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The global existence of small amplitude solutions to the nonlinear acoustic wave equation

By

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1. Introduction

The nonlinear acoustic wave equation in a viscous conducting fluid in n space dimensions is given by (Kuznetsov [4]),

$$\partial_{tt}\varphi - c_0^2 \Delta\varphi = \partial_t \left\{ |\nabla\varphi|^2 + \frac{b_1}{\rho_0}\Delta\varphi + \frac{\nu - 1}{c_0^2}(\partial_t\varphi)^2 \right\}, \tag{1.1}$$

where φ is a wave function, c_0 is a sound velocity of the undisturbed fluid, $\nu = c_p/c_v$, where c_p and c_v are heat capacities of the fluid. $b_1 = \xi + (4\eta)/3 + \kappa(1/c_v - 1/c_p)$, where ξ and η are the coefficients of shear and bulk viscosity respectively, κ is the coefficient of thermal conductivity, and ρ_0 is the density of the undisturbed fluid.

In this paper, we shall consider the following two problems, assuming that $b = b_1/\rho_0 > 0$, $a = (\nu - 1)/c_0^2 > 0$.

P(1): (Initial Boundary Value Problem)

$$\partial_{tt}\varphi - c_0^2\Delta\varphi = \partial_t\{|\nabla\varphi|^2 + b\Delta\varphi + a(\partial_t\varphi)^2\} \text{ in } \Omega \times [0, \infty),$$

$$\varphi(x, 0) = \varphi_0(x),\ \varphi_t(x, 0) = \varphi_1(x) \text{ in } \Omega,\ \varphi = 0 \text{ on } \partial\Omega \times [0, \infty),$$

where Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$.

P(2): (Cauchy Problem)

$$\partial_{tt}\varphi - c_0^2\Delta\varphi = \partial_t\{|\nabla\varphi|^2 + b\Delta\varphi + a(\partial_t\varphi)^2\} \text{ in } R^n \times [0, \infty),$$

$$\varphi(x, 0) = \varphi_0(x),\ \varphi_t(x, 0) = \varphi_1(x) \text{ in } R^n.$$

In this paper, we shall show that there exists one and only one global solution of P(1) and P(2) when $n = 1, 2$ and 3 if φ_0 and φ_1 are sufficiently small. The exponential decay is also shown for P(1). The proofs are based on the usual energy arguments. To get the energy estimates, we proceed differently for P(1) and P(2). For P(1), Poincare's inequality is applicable and the derivation of the estimates is much easier. However, it is not for P(2) and a more careful computation is required. For the one dimensional case, we use iteratively the

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equation itself, while for the higher dimensional cases, we use Gagliardo-Nirenberg's inequality.

There are several works related to P(1). G. F. Webb [7], P. Avile and J. Sandefur [2], D. D. Ang and A. P. N. Dinh [1] studied the following problem which arises from strongly damped Klein-Gordon equation.

P(3):

$$\partial_{tt}\varphi - \Delta\varphi - \lambda\Delta\varphi_t = f(\varphi) \text{ in } \Omega \times [0, \infty),$$

$$\varphi(x, 0) = \varphi_0(x),\ \varphi_t(x, 0) = \varphi_1(x) \text{ in } \Omega,\ \varphi = 0 \text{ on } \partial\Omega \times [0, \infty),$$

where Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$.

Under some restrctions on f, they have proved that there exists a unique solution which decays exponentially. R. Racke and Y. Shibata [6] have studied a similar problem in the theory of one dimensional nonlinear thermoelasticity which is more complicated than P(3). In [6], they also show, under some conditions on initial data, that there exists a unique solution which decays polynomially.

2. Notation

In this paper, $H_0^1(\Omega)$ and $H^m(\Omega)$ denote the usual Sobolev spaces. $\|\cdot\|$ and $\|\cdot\|_m$ denote the L^2-norm and H^m-norm respectively and $(\ ,\ )$ denotes the L^2 inner product. For a given Banach space X and a positive constant T, we denote by $L^2(0, T; X)$, the space of functions f on $(0, T)$ with values in X such that

$$\left(\int \|f(t)\|_X^2\, dt\right)^{\frac{1}{2}} = \|f\|_{L^2(0,T;X)} < \infty,$$

and by $C([0, T); X)$ the space of continuous functions on $[0, T)$ with values in X. $W^{k,p}(\Omega)$ is also the usual Sobolev space. We define $\|\cdot\|_{W^{k,p}}$ as follows:

$$\|f\|_{W^{k,p}} = \sum_{0 \le |\alpha| \le k} \|D_x^\alpha f\|_{L^p}^p,$$

where $\|\cdot\|_{L^p}$ is the L^p-norm, and

$$D_x^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n},\ \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n),$$

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,\ D_i = \frac{\partial}{\partial x_i}.$$

Finally, we define

$$\|f\|_n^* = \sum_{1 \le |\alpha| \le n} \|D_x^\alpha f\|.$$

3. Global existence for one dimensional case

In this section, we shall study P(1) and P(2) for the one dimensional case.

P(1): (Initial Boundary Value Problem)

$$\varphi_{tt} - c_0^2\varphi_{xx} - b\varphi_{xxt} = (\varphi_x^2 + a\varphi_t^2)_t, \tag{3.1}$$

$$\varphi(x, 0) = \varphi_0(x),\ \varphi_t(x, 0) = \varphi_1(x) \text{ in } \Omega,\ \varphi = 0 \text{ on } \partial\Omega \times [0, \infty), \tag{3.2}$$

where Ω is a finite interval in R^1.

P(2): (Cauchy Problem)

$$\varphi_{tt} - c_0^2\varphi_{xx} - b\varphi_{xxt} = (\varphi_x^2 + a\varphi_t^2)_t, \tag{3.3}$$

$$\varphi(x, 0) = \varphi_0(x),\ \varphi_t(x, 0) = \varphi_1(x)\ \ x \in R^1. \tag{3.4}$$

First, we shall discuss the corresponding two linear problems.

LP(1):

$$\varphi_{tt} - c_0^2\varphi_{xx} - b\varphi_{xxt} = f(x, t), \tag{3.5}$$

$$\varphi(x, 0) = \varphi_0(x),\ \varphi_t(x, 0) = \varphi_1(x) \text{ in } \Omega,\ \varphi = 0 \text{ on } \partial\Omega \times [0, \infty). \tag{3.6}$$

LP(2):

$$\varphi_{tt} - c_0^2\varphi_{xx} - b\varphi_{xxt} = f(x, t), \tag{3.7}$$

$$\varphi(x, 0) = \varphi_0(x),\ \varphi_t(x, 0) = \varphi_1(x)\ \ x \in R^1. \tag{3.8}$$

Then, we have the following theorem for LP(1).

