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Note on the behavior spaces on open Riemann surfaces and its applications to multiplicative differentials

By

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§ 1. Introduction

1A. Let R be an arbitrary Riemann surface of genus g which may be infinity and $\{R_n\}$ a canonical exhaustion of R, then we can choose a canonical homology basis $\{A_j, B_j\}_{j=1}^g$ modulo dividing curves such that $\{A_j, B_j\} \cap G_n^k$ is also a canonical homology basis of G_n^k modulo ∂G_n^k for each n and k, where G_n^k denotes a component of $R_{n+1} - \bar{R}_n$ (Ahlfors and Sario [1]). Further, let J be the set which consists of integers $1, 2, \ldots, g$ and $D_K = D_K(J)$ denote a patition of J into mutually disjoint subsets $J_1, J_2, \ldots, J_K$ so that $J = \bigcup_{k=1}^{K} J_k$ and $2 \leqq K \leqq g$. The totality of square integrable complex (resp. real) differentials on R forms a real Hilbert space $\Lambda = \Lambda(R)$ (resp. $\Gamma = \Gamma(R)$) over the real number field with the real Dirichlet inner product. It should be noticed that the meanings of the letter Λ and Γ are different from those in [1]. With these exceptions, we inherit the terminologies and the notations of [1], if not mentioned further. For example, Λ_h, Λ_{hse}, Λ_a, $\Lambda_{ase}, \ldots$ (resp. Γ_h, Γ_{hse}, $\Gamma_{he}, \ldots$) stand for the real Hilbert spaces of complex (resp. real) differentials on R with corresponding restricted properties. Moreover, for simplicity, we use in this paper the following notations and terminologies:

$$\int_{J_p} \Lambda_x = \left\{ \int_{A_j, B_j} \lambda : \lambda \in \Lambda_x \text{ and } j \in J_p \right\}, \text{ where } \Lambda_x \text{ is a subspace of } \Lambda_h,$$

$$\int_{A} \Lambda_x = \left\{ \int_{A_j} \lambda : \lambda \in \Lambda_x \text{ and } j \in J \right\},$$

$L_K = \{L_p\}_{p=1}^{K}$, *where each* L_p *is a straight line on the complex plane* $\mathbf{C}$ *passing through the origin,* $L_p \neq L_q$ *for* $p \neq q$ *and* L_1 *is the real axis,*

$S(J_p) =$ *the space spanned by* $\{\sigma(A_j), \sigma(B_j) : j \in J_p\}$ *over the real number field where* $\sigma(\gamma)$ *is the* γ*-reproducer in* Γ_h*, that is to say,* $\int_\gamma \omega = \langle \omega, \sigma(\gamma)^* \rangle$ *for each* $\omega \in \Gamma_h$,

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$S(A)$ = the space spanned by $\{\sigma(A_j), j \in J\}$ over the real number field,
$\Gamma_1^\perp$ (resp. $\Lambda_1^\perp$) = the orthogonal complement of Γ_1 (resp. Λ_1) in Γ_h (resp. Λ_h) for any subspace Γ_1 (resp. Λ_1) in Γ_h (resp. Λ_h).

Definition 1 (Cf. [2], [6] and [9]). *According to* [9], *we call, in this paper, a closed subspace $\Lambda_K = \Lambda(D_K, \boldsymbol{L}_K)$ of Λ_h an S-behavior space associated with $(D_K, \boldsymbol{L}_K) = (D_K, \{L_p\}_{p=1}^K)$ if the following conditions are satisfied*:

(1) $\Lambda_K = i\Lambda_K^{*\perp} \subset \Lambda_{hse}$ *where* $i = \sqrt{-1}$,

(2) $\int_{J_p} \Lambda_K \in L_p$ *for each* p,

where $\Lambda_K^* = \{\lambda$: the conjugate differential λ^* of λ belongs to $\Lambda_K\}$. Analogously, we call, in this paper, a subspace Λ_B of Λ_h a B-behavior space if Λ_B satisfies the following conditions (Cf. [2]):

(1) $\Lambda_B = i\Lambda_B^{*\perp} \subset \Lambda_{hse}$,

(2) $\int_A \Lambda_B = 0$.

Hereafter, we denote S-behavior space (resp. B-behavior space) simply by S-space (resp. B-space).

Definition 2 (Cf. [9]). *Let Λ_0 be an S-space (or B-space). We call, in this paper, a meromorphic differential ϕ on R has Λ_0-behavior if there exists a compact region $\bar{D}$, $\lambda \in \Lambda_0$ and $\lambda_{e0} \in \Lambda_{e0} \cap \Lambda^1$ such that*

$$\phi = \lambda + \lambda_{e0} \text{ on } R - \bar{D}.$$

A single valued meromorphic function f on R is called, in this paper, to have Λ_0-behavior if df has Λ_0-behavior.

1B. The generalization of the Riemann-Roch theorem in the classical theory of algebraic functions to open Riemann surfaces were studied, at first, by Kusunoki [3] and, afterwards, along his method by many authors, for examples, Baskan [2], Matsui and Nishida [7], Mizumoto [8], Shiba [9] and Yoshida [10]. Above all, Shiba's theorem in [9], an extension of those in [3], [8] and [10], were formulated in terms of differentials with S-behavior, and further, Baskan's result formulated by B-space is somewhat different from [9]. Accordingly, we have the special interest about the notions of S-space and B-space. Whereas we do not know the general existences of these spaces yet, though some restrictive examples of S-spaces were given by [5], [6] and [9]. In this paper, we shall give, in §2, some classes of B-spaces and their application to the Abelian integral theory. In §3, we show some classes of S-spaces $\Lambda_2 = \Lambda(D_2, \boldsymbol{L}_2)$ associated with $(D_2(J), \boldsymbol{L}_2)$. In §4, we consider for arbitrary given $(D_K, \boldsymbol{L}_K)$ $(K \geqq 3)$, a sequence $\{(D_K^n, \boldsymbol{L}_K^n)\}_{n=1}^\infty$ which are properly constructed from $(D_K, \boldsymbol{L}_K)$ and a sequence of the S-spaces $\{\Lambda_n\}_{n=1}^\infty$ with $\Lambda_n = \Lambda(D_K^n, \boldsymbol{L}_K^n)$ (Cf. [5]), and give a condition that the limit of the sequence $\{\Lambda_n\}_{n=1}^{\infty}$ is a behavior space associated with (D_K, L_K). In §5, we consider certain class of open symmetric Riemann surfaces, and on such a surface we show a formulation of a duality theorem (Riemann-Roch type theorem) for multiplicative differentials which is, in case that the surface is symmetric closed, different from Prym-Weyl's theorem (Cf. Weyl [11]). The author wishes to express his hearty thanks to Prof. Y. Kusunoki for his valuable suggestions and ceaseless encouragements.

