; D_x)[f(x)Z(\theta(x), \rho(x))] \\ &= fP^H(x; \theta_x D_\theta + \rho_x D_\rho)Z + \frac{1}{2} fP^{H^{(i,j)}}(x; \theta_x D_\theta + \rho_x D_\rho)(\theta_{x_i x_j} D_\theta + \rho_{x_i x_j} D_\rho)Z \\ &\quad + D_i f P^{H^{(i)}}(x; \theta_x D_\theta + \rho_x D_\rho)Z + fQ^H(x; \theta_x D_\theta + \rho_x D_\rho)Z + \cdots, \end{aligned}$$

for every $f(x)$ *and* $Z(\theta, \rho)$, *where* ... *consists of terms involving the derivatives of* Z *of order less than* $l-1$.

§3. Construction of the formal solution

Suppose that the series of (1.3) solves (C.P.). Operating $L(x; D_x)$ to it by term by term differentiation, we obtain a system of linear equations with respect to $\{g_p^r, h_p^r\}$ which consists of an infinite numbers of couples of partial differential equations. Coefficients are holomorphic thanks to the hypothesis (A.3) (Proposition 3.1). The p-th couple contains $\{g_q^r, h_q^r\}_{q=p}^{p+2m}$ and their derivatives of order at most $2m+p-q$. However, the coefficients of g_{p+2m}^r and h_{p+2m}^r are revealed to be identically equal to zero owing to the choice of (θ^r, ρ^r) (Proposition 3.2). So, the p-th couple is a first order system with two unknowns $\{g_{p+2m-1}^r, h_{p+2m-1}^r\}$ if $\{g_q^r, h_q^r\}_{q=p}^{p+2m-2}$ are already known (Proposition 3.3). Hyperplane $x_0 = 0$ is characteristic with respect to the couple. However, the Cauchy problem has one and only one holomorphic solution $\{g_{p+2m-1}^r, h_{p+2m-1}^r\}$ every time we prescribe arbitrary but holomorphic Cauchy data (Proposition 3.4).

Given a linear differential operator $K(x, D_x)$ of order l, let us define

$$M_k^r(K)(x; D_x)\,, \quad \tilde{M}_k^r(K)(x; D_x)\,, \quad N_k^r(K)(x; D_x) \quad \text{and} \quad \tilde{N}_k^r(K)(x; D_x)\,,$$

linear differential operators of order $l-k$, as follows:

$$\begin{aligned} &K(x; D_x)\{g(x)X_p(\theta_r(x), \rho_r(x))\} \\ &\quad = \sum_{k=0}^{l} \{M_k^r(K)(x; D_x)g(x)\}X_{p-k} + \{N_k^r(K)(x; D_x)g(x)\}\,Y_{p-k} \end{aligned}$$

for every $g(x)$, and

$$\begin{aligned} &K(x; D_x)\{h(x)Y_p(\theta_r(x), \rho_r(x))\} \\ &\quad = \sum_{k=0}^{l} \{\tilde{M}_k^r(K)(x; D_x)h(x)\}X_{p-k} + \{\tilde{N}_k^r(K)(x; D_x)h(x)\}\,Y_{p-k} \end{aligned}$$

for every $h(x)$.

Proposition 3.1. *The coefficients of the operators $M_k^r(L)$, $\tilde{M}_k^r(L)$, $N_k^r(L)$ and $\tilde{N}_k^r(L)$ are all holomorphic in a neighborhood of the origin.*

Proof. Let H be the ring of germs of holomorphic functions of x at the origin, $H[D_x]$ the ring of linear partial differential operators with coefficient in H, and $H\langle\{X_q\}, \{Y_q\}\rangle$ the vector space of finite linear combinations of X_q's and Y_q's with coefficients in H. Using these notations, Proposition 3.1 can be written as;

$$M_k^r(L) \equiv \tilde{M}_k^r(L) \equiv N_k^r(L) \equiv \tilde{N}_k^r(L) \equiv 0 \qquad (\text{mod. } H[D_x])$$
$$(1 \le r \le m, 0 \le k \le 2m)\,.$$

In order to prove this proposition, we use the expression (1.2) of L. Taking account of Lemma 2.1 and (1.8), it is evident that

$$M_k^r(L^-) \equiv \tilde{M}_k^r(L^-) \equiv N_k^r(L^-) \equiv \tilde{N}_k^r(L^-) \equiv 0 \qquad (\text{mod. } H[D_x])$$
$$(1 \le r \le m, 0 \le k \le 2m)\,.$$

So, we have only to consider the term $x_0^{-\kappa}L_\beta^\kappa(x'; x_0D_0)D'^\beta$ for every β and $\kappa > 0$.
Step 1: First we prove that

$$x_0^{-\kappa}L_\beta^\kappa(x'; x_0D_0)X_p(\theta(x), \rho(x)) \equiv 0\,,$$
$$x_0^{-\kappa}L_\beta^\kappa(x'; x_0D_0)Y_p(\theta(x), \rho(x)) \equiv 0 \qquad (\text{mod. } H\langle\{X_q\}, \{Y_q\}\rangle).$$

Here, (θ, ρ) stands for any one of (θ_r, ρ_r)'s. From (1.8), we can regard x_0 as a holomorphic function of θ and x', and we write $\tilde{\rho}(\theta, x') = \rho(x_0(\theta, x'), x')$. Then, applying Lemma A′ in Section 5, we have

$$x_0^{-\kappa} L_\beta^\kappa(x'; x_0 D_0) X_p(\theta(x), \rho(x)) = \sum_{j \geq 0} x_0^{j-\kappa} \{T_j(\lambda, \Delta_\lambda) L_\beta^\kappa(x'; \lambda)\}|_{\lambda = \theta D_\theta} X_p(\theta, \tilde{\rho}(\theta, x'))$$
$$= \sum_{j \geq 0} x_0^{j-\kappa} (T_j L_\beta^\kappa)|_{\lambda = \theta D_\theta + \theta \tilde{\rho}_\theta D_\rho} X_p(\theta, \rho).$$

And using Lemma B, we have

$$x_0^{-\kappa} L_\beta^\kappa(x'; x_0 D_0) X_p(\theta(x), \rho(x)) = \sum_{j \geq 0} \sum_{q \geq 0} \sum_{k \geq q} \frac{1}{q!} x_0^{j-\kappa} \theta^k \{(U_{q,k} \circ T_j) L_\beta^\kappa\}|_{\lambda = \theta D_\theta} D_\rho^q X_p(\theta, \rho)$$