Theorem 3.1. *Suppose* $f \in C([0, T); H^{-1}) \cap L^2(0, T; L^2)$. *And let* $\varphi_0 \in H_0^1 \cap H^2$, $\varphi_1 \in H_0^1$. *Then for each* $T > 0$, *the initial boundary value problem* (3.5) (3.6) *admits a unique solution* $\varphi(x, t)$ *in the following sense*:

I. $\varphi \in C([0, T); H_0^1 \cap H^2)$.

II. $\varphi_t \in C([0, T); H_0^1) \cap L^2(0, T; H^2)$.

III. $\dfrac{d}{dt}(\varphi_t, \psi) + c_0^2\langle\varphi, \psi\rangle + b\langle\varphi_t, \psi\rangle = (f, \psi)$ *for any* $\psi \in H_0^1$,

$\varphi(x, 0) = \varphi_0(x),\ \varphi_t(x, 0) = \varphi_1(x)$

where $\langle\varphi, \psi\rangle = (\varphi_x, \psi_x)$.

Proof. A similar fact was proved in [1]. So we only give a sketch of the proof. Taking the inner product of (3.5) with φ_t and φ_{xxt} in $L^2(\Omega)$, and integrating in t over $(0, T)$ give, after some rearrangements, the following two inequalities.

$$
(3.9)\quad \begin{aligned} &\frac{1}{2}\|\varphi_t\|^2+\frac{c_0^2}{2}\|\varphi_x\|^2+b\int_0^T\|\varphi_{xt}\|^2\mathrm{d}t \\ &\le \frac{1}{2}\int_0^T(\|\varphi_t\|^2+\|f\|^2)\mathrm{d}t+\frac{1}{2}\|\varphi_1\|^2+\frac{c_0^2}{2}\|\varphi_{0x}\|^2. \end{aligned}
$$

$$
(3.10)\quad \begin{aligned} &\frac{1}{2}\|\varphi_{xt}\|^2+\frac{c_0^2}{2}\|\varphi_{xx}\|^2+\frac{b}{2}\int_0^T\|\varphi_{xxt}\|^2\mathrm{d}t \\ &\le \frac{1}{2b}\int_0^T\|f\|^2\mathrm{d}t+\frac{1}{2}\|\varphi_{1x}\|^2+\frac{c_0^2}{2}\|\varphi_{0xx}\|^2. \end{aligned}
$$

Moreover, we note

$$
(3.11)\quad \|\varphi\|^2 \le 2\|\varphi_0\|^2+2\int_0^T\|\varphi_t\|^2\mathrm{d}t.
$$

Combining (3.9), (3.10) and (3.11), and using Grownwall's inequality, we obtain

$$
(3.12)\quad \begin{aligned} &\|\varphi\|_2^2+\|\varphi_t\|_1^2+\int_0^T\|\varphi_{xxt}\|^2\mathrm{d}t \\ &\le C(T)\left(\|\varphi_0\|_2^2+\|\varphi_1\|_1^2+\int_0^T\|f\|^2\mathrm{d}t\right). \end{aligned}
$$

Then, it is easily proved, using a Galerkin approximation scheme, that for each $T>0$ a unique solution exists on $[0, T)$. For the detail, see [1].

Exactly in the same way, we can prove the following theorem for LP(2).

Theorem 3.2. *Suppose* $f\in C([0, T); H^{-1})\cap L^2(0, T; L^2)$. *And let* $\varphi_0\in H^2$, $\varphi_1\in H^1$. *Then for each* $T>0$, *the initial and boundary value problem* (3.7) (3.8) *admits a unique solution* $\varphi(x, t)$ *in the following sense*:

I. $\varphi\in C([0, T); H^2)$.

II. $\varphi_t\in C([0, T); H^1)\cap L^2(0, T; H^2)$.

III. $\frac{d}{dt}(\varphi_t, \varphi)+c_0^2\langle\varphi, \psi\rangle+b\langle\varphi_t, \psi\rangle=(f, \psi)$ *for any* $\psi\in H^1$,

$\varphi(x, 0)=\varphi_0(x)$, $\varphi_t(x, 0)=\varphi_1(x)$

where $\langle\varphi, \psi\rangle=(\varphi_x, \psi_x)$.

Remark. We can get the same results as Theorem 3.1 and Theorem 3.2 for the two and three dimensional cases. We shall use these results in Section 4.

Now, by using Theorem 3.1, we shall prove the local existence.

Theorem 3.3 (Local Existence for P(1)). *There exists a positive number* ε

such that if $\varphi_0 \in H_0^1 \cap H^2$ *and* $\varphi_1 \in H_0^1$ *and if* $\|\varphi_1\|_1^2 \le \varepsilon$, *there exists a constant* $\tau > 0$ *depending only on the norm* $\|\varphi_0\|_2$ *and a unique solution to* (3.1) *and* (3.2) *satisfying*:

$$\varphi \in C([0, \tau); H_0^1 \cap H^2), \tag{3.13}$$
$$\varphi_t \in C([0, \tau); H_0^1) \cap L^2(0, \tau; H^2).$$

Proof. We shall construct local solutions by successive approximations. We first assume that $|\varphi_1| \le 1/4a$ and consider the equation

$$\varphi_{tt}^{(1)} - c_0^2 \varphi_{xx}^{(1)} - b\varphi_{xxt}^{(1)} = 2\varphi_{0x}\varphi_{1x} + 2a\varphi_1 \frac{c_0^2\varphi_{0xx} + b\varphi_{1xx} + 2\varphi_{0x}\varphi_{1x}}{1 - 2a\varphi_1}, \tag{3.14}$$

$$\varphi^{(1)}(x, 0) = \varphi_0(x), \ \varphi_t^{(1)}(x, 0) = \varphi_1(x) \text{ in } \Omega, \ \varphi^{(1)} = 0 \text{ on } \partial\Omega. \tag{3.15}$$

Then for any $T > 0$, a unique solution $\varphi^{(1)}(x, t)$ exists on $[0, T)$ by Theorem 3.1. Suppose by induction $\varphi^{(n+1)}$ is a solution of

$$\varphi_{tt}^{(n+1)} - c_0^2\varphi_{xx}^{(n+1)} - b\varphi_{xxt}^{(n+1)} = 2\varphi_x^{(n)}\varphi_{xt}^{(n)} + 2a\varphi_t^{(n)} \frac{c_0^2\varphi_{xx}^{(n)} + b\varphi_{xxt}^{(n)} + 2\varphi_x^{(n)}\varphi_{xt}^{(n)}}{1 - 2a\varphi_t^{(n)}}, \tag{3.16}$$

$$\varphi^{(n+1)}(x, 0) = \varphi_0(x), \ \varphi_t^{(n+1)}(x, 0) = \varphi_1(x) \text{ in } \Omega, \quad \varphi^{(n+1)} = 0 \text{ on } \partial\Omega. \tag{3.17}$$

If we assume that $|\varphi_t^{(n)}| \le 1/4a$, a unique solution of (3.16) and (3.17) exists by Theorem 3.1.