§2. *B*-behavior space

2A. Existence. In the following, suppose $\Gamma_{hse} \neq \{0\}$.

Lemma 1. *Let Γ_1, Γ_2 be arbitrary subspaces of Γ_{hse}, and set $\Gamma'_k = \Gamma_k^{*\perp}$, $k = 1, 2$. For each complex number z such that $z = e^{i\theta} \neq \pm 1$, we have*

(1) *$\Lambda_0 = \Gamma_1 + z\Gamma_2$ is a closed subspace in Λ_h,*

(2) *a differential λ belongs to the spaces $i\Lambda_0^{*\perp}$ if and only if λ can be written in a form $\lambda = (z\omega'_1 - \omega'_2)/\mathrm{Im}(z)$ where $\omega'_1 = \mathrm{Im}(\lambda) \in \Gamma'_1$ and $\omega'_2 = \mathrm{Im}(\bar{z}\lambda) \in \Gamma'_2$,*

(3) *analytic differential ϕ belongs to $\Lambda_a \cap i\Lambda_0^{*\perp} \cap \Lambda_0^{\perp}$ if and only if* $\mathrm{Im}(\phi) \in \Gamma'_1$ *and* $\mathrm{Im}(\bar{z}\phi) = \mathrm{Im}\{(\phi)\cos\theta - \mathrm{Re}(\phi)\sin\theta\} \in \Gamma'_2$

(4) *if $\Gamma_1 \perp \Gamma_2^*$, we have*

$$\Lambda_h = \Lambda_0 \dotplus i\Lambda_0^* \dotplus \Lambda_a \cap \Lambda_0^{\perp} \cap i\Lambda_0^{*\perp} \dotplus \Lambda_{\bar{a}} \cap \Lambda_0^{\perp} \cap i\Lambda_0^{*\perp}.$$

where $\Lambda_{\bar{a}} = \{\phi$: complex conjugate $\bar{\phi}$ of ϕ belongs to $\Lambda_a\}$.

Proof. (1). Suppose the sequence $\{\lambda_n = x_n + zy_n\}_{n=1}^{\infty}$ with $x_n \in \Gamma_1$ and $y_n \in \Gamma_2$ is a Cauchy sequence whose limit is λ. Since $\{\mathrm{Im}(\lambda_n)\}_{n=1}^{\infty}$ and $\{\mathrm{Re}(\lambda_n)\}_{n=1}^{\infty}$ are convergent, we have $\lim_{n\to\infty} \mathrm{Im}(\lambda_n) = \lim_{n\to\infty} y_n \sin\theta = y \sin\theta \in \Gamma_2$, and so $\lim_{n\to\infty} \{\mathrm{Re}(\lambda_n) - y_n \cos\theta\} = \lim_{n\to\infty} x_n = x \in \Lambda_1$, hence we get $\lim_{n\to\infty} \lambda_n = x + zy = \lambda \in \Lambda_0$.

(2). Let $\lambda = x + iy$ be orthogonal to $i\Lambda_0^*$, then we have $\langle \lambda, i\Gamma_1^* \rangle = \langle \lambda, iz\Gamma_2^* \rangle = 0$ and so $\omega'_1 = \mathrm{Im}(\lambda) = y \in \Gamma'_1$ and $\omega'_2 = \mathrm{Im}(\bar{z}\lambda) = y\cos\theta - x\sin\theta \in \Gamma'_2$. Thus we get $\lambda = x + iy = (z\omega'_1 - \omega'_2)/\mathrm{Im}(z)$ and the converse is also true.

(3). (3) is obvious from (2) and so omitted.

(4). If $\Gamma_1 \perp \Gamma_2^*$, then $\Lambda_0 \perp i\Lambda_0^*$ and so we have

$$\begin{aligned}\Lambda_h &= \Lambda_0 \dotplus i\Lambda_0^* \dotplus \Lambda_0^{\perp} \cap i\Lambda_0^{*\perp} \\ &= \Lambda_0 \dotplus i\Lambda_0^* \dotplus \Lambda_0^{\perp} \cap i\Lambda_0^{*\perp} \cap \Lambda_a \dotplus \Lambda_0^{\perp} \cap i\Lambda_0^{*\perp} \cap \Lambda_{\bar{a}}.\end{aligned}$$

q.e.d.

Theorem 1. *Let Γ_x be a closed space of Γ_{hse} such that*

$$\Gamma_{hm} + S(A) \subset \Gamma_x \subset \Gamma_{hse} \cap S(A)^{*\perp},$$

then, for each complex number $z = e^{i\theta} \neq \pm 1$, the space $\Lambda_x = \Gamma_x + z\Gamma'_x$ is a B-space where $\Gamma'_x = \Gamma_x^{*\perp}$.

Proof. Since $\Gamma_{hm}^{*\perp} = \Gamma_{hse}$, we have $\Gamma_{hm} + S(A) \subset \Gamma'_x = \Gamma_x^{*\perp} \subset \Gamma_{hse} \cap S(A)^{*\perp}$, $\int_A \Lambda_x = 0$ and $\Lambda_{hm} \subset \Lambda_x \subset \Lambda_{hse}$ Next, from $\Gamma'_x = \Gamma_x^{*\perp}$ we have (Cf. Lemma 1)

$$\Lambda_h = \Lambda_x \dotplus i\Lambda_x^* \dotplus \Lambda_x^\perp \cap i\Lambda_x^{*\perp} \cap \Lambda_a \dotplus \Lambda_x^\perp \cap i\Lambda_x^{*\perp} \cap \Lambda_{\bar{a}}.$$

