J. Math. Kyoto Univ. (JMKYAZ)
33-2 (1993) 413–436

Local b-functions of prehomogeneous Lagrangians

By

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0. Introduction

0.1. Let X be a connected non-singular algebraic variety of dimension n over the complex number field $\mathbf{C}$, G a connected linear algebraic group acting on X, $\varpi^{(i)} \in \mathrm{Hom}(G, \mathbf{C}^\times)(1 \le i \le l)$, and f_i $(1 \le i \le l)$ non-constant regular functions on X such that

$$(a) \qquad f_i(gx) = \varpi^{(i)}(g) f_i(x) \quad (g \in G, x \in X).$$

Put $f^\lambda = \prod_{i=1}^{l} f_i^{\lambda_i}$ for $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbf{C}^l$.

M. Kashiwara, T. Kimura and M. Muro [8] proved a functional equation of the form

$$P_\mu(f^\mu \cdot f^\lambda) = b(\lambda) f^\lambda \quad (\lambda \in \mathbf{C}^l, \mu \in \mathbf{N}^l)$$

on the conormal bundle Λ of a G-orbit under certain assumptions including the G-prehomogeneity of Λ. Here P_μ is an invertible microdifferential operator and b_μ is a polynomial. Unfortunately, the manuscript [8] is hardly available and seems still unfinished.

0.2. Inspired by the work of S. Suga [12], the present author started to study a relation between the b-functions of semi-invariants and the irreducibility of certain highest weight modules over the complex semisimple Lie algebras [4]. Thus it becomes necessary to study the functional equations of the above form, where f_i's are semi-invariants corresponding to fundamental weights.

The purpose of this paper is, instead of completing [8], to prove essentially the same assertion by a different argument and to furnish a necessary device for our present study. Our argument is elementary in the sense that it does not use the quantized contact transformation. Several parts are close to [11], especially in §3 and §5.

0.3. In order to state our result more precisely, first let us state our assumptions. To keep the argument from non-essential complication, we assume that

$$(b) \qquad \text{the characters of } \mathrm{Lie}(G) \text{ corresponding to } \varpi^{(i)} (i \in S) \text{ are linearly independent.}$$

Communicated by Prof. T. Hirai, October 22, 1991

Let q_0 be a point of X, T^*X the cotangent bundle of X, and Λ the conormal bundle of the G-orbit $G\cdot q_0$, i.e., the Zariski closure of

$$\{(x, y) = (x_1, \dots, x_n, y_1, \dots, y_n) \in T^*X \mid x \in G\cdot q_0,\ y \perp T(G\cdot q_0)\},$$

where $\{x_i\}$ is a local coordinate of X and $\{y_i\}$ is the corresponding fibre coordinate. We fix an element $\delta = (\delta_1, \dots, \delta_l) \in (\mathbf{Z}_{>0})^l$. Let W' be the Zariski closure of

$$\{(x, s\sum_{i=1}^{l} \delta_i \operatorname{grad}\log f_i(x) \in T^*X \mid x \in \mathbf{C}, f_i(x) \neq 0\}$$

and $W_0' = \{(x, y) \in W' \mid f^\delta(x)y = 0\}$. We assume that

(c) $\qquad \Lambda$ has an open G-orbit, say Λ_0, and

(d) $\qquad \Lambda \subset W_0'$.

We fix an element p_0 in $\Lambda_0 \cap T^*_{q_0}X$.

0.4. Next, let us give definitions necessary to state our result. Let $\zeta = (\zeta_1, \dots, \zeta_l)$ be an l-tuple of independent complex variables, $\mathscr{D} = \mathscr{D}_X$ the sheaf of differential operators on X, and $\mathscr{D}[\zeta] = \mathscr{D} \otimes_{\mathbf{C}} \mathbf{C}[\zeta]$. Let X_0 be a simply connected open dense subset of $\bigcap_i f_i^{-1}(\mathbf{C}^\times)$,

$$X_0 \times \mathbf{C}^l \ni (x, \zeta) \longrightarrow f(x)^\zeta = \prod_i f_i(x)^{\zeta_i}$$

a single-valued branch, $\mathscr{N} = \mathscr{D}[\zeta]f^\zeta$, $f^\lambda = (f^\zeta \bmod \sum_i(\zeta_i - \lambda_i)\mathscr{N})$ for $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathbf{C}^l$, and $\mathscr{N}_0 = \mathscr{N}_0(\lambda) = \mathscr{D}f^\lambda$. Let $\mathscr{E} = \mathscr{E}_X$ be the sheaf of microdifferential operators. We consider f^λ as a section of $\mathscr{E} \otimes_{\mathscr{D}} \mathscr{N}_0$. Let G_{q_0} be the isotropy subgroup of G at q_0. Then G_{q_0} acts on $\Lambda_{q_0} = T^*_{q_0}X \cap \Lambda$ and we have $G_{q_0}\cdot p_0 = \Lambda_{q_0} \cap \Lambda_0$. Since Λ_{q_0} is a vector space, we can identify it with its tangent space. Then $\mathfrak{g}_{q_0} := \operatorname{Lie}(G_{q_0})$ acts on Λ_{q_0}, and $\mathfrak{g}_{q_0}\cdot p_0 = \Lambda_{q_0}$. Especially there exists an element $A \in \mathfrak{g}_{q_0}$ such that $Ap_0 = p_0$. Let $\mathfrak{h} = \{A \in \mathfrak{g}_{q_0} \mid Ap_0 = p_0\}$ and H be the corresponding connected subgroup of G. Then $\mathbf{C}p_0$ gives a non-trivial character of H. Hence the action of a maximal torus, say T, of H on $\mathbf{C}p_0$ is non-trivial. Then we can find an element $A_0 \in \mathfrak{t} := \operatorname{Lie}(T)$ such that

$$A_0 p_0 = p_0.$$

We fix such a torus T and an element $A_0 \in \mathfrak{t}$.

Take an element A_1 of a Cartan subalgebra of $\operatorname{Lie}(G)$ containing $\operatorname{Lie}(T)$ such that $\sum_i \delta_i \varpi^{(i)}(A_1) = 1$. (Here and below, we shall identify $\operatorname{Hom}(G, \mathbf{C}^\times)$ with the corresponding subgroup of $\operatorname{Hom}(\operatorname{Lie}(G), \mathbf{C})$.) The element A_1 of $\operatorname{Lie}(G)$ induces a vector field of X, which we consider as a differential operator (cf. (2.1)). Denote by σ its principal symbol. We can show that for any $\mu \in \mathbf{N}^l$, $f^\mu | W' = \hat{f}^\mu \sigma^m$ for some $m \in \mathbf{N}$ and a regular function $\hat{f}^\mu$ in a neighbourhood of

p_0 such that $\hat{f}^{\mu}|\Lambda \not\equiv 0$. Finally, let Δ_0 be the set of $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbf{N}^l$ such that $\mathrm{GCD}(\alpha_1, \dots, \alpha_l) = 1$.

0.5. Now let us state our result.

Theorem. (1) *For any $\mu = (\mu_1, \dots, \mu_l) \in \mathbf{N}^l$, there exist a microdifferential operator $P_\mu(\zeta) \in \mathscr{E}[\zeta]$ of order $-m$ defined in a neighbourhood of p_0 and a polynomial $b_\mu(\zeta) \in \mathbf{C}[\zeta]$ such that*

$$f^{\mu} \cdot f^{\lambda} = b_\mu(\lambda) P_\mu(\lambda) f^{\lambda} \qquad \text{for any } \lambda \in \mathbf{C}^l,$$

and that the restriction to Λ of the principal symbol of $P_\mu(\zeta)$ is $\hat{f}^{\mu}$.

(2) *The polynomial $b_\mu(\zeta)$ is uniquely determined.*

(3) *There exist $c_\mu \in \mathbf{C}^\times$, a finite subset Δ of Δ_0, $n(\alpha) \in \mathbf{N}\,(\alpha \in \Delta)$, and positive rational numbers $a_{\alpha,j}(\alpha \in \Delta,\ 1 \le j \le n(\alpha))$ such that*

$$b_\mu(\zeta) = c_\mu \prod_{\substack{\alpha \in \Delta \\ 1 \le j \le n(\alpha)}} (\alpha_1 \zeta_1 + \cdots + \alpha_l \zeta_l + a_{\alpha,j})$$

(4) $\deg b_\mu(\zeta) = -\sum_{i=1}^{l} \mu_i \varpi^{(i)}(A_0)$.