By (3.12),

$$\begin{aligned}
&\|\varphi^{(n+1)}\|_2^2 + \|\varphi_t^{(n+1)}\|_1^2 + \int_0^T \|\varphi_{xxt}^{(n+1)}\|^2 \,dt \\
&\le 2aC \int_0^T \left\| \varphi_t^{(n)} \frac{c_0^2\varphi_{xx}^{(n)} + b\varphi_{xxt}^{(n)} + 2\varphi_x^{(n)}\varphi_{xt}^{(n)}}{1 - 2a\varphi_t^{(n)}} \right\|^2 dt \\
&\quad + 2C \int_0^T \|\varphi_x^{(n)}\varphi_{xt}^{(n)}\|^2 \,dt + C(\|\varphi_0\|_2^2 + \|\varphi_1\|_1^2) \\
&\le 4aC \int_0^T \|\varphi_t^{(n)}(c_0^2\varphi_{xx}^{(n)} + b\varphi_{xxt}^{(n)} + 2\varphi_x^{(n)}\varphi_{xt}^{(n)})\|^2 \,dt \\
&\quad + C\,(\|\varphi_0\|_2^2 + \|\varphi_1\|_1^2).
\end{aligned} \tag{3.18}$$

The following Sobolev's inequality is well known:

$$\|f\|_\infty = \sup |f| \le C(n) \|f\|_{[\frac{n}{2}]+1} \tag{3.19}$$

Let $D(M, \varepsilon, \tau)$ be a set of functions $\varphi(x, t)$ on $[0, \tau]$ such that

I. $\varphi \in C([0, \tau); H_0^1 \cap H^2)$, $\varphi_t \in C([0, \tau); H_0^1) \cap L^2(0, \tau; H^2)$.

II. $\|\varphi_t\|_1^2 \leq \varepsilon$.

III. $\|\varphi\|_2^2 + \|\varphi_t\|_1^2 + \int_0^\tau \|\varphi_{xxt}\|^2 \mathrm{d}t \leq M$.

Choose ε so small that $\varepsilon \leq C(1)/32a^2$. Then, using (3.18) and (3.19), an easy caluculation shows that there exists a positive constant $\tau(\varepsilon, \|\varphi_0\|_2)$ such that if $\|\varphi_1\|_1^2 \leq \varepsilon$, $\varphi^{(n)}$ belongs to $D(M, \varepsilon, \tau(\varepsilon, M))$ for some positive constant $M(\varepsilon, \|\varphi_0\|_2)$ for all $n = 1, 2, 3, \ldots$, and that $\varphi^{(n)}$ makes a Cauchy sequence in $C([0, \tau); H_0^1 \cap H^2)$ and $\varphi_t^{(n)}$ makes a Cauchy sequence in $C([0, \tau); H_0^1) \cap L^2(0, \tau; H^2)$.

Thus passing to the limit in (3.16), there exists a φ which satisfies (3.1), (3.2) and (3.13). It is immediate to see that there is at most one solution of P(1).

Similarly, we also get the following theorem for P(2).

Theorem 3.4 (Local Existence for P(2)). *There exists a positive number ε such that if $\varphi_0 \in H^2$ and $\varphi_1 \in H^1$ and if $\|\varphi_1\|_1^2 \leq \varepsilon$, there exists a constant $\tau > 0$ depending only on the norm $\|\varphi_0\|_2^*$ and a unique solution to* (3.3) *and* (3.4) *satisying*:

$$\begin{aligned} &\varphi \in C([0, \tau); H^2), \\ &\varphi_t \in C([0, \tau); H^1) \cap L^2(0, \tau; H^2). \end{aligned} \tag{3.20}$$

Remark. In Theorem 3.4, τ does not depend on $\|\varphi_0\|$.

Now we shall state the main theorem in this section.

Theorem 3.5 (Global Existence and Asymptotic Behavior for P(1)). *There exists a constant $\varepsilon > 0$ such that if $\varphi_0 \in H_0^1 \cap H^2$ and $\varphi_1 \in H_0^1$ and if $\|\varphi_0\|_2^2 + \|\varphi_1\|_1^2 \leq \varepsilon$, then a unique solution to* (3.1) *and* (3.2) *exists and satisfies*

$$\begin{aligned} &\varphi \in C([0, \infty); H_0^1 \cap H^2), \\ &\varphi_t \in C([0, \infty); H_0^1) \cap L^2(0, \infty; H^2). \end{aligned} \tag{3.21}$$

Moreover, there exist $M_0 > 0$ and $\gamma > 0$ such that

$$\|\varphi\|_2^2 + \|\varphi_t\|_1^2 \leq M_0 e^{-\gamma t}. \tag{3.22}$$

Proof. To prove (3.21), we must get global estimates. The following idea is due to Prof. T. Nishida. Here we must use the following Poincare's inequality:

$$\|f\| \leq C\|\nabla f\| \text{ if } f = 0 \text{ on } \partial\Omega. \tag{3.23}$$

Taking the inner product of (3.1) with φ_t and φ_{xxt} in $L^2(\Omega)$ and integrating over $(0, T)$ give, after some rearrangements, the following two inequalities.

$$\begin{aligned} &\int \left(\frac{1}{2} - \frac{a}{3}\varphi_t\right)\varphi_t^2 + \frac{c_0^2}{2}\varphi_x^2 \,\mathrm{d}x + b\int_0^T \int |\varphi_{xt}|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &= 2\int_0^T \int \varphi_t \varphi_x \varphi_{xt} \,\mathrm{d}x\,\mathrm{d}t + \int \left(\frac{1}{2} - \frac{a}{3}\varphi_1\right)\varphi_1^2 + \frac{c_0^2}{2}\varphi_{0x}^2 \,\mathrm{d}x. \end{aligned} \tag{3.24}$$

$$\int \frac{1}{2}\varphi_{xt}^2 + \frac{c_0^2}{2}\varphi_{xx}^2\,\mathrm{d}x + b\int_0^T\int|\varphi_{xxt}|^2\,\mathrm{d}x\,\mathrm{d}t.$$

$$= 2\int_0^T\int \varphi_t\varphi_{xt}\varphi_{xxt}\,\mathrm{d}x\,\mathrm{d}t + \int\frac{1}{2}\varphi_{1x}^2 + \frac{c_0^2}{2}\varphi_{0xx}^2\,\mathrm{d}x. \tag{3.25}$$

$$+ 2a\int_0^T\int\varphi_t\varphi_{tt}\varphi_{xxt}\,\mathrm{d}x\,\mathrm{d}t.$$

Then using (3.19) and (3.23), we have the following inequalities from (3.24) and (3.25).

$$\int\left(\frac{1}{2} - \frac{a}{3}\varphi_t\right)\varphi_t^2 + \frac{c_0^2}{2}\varphi_x^2\,\mathrm{d}x + b\int_0^T\|\varphi_{xt}\|^2\,\mathrm{d}t$$

$$\le C\sup_{0\le t\le T}\|\varphi_x\|_1\cdot\int_0^T\|\varphi_{xt}\|^2\,\mathrm{d}t \tag{3.26}$$

$$+\int\left(\frac{1}{2} - \frac{a}{3}\varphi_1\right)\varphi_1^2 + \frac{c_0^2}{2}\varphi_{0x}^2\,\mathrm{d}x.$$

$$\int\left(\frac{1}{2} - C\sup_{0\le t\le T}\|\varphi_t\|_1\right)\varphi_{xt}^2 + \frac{c_0^2}{2}\varphi_{xx}^2\,\mathrm{d}x + b\int_0^T\|\varphi_{xxt}\|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$\le C\sup_{0\le t\le T}\|\varphi_t\|_1\cdot\int_0^T(\|\varphi_{xt}\|^2 + \|\varphi_{xxt}\|^2)\,\mathrm{d}t \tag{3.27}$$

$$+\int C\varphi_{1x}^2 + \frac{c_0^2}{2}\varphi_{0xx}^2\,\mathrm{d}x.$$

So from (3.26) and (3.27), there exist constants $\varepsilon > 0$ and $C > 0$ such that if $\|\varphi_0\|_2^2 + \|\varphi_1\|_1^2 \le \varepsilon$, then

$$\|\varphi(x,t)\|_2^2 + \|\varphi_t(x,t)\|_1^2 \le C(\|\varphi_0\|_2^2 + \|\varphi_1\|_1^2), \quad \text{for all } t \ge 0. \tag{3.28}$$