On the other hand, for $\phi = -\omega^* + i\omega \in \Lambda_x^\perp \cap i\Lambda_x^{*\perp} \cap \Lambda_a$, we have from Lemma 1

$$\phi = -\omega^* + i\omega = (z\omega - \sigma)/\sin\theta,$$

where $\sigma = \mathrm{Im}(\bar{z}\phi) \in \Gamma_x'^{*\perp} = \Gamma_x$ and $\omega = \mathrm{Im}(\phi) \in \Gamma_x^{*\perp} = \Gamma'_x$. Thus we get $\Gamma'_x \ni \omega \perp \sigma^* \in \Gamma_x^* = \Gamma_x'^{\perp}$ and so from $\mathrm{Re}(\phi \sin\theta) = -\omega^* \sin\theta = \omega\cos\theta - \sigma$, we have the following orthogonal relations:

$$\sigma = \omega\cos\theta \dotplus \omega^* \sin\theta, \text{ and}$$

$$\omega\cos\theta = -\omega^*\sin\theta \dotplus \sigma.$$

Therefore, we have $\|\sigma\|^2 = \|\omega\|^2$ and $\|\omega\|^2\cos^2\theta = (1+\sin^2\theta)\ \|\omega\|^2$, hence $\omega = 0 = \phi$ because $z \neq \pm 1$. Analogously, we can prove $\Lambda_x^\perp \cap i\Lambda_x^{*\perp} \cap \Lambda_{\bar{a}} = \{0\}$, hence $\Lambda_x = i\Lambda_x^{*\perp}$.

Conversely, we have the following

Theorem 2. *Suppose Λ_x is an arbitrary B-space. The necessary and sufficient condition that Λ_x can be written in the form $\Lambda_x = \Gamma_x + z\Gamma'_x$ where $\Gamma_x^{*\perp} = \Gamma'_x$ and $z = e^{i\theta} \neq \pm 1$, is* $\mathrm{Im}(\bar{z}\Lambda_x) \subset \{\mathrm{Im}(\Lambda_x)\}^{*\perp}$.

Proof. To show the necessary condition, we set $\mathrm{Im}(\Lambda_x) = \hat{\Gamma}_x$, $\Gamma_x = \hat{\Gamma}_x^{*\perp}$ and $\Gamma_x^{*\perp} = \Gamma'_x$. Because Λ_x is a B-space, we have $\Lambda_{hm} + S(A) + iS(A) \subset \Lambda_x \subset \Lambda_{hse} \cap S(A)^{*\perp} \cap iS(A)^{*\perp}$ and so $\Lambda_B = \Lambda_x + z\Gamma'_x$ is a B-space (Cf. Theorem 1) and further, from the assumption, we get $\langle \Lambda_x, i\Gamma_x^* \rangle = 0$ and $\langle \Lambda_x, iz\Gamma_x'^* \rangle = \langle \mathrm{Im}(\bar{z}\Lambda_x), \Gamma_x'^* \rangle = 0$, hence we can get $\Lambda_x \subset i\Lambda_B^{*\perp} = \Lambda_B \subset i\Lambda_x^{*\perp} = \Lambda_x$. The converse is evident.

From Tneorem 1, we can obtain infinitely many B-spaces $\{\Gamma_x + z\Gamma'_x\}$ by changing paramter z and subspace Γ_x where $S(A) + \Gamma_{hm} \subset \Gamma_x \subset \Gamma_{hse} \cap S(A)^{*\perp}$ and $\Gamma'_x = \Gamma_x^{*\perp}$. However, next example shows the existence of a B-space which can not be written in the form $\Gamma_x + z\Gamma'_x$.

Example 1. Let G be the interior of a compact bordered Riemann surface and $\partial G = \beta_1 \cup \beta_2 \cup \beta_3$ where each β_j is a border contour. At first, we set

$$\Gamma_{he0}(\partial G - \beta_j) = \{du \in \Gamma_{he}(G) \colon u = 0 \text{ on } \partial G - \beta_j\}, \text{ and}$$

$$\Gamma_{h0}(\beta_j) = \Gamma_{he0}(\partial G - \beta_j)^{*\perp}.$$

Next, for a set of complex numbers $\{t_j = e^{i\theta_j}\}_{j=1}^3$ such that $t_p \neq \pm t_q$ for $p \neq q$, we consider the following families of differentials:

$$\Lambda_x = \left\{\lambda \in \Lambda_{hse} \colon (1) \int_A \lambda = 0,\ (2)\ \mathrm{Im}(\bar{t}_j\lambda) \in \Gamma_{h0}(\beta_j),\ j = 1, 2, 3\right\},$$

$$\Lambda'_x = \text{closure } \{\Lambda_{hm} + S(A) + iS(A) + \sum_j t_j \Gamma_{he0}(\partial G - \beta_j)\}.$$

Since $\Gamma_{he0}(\beta_p) \perp \Gamma_{he0}(\partial G - \beta_p)^*$ and $S(A) \perp S(A)^*$, we have $\Lambda_x \supset i\Lambda_x'^{*\perp} \supset \Lambda'_x \supset i\Lambda_x^{*\perp}$. On the other hand, by using the Kusunoki's formula (for example. Cf. Lemma 6 in [9]), we have $\langle \lambda, i\mu^* \rangle = 0$ for each $\lambda, \mu \in \Lambda_x$. Thus, $\Lambda'_x = \Lambda_x$ is a B-space. Now we choose $du_j \in \Gamma_{he0}(\partial G - \beta_j)$ such that $du_j \in \Gamma_{h0}(\beta_j)$, $j = 1, 2, 3$, then we have $t_1 du_1 + t_2 du_2 + t_3 du_3 = \omega \in \Lambda_x$ and further, ω can not be written in a form $du + zdv$ with $du \in \Gamma_1$ and $dv \in \Gamma'_1 = \Gamma_1^{*\perp}$.

2B. Application. By using B-spaces, we know already the Riemann-Roch theorem for differentials with B-behavior (Cf. [2]). Now besides this, we show in 2B an another application of this space to the Abelian integral theory. We set

$$\Lambda_a^0 = \left\{\phi : (1)\ \phi \in \Lambda_{ase},\ (2)\ \phi \text{ is a Schottky differential},\ (3)\ \phi = 0 \text{ if } \int_A \phi = 0\right\},$$

$$\Gamma_x = \text{closure } \{\Gamma_{hm} + S(A)\},$$

$$\Gamma_A = \Gamma_x \cap \Gamma_{hse}^* + \Gamma_x^* \cap \Gamma_{hse}.$$

By using the B-space $\Lambda_x = \Gamma_x + i\Gamma_x^{*\perp}$, we prove the next proposition which is similar to that of Kusunoki [4].