Conventions. We keep the notation introduced in the introduction.

We denote the complex number field (resp. the rational integer ring) by $\mathbf{C}$ (resp. $\mathbf{Z}$), and put $\mathbf{N} = \{0, 1, 2, \dots\}$.

For a complex manifold Z, $\mathcal{O} = \mathcal{O}_Z$ denotes the sheaf of holomorphic functions. For a coherent sheaf of ideals $\mathscr{I}$ of $\mathcal{O}_Z$, we denote by $V(\mathscr{I})$ the analytic subset of Z defined by $\mathscr{I}$.

We denote the local coordinate of X by $\{x_1, \dots, x_n\}$ and the corresponding fibre coordinate of T^*X by $\{y_1, \dots, y_n\}$. The sheaf of differential (resp. microdifferential) operators is denoted by $\mathscr{D} = \mathscr{D}_X$ (resp. $\mathscr{E} = \mathscr{E}_X$). For a coherent $\mathscr{D}_X$-module $\mathscr{M}$, we denote its characteristic variety (resp. characteristic cycle) by $\mathrm{Ch}\,\mathscr{M}$ (resp. $\mathbf{Ch}\,\mathscr{M}$). For an irreducible analytic subset C of T^*X, we denote the multiplicity of a coherent $\mathscr{D}$-module (or $\mathscr{E}$-module) $\mathscr{M}$ along C by $\mathrm{mlt}(C, \mathscr{M})$.

We put $\mathrm{Lie}(G) = \mathfrak{g}$ and $\mathrm{Lie}(T) = \mathfrak{t}$. For $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathbf{C}^l$ and $A \in \mathfrak{g}$, put $\lambda(A) = \langle \lambda, A \rangle = \sum \lambda_i \varpi^{(i)}(A)$. In other words, we identify $\lambda \in \mathbf{C}^l$ with $\Sigma \lambda_i \varpi^{(i)}$. We identify $\mathrm{Hom}(G, \mathbf{C}^\times)$ with the corresponding subgroup of $\mathrm{Hom}(\mathfrak{g}, \mathbf{C})$ and we use the additive notation for characters of G, e.g., $(\varpi + \varpi')(g) = \varpi(g)\varpi'(g)$ for $g \in G$.

A lowercase Greek letter without a suffix always denotes an l-tuple or the character of $\mathfrak{g}$ (or G) identified with it. (Thus $\varpi^{(i)}$ denote the natural basis elements of $\mathbf{C}^l$, and also the characters identified with them.) There are two exceptions for this convention. One is δ, which denotes the "δ-function" in (4.3) and (4.4). (We do not use the letter δ for this meaning in other places.) The other is σ, which denotes the principal symbol of a (micro-)differential operator or the principal symbol of a local section of a simple holonomic system etc. The i-th component of an l-tuple is denoted by the same letter with the suffix i. The element $\delta \in (\mathbf{Z}_{>0})^l$ is fixed throughout the paper (except in (4.3) and (4.4)).

We shall mainly consider a small neighbourhood of p_0 and often omit to say "in a neighbourhood of p_0". In such a case, we often write $\mathcal{N}$ etc. for $\mathscr{E}\otimes_{\mathscr{D}}\mathcal{N}$ etc.

1. In this section, we collect results which can be obtained without the assumptions (a)–(d).

Lemma 1.1. [9], [5]. *For any $\mu\in\mathbf{N}^l$, there exist a differential operator $P'_\mu(\zeta)\in\Gamma(X, \mathscr{D}_X[\zeta])$ and a non-zero polynomial $B_\mu(\zeta)\in\mathbf{C}[\zeta]$ such that*

$$P'_\mu(\zeta)f^{\zeta+\mu}=B_\mu(\zeta)f^\zeta.$$

Moreover, we can take $B_\mu(\zeta)$ so that

$$B_\mu(\zeta)=\prod_i(\alpha_1^{(i)}\zeta_1+\cdots+\alpha_l^{(i)}\zeta_l+a_i),$$

where $\alpha_j^{(i)}\in\mathbf{N}$, $\mathrm{GCD}(\alpha_1^{(i)},\dots,\alpha_l^{(i)})=1$ and $a_i\in\mathbf{Q}_{>0}$ for any i.

1.2. Take $B_\mu(\zeta)$ and $P'_\mu(\zeta)$ so that B_μ has the special form asserted in the latter half of the above lemma. Put $B(\lambda, s)=B_\delta(\lambda+s\delta)$, and $P'(\lambda, s)=P'_\delta(\lambda+s\delta)$ for $\lambda\in\mathbf{C}^l$. Then

(1.2.1) $$P'(\lambda, s)f^{\lambda+(s+1)\delta}=B(\lambda, s)f^{\lambda+s\delta}.$$

Put $\mathcal{N}'=\mathcal{N}'(\lambda)=\mathscr{D}[s]f^{\lambda+s\delta}$, $f^{\lambda+0\delta}=(f^{\lambda+s\delta} \bmod s\mathcal{N}'(\lambda))$, and $\mathcal{N}'_0=\mathcal{N}'_0(\lambda)=\mathscr{D}f^{\lambda+0\delta}$. Here $f^{\lambda+s\delta}$ is the restriction of f^ζ to $\{(x, \zeta)\in X_0\times(\lambda+\mathbf{C}\delta)\}$ (cf. (0.4)).

Lemma 1.3. *If $B(\lambda, -j)\neq 0$ for any $j=1, 2, 3,\dots$, then $\mathscr{D}f^{\lambda+0\delta}=(\mathscr{D}f^{\lambda+0\delta})[(f^\delta)^{-1}]$.*

The proof is the same as that of [7, Lemma 2.3]. Read the proof replacing $f^\alpha u\to f^{\lambda+0\delta}$, $f^s u\to f^{\lambda+s\delta}$, $\alpha\to 0$ and $f\to f^\delta$.

Lemma 1.4. *For a sufficiently large integer m, $\mathcal{N}'_0(\lambda-m\delta)=\mathcal{N}'_0(\lambda)[(f^\delta)^{-1}]$.*

Proof. We may assume that $B(\lambda-m\delta, s)=B(\lambda, s-m)$. If m is sufficiently large, then $B(\lambda-m\delta, -j)=B(\lambda, -j-m)\neq 0$ for $j=1, 2,\dots$. Hence

$$\begin{aligned}\mathcal{N}'_0(\lambda-m\delta)&=\mathcal{N}'_0(\lambda-m\delta)[(f^\delta)^{-1}], \quad \text{by (1.3)}\\ &=\mathcal{N}'(\lambda-m\delta)[(f^\delta)^{-1}]/s\mathcal{N}'(\lambda-m\delta)[(f^\delta)^{-1}]\\ &=\mathcal{N}'(\lambda)[(f^\delta)^{-1}]/s\mathcal{N}'(\lambda)[(f^\delta)^{-1}]\\ &=\mathcal{N}'_0(\lambda)[(f^\delta)^{-1}].\end{aligned}$$

1.5. A coherent $\mathscr{D}$-module $\mathcal{M}$ is said to be holonomic (resp. subholonomic) if $\dim \mathrm{Ch}(\mathcal{M})\leq\dim X$ (resp. $\leq\dim X+1$).

Lemma 1.6. *For $\lambda\in\mathbf{C}^l$, the $\mathscr{D}_X$-module $\mathcal{N}'(\lambda)$* (resp. $\mathcal{N}'_0(\lambda)$) *is subholonomic (resp. holonomic). Moreover,* $\mathrm{Ch}(\mathcal{N}'(\lambda))=W'$ *and the multiplicity of $\mathcal{N}'(\lambda)$ at W' is one.*

This lemma can be proved in the same way as in [6] (cf. [1]).

Lemma 1.7. *Let* $\lambda \in \mathbf{C}^l$. (1) *The characteristic cycle* $\mathbf{Ch}\,\mathcal{N}_0'(\lambda)$ *is determined solely by* f^δ.

(2) *The characteristic variety of* $\mathcal{N}_0'(\lambda)$ *is* W_0'. (*See* (0.3) *for* W_0'.)