Now we apply Lemma 2.1′. Note that $(U_{q,k} \circ T_j) L_\beta^\kappa$ is divisible by $\lambda(\lambda - 1) \ldots (\lambda - \kappa + k + j + 1)$ because L_β^κ is divisible by $\lambda(\lambda - 1) \ldots (\lambda - \kappa + 1)$ and $U_{q,k}(\Delta_\lambda) \circ T_j(\lambda, \Delta_\lambda)$ is of order at most $k + j$ with respect to Δ_λ. So we can apply Lemma 2.1′ with $l' = \kappa - j - k$. Recalling (1.8), we obtain

$$\sum_{j \geq 0} \sum_{q \geq 0} \sum_{k \geq q} \frac{1}{q!} x_0^{j-\kappa} \theta^k \{(U_{q,k} \circ T_j) L_\beta^\kappa\}|_{\lambda = \theta D_\theta} D_\rho^q X_p(\theta, \rho)$$
$$\equiv \sum_{j \geq 0} \sum_{q \geq 0} \sum_{k \geq q} \frac{1}{q!} x_0^{j-\kappa} \theta^k \left[\sum_{l=1}^{\kappa-j-k-1} \left\{ \left(E_{l-1} \circ \frac{1}{\lambda - l} \circ U_{q,k} \circ T_j \right) L_\beta^\kappa \right\} \Bigg|_{\lambda = c} \theta^l X_{p-q-2l} \right.$$
$$\left. + \sum_{l=1}^{\kappa-j-k-1} \{(E_l \circ U_{q,k} \circ T_j) L_\beta^\kappa\}|_{\lambda = c} \theta^l Y_{p-q-2l-1} \right] \quad (\mathrm{mod.}\, H\langle \{X_q\}, \{Y_q\} \rangle).$$

Note also that $E_{l-1} \circ \dfrac{1}{\lambda - l} \circ U_{q,k} \circ T_j$ and $E_l \circ U_{q,k} \circ T_j$ are of order at most $l + k + j$ and hence at most $\kappa - 1$ if $l \leq \kappa - j - k - 1$. Therefore $\left\{ \left(E_{l-1} \circ \dfrac{1}{\lambda - l} \circ U_{q,k} \circ T_j \right) L_\beta^\kappa \right\} \Bigg|_{\lambda = c} = \{(E_l \circ U_{q,k} \circ T_j) L_\beta^\kappa\}|_{\lambda = c} = 0$ thanks to (A.3)′. Consequently,

$$x_0^{-\kappa} L_\beta^\kappa(x'; x_0 D_0) X_p(\theta_r(x), \rho_r(x)) \equiv 0 \qquad (\mathrm{mod.}\, H\langle \{X_q\}, \{Y_q\} \rangle).$$

Just in the same way, we can prove that

$$x_0^{-\kappa} L_\beta^\kappa(x'; x_0 D_0) Y_p(\theta_r(x), \rho_r(x)) \equiv 0 \qquad (\mathrm{mod.}\, H\langle \{X_q\}, \{Y_q\} \rangle).$$

Step 2: Next we prove that

$$M_k^r(x_0^{-\kappa} L_\beta^\kappa) \equiv \tilde{M}_k^r(x_0^{-\kappa} L_\beta^\kappa) \equiv N_k^r(x_0^{-\kappa} L_\beta^\kappa) \equiv \tilde{N}_k^r(x_0^{-\kappa} L_\beta^\kappa) \equiv 0 \qquad (\mathrm{mod.}\, H[D_x])$$
$$\text{for all } \beta,\ \kappa > 0, \qquad 1 \leq r \leq m, \qquad 0 \leq k \leq 2m.$$

Indeed, by Lemma C,

$$x_0^{-\kappa} L_\beta^\kappa(g(x) X_p) = \sum_{l \geq 0} \frac{1}{l!} x_0^{l-\kappa} D_0^l g \{\Delta_\lambda^l L_\beta^\kappa(x'; \lambda)\}|_{\lambda = x_0 D_0} X_p,$$

and we can verify just in the same way as Step 1 that

$$x_0^{l-\kappa}\{\Delta_\lambda^l L_\beta^\kappa(x';\lambda)\}|_{\lambda=x_0D_0}X_p=0 \qquad (\text{mod. } H\langle\{X_q\},\{Y_q\}\rangle)\,.$$

So we have

$$M_k^r(x_0^{-\kappa}L_\beta^\kappa)\equiv N_k^r(x_0^{-\kappa}L_\beta^\kappa)\equiv 0 \qquad (\text{mod. } H[D_x])\,.$$

For the same reason,

$$\tilde{M}_k^r(x_0^{-\kappa}L_\beta^\kappa)\equiv \tilde{N}_k^r(x_0^{-\kappa}L_\beta^\kappa)\equiv 0 \qquad (\text{mod. } H[D_x])\,.$$

Step 3: To complete the proof of Proposition 3.1, it suffices to prove that

$$M_k^r(x_0^{-\kappa}L_\beta^\kappa D'^\beta)\equiv \tilde{M}_k^r(x_0^{-\kappa}L_\beta^\kappa D'^\beta)\equiv N_k^r(x_0^{-\kappa}L_\beta^\kappa D'^\beta)\equiv \tilde{N}_k^r(x_0^{-\kappa}L_\beta^\kappa D'^\beta)\equiv 0$$

$$(\text{mod. } H[D_x]) \qquad \text{for all } \beta,\ k>0\,, \qquad 1\le r\le m\,, \qquad 0\le k\le 2m\,.$$

However, this is easy because, by Step 2, we have only to check that for $1\le i\le n$,

$$D_iX_p(\theta(x),\rho(x))=\rho_{x_i}X_{p-1}+\theta_{x_i}\theta^{-1}Y_{p-1}\,,$$

$$D_iY_p(\theta(x),\rho(x))=\theta_{x_i}X_{p-1}+\rho_{x_i}Y_{p-1}+c\theta_{x_i}\theta^{-1}Y_p\,,$$

and ρ_{x_i}, θ_{x_i}, $\theta_{x_i}\theta^{-1}\in H$. Q.E.D.

Proposition 3.2. *For each* r $(1\le r\le m)$,

$$M_{2m}^r(L)(x)=N_{2m}^r(L)(x)=\tilde{M}_{2m}^r(L)(x)=\tilde{N}_{2m}^r(L)(x)=0\,.$$

Proof. We can verify by making use of Lemma 2.2 and Lemma 2.3 that

$$M_{2m}^r(L)=\theta_r^{-2m}L^{H1}\,, \qquad N_{2m}^r(L)(x)=\theta_r^{-2m}L^{H2}$$

$$\tilde{M}_{2m}^r(L)=\theta_r^{-2m}\widetilde{L^{H1}}\,, \qquad \tilde{N}_{2m}^r(L)(x)=\theta_r^{-2m}\widetilde{L^{H2}}\,.$$

Since $\widetilde{L^{H1}}=\theta L^{H2}$ and $\widetilde{L^{H2}}=L^{H1}$ it suffices to prove $L^{H1}=L^{H2}=0$. From the definition of θ and ρ, we have

$$L^H(x;(\rho\pm 2\theta^{1/2})_x)=0\,,$$

and by the homogeneity of L^H, we have

$$L^H(x;\theta_x\pm\theta^{1/2}\rho_x)=0\,,$$

that is,

$$\sum_{k=0}^{2m}L_k^H(x;\theta_x,\theta^{1/2}\rho_x)=0 \qquad \text{and} \qquad \sum_{k=0}^{2m}L_k^H(x;\theta_x,-\theta^{1/2}\rho_x)=0\,.$$

By adding or substracting these two equalities, we obtain

$$\sum_{k=0}^{m}L_{2k}^H(x;\theta_x,\theta^{1/2}\rho_x)=0 \qquad \text{and} \qquad \sum_{k=0}^{m-1}L_{2k+1}^H(x;\theta_x,\theta^{1/2}\rho_x)=0\,.$$

Therefore we have

$$L^{H1} = \sum_{k=0}^{m} \theta^{2m-k} L^{H}_{2k}(x; \theta_x, \rho_x) = 0\,,$$

$$L^{H2} = \sum_{k=0}^{m} \theta^{2m-k-1} L^{H}_{2k+1}(x; \theta_x, \rho_x) = 0\,. \qquad \text{Q.E.D.}$$