Proposition 1 (Kusunoki [4]). $\Lambda_a^0 = \{\Gamma_A + i\Gamma_A\} \cap \Lambda_a$.

Proof. At first, we choose the B-space $\Lambda_x = \Gamma_x + i\Gamma_x^{*\perp}$ (Cf. Theorem 1). Each $\phi \in \Lambda_{ase}$ with $\int_A \phi = 0$ has a decomposition of the form

$$\phi = \lambda + i\lambda^* = \sigma + i\tau + i\sigma^* - \tau^*,$$

where $\lambda \in \Lambda_x$, $\mathrm{Re}(\phi) = \sigma \in \Gamma_x$ and $\mathrm{Im}(\phi) = \tau \in \Gamma_x^{*\perp} = \Gamma_{hse} \cap S(A)^{*\perp}$. Terefore, we have $\int_A (\tau + \sigma^*) = \int_A \sigma^* = 0$ and $\int_\gamma (\tau + \sigma^*) = 0 = \int_\gamma \sigma^* = 0$, where γ denotes an arbitrary dividing curves. Thus we can get $\sigma = 0$, $\phi = -\tau^* + i\tau$, and so $\tau \in \Gamma_{hse} \cap \Gamma_{hse}^* \cap S(A)^\perp \cap S(A)^{*\perp} = \Gamma_x^\perp \cap \Gamma_x^{*\perp}$. If $\phi \in (\Gamma_A + i\Gamma_A) \cap \Lambda_a$ with $\int_A \phi = 0$, then we get $\phi = -\tau^* + i\tau$ where $\mathrm{Im}(\phi) = \tau \in (\Gamma_x + \Gamma_x^*) \cap \Gamma_x^\perp \cap \Gamma_x^{*\perp}$ and so $\phi = 0$. Hence $(\Gamma_A + i\Gamma_A) \cap \Lambda_a \subset \Lambda_a^0$, and we get the following

$$\Lambda_a^0 = (\Gamma_A + i\Gamma_A) \cap \Lambda_a + \Gamma_A^\perp \cap i\Gamma_A^\perp \cap \Lambda_a^0.$$

However, for each $\phi = \omega + i\omega^* \in \Gamma_A^\perp \cap i\Gamma_A^\perp \cap \Lambda_a^0 \subset \Lambda_{ase}$ with $\mathrm{Re}(\phi) = \omega$, we have

$$\omega \in \Gamma_A^\perp \cap \Gamma_{hse} \cap \Gamma_{hse}^* = \Gamma_{hse} \cap S(A)^{*\perp} \cap \Gamma_{hse}^* \cap S(A)^\perp.$$

Therefore, we have $\int_A \phi = 0$ and $\phi \in \Lambda_a^0$, hence $\phi = 0$ and $\Lambda_a^0 = (\Gamma_A + i\Gamma_A) \cap \Lambda_a$.

§3. S-behavior space $\Lambda_2 = \Lambda(\{J_k\}_{k=1}^2, \{L_k\}_{k=1}^2)$

Theorem 3. *Let $D_2(J) = \{J_k\}_{k=1}^2$ be a partition of the set $J = \{1, 2, \ldots, g\}$, Γ_1 the space such that $S(J_1) + \Gamma_{hm} \subset \Gamma_1 \subset \Gamma_{hse} \cap S(J_2)^{*\perp}$ and $\Gamma_1' = \Gamma_1^{*\perp}$. Then, for each complex number $z = e^{i\theta} \neq \pm 1$, the space $\Lambda_2 = \Gamma_1 + z\Gamma_1'$ is an S-space associated with $(\{J_k\}_{k=1}^2, \{L_k\}_{k=1}^2)$ where L_1 is the real axis and $L_2 \ni z$.*

Proof. Since $\Gamma_{hm} + S(J_2) \subset \Gamma_1' = \Gamma_1^{*\perp} \subset \Gamma_{hse} \cap S(J_1)^{*\perp}$, we have $\int_{J_1} \Lambda_2 = \int_{J_1} \Gamma_1 \in L_1$ and $\int_{J_2} \Lambda_2 = z \int_{J_2} \Gamma_1' \in L_2$. From $\Gamma_1' = \Gamma_1^{*\perp}$, we get $\Lambda_2 \subset i\Lambda_2^{*\perp}$, and

$$\Lambda_h = \Lambda_2 \dotplus i\Lambda_2^* \dotplus \Lambda_2^\perp \cap i\Lambda_2^{*\perp} \cap \Lambda_a \dotplus \Lambda_2^\perp \cap i\Lambda_2^{*\perp} \cap \Lambda_{\bar{a}}.$$

However, for each $\phi = -\omega^* + i\omega \in \Lambda_2^\perp \cap i\Lambda_2^{*\perp} \cap \Lambda_a$, we have $\operatorname{Im}(\phi) = \omega \in \Gamma_1^{*\perp} = \Gamma_1'$ and $\operatorname{Im}(\bar{z}\phi) = \sigma = \omega^* \sin\theta + \omega\cos\theta \in \Gamma_1'^{*\perp} = \Gamma_1 \perp \omega^* \in \Gamma_1'^*$. Consequently, we obtain $\langle \sigma, \omega^* \rangle = 0 = \|\omega\|^2 \sin\theta$, hence $\phi = 0$ and so $\Lambda_2^\perp \cap i\Lambda_2^{*\perp} \cap \Lambda_a = \{0\}$. Analogously, we can get $\Lambda_2^\perp \cap i\Lambda_2^{*\perp} \cap \Lambda_{\bar{a}} = \{0\}$, and so $\Lambda_2 = i\Lambda_2^{*\perp}$. q.e.d.