Proof. Since $\mathcal{N}'$ is subholonomic, we can apply [3, 2.8.5], and we can see that

$$\begin{aligned}\mathbf{Ch}\,\mathcal{N}_0'(\lambda) = \mathbf{Ch}\,\mathcal{N}'/s\mathcal{N}' = \mathbf{Ch}\,\mathcal{N}'/(s+m)\mathcal{N}' &= \mathbf{Ch}\mathcal{N}_0'(\lambda - m\delta)\\ &= \mathbf{Ch}\,\mathcal{N}_0'(\lambda)[(f^\delta)^{-1}]\end{aligned}$$

for a sufficiently large integer m. Hence we get (1) by the same argument as in [4, 9.3]. By (1), we have

$$\mathbf{Ch}\,\mathcal{N}_0'(\lambda) = \mathbf{Ch}\,\mathcal{N}_0'(0) = \mathbf{Ch}\,\mathcal{D}[s](f^\delta)^s/s\mathcal{D}[s](f^\delta)^s,$$

whose support is known to be W_0' [11, appendix].

Lemma 1.8. *If* $\lambda \in \mathbf{C}$ *and* $B_\mu(\lambda) \neq 0$, *then* $\mathcal{D}(f^\mu \cdot f^\lambda) = \mathcal{D}f^\lambda (= \mathcal{N}_0(\lambda))$ *and* $\mathcal{D}(f^\mu \cdot f^{\lambda + 0\delta}) = \mathcal{D}f^{\lambda + 0\delta}(= \mathcal{N}_0'(\lambda))$.

Proof. These follow from the functional equation of (1.1).

1.9. Let $\{F_j\mathcal{E}\}_{j\in\mathbf{Z}}$ be the order filtration of the sheaf $\mathcal{E}$ of microdifferential operators, $\hat{\mathcal{E}}_p = \varprojlim \mathcal{E}_p/F_j\mathcal{E}_p$ for $p \in T^*X$, and $\mathcal{O} = \mathcal{O}_{T^*X}$ the sheaf of analytic functions on T^*X. ($\mathcal{E}_p$ etc. denotes the stalk.) For $P \in F_j\mathcal{E}$, we denote its principal symbol by $\sigma(P) = \sigma_j(P)$. Let p be a point of T^*X, and let us consider everything in a neighbourhood of p in (1.10) and (1.11). Let $\mathcal{I}$ be a left coherent ideal of $\mathcal{E}$. We denote by $\sigma(\mathcal{I})$ its symbol ideal, i.e., the ideal of $\mathcal{O}_{T^*X}$ generated by $\{\sigma(P) | P \in \mathcal{I}\}$, and put $V = V(\sigma(\mathcal{I}))$.

Lemma 1.10. *Let* $P \in F_k\mathcal{E}_p$. *If* $\sigma(P) \in \sigma(\mathcal{I})$, *then there exists* $Q \in F_k\mathcal{E}_p \cap \mathcal{I}_p$ *such that* $\sigma(P) = \sigma(Q)$.

Proof. Take $a_j \in \mathcal{O}_p$ and $R_j \in F_{m_j}\mathcal{E}_p \cap \mathcal{I}_p$ so that $\sigma_k(P) = \sum_j a_j(x, y)\sigma_{m_j}(R_j)$. ($x = (x_1, \dots, x_n)$ is a local coordinate of the base space X and $y = (y_1, \dots, y_n)$ is the corresponding fibre coordinate of T^*X.) If p lies in the zero section T_X^*X of T^*X, then $a_j(x, y)$ is a finite or infinite sum of analytic functions which are homogeneous polynomials in y. Hence we may assume that $a_j(x, y)$ is a homogeneous in y of degree $k - m_j$ in this case. Next, assume that p lies outside of T_X^*X. Take a hypersurface Y of $T^*X \setminus T_X^*X$ so that $p \in Y$ and the composition of $Y \to T^*X \setminus T_X^*X \to P^*X$ is an open immersion, where P^*X is the bundle of projective spaces obtained from T^*X. Then $a_j | Y$ can be uniquely extended to an analytic function, say a_j', in a neighbourhood of p which is homogeneous in y of degree $k - m_j$. Since $\sigma(P) = \sum a_j'\sigma(R_j)$, we may assume that $a_j(x, y)$ is homogeneous in y of degree $k - m_j$ also in this case. Then in both cases, we can take $S_j \in F_{k-m_j}\mathcal{E}_p$ so that $\sigma(S_j) = a_j$. Put $Q = \sum S_j R_j$. Then $Q \in F_k\mathcal{E}_p \cap \mathcal{I}_p$

and $\sigma_k(P)=\sigma_k(Q)$.

Lemma 1.11. *Assume that* $\sqrt{\sigma(\mathscr{I})}=\sigma(\mathscr{I})$. *Then for any* $P\in\mathscr{E}_p$,

(1) $P\in\mathscr{I}_p$ *or*

(2) *there exists* $Q\in\mathscr{E}_p$ *such that* $P-Q\in\mathscr{I}_p$ *and* $\sigma(Q)\not\equiv 0$ *on* $V=V(\sigma(\mathscr{I}))$.

Proof. Assume that $P_k:=P\in F_k\mathscr{E}_p$. If $\sigma(P_k)\not\equiv 0$ on V, there is nothing to prove. Assume the contrary. Then $\sigma(P_k)\in\sigma(\mathscr{I})$. Take $Q_k\in F_k\mathscr{E}_p\cap\mathscr{I}_p$ so that $\sigma_k(P_k)=\sigma_k(Q_k)$, and put $P_{k-1}=P_k-Q_k$. If $\sigma_{k-1}(P_{k-1})\not\equiv 0$ on V, then we get the desired assertion. If $\sigma_{k-1}(P_{k-1})\equiv 0$ on V, then we can repeat the same argument. If this argument stops after several steps, then we get the desired assertion. Thus we may assume that this argument can be repeated infinitely. Then $P_k=\sum_{j\le k}Q_j$ with some $Q_j\in F_j\mathscr{E}_p\cap\mathscr{I}_p$. (Here the summation has a meaning in $\hat{\mathscr{E}}_p$.) Let $\{J_1,\dots,J_N\}$ be an involutive base [11, 2.9] of $\mathscr{I}$, and $\operatorname{ord}J_i=m_i$. Then by the same argument as in the proof of (1.10), we can show that $1\cdot\sigma(Q_j)=\sum_{i=1}^N r_{ji}(x,y)\sigma(J_i)$ with some analytic functions r_{ji} which are homogeneous of degree $j-m_i$. Applying [11, 2.10] to these relations, we can take $S\in F_0\mathscr{E}_p$ and $R'_{ji}\in F_{j-m_i}\mathscr{E}_p$ so that $\sigma(S)=1$, $\sigma(R'_{ji})=r_{ji}$ and $SQ_j=\sum_{i=1}^N R'_{ji}J_i$. Then $S^{-1}R'_{ji}=:R_{ji}$ satisfy $Q_j=\sum_{i=1}^N R_{ji}J_i$ and $R_{ji}\in F_{j-m_i}\mathscr{E}_p$. Put $R_i=\sum_{j\le k}R_{ji}\in\hat{\mathscr{E}}_p$. Then

$$P_k=\sum_{j\le k}Q_j=\sum_{i=1}^N R_iJ_i\in\mathscr{E}_p\cap\hat{\mathscr{E}}_p\mathscr{I}_p.$$

It is known that $\hat{\mathscr{E}}_p$ is a faithfully flat right $\mathscr{E}_p$-module [10, chapter 2, Theorem 3.4]. Hence by [2, chapter 1, §3, Proposition 8, (2)], $\mathscr{E}_p\cap\hat{\mathscr{E}}_p\mathscr{I}_p=\mathscr{I}_p$. Thus $P=P_k\in\mathscr{I}_p$ and we get the desired assertion.

2. The purpose of this section is to prove the smoothness and the simplicity of characteristic varieties of certain $\mathscr{D}$-modules. From now on, we assume the assumptions (a)–(d).

2.1. For $A\in\mathfrak{g}$, define the vector field $P(A)$ on X by

$$(P(A)F)(x)=\frac{d}{dt}F(e^{tA}x)|_{t=0}.$$

We shall consider $P(A)$ as a (micro-)differential operator on X.

Lemma 2.2. *For* $A\in\mathfrak{g}$, *the principal symbol* $\sigma(P(A))$ *of* $P(A)$ *is* $\langle Ax,y\rangle$, *where* $\langle\ ,\ \rangle$ *is the natural pairing of the tangent bundle* TX *and the cotangent bundle* T^*X.