Proposition 3.3. *We put*

$$\begin{aligned} M^r_{2m-1}(L)(x; D_x) &= \sum_{i=0}^{m} P^r_i(x) D_i + Q^r(x)\,, \\ N^r_{2m-1}(L)(x; D_x) &= \sum_{i=0}^{m} R^r_i(x) D_i + S^r(x)\,, \\ \tilde{M}^r_{2m-1}(L)(x; D_x) &= \sum_{i=0}^{m} \tilde{P}^r_i(x) D_i + \tilde{Q}^r(x)\,, \\ \tilde{N}^r_{2m-1}(L)(x; D_x) &= \sum_{i=0}^{m} \tilde{R}^r_i(x) D_i + \tilde{S}^r(x)\,. \end{aligned} \tag{3.1}$$

Then we have

$$\begin{aligned} &P^r_0 = O(x_0)\,, \\ &R^r_0 = d_r(x') + O(x_0)\,, \\ &\tilde{P}^r_0 = 2e_r(x')x_0 + O(x_0^2)\,, \qquad \tilde{P}^r_i = O(x_0) \qquad (i = 1, \dots, n)\,, \\ &\tilde{Q}^r_0 = e_r(x') + O(x_0)\,, \\ &\tilde{R}^r_0 = O(x_0)\,, \end{aligned}$$

where $d_r(x')$ and $e_r(x')$ are holomorphic functions of x' in a neighborhood of the origin and

$$d_r(0) \neq 0\,, \qquad e_r(0) \neq 0\,.$$

Proof. We can check the following by simple calculation using Lemma 2.2 and Lemma 2.3:

$$\begin{aligned} \theta^{2m} P_i &= \theta L^{H(i)1} \\ \theta^{2m} R_i &= \theta L^{H(i)2} \\ \theta^{2m} S &= L^{H4} + \theta R^{H2} + \frac{1}{2}\theta\theta_{x_i x_j} L^{H(i,j)1} + \frac{1}{2}\theta^2 \rho_{x_i x_j} L^{H(i,j)2} \\ \theta^{2m} \tilde{P}_i &= \theta^2 L^{H(i)2} \\ \theta^{2m} \tilde{Q} &= \widetilde{L^{H3}} + \theta^2 R^{H2} + \frac{1}{2}\theta^2\theta_{x_i x_j} L^{H(i,j)1} + \frac{1}{2}\theta^3 \rho_{x_i x_j} L^{H(i,j)2} \\ \theta^{2m} \tilde{R}_i &= \theta L^{H(i)1} \qquad (i = 0, 1, \dots, n)\,, \end{aligned} \tag{3.2}$$

where $R^H(x; \xi)$ is the principal symbol of $R(x; \xi) = L(x; \xi) - L^H(x; \xi)$.

Step 1: We have

$$\begin{aligned} L^{H(0)1} &= O(x_0^{2m}), & L^{H(i)2} &= O(x_0^{2m-1}) \\ \theta_{x_i x_j} L^{H(i,j)1} &= O(x_0^{2m-1}), & L^{H(i,j)2} &= O(x_0^{2m-1}) \end{aligned} \tag{3.3}$$

We prove only the second and the third because others can be proved in a similar way. From Lemma 2.2, we have

$$L^{H(i)2} = \sum_{k=0}^{m-1} \theta^{2m-2-k} L_{2k+1}^{H(i)}(x; \theta_x, \rho_x).$$

$L_{2k+1}^{H(i)}$ is a polynomial of θ_x and ρ_x, and of degree $2k+1$ in θ_x. Pay attention to the terms of degree k_1 in θ_{x_0} and put $2k+1 = k_1 + k_2$. From (1.9), their coefficients are at least of $O(x_0^{[(k_1+1)/2]})$, and especially of $O(x_0^{[(k_1+2)/2]})$ if $i = 0$. On the other hand, since $\theta_{x'} = O(x_0)$, they are at least of $O(x_0^{k_2})$. Therefore, $L_{2k+1}^{H(i)} = O(x_0^{[(2k+1+1)/2]}) = O(x_0^{k+1})$, and $L^{H(i)2} = O(x_0^{2m-1})$. Here remark that the terms in $L_{2k+1}^{H(0)}$ of degree less than $2k+1$ with respect to θ_{x_0} (i.e. $k_1 < 2k+1$) are of $O(x_0^{k+2})$.

$$L^{H(i,j)1} = \sum_{k=0}^{m-1} \theta^{2m-2-k} L_{2k}^{H(i,j)}.$$

In the case $i \neq 0$, $j \neq 0$, we can observe in a similar way as above that $L^{H(i,j)1} = O(x_0^{2m-2})$. However, $\theta_{x_i x_j} = O(x_0)$ and hence $\theta_{x_i x_j} L^{H(i,j)1} = O(x_0^{2m-1})$. If either $i = 0$ or $j = 0$, $L^{H(i,j)1}$ itself is of $O(x_0^{2m-1})$.

Step 2: We prove that

$$d(x') = 2 \frac{\partial \widehat{L^H}}{\partial \eta_0}(0, x'; 2\theta_{x_0}(0, x')^{1/2}, 1, 0, \dots, 0) \theta_{x_0}(0, x')^{-1/2},$$

$$e(x') = \frac{\partial \widehat{L^H}}{\partial \eta_0}(0, x'; 2\theta_{x_0}(0, x')^{1/2}, 1, 0, \dots, 0) \theta_{x_0}(0, x')^{1/2},$$

and $d(0) \neq 0$, $e(0) \neq 0$. As for $\tilde{P}_0$ and R_0, recalling the above remark in step 1 and the fact $\rho_{x_k}(0, x') = \delta_{1,k}$, we obtain

$$L_{2k+1}^{H(0)}(x; \theta_x, \rho_x) = (2k+2) \tilde{a}_{2k+2,(2m-2k-2,0,\dots,0)}(0, x')^{2k+1} x_0^{k+1} + O(x_0^{k+2}).$$

Therefore

$$\tilde{P}_0 = \sum_{k=0}^{m} 2k \tilde{a}_{2k,(2m-2k,0,\dots,0)}(0, x')^k x_0 + O(x_0^2),$$

and if we put

$$e(x') = \sum_{k=0}^{m} k \tilde{a}_{2k,(2m-2k,0,\dots,0)}(0, x')^k,$$

then

$$\tilde{P}_0 = 2e(x') x_0 + O(x_0^2), \qquad R_0 = d(x') + O(x_0).$$

On the other hand,

$$\widehat{L^H}(y;\tilde{\phi}_y) = L^H(y_0^2, y'; \frac{1}{2y_0}\tilde{\phi}_{y_0}, \tilde{\phi}_{y'})$$

$$= \sum_{k+|\beta|\leq 2m} a_{k,\beta}(y_0^2, y')\left(\frac{1}{2y_0}\tilde{\phi}_{y_0}\right)^k \tilde{\phi}_{y'}^{\beta}$$

and taking account of the fact $a_{2k+1,\beta}(x) = O(x_0^{k+1})$, we obtain

$$\widehat{L^H}(0, y'; \tilde{\phi}_{y_0}(0, y'), 1, 0, \dots, 0) = \sum_{k=0}^{m} \frac{1}{2^{2k}} \tilde{a}_{2k,(2m-2k,0,\dots,0)}(0, y')\tilde{\phi}_{y_0}(0, y')^{2k},$$

$$\frac{\partial \widehat{L^H}}{\partial \eta_0}(0, y'; \tilde{\phi}_{y_0}(0, y'), 1, 0, \dots, 0) = \sum_{k=0}^{m} \frac{k}{2^{2k}} \tilde{a}_{2k,(2m-2k,0,\dots,0)}(0, y')\tilde{\phi}_{y_0}(0, y')^{2k-1}.$$