Conversely, we have the following

Theorem 4. *Suppose Λ_K is an S-behavior space associated with $(D_K, L_K) = (\{J_k\}_{k=1}^K, \{L_k\}_{k=1}^K)$ where L_1 is the real axis. The necessary and sufficient condition that Λ_K can be written in the form $\Gamma_1 + z\Gamma_1'$ where $\Gamma_{hm} + S(J_1) \subset \Gamma_1'^{*\perp} = \Gamma_1 \subset \Gamma_{hse} \bigcap_{k=2}^K S(J_k)^{*\perp}$ and $z = e^{i\theta} \neq \pm 1$, is*

$$\operatorname{Im}(\bar{z}\Lambda_K) \subset \operatorname{Im}(\Lambda_K)^{*\perp}.$$

Proof. At first, we set $\operatorname{Im}(\Lambda_K) = \hat{\Gamma}_1$, $\hat{\Gamma}_1^{*\perp} = \Gamma_1$ and $\Gamma_1' = \Gamma_1^{*\perp}$. Since $\langle \Lambda_K, iz\Gamma_1'^* \rangle = \langle \Lambda_K, i\Gamma_1^* \rangle = 0$, we have $\Lambda_2 \subset i\Lambda_K^{*\perp} = \Lambda_K \subset i\Lambda_2^{*\perp}$ where $\Lambda_2 = \Gamma_1 + z\Gamma_1'$. On the other hand, from $\Lambda_K \subset \Lambda_{hse}$ and $\int_{J_1} \Lambda_K \in L_1$, we get $\Gamma_1' \subset \Gamma_{hse} \cap S(J_1)^{*\perp}$ and $\Gamma_1 \supset \Gamma_{hm} + S(J_1)$. Further, since $\Lambda_K \subset \Lambda_{hse} \bigcap_{k=2}^K iz_k S(J_k)^{*\perp}$ where $0 \neq z_k \in L_k$, $k = 2, 3, \ldots, K$, we have $\Lambda_K = i\Lambda_K^{*\perp} \supset \sum_{k=2}^K z_k S(J_k) + \Lambda_{hm}$, and so $\Gamma_1' \supset \Gamma_{hm} + \sum_{k=2}^K S(J_k)$. Therefore, we can get $\Gamma_{hm} + S(J_1) \subset \Gamma_1 = \Gamma_1'^{*\perp} \subset \Gamma_{hse} \bigcap_{k=2}^K S(J_k)^{*\perp}$, hence $i\Lambda_2^{*\perp} = \Lambda_2 = \Lambda_K$ from Theorem 3. q.e.d.

Next, we give, analogously as in 2A, an example of an S-space $\Lambda_2 = \Lambda(\{J_k\}_{k=1}^2, \{L_k\}_{k=1}^2)$ which can not be written in the form $\Gamma_1 + z\Gamma_1'$ where $\Gamma_1' = \Gamma_1^{*\perp}$.

Example 2. Let G be the interior of the compact bordered surface and $\partial G = \beta_1 \cup \beta_2 \cup \beta_3$ where each β_j is a contour. We use here the same notations as in the example 1, and consider the following subspaces of differentials on G:

$$\Lambda_2 = \left\{\lambda \in \Lambda_{hse}: \text{(a)} \int_{J_k} \lambda \in L_k,\ k = 1, 2 \text{ where } L_1 \text{ is the real axis,} \right.$$

$$\left. \text{(b) } \operatorname{Im}(\bar{t}_r \lambda) \in \Gamma_{h0}(\beta_r),\ r = 1, 2, 3 \right\},$$

$$\Lambda'_2 = \text{closure}\left\{\Lambda_{hm} + \sum_{k=1}^{K} t_k \Gamma_{he0}(\partial G - \beta_k) + S(J_1) + \zeta S(J_2)\right\},$$

where each t_j is a complex number such that $t_p \neq \pm t_q$ for $p \neq q$ and ζ is a complex number such that $\zeta \neq 0$ and $\zeta \in L_2$. By the same way as in the example 1, we can prove $\Lambda_2 = \Lambda'_2 = i\Lambda_2^{*\perp}$. Next, the differential ω in the example 1 belongs to Λ_2, and further, ω can not be written in the form $du + zdv$ with $du, dv \in \Gamma_h$.

§4. S-behavior space $\Lambda_K = \Lambda(D_K, \boldsymbol{L}_K)$ $(3 \leqq K)$

4A. Λ_K and Λ'_K. Let Λ_K be an arbitrary S-space associated with $(D_K, \boldsymbol{L}_K)$ where $D_K = \{J_k\}_{k=1}^K$ and $\boldsymbol{L}_K = \{L_K\}_{k=1}^K$. We set $\operatorname{Im}(\bar{z}_j \Lambda_K) = \Gamma'_j$, $\Gamma_j'^{*\perp} = \Gamma_j$, and $\Lambda'_K = \text{closure}\ \{\sum_{k=1}^K z_k \Gamma_K\}$, where $z_k \in L_k$ and $|z_k| = 1$, $k = 1, 2, \ldots, K$.

Lemma 2. *In order that Λ_K be equal to Λ'_K, it is necessary and sufficient that Λ'_K is an S-space.*

Proof. Since $\langle \Lambda_K, iz_j \Gamma_j^* \rangle = 0$ for each j, we obtain the relation $\Lambda'_K \subset i\Lambda_K^{*\perp} = \Lambda_K \subset i\Lambda_K'^{*\perp}$. Therefore, $\Lambda_K = \Lambda'_K$ means $\Lambda'_K = i\Lambda_K'^{*\perp}$.

Remark. By the same way as in the former examples, we can give an example of Λ_K such that $\Lambda_K \neq \Lambda'_K$.

4B. Λ^n, Λ_χ and Λ'_χ. Let $\beta = \bigcup_{r=1}^{s} \beta_r$ be a regular partition of the Stoilow's ideal boundary β of an open Riemann surface R, $\{R_n\}$ a regular canonical exhaustion of R and $R - R_n = \bigcup_{r=1}^{s} W_n^r$ where W_n^r is an end towards β_r. We set

$\Gamma_{e0}^1(\beta_r) = \{df$: (a) $df \in \Gamma_e^1$, (b) there exists an end towards β_r which is disjoint with the support of $f\}$,

$\Gamma_{he0}(\beta_r) = \Gamma_h \cap \text{closure}\ \{\Gamma_{e0}^1(\beta_r)\}$,

$\Gamma_{h0}(\beta - \beta_r) = \Gamma_{he0}(\beta_r)^{*\perp}$.