From the inequality (3.28) and Theorem 3.2, we obtain (3.21). To prove (3.22), we first assume that $|\varphi_t| \le 1/4a$ for all $t \ge 0$. Put $M_1^2 = \|\varphi_0\|_2^2 + \|\varphi_1\|_1^2$. Taking the inner product of (3.1) with $e^{\gamma t}\varphi$, $e^{\gamma t}\varphi_t$, $e^{\gamma t}\varphi_{xx}$ and $e^{\gamma t}\varphi_{xxt}$ in $L^2(\Omega)$, and integrating from 0 to T give, after some rearrangements, and using (3.19) and (3.23)

$$\frac{b}{4}\|\varphi_x\|^2e^{\gamma t} + c_0^2\int_0^T\|\varphi_x\|^2e^{\gamma t}\,\mathrm{d}t$$

$$\le C\gamma\int_0^T(\|\varphi_x\|^2 + \|\varphi_{xt}\|^2)e^{\gamma t}\,\mathrm{d}t + C\|\varphi_{xt}\|^2 \tag{3.29}$$

$$+ C(M_1+1)\int_0^T\|\varphi_{xt}\|^2e^{\gamma t}\,\mathrm{d}t + CM_1^2.$$

$$\begin{aligned}(3.30)\qquad & \frac{1}{3}\|\varphi_t\|^2 e^{\gamma t} + \frac{c_0^2}{2}\|\varphi_x\|^2 e^{\gamma t} + b\int_0^T \|\varphi_{xt}\|^2 e^{\gamma t}\,\mathrm{d}t \\ & \le C\gamma \int_0^T (\|\varphi_x\|^2 + \|\varphi_{xt}\|^2) e^{\gamma t}\,\mathrm{d}t \\ & \quad + CM_1 \int_0^T \|\varphi_{xt}\|^2 e^{\gamma t}\,\mathrm{d}t + CM_1^2.\end{aligned}$$

$$\begin{aligned}(3.31)\qquad & \frac{b}{4}\|\varphi_{xx}\|^2 e^{\gamma t} + \frac{c_0^2}{2}\int_0^T \|\varphi_{xx}\|^2 e^{\gamma t}\,\mathrm{d}t \\ & \le C(M_1+\gamma)\int_0^T (\|\varphi_{xt}\|^2 + \|\varphi_{xx}\|^2) e^{\gamma t}\,\mathrm{d}t + C\|\varphi_{xt}\|^2 \\ & \quad + C\int_0^T (\|\varphi_{xt}\|^2 + \|\varphi_{xxt}\|^2) e^{\gamma t}\,\mathrm{d}t + CM_1^2.\end{aligned}$$

$$\begin{aligned}(3.32)\qquad & \frac{1}{4}\|\varphi_{xt}\|^2 e^{\gamma t} + \frac{c_0^2}{2}\|\varphi_{xx}\|^2 e^{\gamma t} + b\int_0^T \|\varphi_{xxt}\|^2 e^{\gamma t}\,\mathrm{d}t \\ & \le C\gamma \int_0^T (\|\varphi_{xt}\|^2 + \|\varphi_{xx}\|^2) e^{\gamma t}\,\mathrm{d}t \\ & \quad + CM_1 \int_0^T (\|\varphi_{xt}\|^2 + \|\varphi_{xxt}\|^2) e^{\gamma t}\,\mathrm{d}t + CM_1^2.\end{aligned}$$

Combining (3.29), (3.30), (3.31) and (3.32), we have

$$\begin{aligned}(3.33)\qquad & (\|\varphi\|_2^2 + \|\varphi_t\|_1^2) e^{\gamma t} \\ & + \int_0^T (\|\varphi_x\|_1^2 + \|\varphi_{tx}\|_1^2) e^{\gamma t}\,\mathrm{d}t \\ & \le C_1(M_1+\gamma)\int_0^T (\|\varphi_x\|_1^2 + \|\varphi_{tx}\|_1^2) e^{\gamma t}\,\mathrm{d}t + CM_1^2.\end{aligned}$$

If we choose $M_1 > 0$ and $\gamma > 0$ such that $C_1(M_1+\gamma) < 1$, and put a constant $M = C \cdot M_1^2$, we get (3.22) from (3.33).

For P(2), we can't prove the global existence in the same way. The main difficulty is that we can't use Poincare's inequality. However, a nice structure of the equation (3.3) allows us to prove the global existence for P(2).

Theorem 3.6 (Global Existence for P(2)). *There exists a constant $\varepsilon > 0$ such that if $\varphi_0 \in H^2$ and $\varphi_1 \in H^1$ and if $\|\varphi_0\|_2^{*2} + \|\varphi_1\|_1^2 \le \varepsilon$, there exists a unique solution to* (3.3) *and* (3.4) *satisfying*:

$$\varphi \in C([0, \infty); H^2), \quad \varphi_t \in C([0, \infty); H^1) \cap L^2(0, \infty; H^2). \tag{3.34}$$

Proof. Taking the inner product of (3.3) with φ_t and φ_{xxt} in $L^2(\Omega)$ and integrating in t over $(0, T)$ give, just as in the proof of Theorem 3.5, after some rearrangements, the following two inequalities.

$$\begin{aligned}\int\left(\frac{1}{2} - \frac{a}{3}\varphi_t\right)\varphi_t^2 + \frac{c_0^2}{2}\varphi_x^2 \,\mathrm{d}x + b\int_0^T\int|\varphi_{xt}|^2\,\mathrm{d}x\,\mathrm{d}t \\ = 2\int_0^T\int\varphi_t\varphi_x\varphi_{xt}\,\mathrm{d}x\,\mathrm{d}t + \int\left(\frac{1}{2} - \frac{a}{3}\varphi_1\right)\varphi_1^2 + \frac{c_0^2}{2}\varphi_{0x}^2\,\mathrm{d}x.\end{aligned} \tag{3.35}$$

$$\begin{aligned}\int\frac{1}{2}\varphi_{xt}^2 + \frac{c_0^2}{2}\varphi_{xx}^2\,\mathrm{d}x + b\int_0^T\int|\varphi_{xxt}|^2\,\mathrm{d}x\,\mathrm{d}t \\ = 2\int_0^T\int\varphi_x\varphi_{xt}\varphi_{xxt}\,\mathrm{d}x\,\mathrm{d}t + \int\frac{1}{2}\varphi_{1x}^2 + \frac{c_0^2}{2}\varphi_{0xx}^2\,\mathrm{d}x \\ + 2a\int_0^T\int\varphi_t\varphi_{tt}\varphi_{xxt}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.36}$$