Here,

$$\tilde{\phi}_{y_0}(0, y') = \tilde{\rho}_{y_0}(0, y') \pm \tilde{\theta}^{-1/2}(0, y')\tilde{\theta}_{y_0}(0, y').$$

Since $\tilde{\rho}$ is even with respect to y_0, we have $\tilde{\rho}_{y_0}(0, y') = 0$, and since $\tilde{\theta} = x_0\sigma(x) = y_0^2\sigma(x)$ and $\sigma(0, x') = \theta_{x_0}(0, x')$, we have

$$\tilde{\theta}^{-1/2}(0, y')\tilde{\theta}_{y_0}(0, y') = \sigma(0, x')^{-1/2} y_0^{-1}\tilde{\theta}_{y_0}(0, y') = 2\sigma(0, x')^{-1/2}\theta_{x_0}(0, x') = 2\theta_{x_0}(0, x')^{1/2}.$$

And hence, we have

$$\frac{\partial \widehat{L^H}}{\partial \eta_0}(0, x'; 2\theta_{x_0}(0, x'), 1, 0, \dots, 0) = \pm \sum_{k=0}^{m} k\tilde{a}_{2k,(2m-2k,0,\dots,0)}(0, x')\tilde{\theta}_{x_0}(0, x')^{k-1/2}.$$

Therefore,

$$e(x') = \frac{\partial \widehat{L^H}}{\partial \eta_0}(0, x'; 2\theta_{x_0}(0, x')^{1/2}, 1, 0, \dots, 0)\theta_{x_0}(0, x')^{1/2}.$$

Next, we consider $\tilde{Q}$. Compare $\tilde{Q}$ with S in (3.2). Since $\theta^{2m}S = O(x_0^{2m})$ by Proposition 3.1 and $\frac{1}{2}\theta\theta_{x_ix_j}L^{H(i,j)1} + \frac{1}{2}\theta^2\rho_{x_ix_j}L^{H(i,j)2}$ is of $O(x_0^{2m+1})$ by Step 1, $L^{H4} + \theta R^{H2}$ must be of $O(x_0^{2m})$. On the other hand, we have

$$L^{H4} = \sum_{k=0}^{m} k(c-k)\theta^{2m-1-k}L_{2k}^H,$$

$$\widetilde{L^{H3}} = \sum_{k=0}^{m} k(c+1-k)\theta^{2m-k}L_{2k}^H.$$

So we conclude

$$\widetilde{L^{H3}} + \theta^2 R^{H2} = \sum_{k=0}^{m} k\theta^{2m-k}L_{2k}^H + O(x_0^{2m+1})$$

We can easily verify that

$$L_{2k}^H = \tilde{a}_{2k,(2m-2k,0,\dots,0)}(0, x')\theta_{x_0}(0, x')^{2k}x_0^k + O(x_0^{k+1})$$

in a similar way as above. And hence

$$\widetilde{L^{H3}} + \theta^2 R^{H2} = \sum_{k=0}^{m} k\tilde{a}_{2k,(2m-2k,0,\dots,0)}(0, x')\theta_{x_0}(0, x')^{2m+k}x_0^{2m} + O(x_0^{2m+1})$$

$$= \frac{\partial \widehat{L^H}}{\partial \eta_0}(0, x'; 2\theta_{x_0}(0, x')^{1/2}, 1, 0, \dots, 0)\theta_{x_0}(0, x')^{2m+1/2}x_0^{2m} + O(x_0^{2m+1}) .$$

Therefore,

$$\tilde{Q} = e(x') + O(x_0) .$$

Step 3: Finally we prove that $d(0) \neq 0$ and $e(0) \neq 0$. Since $\theta_{x_0}(0) \neq 0$, it suffices to prove that

$$\frac{\partial \widehat{L^H}}{\partial \eta_0}(0; 2\theta_{x_0}(0)^{1/2}, 1, 0, \dots, 0) \neq 0 .$$

However, this follows immediately from (A.2), so the proof of Proposition 3.3 is completed. Q.E.D.

Now we construct the formal solution of (C.P.). From Proposition 3.2 and Proposition 3.3, g_p^r and h_p^r will be determined inductively in p by solving a first order Fuchsian partial differential system. First we consider the initial conditions of this system.

Lemma 3.1. *All $g_p^r(0, x')$ and $h_p^r(0, x')$ $(2l - \alpha \leqq p)$ are uniquely determined from $w(x')$ by a linear algebraic system of the following form,*

$$\sum_{r=1}^{m} \theta_{rx_0}(0, x')^k g_p^r(0, x') = U_k(\{D_0^\gamma \theta_r(0, x'), D_0^\gamma \rho_r(0, x')\}, \{D_0^t g_{p-s}^r(0, x'), D_0^t h_{p-s}^r(0, x')\})$$
$$+ \delta_{l,k}\delta_{p-2k,-\alpha} w(x') \qquad (k = 0, 1, \dots, m-1) , \tag{3.4}$$

where U_k $(k = 0, 1, \dots, m-1)$ are polynomials of the elements with $1 \leqq r \leqq m$, $\gamma \leqq k$, $1 < s \leqq 2k$ and $2t \leqq s$.

Proof. (3.4) follows from the fact that

$$D_\theta^k X_p(0, \rho) = (-1)^k \frac{\Gamma(c-k)}{\Gamma(c)} k_{p-2k}(\rho) ,$$

$$D_\theta^k Y_p(0, \rho) = (-1)^k k \frac{\Gamma(c-k)}{\Gamma(c)} k_{p-2k+1}(\rho) ,$$

which can be checked by simple calculation. And noting that $\theta_{rx_0}(0) = \frac{1}{4}\tilde{\phi}_{ry_0}(0)^2$, (A.2) guarantees that the $m \times m$ matrix

$$\begin{bmatrix} 1 & \cdots & 1 \\ \theta_{1x_0}(0) & \cdots & \theta_{mx_0}(0) \\ & \cdots & \\ \theta_{1x_0}(0)^{m-1} & \cdots & \theta_{mx_0}(0)^{m-1} \end{bmatrix}$$

is invertible, that is, (3.4) is uniquely solvable. Q.E.D.