Besides $J = \{J_k\}_{k=1}^K$ and $\boldsymbol{L} = \{L_k\}_{k=1}^K$, we consider another partition of J, the family of lines and the subspaces of differentials on R such that:

$J_{k,n} = \{j : j \leqq g_n$ and $j \in J_k$, where g_n is the genus of $R_n\}$,

$J_{k+r,n} = \{j : j > g_n$ and $A_j, B_j \subset W_n^r\}$,

$l = \{l_r\}_{r=1}^{s}$, where each l_r is a line passing through the origin on the z plane,

$D'_n(J) = \{J_{k,n}\}_{k=1}^{K+s}$,

$\boldsymbol{L}'_n = \{L_k\}_{k=1}^{K+s}$, where $L_{K+r} = il_r$, $r = 1, 2, \dots, s$,

$$\Lambda'' = \left\{\lambda \in \Lambda_{hse} : \text{(a)} \int_{J_{k,n}} \lambda \in L_k,\ k = 1, 2, \dots, K + s,\ \text{(b)}\ \operatorname{Im}(\bar{\zeta}_r \lambda) \in \Gamma_{h0}(\beta_r),\right.$$
$$\left. r = 1, 2, \dots, s\right\},$$

where il_r means the set $\{i\zeta : \zeta \in l_r\}$ and ζ_r is a complex number such that $0 \neq \zeta_r \in L_{K+r}$, $r = 1, 2, \dots, s$.

Lemma 3. *Λ'' is an S-behavior space associated with D'_n and $\boldsymbol{L}'_n$.*

Proof. Omitted. (Cf. Lemma 4.3 in [5]).

Further, we set

$\Lambda'_\chi = \{\lambda$: there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ with $\lambda_n \in \Lambda''$ such that $\|\lambda - \lambda_n\| \to 0$ as $n \to \infty\}$,

$\Lambda_\chi = \{\lambda$: λ is equal to the limit of a locally uniformly convergent subsequence of $\{\lambda_n\}_{n=1}^{\infty}$ with $\lambda_n \in \Lambda''$ and $\sup_n \{\|\lambda_n\|\} < \infty\}$.

Then we have the following

Proposition 2. (a) $i\Lambda_\chi^{*\perp} = \Lambda'_\chi \subset \Lambda_\chi = i\Lambda_\chi'^{*\perp}$.

(b) *$\Lambda_\chi = \Lambda'_\chi$ is equivalent to $\Lambda'_\chi = i\Lambda_\chi'^{*\perp}$.*

Proof. At first, we prove $\Lambda_\chi \subset i\Lambda_\chi'^{*\perp}$. From the difinition of Λ_χ and Λ'_χ, we can find for each $\lambda \in \Lambda_\chi$ and each $\mu \in \Lambda'_\chi$ a Cauchy sequence $\{\mu_n\}_{n=1}^{\infty}$ with μ_n Λ'' and a locally uniformly convergent sequence $\{\lambda_{n_\alpha}\}_{\alpha=1}^{\infty}$ (which we denote $\{\lambda_\alpha\}$ hereafter) with $\lambda_\alpha \in \Lambda^\alpha$ and $\sup_\alpha \{\|\lambda_\alpha\|\} < K$ such that $\|\mu_n - \mu\| \to 0$ as $n \to \infty$ and $\lambda_\alpha \to \lambda$ as $\alpha \to \infty$, locally uniformly on R.

Further, for arbitrary small positive number ε, we can choose a compact region $\bar{G}$ on R and a sufficiently large integer N such that $\|\lambda\|_{R-G} + \|\mu\|_{R-G} < \varepsilon$ and $\|\lambda - \lambda_n\|_G + \|\mu - \mu_n\| < \varepsilon$ for each $n > N$.
Therefore, for each $k > N$, we have

$$|\langle \lambda, i\mu^* \rangle| < |\langle \lambda, i\mu^* \rangle_G| + \varepsilon\|\mu\| < |\langle \lambda - \lambda_k, i\mu^* \rangle_G| + |\langle \lambda_k, i\mu^* \rangle_G| + \varepsilon\|\mu\|$$
$$< |\langle \lambda_k, i\mu_k^* \rangle_G| + (2\|\mu\| + K)\varepsilon < |\langle \lambda_k, i\mu_k^* \rangle_{R-G}| + (2\|\mu\| + K)\varepsilon$$
$$< (3K + 2\|\mu\|)\varepsilon.$$

Thus we get $\Lambda_\chi \subset i\Lambda_\chi'^{*\perp}$. Next, we show $i\Lambda_\chi'^{*\perp} \subset \Lambda_\chi$. Let $\lambda \in \Lambda_h$ be a differential such that $\langle \lambda, i\Lambda_\chi^* \rangle = 0$. λ has a decomposition of the form

$$\lambda = \lambda_n + i\mu_n^*, \text{ where } \lambda_n,\ \mu_n \in \Lambda'' \text{ (Cf. Lemma 3).}$$

and so there exists a sequence $\{n_\alpha\}_{\alpha=1}^{\infty}$ (which we denote here simply $\{\alpha\}$) such that $\lambda_\alpha \to \lambda_\chi$ and $\mu_\alpha \to \mu_\chi$, locally uniformly on R. On the other hand, there exists, for each $\varepsilon > 0$, a compact region G such that $\|\lambda\|_{R-G} < \varepsilon$ and so from $= \langle \lambda, i\mu_\chi^* \rangle = 0$, we have

$$|\langle \lambda, i\mu_n^* \rangle_G + \langle \lambda, i\mu_\chi^* - i\mu_n^* \rangle_G| = |\langle \lambda, i\mu_\chi^* \rangle_{R-G}| < \varepsilon \|\mu_\chi\|, \text{ and so}$$

$$|\langle \lambda, i\mu_n^* \rangle_G| < \varepsilon \|\mu_\chi\| + \|\lambda\| \, \|\mu_\chi - \mu_n\|_G.$$

Therefore, for each $n > N'$ which is sufficiently large integer, we have

$$|\langle \lambda, i\mu_n^* \rangle_G| < \varepsilon(\|\mu_\chi\| + \|\lambda\|), \text{ hence}$$

$$|\langle \lambda, i\mu_n^* \rangle| = \|\mu_n\|^2 < \varepsilon\|\lambda\| + \varepsilon(\|\mu_\chi\| + \|\lambda\|).$$

Consequently, we can choose $\|\lambda - \lambda_n\|^2 = \|\mu_n\|^2 \to 0$ as $n \to \infty$, hence $\lambda \in \Lambda'_\chi$, and $i\Lambda_\chi^{*\perp} \subset \Lambda'_\chi$. (b) is evident.