Proof. If $e^{tA}x=(a_1(t,x),\dots,a_n(t,x))$, then

$$(P(A)F)(x)=\frac{d}{dt}F(a_1(t,x),\dots)|_{t=0}=\sum_{i=1}^n\frac{\partial a_i}{\partial t}(0,x)\frac{\partial F}{\partial x_i}(x),$$

$$P(A) = \sum_{i=1}^{n} \frac{\partial a_i}{\partial t}(0, x)\frac{\partial}{\partial x_i}, \text{ and}$$

$$\sigma(P(A)) = \sum_{i=1}^{n} \frac{\partial a_i}{\partial t}(0, x)y_i = \langle Ax, y\rangle.$$

2.3. Let W be the Zariski closure of

$$\{(x, \sum_{i=1}^{l} s_i \operatorname{grad} \log f_i(x)) \in T^*X \mid x \in X, s_i \in \mathbf{C}, f_i(x) \neq 0\}$$

in T^*X. Note that $\Lambda \subset W' \subset W$ (cf. the assumption (d)).

Lemma 2.4. *W is an irreducible variety of dimension $n + l$.*

Proof. It suffices to prove that $\{\operatorname{grad} \log f_i(x)\}_{1 \le i \le l}$ are linearly independent for generic x. Assume the contrary. Then there are (local) regular functions $a_1, \ldots, a_n$ such that $\sum_{i=1}^{l} a_i(x)\frac{\partial f_i}{\partial x_j} = 0$ $(1 \le j \le n)$. Then for any vector field V, $\sum_{i=1}^{l} a_i V(f_i) = 0$. Taking $V = P(A)$ $(A \in \mathfrak{g})$, we get $\sum_{i=1}^{l} a_i(x)\varpi^{(i)}(A)f_i(x) = 0$. By the assumption (b), we get $a_i(x)f_i(x) = 0$ and $a_i = 0$. Thus we get the linear independence.

2.5. Let

$$\mathfrak{g}_0 = \{B \in \mathfrak{g} \mid \varpi^{(i)}(B) = 0 \ (1 \le i \le l)\},$$
$$\mathfrak{g}_0' = \{B \in \mathfrak{g} \mid \delta(B) = 0\},$$

$\{B_i\}$ be a linear basis of $\mathfrak{g}_0$, and take $C_j \in \mathfrak{g}$ $(1 \le j \le l)$ so that $\varpi^{(i)}(C_j) = \delta_{ij}$ (cf. the assumption (b)). Then $\{B_i\} \cup \{C_j\}$ gives a linear basis of $\mathfrak{g}$.

Lemma 2.6. *If $B \in \mathfrak{g}_0$ (resp. $B \in \mathfrak{g}_0'$), then $\sigma(P(B))$ vanishes identically on W (resp. W').*

Proof. Assume that $B \in \mathfrak{g}_0$. Then

$$0 = P(B)f^{\zeta} = \sum_{i=1}^{l} \zeta_i \frac{P(B)f_i}{f_i} f^{\zeta} = \sum_{i=1}^{l} \zeta_i P(B)(\log f_i) \cdot f^{\zeta}.$$

If $P(B) = \sum_{j=1}^{n} a_j(x)\frac{\partial}{\partial x_j}$ in a local coordinate system $\{x_j\}$, then

$$\sum_{i=1}^{l} \sum_{j=1}^{n} \zeta_i a_j(x) \frac{\partial}{\partial x_j}(\log f_i) = 0, \text{ i.e.,}$$

$$\sigma(P(B)) = \sum_{j=1}^{n} a_j(x)y_j = 0 \text{ for } (y_1, \ldots, y_n) = \sum_{i=1}^{l} \zeta_i \operatorname{grad} \log f_i.$$

Thus we get the one half. The other half can be proved in the same way.

2.7. Recall that

$$\mathcal{N} = \mathcal{D}[\zeta]f^{\zeta},\ \mathcal{N}_0 = \mathcal{N}_0(\lambda) = \mathcal{N}/\sum_{i=1}^{l}(\zeta_i - \lambda_i)\mathcal{N} = \mathcal{D}f^{\lambda},$$

$$\mathcal{N}' = \mathcal{N}'(\lambda) = \mathcal{D}[s]f^{\lambda+s\delta},\ \mathcal{N}_0' = \mathcal{N}_0'(\lambda) = \mathcal{N}'/s\mathcal{N}' = \mathcal{D}f^{\lambda+0\delta}$$

for $\lambda\in\mathbf{C}^l$. Put

$$\mathcal{N}_0'' = \mathcal{N}_0''(\lambda) = \mathcal{D}/(\sum_i \mathcal{D}P(B_i) + \sum_{j=1}^{l}\mathcal{D}(P(C_j) - \lambda_j)).$$

From now on, we shall consider everything in a neighbourhood of p_0 (cf. (0.3)). Note that $\mathcal{N} = \mathcal{D}f^{\zeta}$ etc., since $P(C_j)f^{\zeta} = \zeta_j f^{\zeta}$. Hence $\mathcal{N}$ and $\mathcal{N}'$ are coherent $\mathcal{D}$-modules ($\mathcal{E}$-modules), and we can consider their characteristic varieties etc.

Lemma 2.8. *The $\mathcal{E}$-modules $\mathcal{N}_0$, $\mathcal{N}_0'$ and $\mathcal{N}_0''$ are naturally isomorphic to each other (in a neighbourhood of $p_0\in\Lambda_0$). They are simple holonomic systems* [11, 2.8]. *Especially, their multiplicity along Λ is one.*

Proof. Since $\mathcal{N}_0''$ is a simple holonomic system (cf. [11, 4.8]), it suffices to prove the first assertion. The natural surjection

$$\mathcal{E}[\zeta, s]f^{\zeta}/\sum_{i=1}^{l}(\zeta_i - \lambda_i - s\delta_i)\mathcal{E}[\zeta, s]f^{\zeta} \longrightarrow \mathcal{E}[s]f^{\lambda+s\delta}$$

induces a surjection

$$\mathcal{N}_0 = \mathcal{E}[\zeta]f^{\zeta}/\sum_{i=1}^{l}(\zeta_i - \lambda_i)\mathcal{E}[\zeta]f^{\zeta} \longrightarrow \mathcal{E}[s]f^{\lambda+s\delta}/s\mathcal{E}[s]f^{\lambda+s\delta} = \mathcal{N}_0'.$$

The natural surjection

$$\mathcal{E}/\sum_j \mathcal{E}P(B_j) \longrightarrow \mathcal{E}f^{\zeta} = \mathcal{E}[\zeta]f^{\zeta}$$

induces a surjection

$$\begin{aligned}\mathcal{N}_0'' &= \mathcal{E}/(\textstyle\sum\mathcal{E}P(B_i) + \sum_{j=1}^{l}\mathcal{E}(P(C_j) - \lambda_j))\\ &\longrightarrow \mathcal{E}f^{\zeta}/\sum_{j=1}^{l}\mathcal{E}(P(C_j) - \lambda_j)f^{\zeta} = \mathcal{E}[\zeta]f^{\zeta}/\sum_{j=1}^{l}\mathcal{E}[\zeta](\zeta_j - \lambda_j)f^{\zeta} = \mathcal{N}_0.\end{aligned}$$

By the assumption (d) and (1.7, (2)), $\mathcal{N}_0' \neq 0$. Since the multiplicity of $\mathcal{N}_0''$ along Λ is one, the composition of the surjections $\mathcal{N}_0'' \to \mathcal{N}_0 \to \mathcal{N}_0'$ is an isomorphism. Hence these morphisms are isomorphisms.

2.9. Let us fix linear forms $\alpha^{(i)}(\zeta) = \sum_{j=1}^{l}\alpha_j^{(i)}\zeta_j$ $(1 \leq i \leq l)$ which are linearly independent, and $a_i\in\mathbf{C}$ $(1 \leq i \leq l)$. Along with the $\mathcal{D}$-modules given in (2.7), we also consider the $\mathscr{D}$-modules

$$\mathscr{N}_k = \mathscr{N}/\sum_{i>k}(\alpha^{(i)}(\zeta)-a_i)\mathscr{N}, \text{ and}$$

$$\mathscr{N}_k'' = \mathscr{D}/(\sum_{B\in\mathfrak{g}_0}\mathscr{D}P(B)+\sum_{i>k}\mathscr{D}P(\alpha^{(i)}(C))-a_i)),$$

where $\alpha^{(i)}(C)=\sum_{j=1}^{l}\alpha_j^{(i)}C_j$. If we need to make explicit the dependence on $\alpha^{(i)}$ and/or a_i, we write $\mathscr{N}_k=\mathscr{N}_k(\alpha)=\mathscr{N}_k(a)=\mathscr{N}(\alpha;a)$ etc. Thus $\mathscr{N}_0(\lambda)$ in (2.7) is $\mathscr{N}_0(\varpi^{(i)},\dots,\varpi^{(l)};\lambda)$. On the other hand, if $\zeta=\lambda$ is the (unique) solution of $\alpha^{(i)}(\zeta)-a_i=0$ $(1\le i\le l)$, then $\mathscr{N}_0$ and $\mathscr{N}_0''$ defined here coincide with $\mathscr{N}_0(\lambda)$ and $\mathscr{N}_0''(\lambda)$ given in (2.7). Note also that $\mathscr{N}_l$ coincides with $\mathscr{N}$ in (2.7).