Next, we consider the differential system which g_p^r and h_p^r satisfy. From Proposition 3.2, they are written in the following form for each r $(1 \leqq r \leqq m)$;

$$\begin{cases} M_{2m-1}^r(L)(x;D)g_p^r + \tilde{M}_{2m-1}^r(L)(x;D)h_p^r \\ \quad = -\sum_{k=2}^{m} (M_{2m-k}^r(L)(x;D)g_{p-k+1}^r + \tilde{M}_{2m-k}^r(L)(x;D)h_{p-k+1}^r)\,, \\ N_{2m-1}^r(L)(x;D)g_p^r + \tilde{N}_{2m-1}^r(L)(x;D)h_p^r \\ \quad = -\sum_{k=2}^{m} (N_{2m-k}^r(L)(x;D)g_{p-k+1}^r + \tilde{N}_{2m-k}^r(L)(x;D)h_{p-k+1}^r)\,. \end{cases} \tag{3.5}$$

Proposition 3.1 and Proposition 3.3 imply that the system (3.5) is of the form

$$\begin{cases} \left\{O(x_0)D_0 + \sum_{i=1}^{n} O(1)D_i + O(1)\right\}g + \left\{2x_0D_0 + 1 + \sum_{i=1}^{n} O(x_0)D_i\right\}h = F(x)\,, \\ \left\{D_0 + \sum_{i=0}^{n} O(1)D_i + O(1)\right\}g + \left\{O(x_0)D_0 + \sum_{i=1}^{n} O(1)D_i + O(1)\right\}h = G(x)\,, \end{cases} \tag{3.6}$$

where $g(x)$ and $h(x)$ are unknown functions and $F(x)$ and $G(x)$ are given functions.

Proposition 3.4. *Given $g(0, x') = v(x')$ a holomorphic function at $x' = 0$, the system* (3.6) *has a unique holomorphic solution* $(g(x), h(x))$.

Proof. The system of (3.6) can be written in the form

$$[x_0D_0 - A(x; D')]\begin{pmatrix} g(x) \\ h(x) \end{pmatrix} = \begin{pmatrix} x_0H_1(x) \\ H_2(x) \end{pmatrix},$$

where $A(x; D')$ is a 2×2 matrix whose elements are differential operators with respect to x' of order at most 1 with holomorphic coefficients and $H_1(x)$ and $H_2(x)$ are holomorphic functions. We can easily check that

$$A|_{x_0=0} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

This means that (3.6) is a Fuchsian system in the sense of H. Tahara [10] with characteristic eigenvalues 0, $-\frac{1}{2}$. So we can apply Theorem 1.2.10 of [10] with $\eta_0 = 1$. If we give $g(0, x')$, $h(0, x')$ as Cauchy data, the compatibility condition is

$$-A|_{x_0=0}\begin{pmatrix} g(0, x') \\ h(0, x') \end{pmatrix} = \begin{pmatrix} 0 \\ H_2(0, x') \end{pmatrix},$$

that is,

$$h(0, x') = 2H_2(0, x')\,.$$

Then, Proposition 3.4 is a direct consequence of Theorem 1.2.10 of [10].

Q.E.D.

To conclude, first of all we determine $g^r_{-\alpha+2l}(0, x')$ $(1 \leqq r \leqq m)$ by means of (3.4) and next $g^r_{-\alpha+2l}(x)$ and $h^r_{-\alpha+2l}(x)$ by means of (3.5). Then we can determine $g^r_{-\alpha+2l+1}(0, x')$ $(1 \leqq r \leqq m)$ again by means of (3.4). Repeating this procedure, we can determine uniquely all $g^r_p(x)$ and $h^r_p(x)$. Thus we have completed the construction of the formal solution.

§ 4. Convergence of the formal solution

After the construction of the formal solution which has been done in the previous section, it remains for us to verify the convergence of the formal solution. We prove it by the method of the majorant function. To do so, we prepare a family of scale functions $\Phi_p(s, z)$ and $\Psi_p(s, z)$. We define them as follows,

$$\Phi_p(s, z) = \sum_{j \geqq 0} \frac{(2j+p)!(\rho s)^j}{(2j)!(R-z)^{2j+1+p}} = D_z^p \Phi_0 , \qquad \Phi_0(s, z) = \frac{R-z}{(R-z)^2 - \rho s}$$

$$\Psi_p(s, z) = \sum_{j \geqq 0} \frac{(j+1)(2j+p)!(\rho s)^j}{(2j)!(R-z)^{2j+1+p}} = D_z^p \Psi_0 , \qquad \Psi_0 = D_s(s\Phi_0) ,$$

where ρ and R are some positive constants $(\rho > 1)$.

The following proposition can be easily checked.

Proposition 4.1.

(i) $D_z \Phi_p = \Phi_{p+1}$, $\quad D_z \Psi_p = \Psi_{p+1}$,

(ii) $$(2sD_s + 1)\Phi_p \gg \begin{cases} Ks^d D_s^l \Phi_{p-r} & \text{if } r \geqq \max(2(l-d), l-1) , \\ Ks^d D_s^l \Psi_{p-r} & \text{if } r \geqq \max(2(l-d), l) , \end{cases}$$

(iii) $$D_s \Psi_p \gg \begin{cases} Ks^d D_s^l \Phi_{p-r} & \text{if } r \geqq \max(2(l-d-1), l-2) , \\ Ks^d D_s^l \Psi_{p-r} & \text{if } r \geqq \max(2(l-d-1), l-1) , \end{cases}$$

(iv) $$\frac{1}{R-z}\Phi_p \gg \frac{1}{R}\Phi_p , \qquad \frac{1}{R-z}\Psi_p \gg \frac{1}{R}\Psi_p ,$$

(v) $$\frac{1}{(R'-R)(R''-R)}\Phi_p \gg \frac{1}{(R'-s)(R''-z)}\Phi_p ,$$

$$\frac{1}{(R'-R)(R''-R)}\Psi_p \gg \frac{1}{(R'-s)(R''-z)}\Psi_p , \qquad (R' > R, R'' > R)$$

where $K = K(R, \rho)$ is a positive constant independent of p.

We write the Cauchy problem (3.5) with initial deta $g^r_p(0, x') = w^r_p(x')$, where $w^r_p(x')$ are determined by (3.4), in the following form,

$$\begin{cases} e_r(x')(2x_0D_0+1)h_p^r \\ \quad = -M_{2m-1}^r(L)(x;D)g_p^r - \{\tilde{M}_{2m-1}^r(L)(x;D) - e_r(x')(2x_0D_0+1)\}h_p^r \\ \quad - \sum_{k=2}^{m} \{M_{2m-k}^r(L)(x;D)g_{p-k+1}^r + \tilde{M}_{2m-k}^r(L)(x;D)h_{p-k+1}^r\} & (4.1) \\ d_r(x')D_0g_p^r \\ \quad = -\{N_{2m-1}^r(L)(x;D) - d_r(x')D_0\}g_p^r - \tilde{N}_{2m-1}^r(L)(x;D)h_p^r \\ \quad - \sum_{k=2}^{m} \{N_{2m-k}^r(L)(x;D)g_{p-k+1}^r + \tilde{N}_{2m-k}^r(L)(x;D)h_{p-k+1}^r\} & (4.2) \\ g_p^r(0,x') = w_p^r(x') \qquad (r=1,\dots,m) & (4.3) \end{cases}$$

In the following proposition, we use the notation as follows.

$$f^{\#}(x) = \sum_{\alpha} |f_\alpha| x^\alpha$$

for $f(x) = \sum_\alpha f_\alpha x^\alpha$, $\alpha = (\alpha_0, \dots, \alpha_n)$, and

$$K^{\#}(x, D) = \sum_{\alpha} K_\alpha^{\#}(x) D^\alpha$$

for $K(x, D) = \sum_\alpha K_\alpha(x) D^\alpha$.