Note. In the case the genus of R is infinite, we do not know the existence of S-behavior space associated with arbitrary partition $D_K(J) = \{J_k\}_{k=1}^{K}$ of $J = \{1, 2, \ldots\}$ where $K \geqq 3$. The author supposes that, in Proposition 2, Λ_χ would be equal to Λ'_χ, hence $\Lambda_\chi = i\Lambda_\chi^{*\perp}$ is an S-space associated with $(\{J_k\}_{k=1}^{K}, \{L_k\}_{k=1}^{K})$, but this problem is not affirmatively proved yet.

§5. Multiplicative differentials and Riemann-Roch's type theorem

5A. Multiplicative differentials on $\hat{R}$. Concerning the extension of the duality theorem of Riemnn-Roch's type for Prym differentials (multiplicative differemtials) (Cf. Weyl [11]) to open surfaces, no results are known up to now, and so it seems not to be meaningless to give such a sort of theorem for multiplicative differentials on a specific symmetric open surface.

Suppose that R is a non compact bordered surface whose border ∂R consists of a finite number of contours $\{C_q\}_{q=1}^{m}$ and $P(\beta) = \bigcup_{r=1}^{s} \beta_r \bigcup_{q=1}^{m} C_q$ is a regular partition of the Stoilow's ideal boundary β of R. Let G be a canonical region of $R \cup \partial R$ such that $\partial G \supset \partial R$, and set $R - \bar{G} = \bigcup_{r=1}^{s} W_r$ where W_r is an end towards β_r. Next, we divide each C_q into α_{2q-1} and $\bar{\alpha}_{2q}$ where each of α_{2q-1}, α_{2q} is an open are on C_q. Further, we associate each α_k (resp. β_k) with a complex number z_k (resp. ζ_r) such that $|z_k| = 1$ and $|\zeta_r| = 1$, and denote the set $\{z_1, z_2, \ldots, z_{2m}\}$ (resp. $\{\zeta_1, \zeta_2, \ldots, \zeta_s\}$) by Z (resp. S). For Z and S we consider the following subspaces of differentials on R:

$$\Lambda_0 = \Lambda_{0G} = \left\{ \lambda \in \Lambda_{hse} \colon \text{(a)} \int_{A_j, B_j} \lambda \in L_j,\ j \leqq g_G = \text{the genus of } G, \text{ and} \right.$$

$$\int_{A_j, B_j} \lambda \in i l_r, \text{ for } A_j, B_j \subset W_r,\ r = 1, 2, \ldots, s,$$

(b) $\operatorname{Im}(\bar{z}_q\lambda)\in\Gamma_{h0}(\alpha_q)$, $q=1,2,\ldots,2m$

(c) $\operatorname{Im}(\bar{\zeta}_r\lambda)\in\Gamma_{h0}(\beta_r)$, $r=1,2,\ldots,s\}$,

where $\{l_r\}_{r=1}^{s}$ is a family of lines on the z plane passing through the origin and $\zeta_r\in l_r$ for each r. Then we have

Lemma 4. *Λ_0 is an S-behavior space.*

Proof. Cf. Lemma 4.3 in [5] and Lemma 2.6 in [6].

We set

$D(\Lambda_0, \delta, R)=\{\phi$: (a) ϕ has Λ_0-behavior, (b) divisor (ϕ) of ϕ is a multiple of $\delta\}$,

$S(\Lambda_0, \delta, R)=\{f$: (b) function f has Λ_0-behavior, (b) divisor (f) of f is a multiple of $\delta\}$,

where $\delta=\delta_a/\delta_b$ and δ_a, δ_b are finite integral disjoint divisors. Then

Lemma 5. $\dim S(\Lambda_0, 1/\delta, R)=2[\operatorname{ord}\delta_a+1-\min(\operatorname{ord}\delta_b, 1)]$
$-\dim[D(\bar{\Lambda}_0, 1/\delta_b, R)/D(\bar{\Lambda}_0, \delta, R)]$.

Proof. Since Λ_0 is an S-behavior space, from [9] we have the conclusion.

Next, we show an example of a duality theorem for multiplicative differentials on $\hat{R}$ by considering Lemma 5 on the surface $\hat{R}$, where $\hat{R}$ is the double of R with respect to ∂R. We set

J = the involutory mapping $\hat{R}$ onto itself,

$\omega^{\sim}$ = the differential on $\hat{R}$ associated with J and $\omega\in\Gamma_h(\hat{R})$ such that if ω is expressed as $\omega=a(z)dx+b(z)dy$ in local parameter $z=h(p)$ in V, then $\omega^{\sim}=a(\bar{z})dx-b(\bar{z})dy$ in $z=\bar{h}(J(p))$ in $J(V)$ where V is a parametric disc.

Next, let P (resp. Y_k) be a point on R (resp. α_k), $\gamma(k)$ an analytic arc on $R\cup\partial R$ such that $\partial\gamma(k)=Y_k-P$, and set

$$\hat{\gamma}_k=\gamma(k)\cup\{-J\gamma(k)\},$$
$$\hat{\gamma}_{kn}=\hat{\gamma}_k\cup(-\hat{\gamma}_n).$$

Lemma 6. *The analytic continuation ϕ_k of $\phi\in D(\Lambda_0, \delta, R)$ from a point P to $J(P)$ along $\hat{\gamma}_k$, is $z_k^2\bar{\phi}^{\sim}$, and so the analytic continuation of ϕ along $\hat{\gamma}_{pq}$ from a point P to P is $z_q^2\bar{z}_p^2\phi$. Consequently, we have*

$$\int_{\substack{JA_h\\ JB_h}}\phi_k\in z_k^2\bar{L}_h \text{ for each } h, \text{ and}$$

$$\operatorname{Im}(z_n\bar{z}_k^2\phi_k)=0 \text{ along } J(\alpha_n) \text{ for each } n,$$

where $\bar{L}_h$ *means the symmetric line of* L_h *with respect to the real axis.*