In the proof of Theorem 3.5, we used (3.19) (Poincare's inequality) to estimate the nonlinear term $\int_0^T\int\varphi_t\varphi_x\varphi_{xt}\,\mathrm{d}x\,\mathrm{d}t$ in (3.24). Here, we must estimate it more carefully. First, by integration by parts,

$$\begin{aligned}2\int_0^T\int\varphi_t\varphi_x\varphi_{xt}\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int(\varphi_t^2)_x\varphi_x\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\int\varphi_t^2\varphi_{xx}\,\mathrm{d}x\,\mathrm{d}t \\ &= -\frac{1}{c_0^2}\int_0^T\int\varphi_t^2\{\varphi_{tt} - b\varphi_{xxt} - (\varphi_x^2 + a\varphi_t^2)_t\}\,\mathrm{d}x\,\mathrm{d}t \\ &= -\frac{1}{c_0^2}\int_0^T\int\left(\frac{\varphi_t^3}{3}\right)_t - \left(\frac{a\varphi_t^4}{2}\right)_t + 2b\varphi_t\varphi_{xt}^2 - 2\varphi_t^2\varphi_x\varphi_{xt}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.37}$$

The following inequality is easily proved.

$$\|\varphi_t\|_\infty^2 \le 2\|\varphi_t\|\,\|\varphi_{xt}\|. \tag{3.38}$$

Using (3.38) to (3.37) give, after some rearrangements,

$$\begin{aligned}2\left|\int_0^T\int\varphi_t\varphi_x\varphi_{xt}\,\mathrm{d}x\,\mathrm{d}t\right| &\le \left|\frac{1}{c_0^2}\left(\int\frac{\varphi_t^3}{3} - \frac{a\varphi_t^4}{2}\,\mathrm{d}x - \int\frac{\varphi_1^3}{3} - \frac{a\varphi_1^4}{2}\,\mathrm{d}x\right)\right| \\ &+ \frac{4}{c_0^2}\sup_{0\le t\le T}(\|\varphi_t\|\,\|\varphi_x\|)\cdot\int_0^T\|\varphi_{xt}\|^2\,\mathrm{d}t \\ &+ \frac{2b}{c_0^2}\sup_{0\le t\le T}(\|\varphi_t\|_\infty)\cdot\int_0^T\|\varphi_{xt}\|^2\,\mathrm{d}t.\end{aligned} \tag{3.39}$$

So from (3.36) and (3.39), there exist constants $\varepsilon > 0$ and $C^* > 0$ such that if $\|\varphi_0\|_2^{*2} + \|\varphi_1\|_1^2 \leq \varepsilon$, then

$$\|\varphi(x, t)\|_2^{*2} + \|\varphi_t(x, t)\|_1^2 \leq C^*(\|\varphi_0\|_2^{*2} + \|\varphi_1\|_1^2), \tag{3.40}$$

for all $t \geq 0$. Now, (3.34) follows from the inequality (3.40) and Theorem 3.4.

4. Global existence for two and three dimensional cases

In this section, we shall study P(1) and P(2) for the two and three dimensional cases.

P(1): (Initial Boundary Value Problem)

$$\varphi_{tt} - c_0^2 \Delta \varphi - b \Delta \varphi_t = (|\nabla \varphi|^2 + a\varphi_t^2)_t, \tag{4.1}$$

$$\varphi(x, 0) = \varphi_0(x), \ \varphi_t(x, 0) = \varphi_1(x) \text{ in } \Omega, \ \varphi = 0 \text{ on } \partial\Omega \times [0, \infty), \tag{4.2}$$

where Ω is a bounded domain in R^n ($n = 2, 3$) with a smooth boundary $\partial\Omega$.

P(2): (Cauchy Problem)

$$\varphi_{tt} - c_0^2 \Delta \varphi - b \Delta \varphi_t = (|\nabla \varphi|^2 + a\varphi_t^2)_t, \tag{4.3}$$

$$\varphi(x, 0) = \varphi_0(x), \ \varphi_t(x, 0) = \varphi_1(x) \ x \in R^n \ n = 2, 3. \tag{4.4}$$

As in the previous section, we shall start from the corresponding linear problems.

LP(1):

$$\varphi_{tt} - c_0^2 \Delta \varphi - b \Delta \varphi_t = f(x, t), \tag{4.5}$$

$$\varphi(x, 0) = \varphi_0(x), \ \varphi_t(x, 0) = \varphi_1(x) \text{ in } \Omega, \ \varphi = 0 \text{ on } \partial\Omega \times [0, \infty). \tag{4.6}$$

LP(2):

$$\varphi_{tt} - c_0^2 \Delta \varphi - b \Delta \varphi_t = f(x, t), \tag{4.7}$$

$$\varphi(x, 0) = \varphi_0(x), \ \varphi_t(x, 0) = \varphi_1(x) \ x \in R^n \ n = 2, 3. \tag{4.8}$$

Theorem 4.1. *Suppose* $f \in C([0, T); L^2) \cap L^2(0, T; H^1)$ *and* $f_t \in C([0, T); H^{-1}) \cap L^2(0, T; L^2)$. *And let* $\varphi_0 \in H_0^1 \cap H^3$, $\varphi_1 \in H_0^1 \cap H^2$. *Then for each* $T > 0$, *the initial boundary value problem admits a unique solution* $\varphi(x, t)$ *to* (4.5) *and* (4.6) *satisfying*:

$$\varphi \in C([0, T); H_0^1 \cap H^3),$$
$$\varphi_t \in C([0, T); H_0^1 \cap H^2) \cap L^2(0, T; H^3).$$

Proof. From the remark after Theoem 3.1, there exists a unique solution to (4.5) amd (4.6) satisfying

$$\varphi \in C([0, T); H_0^1 \cap H^2),$$
$$\varphi_t \in C([0, T); H_0^1) \cap L^2(0, T; H^2).$$

If we differentiate (4.5) with respect to t, we get

$$\varphi_{ttt} - c_0^2 \Delta \varphi_t - b \Delta \varphi_{tt} = f_t.$$

Let us consider the following linear problem.

$$v_{tt} - c_0^2 \Delta v - b \Delta v_t = f_t(x, t), \tag{4.9}$$

$$v(x, 0) \in H_0^1 \cap H^2, \; v_t(x, 0) \in L^2, \; v = 0 \text{ on } \partial\Omega. \tag{4.10}$$

Then there exists a unique solution to (4.9) and (4.10) satisfying

$$v \in C([0, T); H_0^1 \cap H^2),$$
$$v_t \in C([0, T); L^2) \cap L^2(0, T; H_0^1).$$

The proof of the above fact is similar to that of Theorem 3.1.