We define two functions $g_p^*(x)$ and $h_p^*(x)$ which will be majorants of the solutions $g_p^r(x)$ and $h_p^r(x)$ $(r = 1, \dots, m)$ of the Cauchy problem (4.1), (4.2), (4.3) respectively.

$$g_p^*(x) = GC^p \Psi_{p+2l-\alpha}(x_0, x_1 + \cdots + x_n)$$

$$h_p^*(x) = HC^p \Phi_{p+1+2l-\alpha}(x_0, x_1 + \cdots + x_n)$$

where G, H and C are some positive numbers.

Proposition 4.2. *There exist $\rho > 1$ and positive numbers R, G, H and C such that the following majorant relations hold,*

$$\begin{cases} (2x_0D_0+1)h_p^* \gg (e_r^{-1}M_{2m-1}^r)^{\#}g_p^* + \{e_r^{-1}\tilde{M}_{2m-1}^r - (2x_0D_0+1)^{\#}\}h_p^* \\ \qquad + \sum_{k=2}^{m} \{(e_r^{-1}M_{2m-k}^r)^{\#}g_{p-k+1}^* + (e_r^{-1}\tilde{M}_{2m-k}^r)^{\#}h_{p-k+1}^*\} & (4.4) \\ D_0g_p^* \gg (d_r^{-1}N_{2m-1}^r - D_0)^{\#}g_p^* + (d_r^{-1}\tilde{N}_{2m-1}^r)^{\#}h_p^* \\ \qquad + \sum_{k=2}^{m} \{(d_r^{-1}N_{2m-1}^r)^{\#}g_{p-k+1}^* + (d_r^{-1}\tilde{N}_{2m-k}^r)^{\#}h_p^* & (4.5) \\ g_p^*(0,x') \gg w_p^r(x') \qquad r = 1, \dots, m & (4.6) \end{cases}$$

and $g_p^*(x)$ and $h_p^*(x)$ are majorants of $g_p^r(x)$ and $h_p^r(x)$ $(r = 1, \dots, m)$ respectively.

Proof. First we investigate M_{2m-k}^r, $\tilde{M}_{2m-k}^r$, N_{2m-k}^r and $\tilde{N}_{2m-k}^r$ in detail. Calculating $x_0^{d'}D_0^{l'}D'^{\beta'}(g_p(x)X_p(\theta(x), \rho(x)))$ and $x_0^{d'}D_0^{l'}D'^{\beta'}(h_p(x)Y_p(\theta(x), \rho(x)))$ using Lemma 2.1, we find that the former is a linear combination of

$$x_0^{d'-l_1+j}D_0^{l'-l_1}D'^{\beta_1}g_pX_{p-(\beta_2+2j)} \quad \text{with} \quad \beta_1+\beta_2 \leqq \beta',\ 2j \leqq l_1 \leqq l'$$

and

$$x_0^{d'-l_1+j}D_0^{l'-l_1}D'^{\beta_1}h_pY_{p-(\beta_2+2j+1)} \quad \text{with} \quad \beta_1+\beta_2 \leqq \beta',\ 2j+1 \leqq l_1 \leqq l'$$

and the latter is a linear combination of

$$x_0^{d'-l_1+j}D_0^{l'-l_1}D'^{\beta_1}g_pX_{p-(\beta_2+2j-1)} \quad \text{with} \quad \beta_1+\beta_2 \leqq \beta',\ 2j-1 \leqq l_1 \leqq l'$$

and

$$x_0^{d'-l_1+j}D_0^{l'-l_1}D'^{\beta_1}h_pY_{p-(\beta_2+2j)} \quad \text{with} \quad \beta_1+\beta_2 \leqq \beta',\ 2j \leqq l_1 \leqq l'$$

with holomorphic coefficients. It follows from these and the condition (A.1) ($|\beta| \leqq 2(m-l'+d')$ and $l'+|\beta'| \leqq 2m$) that M^r_{2m-k}, $\tilde{M}^r_{2m-k}$, N^r_{2m-k} and $\tilde{N}^r_{2m-k}$ are linear combinations of

$$\begin{aligned} & x_0^dD_0^lD'^\beta \quad \text{with} \quad \max(2(l-d), l)+|\beta| \leqq k\,, \\ & x_0^dD_0^lD'^\beta \quad \text{with} \quad \max(2(l-d)+1, l-1)+|\beta| \leqq k\,, \\ & x_0^dD_0^lD'^\beta \quad \text{with} \quad \max(2(l-d)-1,\ l+1)+|\beta| \leqq k\,, \\ \text{and} \quad & x_0^dD_0^lD'^\beta \quad \text{with} \quad \max(2(l-d), l)+|\beta| \leqq k \end{aligned}$$

respectively with holomorphic coefficients.

Now we choose constants R, ρ, G, H and C so that (4.4), (4.5), (4.6) may hold. Note that $w^r_p(x')$ which are determined by (3.3) have a common radius of convergence $\tilde{R} \geqq 0$ and $\sup_{|x'|\leqq \tilde{R}/2}|w^r_p(x')| \leqq C'^{p+1}p!$ for sufficiently large constant $C'>0$. And hence, (4.6) holds for sufficiently large G and C. In (4.4) and (4.5), we can check the following using the above property of M^r_{2m-k}, $\tilde{M}^r_{2m-k}$, N^r_{2m-k}, M^r_{2m-k} and Proposition 4.1:

$$(2x_0D_0+1)h^*_p \gg (e_r^{-1}M^r_{2m-1})^\# g^*_p$$

and

$$D_0g^*_p \gg (d_r^{-1}\tilde{N}^r_{2m-1})^\# h^*_p$$

for sufficiently small $R>0$,

$$(2x_0D_0+1)h^*_p \gg \{e_r^{-1}\tilde{M}^r_{2m-1}-(2x_0D_0+1)\}^\# h^*_p$$

and

$$D_0g^*_p \gg (d_r^{-1}N^r_{2m-1}-D_0)^\# g^*_p$$

for sufficiently large $\rho>1$, and

$$(2x_0D_0+1)h^*_p \gg \sum_{k=2}^{m}\{(e_r^{-1}M^r_{2m-k})^\# g^*_{p-k+1}+(e_r^{-1}\tilde{M}^r_{2m-k})^\# h^*_{p-k+1}\}$$

and

$$D_0 g_p^* \gg \sum_{k=2}^{m} \{(d_r^{-1} N_{2m-k}^r)^\# g_{p-k+1}^* + (d_r^{-1} \tilde{N}_{2m-k}^r)^\# g_{p-k+1}^*\}$$

for sufficiently large $C > 0$.

Thus we have proved the first part of Proposition 4.2. The second part is not difficult to verify. Q.E.D.

Therefore, we know the holomorphic coefficients have the estimates $|g_p|$, $|h_p| \leqq Kp!T^p$, where K and T are positive constants independent of p, in the common existence domain which is a neighborhood of the origin of $\mathbf{C}^{n+1}$. On the other hand, we have the estimate $|X_p|$, $|Y_p| \leqq \frac{1}{p!} C_Q r^p$ on every compact set Q in the universal covering space over $D_r - (\bigcup_{r=1}^m V_r) \cup S$ where $D_r = \{x \in \mathbf{C}^{n+1}; |x| \leqq r\}$ and C_Q is a constant which depends only on Q and is independent of p (see [13]). Thus choosing r such that $r < T$ and $\omega = D_r$, we prove the convergence of the formal solution.

§ 5. Appendix

In this section, we prove some fundamental lemmas which we have used in previous sections.