Let u_k (resp. u_k'') be the section of $\mathscr{N}_k$ (resp. $\mathscr{N}_k''$) corresponding to $f^\zeta\in\mathscr{N}$ (resp. $1\in\mathscr{D}$), and $\mathscr{T}_k$ (resp $\mathscr{T}_k''$) its annihilator in $\mathscr{E}$

Lemma 2.10. *Put* $\sigma^{(j)}=\sigma(P(\alpha^{(j)}(C)))$ *and* $\mathscr{K}_k=\sum_i\mathscr{O}\sigma(P(B_i))+\sum_{j>k}\mathscr{O}\sigma^{(j)}$, *where* $\mathscr{O}=\mathscr{O}_{T^*X}$. *Then* $\sqrt{\mathscr{K}_k}=\mathscr{K}_k$ *and* $V(\mathscr{K}_k)$ *is a non-singular manifold of dimension* $n+k$.

Proof. *Let* $K_k=\{dF(p_0)|F\in\mathscr{K}_k\}$. Since Λ_0 is G-homogeneous and $\dim\Lambda_0=n$, $\dim K_0=n$ (cf. (2.2)). By (2.4) and (2.6), we have $2n-\dim K_l\ge\dim W=n+l$, i.e., $\dim K_l\le n-l$. On the other hand, $\dim K_k\ge\dim K_{k+1}\ge\dim K_k-1$. By these relations we get $\dim K_k=n-k$. Since K_k is the **C**-linear span of $\{d\sigma(P(B_i))(p_0)\}\cup\{d\sigma^{(j)}(p_0)|k<j\le l\}$, we can rearrange $\{B_i\}$ so that K_l is spanned by $\{d\sigma(P(B_i))(p_0)|1\le i\le n-l\}$. Put $\mathscr{K}_k'=\sum_{i=1}^{n-l}\mathscr{O}\sigma(P(B_i))+\sum_{j>k}\mathscr{O}\sigma^{(j)}$. Then $K_k=\{dF(p_0)|F\in\mathscr{K}_k'\}$ for any k. Hence $V(\mathscr{K}_k')$ is a non-singular manifold of dimension $n+k$ and $\sqrt{\mathscr{K}_k'}=\mathscr{K}_k'$. Especially, $\mathscr{K}_k'$ is the sheaf of functions vanishing identically on $V(\mathscr{K}_k')$.

Suppose that $V(\mathscr{K}_k')=V(\mathscr{K}_k)$ for some k. Since a function in $\mathscr{K}_k$ vanishes identically on $V(\mathscr{K}_k')=V(\mathscr{K}_k)$, $\mathscr{K}_k\subset\mathscr{K}_k'$. Hence $\mathscr{K}_k=\mathscr{K}_k'$ and we get the desired assertion. Thus it suffices to prove the coincidence of these two varieties.

First, let us consider the case where $k=0$. Then by (2.8),

$$V(\mathscr{K}_0')\supset V(\mathscr{K}_0)\supset V(\sigma(\mathscr{T}_0''))=\mathrm{Ch}\,\mathscr{N}_0''=\mathrm{Ch}\,\mathscr{N}_0'.$$

By the assumption (d) and (1.7, (2)), $\mathrm{Ch}\,\mathscr{N}_0'=\Lambda$ (in a neighbourhood of $p_0\in\Lambda_0$). Since $V(\mathscr{K}_0')$ is a non-singular manifold of dimension n and Λ is also of dimension n, we get

$$V(\mathscr{K}_0')=V(\mathscr{K}_0)=\Lambda. \tag{2.10.1}$$

Next, let us consider the case where $k=l$. By (2.6), $V(\mathscr{K}_l')\supset V(\mathscr{K}_l)\supset W$. Since $W\supset W'\supset W_0'=\Lambda\ni p_0$, W is a variety of dimension $n+l$ in a neighbourhood of p_0(cf. (2.4)). On the other hand, $V(\mathscr{K}_l')$ is a non-singular manifold of the same dimension $n+l$. Hence

$$V(\mathscr{K}_l')=V(\mathscr{K}_l)=W. \tag{2.10.2}$$

Let us consider the general case. Assume that $V(\mathscr{K}_k')=V(\mathscr{K}_k)$. Note that $V(\mathscr{K}_{k-1})$ is the subset of $V(\mathscr{K}_k)$ defined by $\sigma^{(k)}=0$ and that $V(\mathscr{K}_k)\supset V(\mathscr{K}_0)=\Lambda$ by (2.10.1). Hence

$$n+k-1=\dim V(\mathscr{K}_{k-1}')\geq \dim V(\mathscr{K}_{k-1})\geq$$
$$\dim V(\mathscr{K}_k)-1=\dim V(\mathscr{K}_k')-1=n+k-1.$$

Since $V(\mathscr{K}_{k-1}')$ is a non-singular manifold and $V(\mathscr{K}_{k-1})$ is its subvariety of the same dimension, $V(\mathscr{K}_{k-1}')=V(\mathscr{K}_{k-1})$. Thus we get the desired assertion by the descending induction on k starting from (2.10.2).

In the proof of the above lemma, we have also get the following assertion.

Lemma 2.11. *$V(\mathscr{K}_l)=W$ and $V(\mathscr{K}_{k-1})$ is the hypersurface of $V(\mathscr{K}_k)$ defined by $\sigma^{(k)}=0$. More precisely, a holomorphic function on $V(\mathscr{K}_k)$ vanishing identically on $V(\mathscr{K}_{k-1})$ is divisible by $\sigma^{(k)}$.*

Lemma 2.12. *The characteristic variety of $\mathscr{N}=\mathscr{D}[\zeta]f^{\zeta}$ contains W.*

Proof. Since $\mathscr{D}f^{\lambda+s\delta'}(\delta'\in(\mathbf{Z}_{>0})^l)$ is a quotient of $\mathscr{N}$, $\mathrm{Ch}\,\mathscr{N}$ contains $\mathrm{Ch}\,\mathscr{D}f^{\lambda+s\delta'}$. Since

$$\mathrm{Ch}\,\mathscr{D}f^{\lambda+s\delta'}\supset\{(x,\ s\textstyle\sum\delta_i'\operatorname{grad}\log f_i(x))\,|\,x\in X,\ s\in\mathbf{C},\ f_i(x)\neq 0\}$$

(cf. (1.6)), $\mathrm{Ch}\,\mathscr{N}$ contains the union of the right hand side for various $\delta'\in(\mathbf{Z}_{>0})^l$ and also contains its Zariski closure, which is W.

Lemma 2.13. $\sigma(\mathscr{T}_k)=\sigma(\mathscr{T}_k'')=\mathscr{K}_k$.

Proof. Since $\mathscr{N}_k$ is a quotient of $\mathscr{N}_k''$,

$$\mathscr{T}_k\supset\mathscr{T}_k''\supset\sum_{B\in\mathfrak{g}_0}\mathscr{E}P(B)+\sum_{i>k}\mathscr{E}(P(\alpha^{(i)}(C))-a_i).$$

Hence $\sigma(\mathscr{T}_k)\supset\sigma(\mathscr{T}_k'')\supset\mathscr{K}_k$. Since $\sqrt{\mathscr{K}_k}=\mathscr{K}_k$ by (2.10), it is enough to show that $V(\sigma(\mathscr{T}_k))=V(\mathscr{K}_k)$, which we shall prove by the descending induction on k. By (2.11) and (2.12), we have

(2.13.1) $$W\subset\mathrm{Ch}\,\mathscr{N}=V(\sigma(\mathscr{T}_l))\subset V(\mathscr{K}_l)=W,$$

and we get the equality for $k=l$.