Let $v(x, 0) = \varphi_1$, $v_t(x, 0) = \varphi_{tt}(x, 0) = c_0^2 \Delta \varphi_0 + b \Delta \varphi_1 + f(x, 0)$ and $w = \varphi_0 + \int_0^t v(x, \tau)\, d\tau$. Integrating (4.9) from 0 to t, we get

$$w_{tt} - c_0^2 \Delta w - b \Delta w_t = f, \; w(x, 0) = \varphi_0, \; w_t(x, 0) = \varphi_1.$$

From (4.5) and (4.6), we can conclude that $w \equiv \varphi$. Then, it follows that

$$\varphi_t \in C([0, T); H_0^1 \cap H^2),$$
$$\varphi_{tt} \in C([0, T); L^2) \cap L^2(0, T; H_0^1).$$

Now we must obtain more x differentiability for φ. We do this using the theory of elliptic boundary value problems.

From (4.5),

$$c_0^2 \Delta \varphi + b \Delta \varphi_t = f - \varphi_{tt}. \tag{4.11}$$

From above arguments, the right side of (4.11) belongs to $L^2(0, T; H^1)$. If we put $\Delta \varphi = h$ and $f - \varphi_{tt} = H$,

$$h_t + \frac{c_0^2}{b} h = H.$$

Then,

$$h = e^{-\frac{c_0^2}{b} t} h(0) + \int_0^t e^{-\frac{c_0^2}{b}(t-s)} H(s)\, ds. \tag{4.12}$$

Then an easy computation shows that h belongs to H^1. As $\partial\Omega$ is smooth, φ belongs to H^3. So φ_t belongs to $L^2(0, T; H^3)$.

We also have the following theorem for LP(2) in the same way.

Theorem 4.2. *Suppose* $f \in C([0, T); L^2) \cap L^2(0, T; H^1)$ *and* $f_t \in C([0, T); H^{-1})$

$\cap L^2(0, T; L^2)$. *And let* $\varphi_0 \in H^3$, $\varphi_1 \in H^2$. *Then for each* $T > 0$, *the initial boundary value problem admits a unique solution* $\varphi(x, t)$ *to* (4.7) *and* (4.8) *satisfying*:

$$\varphi \in C([0, T); H^3),$$
$$\varphi_t \in C([0, T); H^2) \cap L^2(0, T; H^3).$$

Now we shall state the local results for P(1) and P(2).

Theorem 4.3 (Local Existence for P(1). *There exists a positive number* ε *such that if* $\varphi_0 \in H_0^1 \cap H^3$ *and* $\varphi_1 \in H_0^1 \cap H^2$ *and if* $\|\varphi_1\|_2^2 \leq \varepsilon$, *there exists a constant* $\tau > 0$ *depending only on the norm* $\|\varphi_0\|_3$ *and a unique solution to* (4.1) *and* (4.2) *satisfying*:

$$\varphi \in C([0, \tau); H_0^1 \cap H^3),$$
$$\varphi_t \in C([0, \tau); H_0^1 \cap H^2) \cap L^2(0, \tau; H^3).$$

Theorem 4.4 (local Existence for P(2)). *There exists a positive number* ε *such that if* $\varphi_0 \in H^3$ *and* $\varphi_1 \in H^2$ *and if* $\|\varphi_1\|_2^2 \leq \varepsilon$, *there exists a constant* $\tau > 0$ *depending only on the norm* $\|\varphi_0\|_3^*$ *and a unique solution to* (4.3) *and* (4.4) *satisfying*:

$$\varphi \in C([0, \tau); H^3),$$
$$\varphi_t \in C([0, \tau); H^2) \cap L^2(0, \tau; H^3).$$

Using Theorem 4.1 and Theorem 4.2, the proof of these theorems is almost similar to that of Theorem 3.2. But in order to get local estimates, we must use Gagliardo-Nirenberg's inequality. The derivation of local estimates is almost similar to that of global estimates in the next theorem.

Now we shall state the main theorem in this section.

Theorem 4.5 (Global Existence and Asymptotic Behavior for P(1)). *There exists a constant* $\varepsilon > 0$ *such that if* $\varphi_0 \in H_0^1 \cap H^3$ *and* $\varphi_1 \in H_0^1 \cap H^2$ *and if* $\|\varphi_0\|_3^2 + \|\varphi_1\|_2^2 \leq \varepsilon$, *there exists a unique solution to* (4.1) *and* (4.2) *satisfying*:

$$\begin{aligned} &\varphi \in C([0, \infty); H_0^1 \cap H^3), \\ &\varphi_t \in C([0, \infty); H_0^1 \cap H^2) \cap L^2(0, \infty; H^3). \end{aligned} \tag{4.13}$$

Moreover, there exist $M_0' > 0$ *and* $\gamma' > 0$ *such that the asymptotic behavior is given by*

$$\|\varphi\|_3^2 + \|\varphi_t\|_2^2 \leq M_0' e^{-\gamma' t}. \tag{4.14}$$

Proof. In order to prove Theorem 4.3, we must use the following well-known inequality.

Lemma 4.6 (Gagliardo-Nirenberg).

$$\|u\|_{W^{k,p}} \leq C \|u\|_{W^{m,q}}^{\theta} \cdot \|u\|_{L^r}^{1-\theta} \tag{4.15}$$

if $p \geq q$, $p \geq r$, $0 \leq \theta \leq 1$ *and*

$$k - \frac{n}{p} \leq \theta\left(m - \frac{n}{q}\right) - \frac{n(1-\theta)}{r}$$

with strict inequality if q or r = 1.

$$\text{Set } \varphi_0^* = \varphi_{tt}(x, 0) = \frac{c_0^2 \Delta\varphi_0 - b\Delta\varphi_1 + 2\varphi_0\varphi_1}{1 - 2a\varphi_1}.$$

Taking the inner product of (4.1) with φ_t, $\Delta\varphi$ and $\Delta\varphi_t$ in $L^2(\Omega)$ and integrating over $(0, T)$ give, after some rearrangements, the following three inequalities.

$$\begin{aligned} &\left(\frac{1}{2} - \frac{a}{3}\varphi_t\right)\|\varphi_t\|^2 + \frac{c_0^2}{2}\|\nabla\varphi\|^2 + b\int_0^T \|\nabla\varphi_t\|^2 \,\mathrm{d}t \\ &= \left(\frac{1}{2} - \frac{a}{3}\varphi_1\right)\|\varphi_1\|^2 + \frac{c_0^2}{2}\|\nabla\varphi_0\|^2 \\ &+ 2\int_0^T \int \varphi_t \nabla\varphi \cdot \nabla\varphi_t \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.16}$$

$$\begin{aligned} &\frac{b}{2}\|\Delta\varphi\|^2 + c_0^2 \int_0^T \|\Delta\varphi\|^2 \,\mathrm{d}t \\ &= \int_0^T \int - \varphi_{tt}\Delta\varphi + 2\nabla\varphi_t \cdot \nabla\varphi\Delta\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \int 2a\varphi_t\varphi_{tt}\Delta\varphi \,\mathrm{d}x\,\mathrm{d}t + \frac{b}{2}\|\Delta\varphi_0\|^2. \end{aligned} \tag{4.17}$$

$$\begin{aligned} &\frac{1}{2}\|\nabla\varphi_t\|^2 + \frac{c_0^2}{2}\|\Delta\varphi\|^2 + b\int_0^T \|\Delta\varphi_t\|^2 \,\mathrm{d}t \\ &= \frac{1}{2}\|\nabla\varphi_1\|^2 + \frac{c_0^2}{2}\|\Delta\varphi_0\|^2 \\ &+ 2\int_0^T \int \nabla\varphi_t \cdot \nabla\varphi\Delta\varphi_t \,\mathrm{d}x\,\mathrm{d}t + 2a\int_0^T \int \varphi_t\varphi_{tt}\Delta\varphi_t \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.18}$$