Let $s \to t = t(s)$ be a change of variable which is holomorphic at $s = 0$ and satisfying

$$t(0) = 0 \quad \text{and} \quad t'(0) \neq 0\,. \tag{5.1}$$

Given a polynomial $K(\sigma)$ of single variable, there exists one and only one $\tilde{K}(s, \sigma)$, which is polynomial with respect to σ and holomorphic with respect to s at $s = 0$, such that

$$\left\{K\left(t\frac{d}{dt}\right)[u(s(t))]\right\}\Bigg|_{t=t(s)} = \tilde{K}\left(s, s\frac{d}{ds}\right)u(s) \tag{5.2}$$

holds for any smooth function $u(s)$.

We are going to show an algebraic procedure to compute $\tilde{K}(s, \sigma)$.

Lemma A. *There exists a family of polynomials $\{S_j(\sigma, \lambda)\}_{j=0}^{\infty}$ of two variables (σ, λ) satisfying the conditions:*

i. *$S_j(\sigma, \lambda)$ is of degree at most j with respect to each of σ, λ;*

ii. *Given a polynomial $K(\sigma)$, then (5.2) holds for*

$$\tilde{K}(s, \sigma) = \sum_{j=0}^{\infty} s^j S_j(\sigma, \Delta_\sigma) K(\sigma)\,, \tag{5.3}$$

where Δ_σ is the operator defined to be $(\Delta_\sigma K)(\sigma) = K(\sigma + 1) - K(\sigma)$.

Remark 5.1. σ and λ are assumed to be non-commutative, more precisely,

$$(\lambda + 1)^q \sigma^p = (\sigma + q)^p (\lambda + 1)^q \tag{5.4}$$

for non-negative integers p, q. And (5.3) should be interpreted in the following way:

$$\text{If}\quad S_j(\sigma, \lambda) = \sum_{p,q=0}^{j} c_{pq}\sigma^p\lambda^q \quad (c_{pq}\text{'s are constants}),$$

$$\text{then}\quad S_j(\sigma, \Delta_\sigma)K(\sigma) = \sum_{p,q=0}^{j} c_{pq}\sigma^p(\Delta_\sigma^q K)(\sigma). \tag{5.5}$$

Proof. By hypothesis (5.1), t (or s) is represented as a convergent power series in a neighborhood of $s = 0$ (resp. $t = 0$):

$$t = \sum_{k=0}^{\infty} a_k s^{k+1}, \tag{5.6}$$

$$s = \sum_{k=0}^{\infty} b_k t^{k+1}, \tag{5.6'}$$

where $a_0 = t'(0) \neq 0$ and $b_0 = \dfrac{1}{a_0} \neq 0$. Therefore, we have

$$t^\sigma = a_0^\sigma \sum_{p=0}^{\infty} s^{\sigma+p}A_p(\sigma), \tag{5.7}$$

$$s^\sigma = b_0^\sigma \sum_{q=0}^{\infty} t^{\sigma+q}B_q(\sigma). \tag{5.7'}$$

$A_p(\sigma)$'s and $B_q(\sigma)$'s are polynomials of σ. We may verify that

$$\deg A_p(\sigma) \leqq p, \qquad \deg B_q(\sigma) \leqq q, \tag{5.8}$$

especially,

$$A_0(\sigma) = B_0(\sigma) = 1.$$

Given a polynomial $K(\sigma)$, let $\tilde{K}(s, \sigma)$ be the function with which (5.2) holds. Applying (5.2) to $u(s) = s^\sigma$ and taking account of (5.7)$'$, we have

$$\tilde{K}(s, \sigma) = s^{-\sigma}b_0^\sigma \sum_{q=0}^{\infty} t^{\sigma+q}B_q(\sigma)K(\sigma + q). \tag{5.2'}$$

(If (5.2)$'$ holds for all complex numbers σ, then it does in particular for non-negative integers. So, (5.2) holds for polynomials and hence for any smooth function $u(s)$. Therefore, (5.2)$'$ is equivalent to (5.2)). Representing conversely $t^{\sigma+q}$ as power series in s by making use of (5.7) we have

$$\tilde{K}(s, \sigma) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_0^q s^{p+q}A_p(\sigma + q)B_q(\sigma)K(\sigma + q).$$

The right-hand side may be rearranged as

$$\tilde{K}(s, \sigma) = \sum_{j=0}^{\infty} s^j H_j(\sigma), \tag{5.9}$$

where

$$H_j(\sigma) = \sum_{q=0}^{j} a_0^q A_{j-q}(\sigma + q) B_q(\sigma) K(\sigma + q)$$

$$= \sum_{q=0}^{j} a_0^q A_{j-q}(\sigma + q) B_q(\sigma) (\Delta_\sigma + 1)^q K(\sigma)$$

because $K(\sigma + q) = (\Delta_\sigma + 1)^q K(\sigma)$. So, if we define $S_j(\sigma, \lambda)$ to be

$$S_j(\sigma, \lambda) = \sum_{q=0}^{j} a_0^q A_{j-q}(\sigma + q) B_q(\sigma) (\lambda + 1)^q, \tag{5.10}$$

then we have

$$H_j(\sigma) = S_j(\sigma, \Delta_\sigma) K(\sigma). \tag{5.11}$$

Therefore, (5.3) holds. $S_j(\sigma, \lambda)$ is polynomial of degree at most j with respect to each of σ and λ because of (5.8). Q.E.D.

By expanding s into power series of t and taking account of the condition (5.1), we obtain the following.

Lemma A′. *There exists a family of polynomials* $\{T_j(\sigma, \lambda)\}_{j=0}^{\infty}$ *of two variables* (σ, λ) *satisfying the conditions*:

i. $T_j(\sigma, \lambda)$ *is of degree at most* j *with respect to each of* σ, λ;
ii. *Given a polynomial* $K(\sigma)$, *then* (5.2) *holds for*

$$\tilde{K}(s, \sigma) = \sum_{j=0}^{\infty} t^j T_j(\sigma, \Delta_\sigma) K(\sigma),$$

Lemma B. *Let* $g(t)$ *be a holomorphic function which satisfies* $g(0) = 0$. *Then there exist polynomials* $U_{q,j}(\lambda)$ $(q \geqq 0,\ j \geqq q)$ *of degree* j *independent of* K *such that*

$$K(tD_t + g(t)D_z)u = \sum_{q \geqq 0} \frac{1}{q!} D_z^q \sum_{j \geqq q} t^j \{U_{q,j}(\Delta_\tau) K(\tau)\}|_{\tau = tD_t} u \qquad \textit{for any } u = u(t, z). \tag{5.12}$$

Proof. By linearity of the operator $K(tD_t + g(t)D_z)$, we have only to prove (5.12) for $K(\tau) = \tau^n$ $(n = 0, 1, \dots)$, that is, to prove

$$(tD_t + g(t)D_z)^n u = \sum_{q \geqq 0} \frac{1}{q!} D_z^q \sum_{j \geqq q} t^j \{U_{q,j}(\Delta_\tau) \tau^n\}|_{\tau = tD_t} u. \tag{5.13}$$

We shall show first the uniqueness of the polynomial $U_{q,j}(\lambda)$ satisfying (5.13) and next that $U_{q,j}(\lambda)$ is of degree at most j. More precisely, we try a formal series $\sum_{l \geqq 0} U_{q,j}^l \lambda^l$ for $U_{q,j}(\lambda)$ and rewrite (5.13) as

$$(tD_t + g(t)D_z)^n u = \sum_{q \geqq 0} \frac{1}{q!} D_z^q \sum_{j \geqq q} t^j \left\{ \sum_{l \geqq 0} U_{q,j}^l \Delta_\tau^l \tau^n \right\} \Bigg|_{\tau = tD_t} u. \tag{5.13′}$$

The coefficients $U_{q,j}^l$'s will be uniquely determined successively from (5.12). And then, we shall show that $U_{q,j}^l$ are 0 if $l > j$.