Assume that $V(\sigma(\mathscr{T}_k))=V(\mathscr{K}_k)$. Since $\mathscr{E}[\zeta]f^{\zeta}=\mathscr{E}f^{\zeta}$, $\mathscr{N}_k=\mathscr{E}u_k$. (See (2.9) for u_k.) Define the $\mathscr{E}$-endomorphism F_k of $\mathscr{N}_k$ by

$$F_k(u_k)=(P(\alpha^{(k)}(C))-a_k)u_k=(\alpha^{(k)}(\zeta)-a_k)u_k.$$

Then $\mathscr{N}_{k-1}=\mathscr{N}_k/F_k(\mathscr{N}_k)$, $\sigma(P(\alpha^{(k)}(C))-a_k)=\sigma^{(k)}$, $\operatorname{supp}\mathscr{N}_k=V(\mathscr{K}_k)$ and the multiplicity of $\mathscr{N}_k$ along $V(\mathscr{K}_k)$ is one. Hence

$$V(\sigma(\mathscr{T}_{k-1}))=\operatorname{supp}\mathscr{N}_{k-1}=\operatorname{supp}\mathscr{N}_k/F_k(\mathscr{N}_k)$$
$$=\{(x,y)\in\operatorname{supp}\mathscr{N}_k\,|\,\sigma^{(k)}(x,y)=0\}\qquad\text{by [11, Proposition A.4]}$$

$= \{(x, y) \in V(\sigma(\mathscr{T}_k)) | \sigma^{(k)} = 0\}$

$= \{(x, y) \in V(\mathscr{K}_k) | \sigma^{(k)} = 0\}$ by the induction hypothesis

$= V(\mathscr{K}_{k-1})$ by (2.11).

From (2.10), (2.11) and (2.13), we get the following assertion.

Lemma 2.14. *Put $\Lambda_k = V(\mathscr{K}_k)$. Then Λ_k is a non-singular manifold of dimension $n+k$,* supp $\mathscr{N}_k = \Lambda_k$, *and the multiplicity of $\mathscr{N}_k$ along Λ_k is one. Especially* supp $\mathscr{N} = W$ *(in a neighbourhood of p_0).*

3. Order

The purpose of this section is the calculation of orders. The main results are (3.3) and (3.6). First, let us show the existence of a local coordinate system suitable for our calculation.

Lemma 3.1. *Let X_0 be a smooth algebraic variety over an algebraically closed field K, T a torus acting on X_0, $X_0 \supset X_1 \supset \cdots \supset X_k$ T-stable smooth subvarieties of X_0 (locally closed in X_0), $p \in \bigcap_i X_i$, and $d_i = \dim X_i$. Then there exists a local coordinate system $\{x_1, \ldots, x_{d_0}\}$ in a neighbourhood of p such that all the x_i's are relative T-invariants and $x_j \equiv 0$ on X_i for $j > d_i$.*

Proof. Here in this proof, we do not follow Conventions. By [13, Corollary 2], every point of X_0 admits a (Zariski open) T-stable affine neighbourhood. Hence we may assume from the beginning that X_0 is an affine variety. Let $\sigma: T \times X_0 \to X_0$ be the morphism defining the T-action on X_0. Note that $\mathrm{Hom}(T, K^\times)$ gives a K-linear basis of the regular function ring $K[T]$. Hence for any function $f \in K[X_0]$, there exist $\alpha_i \in \mathrm{Hom}(T, K^\times)$ and $f_i \in K[X_0]$ $(1 \le i \le n)$ such that $\sigma^* f = \sum_{i=1}^n \alpha_i \otimes f_i$ and $\alpha_i \ne \alpha_j$ $(i \ne j)$. Moreover, f_i's are uniquely determined and relative T-invariants corresponding to the characters α_i. Note also that $f = \sum_{i=1}^n f_i$.

As is seen by this fact, $K[X_0]$ is generated by some relatively T-invariant regular functions f_i $(1 \le i \le N)$. Let J be the ideal of the polynomial ring $K[z_1, \ldots, z_N]$ consisting of polynomials $\varphi(z)$ such that $\varphi(f_1, \ldots, f_N) = 0$. Then $K[X_0] = K[z_1, \ldots, z_N]/J$, i.e., $x \to (f_1(x), \ldots, f_N(x))$ gives a closed immersion $X_0 \to K^N$.

Let $\{y_1, \ldots, y_{d_0}\}$ be a local coordinate system of X_0 in a neighbourhood of p. (In other words, $X_0 \ni x \to (y_1(x), \ldots) \in K^{d_0}$ is étale in a neighbourhood of p. We do not assume that $y_i(p) = 0$.) Since rank $\left(\frac{\partial z_i}{\partial y_j}(p)\right)_{1 \le i \le N, 1 \le j \le d_0} = d_0$, we may assume that $\det\left(\frac{\partial z_i}{\partial y_j}(p)\right)_{1 \le i, j \le d_0} \ne 0$ by rearranging $\{z_1, \ldots, z_N\}$, if necessary. Then $\{z_1, \ldots, z_{d_0}\}$ gives a local coordinate system of X_0 in a neighbourhood of p. Since $z_i | X_0 = f_i$ are relative T-invariants, projecting to $K^{d_0} (\subset K^N)$, we may assume from the beginning that $X_0 = K^{d_0}$ on which T acts diagonally.

Let $g_1, \dots, g_{d_0} \in K[X_0] = K[z_1, \dots, z_{d_0}]$ be polynomials such that

$$g_i \in \{z_1, \dots, z_{d_0}\} \ (i \le d_1),$$

$$g_i | X_1 \equiv 0 \ (i > d_1),$$

$$\operatorname{rank}\left(\frac{\partial g_i}{\partial z_j}(p)\right)_{d_1 < i \le d_0, 1 \le j \le d_0} = d_0 - d_1, \text{ and}$$

$$\det\left(\frac{\partial g_i}{\partial z_j}(p)\right)_{1 \le i, j \le d_0} \ne 0.$$

Let $\sigma^* g_i = \sum_{j=1}^{k_i} \alpha_{ij} \otimes g_{ij}$, $\alpha_{ij} \in \operatorname{Hom}(T, K^\times)$ and $g_{ij} \in K[X_0]$, where $\alpha_{ij} \ne \alpha_{ij'} (j \ne j')$. Since X_1 is T-stable, $g_{ij} | X_1 \equiv 0$ if $i > d_1$. Hence $T_p X_1^\perp (= \{\xi \in T_p^* X \,|\, \xi \perp T_p X_1\})$ is equal to

$$\sum_{i > d_1} K(dg_i)(p) = \sum_{i > d_1} \sum_{j=1}^{k_i} K(dg_{ij})(p).$$

Choose $(d_0 - d_1)$-elements $h_{d_1+1}, \dots, h_{d_0}$ from $\{g_{ij} | d_1 < i \le d_0, \ 1 \le j \le k_i\}$ so that $T_p X_1^\perp = \sum_{d_1 < i \le d_0} K(dh_i)(p)$. Then $\{g_1, \dots, g_{d_1}, h_{d_1+1}, \dots, h_{d_0}\}$ gives a local coordinate system of X_0 in a neighbourhood of p such that all the coordinate functions are relative T-invariants and $h_i \equiv 0$ on X_1 for $d_1 < i \le d_0$. Repeating this procedure, we get the desired coordinate system.

3.2. Let $\operatorname{codim}_X G \cdot q_0 = c$. Applying (3.1) to the torus T (cf. (0.4)) and $X \supset G \cdot q_0 \supset \{q_0\}$, we get a local coordinate system $\{x_1, \dots, x_n\}$ of X in a neighbourhood of q_0 such that $x_i(tv) = \beta^{(i)}(t) x_i(v)$ $(t \in T, v \in X)$ with some characters $\beta^{(i)} \in \operatorname{Hom}(T, \mathbf{C}^\times)$, that $x_1 = \cdots = x_c = 0$ gives a system of defining equations of $G \cdot q_0$, and that $x_i(q_0) = 0$ for any i.

By (2.10) and (2.13), $\mathcal{M}_0 = \mathcal{E} f^\lambda$ is a simple holonomic system whose characteristic variety is Λ (in a neighbourhood of p_0), and hence we can consider the principal symbol $\sigma_\Lambda(f^\lambda)$ and the order $\operatorname{ord}_\Lambda(f^\lambda)$. (See [11, §3] for their definitions.) Let us calculate these invariants using the local coordinate system introduced above.

Lemma 3.3. $\operatorname{ord}_\Lambda f^\lambda = \lambda(A_0) - \operatorname{tr}(A_0 | \Lambda_{q_0}) + \frac{1}{2} \dim \Lambda_{q_0}$. *(See* (0.4) *for* A_0.)