By differentiation of (4.1) with respect to t,

$$\varphi_{ttt} - c_0^2\Delta\varphi_t - b\Delta\varphi_{tt} = (a\varphi_t^2 + |\nabla\varphi|^2)_{tt}. \tag{4.19}$$

Taking the inner product of (4.19) with φ_{tt} in $L^2(\Omega)$ and integrating from 0 to T give, after some rearrangements,

$$\left(\frac{1}{2} - a\varphi_t\right)\|\varphi_{tt}\|^2 + \frac{c_0^2}{2}\|\nabla\varphi_t\|^2 + b\int_0^T \|\nabla\varphi_{tt}\|^2 \,\mathrm{d}t$$

$$
\begin{aligned}
&= \left(\frac{1}{2} - a\varphi_1\right)\|\varphi_0^*\|^2 + \frac{c_0^2}{2}\|\varphi_1\|^2 \\
(4.20) \quad &+ 2\int_0^T\int \nabla\varphi_t \cdot \nabla\varphi_t\varphi_{tt}\,\mathrm{d}x\,\mathrm{d}t + 2\int_0^T\int \nabla\varphi \cdot \nabla\varphi_{tt}\varphi_{tt}\,\mathrm{d}x\,\mathrm{d}t \\
&+ a\int_0^T\int \varphi_{tt}^3\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Differentiation of (4.1) with respect to x yields

$$
(4.21) \qquad D_x\varphi_{tt} - c_0^2 D_x\Delta\varphi - bD_x\Delta\varphi_t = D_x(|\nabla\varphi|^2 + a\varphi_t^2)_t.
$$

Taking the inner product of (4.21) with $D_x\Delta\varphi$ and $D_x\Delta\varphi_t$ in $L^2(\Omega)$, and integrating from 0 to T give, after some rearrangements,

$$
\begin{aligned}
&\frac{b}{2}\|D_x\Delta\varphi\|^2 + \frac{c_0^2}{2}\int_0^T \|D_x\Delta\varphi\|^2\,\mathrm{d}t \\
&\le \frac{4}{c_0^2}\int_0^T \|D_x\varphi_{tt}\|^2\,\mathrm{d}t + C\int_0^T\int (|\Delta\varphi|^4 + |\nabla\varphi_t|^4)\,\mathrm{d}x\,\mathrm{d}t \\
(4.22) \quad &+ C\int_0^T\int |\varphi_{tt}|^3\,\mathrm{d}x\,\mathrm{d}t + C\int_0^T\int |\nabla\varphi_t|^6\,\mathrm{d}x\,\mathrm{d}t \\
&+ C\int_0^T\int |\varphi_t D_x\varphi_{tt}|^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

$$
\begin{aligned}
&\frac{c_0^2}{2}\|D_x\Delta\varphi\|^2 + \frac{b}{2}\int_0^T \|D_x\Delta\varphi_t\|^2\,\mathrm{d}t \\
&\le \frac{4}{b}\int_0^T \|D_x\varphi_{tt}\|^2\,\mathrm{d}t + C\int_0^T\int (|\Delta\varphi|^4 + |\nabla\varphi_t|^4)\,\mathrm{d}x\,\mathrm{d}t \\
(4.23) \quad &+ C\int_0^T\int |\varphi_{tt}|^3\,\mathrm{d}x\,\mathrm{d}t + C\int_0^T\int |\nabla\varphi_t|^6\,\mathrm{d}x\,\mathrm{d}t \\
&+ C\int_0^T\int |\varphi_t D_x\varphi_{tt}|^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Using Lemma 4.6, we have

$$
\begin{aligned}
&\int_0^T\int |\varphi_{tt}|^3\,\mathrm{d}x\,\mathrm{d}t \le C\int_0^T \|\varphi_{tt}\|_1^2\|\varphi_{tt}\|\,\mathrm{d}t \\
(4.24) \quad &\le C(\sup_{0\le t\le T}\|\varphi_{tt}\|)\cdot\int_0^T \|\varphi_{tt}\|_1^2\,\mathrm{d}t.
\end{aligned}
$$

$$(4.25)\qquad \begin{aligned}\int_0^T\int |\Delta\varphi|^4\,\mathrm{d}x\,\mathrm{d}t &\le C\int_0^T \|\Delta\varphi\|_1^3\|\Delta\varphi\|\,\mathrm{d}t\\ &\le C(\sup_{0\le t\le T}\|\Delta\varphi\|_1\|\Delta\varphi\|)\cdot\int_0^T\|\Delta\varphi\|_1^2\,\mathrm{d}t.\end{aligned}$$

$$(4.26)\qquad \begin{aligned}\int_0^T\int |\nabla\varphi_t|^4\,\mathrm{d}x\,\mathrm{d}t &\le C\int_0^T \|\nabla\varphi_t\|_1^3\|\nabla\varphi_t\|\,\mathrm{d}t\\ &\le C(\sup_{0\le t\le T}\|\nabla\varphi_t\|_1\|\nabla\varphi_t\|)\cdot\int_0^T\|\nabla\varphi_t\|_1^2\,\mathrm{d}t.\end{aligned}$$

$$(4.27)\qquad \begin{aligned}\int_0^T\int |\nabla\varphi_t|^6\,\mathrm{d}x\,\mathrm{d}t &\le C\int_0^T \|\nabla\varphi_t\|_1^6\,\mathrm{d}t\\ &\le C(\sup_{0\le t\le T}\|\nabla\varphi_t\|_1^4)\cdot\int_0^T\|\nabla\varphi_t\|_1^2\,\mathrm{d}t.\end{aligned}$$

Applying (3.19), (3.23), (4.24), (4.25), (4.26) and (4.27) to (4.16), (4.17), (4.18), (4.20), (4.22) and (4.23), and then combining these inequalities, we obtain in the same way as Theorem 3.3 that there exist constants $\varepsilon>0$ and $C>0$ such that if $\|\varphi_0\|_3^2+\|\varphi_1\|_2^2\le\varepsilon$, then

$$(4.28)\qquad \|\varphi(x,t)\|_3^2+\|\varphi_t(x,t)\|_2^2\le C(\|\varphi_0\|_3^2+\|\varphi_1\|_2^2),$$

for all $t\ge 0$. From the inequality (4.28) and Theorem 4.2, we obtain (4.13). The proof of (4.14) is of almost the same type as that of Theorem 3.3.

Without using (3.23) (Poincare's inequality), we can obtain local estimates which are necessary for the proof of Theorem 4.3 and Theorem 4.4. But we can't prove the global existence of the solution of (5.3) and (5.4) just in the same way as in Section 3.