First, substituting $n=0$ into (5.13), we obtain

$$u = \sum_{q\geqq 0} \frac{1}{q!} D_z^q \sum_{j\geqq q} t^j U_{q,j}^0 u .$$

It follows that

$$U_{q,j}^0 = \delta_{0,q} \cdot \delta_{0,j} . \tag{5.14}$$

Next, operating $tD_t + g(t)D_z$ on both sides of (5.13)′, we obtain

$$\begin{aligned}(tD_t + g(t)D_z)^{n+1}u &= \sum_{q\geqq 0} \frac{1}{q!} D_z^q \sum_{j\geqq q} t^j \left\{ (\tau + j) \sum_{l\geqq 0} U_{q,j}^l \Delta_\tau^l \tau^n \right\}\Bigg|_{\tau=tD_t} u \\ &+ \sum_{q\geqq 0} \frac{q}{q!} D_z^q \sum_{k\geqq 0} \sum_{j\geqq q+k} g_{k+1} t^j \left\{ \sum_{l\geqq 0} U_{q-1,j-k-1}^l \Delta_\tau^l \tau^n \right\}\Bigg|_{\tau=tD_t} u ,\end{aligned}$$

where we put $g(t) = \sum_{k=1}^\infty g_k t^k$. On the other hand, replacing n by $n+1$ in (5.13), we have

$$(tD_t + g(t)D_z)^{n+1}u = \sum_{q\geqq 0} \frac{1}{q!} D_z^q \sum_{j\geqq q} t^j \left\{ (\tau + j) \sum_{l\geqq 0} U_{q,j}^l \Delta_\tau^l \tau^{n+1} \right\}\Bigg|_{\tau=tD_t} u .$$

Using a formula

$$\Delta_\tau^l \tau^{n+1} = ((\tau + l)\Delta_\tau^l + l\Delta_\tau^{l-1})\tau^n , \tag{5.15}$$

we obtain

$$(l+1)U_{q,j}^{l+1} = (j-l)U_{q,j}^l + \sum_{k=0}^{j-q} q g_{k+1} U_{q-1,j-k-1}^l . \tag{5.16}$$

We see that all $U_{q,j}^l$ ($j \geqq q \geqq 0, l \geqq 0$) are uniquely determined successively by (5.14) and (5.16), and the formal series $U_{q,j}(\lambda) = \sum_{l\geqq 0} U_{q,j}^l \lambda^l$ satisfies (5.13). Furthermore, if we substitute $l = j$ in (5.16), we get

$$(j+1)U_{q,j}^{j+1} = \sum_{k=0}^{j-q} q g_{k+1} U_{q-1,j-k-1}^j . \tag{5.17}$$

It follows from (5.17) by induction that $U_{q,j}^l$ with $l > j$ is determined only by $U_{q',j'}^0$ with $j' < 0$ which are 0. Therefore,

$$U_{q,j}^l = 0 \qquad \text{for } l > j ,$$

that is, $U_{q,j}(\lambda)$ is a polynomial of degree j. Q.E.D.

Lemma C. *Let $K(\tau)$ be a polynomial of single variable, then,*

$$K(tD_t)(f \cdot g) = \sum_{l\geqq 0} \frac{1}{l!} t^l D_t^l f (\Delta_\tau^l K)|_{\tau=tD_t} g \tag{5.18}$$

for any $f(t)$ and $g(t)$.

Proof. By linearity of the operator $K(tD_t)$, we have only to prove (5.18) for $K(\tau)=\tau^n$ $(n=0,1,\ldots)$, that is, to prove

$$(tD_t)^n(f\cdot g)=\sum_{l\geqq 0}\frac{1}{l!}t^lD_t^lf(\Delta_\tau^l\tau^n)|_{\tau=tD_t}g \tag{5.19}$$

Let us prove (5.19) by induction with respect to n. It is true for $n=0$. Assuming (5.19) for n, let us prove it for $n+1$. Operating tD_t on both sides of (5.19), we have

$$\begin{aligned}(tD_t)^{n+1}(f\cdot g)&=\sum_{l\geqq 0}\frac{l}{l!}(t^{l+1}D_t^{l+1}+lt^lD_t^l)f(\Delta_\tau^l\tau^n)|_{\tau=tD_t}g+\sum_{l\geqq 0}\frac{1}{l!}t^lD_t^lf(\tau\Delta_\tau^l\tau^n)|_{\tau=tD_t}g\\&=\sum_{l\geqq 0}\frac{1}{l!}t^lD_t^lf\{(l\Delta_\tau^{l-1}+(\tau+l)\Delta_\tau^l)\tau^n\}|_{\tau=tD_t}g\,.\end{aligned}$$

It follows from (5.15) that (5.19) is also true for $n+1$. Q.E.D.

Lemma D. *Let $K(\tau)$ be of degree n. Then the following formula holds for* $j=1,2,\ldots.$

$$K(\tau-j)=\sum_{l=0}^{n}(-1)^l\binom{l+j-1}{j-1}\Delta_\tau^lK(\tau)\,. \tag{5.20}$$

Proof. First, let us prove

$$K(\tau-1)=\sum_{l=0}^{n}(-1)^l\Delta_\tau^lK(\tau) \tag{5.21}$$

by induction with respect to $n=\deg K$. It is true for $n=0$. Assuming (5.21) for n, let us prove it for $n+1$. Let $K(\tau)$ be of degree $n+1$. Applying (5.21) to $\Delta_\tau K(\tau)$, which is of degree n, we have

$$K(\tau)-K(\tau-1)=\sum_{l=0}^{n}(-1)^l\Delta_\tau^{l+1}K(\tau)\,.$$

And hence (5.21) is true also for $n+1$.

(5.21) may be rewritten symbolically as

$$K(\tau-1)=(1+\Delta_\tau)^{-1}K(\tau)\,.$$

So, by induction, we have

$$K(\tau-j)=(1+\Delta_\tau)^{-j}K(\tau)\,,$$

for $j\geqq 1$. By power series expansion of $(1+\Delta_\tau)^{-j}$, we obtain (5.20) because

$$\Delta_\tau^lK(\tau)=0\qquad\text{if } l\geqq\deg K\,.$$

Q.E.D.

Lemma E.

$$K(tD_t)(t^{-j}u(t))=t^{-j}K(tD_t-j)u(t)$$

for any $u=u(t)$ and positive integer j.

Proof. By Lemma C and Lemma D, we have

$$\begin{aligned} K(tD_t)(t^{-j}u) &= \sum_{l\geqq 0} \frac{1}{l!} t^l (D_t^l t^{-j})(\varDelta_\tau^l K)|_{\tau=tD_t} u \\ &= \sum_{l\geqq 0} (-1)^l \binom{l+j-1}{j-1} t^{-j} (\varDelta_\tau^l K)|_{\tau=tD_t} u \\ &= t^{-j} K(tD_t - j)u . \end{aligned}$$

Q.E.D.

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