Proof. Using the local coordinate system given in (3.2), we have

$$(P(A)F)(x_1, \dots, x_n) = \frac{d}{dt} F(\beta^{(1)}(e^{tA}) x_1, \dots)|_{t=0} = \left(\sum_{i=1}^n \beta^{(i)}(A) x_i \frac{\partial}{\partial x_i}\right) F \tag{3.3.1}$$

for $A \in \mathfrak{t}$. Let $\{y_1, \dots, y_n\}$ be the fibre coordinate of T^*X correponding to the coordinate $\{x_1, \dots, x_n\}$ of the base space. Then

$$\sigma_\Lambda(f^\lambda) = F(x, y) \sqrt{dy_1 \cdots dy_c dx_{c+1} \cdots dx_n} / \sqrt{dx_1 \cdots dx_n}$$

with some function $F(x, y)$ on Λ. Because of the relative G-invariance of $\sigma_\Lambda(f^\lambda)$, $F^{-1}(0)$ is G-stable. Hence F does not vanish at the point p_0 of the open G-orbit Λ_0. The vector field on Λ induced by the Hamiltonian vector field defined by the principal symbol of $P(A) = \sum_{i=1}^n \beta^{(i)}(A)x_i \frac{\partial}{\partial x_i}$ is

$$H_{\sigma(P(A))}|\Lambda = -\sum_{i=1}^{c} \beta^{(i)}(A)y_i \frac{\partial}{\partial y_i} + \sum_{i=c+1}^{n} \beta^{(i)}(A)x_i \frac{\partial}{\partial x_i}.$$

Put

$$L_{P(A)-\lambda(A)} = (H_{\sigma(P(A))}|\Lambda) + \left(-\lambda(A) - \frac{1}{2}\sum_{i=1}^{n} \beta^{(i)}(A)\right).$$

Since $(P(A) - \lambda(A))f^\lambda = 0$ for any $A \in \mathfrak{t}$, we get

$$\begin{aligned} 0 &= L_{P(A)-\lambda(A)}(\sigma_\Lambda(f^\lambda)\sqrt{dx_1 \cdots dx_n}) \\ &= \left\{\left(H_{\sigma(P(A))} - \lambda(A) - \frac{1}{2}\sum_{i=1}^{n} \beta^{(i)}(A)\right)F + \left(-\frac{1}{2}\sum_{i=1}^{c} \beta^{(i)}(A) + \frac{1}{2}\sum_{i=c+1}^{n} \beta^{(i)}(A)\right)F\right\} \\ &\quad \times \sqrt{dy_1 \cdots dy_c dx_{c+1} \cdots dx_n} \end{aligned}$$

by the definition of the principal symbol. Hence

$$\left(-\sum_{i=1}^{c} \beta^{(i)}(A)y_i \frac{\partial}{\partial y_i} + \sum_{i=c+1}^{n} \beta^{(i)}(A)x_i \frac{\partial}{\partial x_i} - \lambda(A) - \sum_{i=1}^{c} \beta^{(i)}(A)\right)F(x, y) = 0. \tag{3.3.2}$$

By the choice of p_0, A_0 and our local coordinate system, $p_0 = (0, \ldots, 0; y_1(p_0), \ldots, y_c(p_0), 0, \ldots, 0)$ and $-\beta^{(i)}(A_0)y_i(p_0) = y_i(p_0)$ for $1 \le i \le c$. Hence the value of (3.3.2) for $A = A_0$ and $(x, y) = p_0$, which is zero, is also equal to the value of

$$\left(\sum_{i=1}^{c} y_i \frac{\partial}{\partial y_i} - \lambda(A_0) - \sum_{i=1}^{c} \beta^{(i)}(A_0)\right)F(p_0) = (\deg_y F - \lambda(A_0) - \sum_{i=1}^{c} \beta^{(i)}(A_0))F(p_0)$$

by the Euler's identity for homogeneous functions. Since $F(p_0) \neq 0$, we get

$$\operatorname{ord}_\Lambda f^\lambda = \deg_y F + \frac{1}{2}c = \lambda(A_0) + \sum_{i=1}^{c} \beta^{(i)}(A_0) + \frac{1}{2}c.$$

Thus we get the desired expression for $\operatorname{ord}_\Lambda f^\lambda$.

Remark 3.4. Let

$$\mathscr{P} = \{p_0 \in \Lambda_{q_0} | Ap_0 = p_0 \text{ for some } A \in \mathfrak{t}\}$$

and consider the condition that

(f) $\qquad G\mathscr{P}$ is a dense subset of Λ.

If $\mathscr{E}f^\lambda$ is known to be simple holonomic on an open dense subset of Λ, then

(3.3) still holds under the weaker assumption (a) + (f).

3.5. By the assumption (b), $\delta=\sum\delta_i\varpi^{(i)}\neq 0$. Let A_1 be an element of a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$ such that $\delta(A_1)=1$. (Note that $[A, A_1]=0$ for any $A\in\mathfrak{t}$.) Put $\sigma=\sigma(P(A_1))$ and define $\mathfrak{g}_0'$ as in (2.5). Taking $\alpha^{(i)}(\zeta)$ in (2.9) suitably, we may assume that $V(\mathscr{K}_1)=W'$ and $\sigma^{(1)}=\sigma$. Hence $\sigma=0$ is a defining equation of Λ in W'. Let m_i be the largest integer such that σ^{m_i} divides f_i as elements of $\mathcal{O}_{W',p_0}$. Put $\hat{f}_i=f_i\sigma^{-m_i}$. Then $\hat{f}_i$ is a regular function on W' in a neighbourhood of p_0.

Lemma 3.6. $m_i=-\varpi^{(i)}(A_0)$.

Proof. The proof goes in a similar way as in (3.3). We keep the notation there. By (3.3.1), we get $\left(\sum_{i=1}^n\beta^{(i)}(A_0)x_i\frac{\partial}{\partial x_i}-\varpi^{(i)}(A_0)\right)f_i=0$. Since $[A_0, A_1]=0$, we have $[P(A_0), P(A_1)]=0$, $\{\sigma(P(A_0)), \sigma(P(A_1))\}=0$, where $\{\ ,\ \}$ denotes the Poisson bracket, and $H_{\sigma(P(A_0))}\sigma(P(A_1))=0$. Since $H:=H_{\sigma(P(A_0))}=\sum_{i=1}^n\beta^{(i)}(A_0)\left(x_i\frac{\partial}{\partial x_i}-y_i\frac{\partial}{\partial y_i}\right)$ also satisfies $Hf_i=\varpi^{(i)}(A_0)f_i$, we have $H\hat{f}_i=\varpi^{(i)}(A_0)\hat{f}_i$ on Λ, i.e.,

$$\left(-\sum_{i=1}^{c}\beta^{(i)}(A_0)y_i\frac{\partial}{\partial y_i}+\sum_{i=c+1}^{n}\beta^{(i)}(A_0)x_i\frac{\partial}{\partial x_i}-\varpi^{(i)}(A_0)\right)\hat{f}_i(x,y)=0$$

on Λ. If we can show that

(3.6.1) $$\hat{f}_i(p_0)\neq 0,$$

then we get $\deg_y\hat{f}_i=\varpi^{(i)}(A_0)$ in the same way as in (3.3). Since $\deg_y f_i=0$ and $\deg_y\sigma=1$, we get

$$m_i=-\deg_y\hat{f}_i=-\varpi^{(i)}(A_0).$$

Thus it remains to prove (3.6.1). Note that f^δ corresponds to the character δ, which is non-trivial by the assumption (b). Hence f^δ is not locally constant, i.e., grad $\log f^\delta(x)\not\equiv 0$. Thus the projection of W' on X is the whole space, and f_i is not identically zero on W'. Let $C_1,\dots,C_N$ be the irreducible components of $W'\cap f_i^{-1}(0)$ containing p_0. These are all G-stable hypersurfacs of W'. Since $\sigma=0$ is the defining equation of Λ, we have $W'\cap\hat{f}_i^{-1}(0)=\bigcup_{C_j\neq\Lambda}C_j$, and hence $\Lambda\cap\hat{f}_i^{-1}(0)=\bigcup_{C_j\neq\Lambda}(C_j\cap\Lambda)$, which can not contain the element p_0 of the open G-orbit Λ_0. Hence $p_0\notin\hat{f}_i^{-1}(0)$.