Proof. At first, we set $\bar{z}_k\phi = \omega + i\omega^*$ on R where $\omega = \mathrm{Re}(\bar{z}_k\phi)$, then $\omega^* = 0$ along α_k, hence the differential $\hat{\omega}$ on $\hat{R} - (\partial R - \alpha_K)$ such that

$$\begin{aligned}\hat{\omega} &= \omega \text{ on } R, \text{ and}\\ &= \omega^{\sim} \text{ on } \hat{R} - (\partial R - \alpha_k) - R,\end{aligned}$$

is even for J_k and $\hat{\omega}^*$ is odd for J_k where J_k means the restriction of J to $\hat{R} - (\partial R - \alpha_k)$. Therefore, we see that the analytic continuation of $\Phi = \bar{z}_k\phi$ to $\hat{R} - \bar{R} + \alpha_k$ is equal to $\bar{\Phi}^{\sim} = z_k\bar{\phi}^{\sim}$, and so the analytic continuation ϕ_k of $\phi = z_k\Phi$ is equal to $z_k\bar{\Phi}^{\sim} = z_k^2\bar{\phi}^{\sim}$. Consequently, we have

$$\int_{JA_h, JB_h} \phi \in \bar{z}_k^2 L_h \text{ for each } h, \text{ and}$$

$$\mathrm{Im}(z_n\bar{z}_k^2\phi_k) = 0 \text{ along } J\alpha_n \text{ for each } n.$$

Obviously, the analytic continuation of ϕ along γ_{pq} is equal to $z_q^2\bar{z}_p^2\phi$. q.e.d.

Hereafter, we consider, for each $\phi \in D(A_0, \delta, R)$ and each $f \in S(A_0, \delta, R)$, the following elements and families:

$$\begin{aligned}\phi(j_{2k+1}, J_{2k}, \dots, j_2, j_1) &= Z(j_{2k})\phi \text{ on } \bar{R} = R \cup \partial R\\ &= z(j_{2k+1})^2\overline{Z(j_{2k})\phi^{\sim}} \text{ on } \hat{R} - \bar{R},\end{aligned}$$

where we set

$$z(p) = z_p, \text{ and } z(j_{2k})^2\overline{z(j_{2k-1})}^2 \cdots z(j_2)^2\overline{z(j_1)}^2 = Z(j_{2k}),$$

$$\hat{\phi} = \{\phi(j_{2k+1}, j_{2k}, \dots, j_1) : j_p = 1, 2, \dots, 2m, \text{ for } p = 1, 2, \dots, k \text{ and } k = 1, 2, \dots\},$$

$$\hat{D}(A_0, \hat{\delta}, \hat{R}) = \{\hat{\phi} : \phi \in D(A_0, \delta, R)\},$$

$$\hat{S}(A_0, \hat{\delta}, \hat{R}) = \left\{\int \widehat{(df)} : f \in S(A_0, \delta, R)\right\},$$

where $\hat{\delta} = \delta \cup J\delta$ Obviously, $\hat{\phi}$ is a multiplicative differential on $\hat{R}$. Then we have the following proposition 3 which is different from Weyl's theorem (Cf. Weyl [11]).

Proposition 3. *Let* δ_a/δ_b *be a divisor on* R *such that* $\delta_a \cap \delta_b = 0$ *and* δ_a, δ_b *are the finite disjoint divisors. Then, we have*

$$\begin{aligned}\dim \hat{S}(A_0, 1/\hat{\delta}, \hat{R}) &= \mathrm{order}(\hat{\delta}_a) + 2 - 2\min[\{\mathrm{order}(\hat{\delta}_b)\}/2, 1]\\ &\quad - \dim[\hat{D}(\bar{A}_0, 1/\hat{\delta}_b, \hat{R})/\hat{D}(\bar{A}_0, \hat{\delta}, \hat{R})],\end{aligned}$$

where $\hat{\delta} = \delta \cup J\delta$. *If* g, *the genus of* R, *is finite, the genus* $\hat{R}$ *is equal to* $2g + m - 1 = \hat{g}$ *and so*

$$\dim \hat{S}(A_0, 1/\hat{\delta}, \hat{R}) - \dim \hat{D}(\hat{A}_0, \hat{\delta}, \hat{R}) = \mathrm{order}(\hat{\delta}) - \hat{g} + m + 1.$$

Proof. Cf. Lemma 5 and Lemma 4.

5B. Application of Proposition 3. For the case that the order of a given divisor is infinite, several authors investigated the duality theorems of Riemann-Rch type for Abelian differentials under the respective conditions. In the following, we formulate as an application of Proposition 3, a Riemann-Roch theorem of another type with infinite divisor on a covering surface $\tilde{R}$ of R.

At first, we take an infinite number of copies $\{R^n_{1,p,\dots,r}\}$ of R and adjoin R^1_1 with $R^2_{1,k}$ along α_k, $k=1, 2,\dots, 2m$. Next, thus constructed $R^2 = R^1_1 \bigcup_{q=1}^{2m} R^2_{1,k}$ has $2m-1$ number of α_p for p, and so we adjoin $R^2_{1,q}$ of R^2 with $R^3_{1,q,r}$ along α_r, $r=1, 2,\dots, 2m$, $r \neq q$, and so on. Thus we can get a covering surface $\tilde{R} = \lim_{n\to\infty} R^n$. Now we take a differentials $\phi \in D(A_0, \delta, R)$ (resp. df where $f \in S(A_0, \delta, R)$) and denote the analytic continuation of ϕ (resp. df) to $\tilde{R}$ by $\tilde{\phi}$ (resp. $(\widetilde{df})$) (Cf. Lemma 6). We set

$$\tilde{D}(A_0, \tilde{\delta}, \tilde{R}) = \{\tilde{\phi} : \phi \in D(A_0, \delta, R)\},$$

$$\tilde{S}(A_0, \tilde{\delta}, \tilde{R}) = \left\{\int (\widetilde{df}) : f \in S(A_0, \delta, R)\right\},$$

where $\tilde{\delta}$ is the divisor such that the restriction of $\tilde{\delta}$ to each $R = \delta$. Then

Proposition 5. $\dim \tilde{S}(A_0, 1/\tilde{\delta}, \tilde{R}) = 2[\text{order}(\delta_a) + 1 - \min\{\text{order}(\delta_b), 1\}]$
$- \dim [\tilde{D}(\bar{A}_0, 1/\tilde{\delta}_b, \tilde{R})/\tilde{D}(\bar{A}_0, \tilde{\delta}, \tilde{R})]$,

where $\delta = \delta_a/\delta_b$ is a finite divisor. If g, the genue of R, is finite, *then*

$$\dim \tilde{S}(A_0, 1/\tilde{\delta}, \tilde{R}) - \dim \tilde{D}(A_0, \tilde{\delta}, \tilde{R}) = 2\{\text{order}(\delta) - g + 1\}.$$

Proof. Cf. Lemma 4 and Lemma 5.

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