Theorem 4.7 (Global Existence for P(2)). *There exists a constant* $\varepsilon>0$ *such that if* $\varphi_0\in H^3$ *and* $\varphi_1\in H^2$ *and if* $\|\varphi_0\|_3^{*2}+\|\varphi_1\|_2^2\le\varepsilon$, *there exists a unique solution to* (4.3) *and* (4.4) *satisfying*:

$$(4.29)\qquad \begin{aligned}&\varphi\in C([0,\infty);H^3),\\ &\varphi_t\in C([0,\infty;H^2)\cap L^2(0,\infty;H^3).\end{aligned}$$

Proof. Just as in Theorem 4.5, we must use (4.16), (4.17), (4.18), (4.20), (4.23) and (4.23). In Theorem 4.5, we used (3.19) (Poincare's inequality) to estimate the nonlinear term $\int_0^T\int\nabla\varphi_t\cdot\nabla\varphi\varphi_t\,\mathrm{d}x\,\mathrm{d}t$ in (4.16). Here, we must proceed more carefully.

First, we shall estimate it for the two dimensional case. The following inequality can be proved easily.

Lemma 4.8. *For all $\varphi \in H^1(R^2)$, the following inequality holds*:

$$\int_{R^2} \varphi^4 \, dx \le 2 \|\varphi\|^2 \cdot \|\nabla\varphi\|^2. \tag{4.30}$$

An easy computation shows that

$$\left|\int_0^T \int \nabla\varphi_t \cdot \nabla\varphi\varphi_t \, dx \, dt\right| \le \frac{b}{4}\int_0^T \|\nabla\varphi_t\|^2 dt + \frac{1}{b}\int_0^T \int |\nabla\varphi|^2 \varphi_t^2 \, dx \, dt$$

$$\le \frac{b}{4}\int_0^T \|\nabla\varphi_t\|^2 dt + \frac{1}{b}\int_0^T \left(\int |\nabla\varphi|^4 dx\right)^{\frac{1}{2}} \left(\int |\varphi_t|^4 dx\right)^{\frac{1}{2}} dt.$$

Using Lemma 4.8, we get

$$\le \frac{b}{4}\int_0^T \|\nabla\varphi_t\|^2 dt + \frac{1}{2b}\int_0^T (\|\varphi_t\|^2 \|\nabla\varphi_t\|^2)^{\frac{1}{2}} \cdot (\|\nabla\nabla\varphi\|^2 \|\nabla\varphi\|^2)^{\frac{1}{2}} dt.$$

Finally, after some rearrangements,

$$\left|\int_0^T \int \nabla\varphi_t \cdot \nabla\varphi\varphi_t \, dx \, dt\right|$$

$$\le \frac{b}{4}\int_0^T \|\nabla\varphi_t\|^2 dt + \frac{1}{4b}\sup_{0\le t\le T} \|\varphi_t\|^2 \cdot \int_0^T \|\nabla\varphi_t\|^2 dt \tag{4.31}$$

$$+ C \sup_{0\le t\le T} \|\nabla\varphi\|^2 \cdot \int_0^T \|\Delta\varphi\|^2 dt.$$

Now we shall estimate it for the three dimensional case. In this case, we must use the following well-known inequality.

Lemma 4.9 (Sobolev's inequality).

$$\|\varphi\|_{L^p} \le C(n, q) \|\nabla\varphi\|_{L^q} \quad \textit{with } \frac{1}{p} = \frac{1}{q} - \frac{1}{n} \textit{ and } 1 < q < n. \tag{4.32}$$

If we put $p = 6$, $q = 2$ and $n = 3$ in (4.32),

$$\|\varphi\|_{L^6} \le C \|\nabla\varphi\|. \tag{4.33}$$

If we use (4.33), we get

$$\left|\int_0^T \int \nabla\varphi_t \cdot \nabla\varphi\varphi_t \, dx \, dt\right|$$

$$\le \int_0^T \|\nabla\varphi_t\| \cdot \left(\int |\varphi_t|^6 dx\right)^{\frac{1}{6}} \cdot \left(\int |\nabla\varphi|^3 dx\right)^{\frac{1}{3}} dt$$

$$\leq C\int_0^T \|\nabla\varphi_t\|^2\cdot\left(\int|\nabla\varphi|^3\,dx\right)^{\frac{1}{3}}dt.$$

$$\leq C\int_0^T \|\nabla\varphi_t\|^2\cdot(\|\nabla\varphi\|_1^2\,\|\nabla\varphi\|)^{\frac{1}{3}}dt. \qquad \text{(by Lemma 4.6)}$$

Finally,

$$\left|\int_0^T\int\nabla\varphi_t\cdot\nabla\varphi\varphi_t\,dx\,dt\right| \tag{4.34}$$
$$\leq C\sup_{0\leq t\leq T}(\|\nabla\varphi\|_1^2\,\|\nabla\varphi\|)^{\frac{1}{3}}\int_0^T\|\nabla\varphi_t\|^2\,dt.$$

We also used (3.19) (Poincare's inequality) to estimate $\int\Delta\varphi\varphi_{tt}\,dx$ in (4.17). Here, we use the integration by parts.

$$\int\Delta\varphi\varphi_{tt}\,dx = \int\varphi_t\Delta\varphi\,dx - \int\varphi_1\Delta\varphi_0\,dx - \int_0^T\int\varphi_t\Delta\varphi_t\,dx\,dt$$

$$= \int\varphi_t\Delta\varphi\,dx - \int\varphi_1\Delta\varphi_0\,dx + \int_0^T\int|\nabla\varphi_t|^2\,dx\,dt.$$

Finally,

$$\left|\int_0^T\int\varphi_{tt}\Delta\varphi\,dx\,dt\right| \leq \frac{b}{4}\|\Delta\varphi\|^2 + \frac{4}{b}\|\varphi_t\|^2 \tag{4.35}$$
$$+\int_0^T\|\nabla\varphi_t\|^2\,dt + \left|\int\varphi_1\Delta\varphi_0\,dx\right|.$$

We also used (3.19) (Poincare's Inequality) to estimate φ_{tt} in (4.17), (4.20), (4.22) and (4.23). Here we use (4.3),

$$\|\varphi_{tt}\|^2 \leq C(\|\Delta\varphi\|^2 + \|\Delta\varphi_t\|^2 + (\sup|\nabla\varphi|^2)\cdot\|\nabla\varphi_t\|^2). \tag{4.36}$$

Applying (4.31) (or (4.34)), (4.35) and (4.36) to (4.16), (4.17), (4.20), (4.22) and (4.23), we obtain, just as Theorem 4.6, constants $\varepsilon > 0$ and $C^* > 0$ such that if $\|\varphi_0\|_3^{*2} + \|\varphi_1\|_2^2 \leq \varepsilon$, then

$$\|\varphi(x, t)\|_3^{*2} + \|\varphi_t(x, t)\|_2^2 \leq C^*(\|\varphi_0\|_3^{*2} + \|\varphi_1\|_2^2), \tag{4.37}$$

for all $t \geq 0$. From the inequality (4.37) and Theorem 4.4, we obtain (4.29).

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