4. The purpose of this section is to prove (4.4) and (4.5).

4.1. For $\mathscr{D}$-modules $\mathscr{M}_1$ and $\mathscr{M}_2$, we denote by $\underline{\mathrm{Hom}}_{\mathscr{D}}(\mathscr{M}_1,\mathscr{M}_2)=\underline{\mathrm{Hom}}(\mathscr{M}_1,\mathscr{M}_2)$ the sheaf of local homomorphisms. Let $R\,\underline{\mathrm{Hom}}(\mathscr{M}_1,\mathscr{M}_2)$ be its derived functor and $\underline{\mathrm{Ext}}^i(\mathscr{M}_1,\mathscr{M}_2)$ its i-th cohomology. For a complex $A^\bullet$

$= (\cdots \to A^i \xrightarrow{d^i} A^{i+1} \to \cdots)$, let $\sigma_{\geq i} A^\bullet = (\cdots \to 0 \to d^{i-1} A^{i-1} \to A^i \to A^{i+1} \to \cdots)$. For a coherent $\mathscr{D}_X$-module $\mathscr{M}$, put

$$T_i(\mathscr{M}) = \{u \in \mathscr{M} \mid \dim \mathrm{Ch}\,(\mathscr{D}u) \leq n + i\}, \tag{4.1.1}$$

where $n = \dim X$. Then

$$T_i(\mathscr{M}) = \underline{\mathrm{Ext}}^0_{\mathscr{D}}(\sigma_{\geq n-i} R\, \underline{\mathrm{Hom}}_{\mathscr{D}}(\mathscr{M}, \mathscr{D}), \mathscr{D}) \tag{4.1.2}$$

by [6, Theorem (2.10)]. For a coherent $\mathscr{E}_X$-module $\mathscr{M}$, we also put

$$T_i(\mathscr{M}) = \underline{\mathrm{Ext}}^0_{\mathscr{E}}(\sigma_{\geq n-i} R\, \underline{\mathrm{Hom}}_{\mathscr{E}}(\mathscr{M}, \mathscr{E}), \mathscr{E}). \tag{4.1.3}$$

Then

$$T_i(\mathscr{M}) = \{u \in \mathscr{M} \mid \dim \mathrm{supp}\,(\mathscr{E}u) \leq n + i\}. \tag{4.1.4}$$

For a coherent $\mathscr{D}$-module $\mathscr{M}$, we have

$$T_i(\mathscr{E} \otimes_{\mathscr{D}} \mathscr{M}) = \mathscr{E} \otimes_{\mathscr{D}} T_i(\mathscr{M}). \tag{4.1.5}$$

Lemma 4.2. *Let* $\mathscr{N}_k$ *be as in* (2.9). (1) $\underline{\mathrm{Ext}}^i_{\mathscr{E}}(\mathscr{N}_k, \mathscr{E}) = 0$ *for* $i \neq n - k$. (2) $T_k(\mathscr{N}_k) = \mathscr{N}_k$. (3) $T_{k-1}(\mathscr{N}_k) = 0$.

Proof. Let $\{F_j \mathscr{E}\}_{j \in \mathbf{Z}}$ be the order filtration of $\mathscr{E}$, $F_j \mathscr{N}_k = (F_j \mathscr{E}) u_k$, $\mathrm{gr}^F = \bigoplus_{j \in \mathbf{Z}} F_j / F_{j-1}$, and $\bar{\mathscr{N}}_k = \mathscr{O}_{T^*X} \otimes_{\mathrm{gr}^F \mathscr{E}} \mathrm{gr}^F \mathscr{N}_k$. Then

$$\mathrm{supp}\ \underline{\mathrm{Ext}}^i_{\mathscr{E}}(\mathscr{N}_k, \mathscr{E}) \subset \mathrm{supp}\ \underline{\mathrm{Ext}}^i_{\mathscr{O}}(\bar{\mathscr{N}}_k, \mathscr{O})$$

(cf. the proof of [6, Theorem (2.3)]). Since $\bar{\mathscr{N}}_k = \mathscr{O}/\sigma(\mathscr{T}_k)$, we get $\underline{\mathrm{Ext}}^i_{\mathscr{O}}(\bar{\mathscr{N}}_k, \mathscr{O}) = 0$ for $i \neq n - k$ by (2.10) and (2.13). Hence we get (1). The remaining assertions follow from (1).

4.3. Let $t_1', \ldots, t_h'$ be new complex variables and

$$\mathscr{M} = \mathscr{E}\delta(t_1', \ldots, t_h') \hat{\boxtimes} \mathscr{N}_k'' = \mathscr{E}_{\mathbf{C}^h \times X} \delta(t_1') \cdots \delta(t_h') u_k''.$$

(See (2.9) for $\mathscr{N}_k''$ and u_k''.) By the change of variables $t_i' = t_i - f_i(x)$, $\mathscr{M}$ can be expressed as

$$\mathscr{M} = \mathscr{E}_{\mathbf{C}^h \times X} \delta(t_1 - f_1(x)) \cdots \delta(t_h - f_h(x)) u_k''.$$

In the same way as (4.2), we can show that (1) $\underline{\mathrm{Ext}}^i_{\mathscr{E}}(\mathscr{M}, \mathscr{E}) = 0$ for $i \neq h + n - k$, (2) $T_k(\mathscr{M}) = \mathscr{M}$, and (3) $T_{k-1}(\mathscr{M}) = 0$.

Lemma 4.4. *Let* $v(\zeta)$ *be a local section of* $\mathscr{N} = \mathscr{E}[\zeta] f^\zeta$. *Assume that the image of* $v(\zeta)$ *in* $\mathscr{N} / \sum_{i=1}^{l} (\zeta_i - \lambda_i) \mathscr{N} = \mathscr{E} f^\lambda$ *is zero for any* $\lambda \in \mathbf{C}^l$. *Then* $v(\zeta) = 0$.

Proof. Let

$$\mathscr{N}_k = \mathscr{N}_k(\lambda) = \mathscr{N} / \sum_{i>k} (\zeta_i - \lambda_i)\mathscr{N} \ (= \mathscr{N}_k(\varpi^{(1)},\dots,\varpi^{(l)}; \lambda))$$

and $v_k = v_k(\lambda) = v(\zeta_1,\dots,\zeta_k, \lambda_{k+1},\dots,\lambda_l)$ be the image of $v(\zeta)$ in $\mathscr{N}_k$. Assume that $v_k = v_k(\lambda) \neq 0$ for some $\lambda \in \mathbf{C}^l$, and $v_{k-1}(\zeta_1,\dots,\zeta_{k-1}, \lambda'_k, \lambda_{k+1},\dots,\lambda_l) = 0$ for any $\lambda'_k \in \mathbf{C}$. Let us show that a contradiction arises.

Assume that $\varpi^{(k)}(A_0) = 0$ (see (0.4) for A_0). By (3.6), $f_k \not\equiv 0$ on Λ. Until we get (4.4.3) below, let us consider everything on the open set $X \backslash f_k^{-1}(0)$. Put $\zeta' = \sum_{i\neq k} \zeta_i \varpi^{(i)}$ and $f^{\zeta'} = \prod_{i\neq k} f_i^{\zeta_i}$. Let $F_j\mathscr{N}_l = F_j(\mathscr{E}[\zeta]f^{\zeta}) := (F_j\mathscr{E})[\zeta]f^{\zeta}$ and $F_j\mathscr{N}_k$ be its image in $\mathscr{N}_k$. Define a sheaf homomorphism Φ: $\mathscr{E}[\zeta]f^{\zeta} \to \mathscr{E}[\zeta]f^{\zeta'}$ by $P(\zeta)f^{\zeta} \to (f_k^{-\zeta_k}P(\zeta)f_k^{\zeta_k})f^{\zeta'}$. (This homomorphism is well-defined on $X \backslash f_k^{-1}(0)$.) Put $\mathscr{M}_l = \mathscr{E}[\zeta]f^{\zeta'}$, $\mathscr{M}_k = \mathscr{M}_l/\sum_{i>k}(\zeta_i - \lambda_i)\mathscr{M}_l$, and define $F_j\mathscr{M}_k$ in the same way as above. Then Φ induces sheaf homomorphisms $F_j\mathscr{N}_k \to F_j\mathscr{M}_k$ and $\mathrm{gr}^F(\mathscr{N}_k) \to \mathrm{gr}^F(\mathscr{M}_k)$. Moreover, the latter is a $\mathrm{gr}^F(\mathscr{E})[\zeta]$-isomorphism. By (4.1.4), (4.2, (3)) and by our assumption, $v_k(\lambda) \neq 0$ even as a section on $\Lambda \backslash f_k^{-1}(0)$. Hence we can find an integer j such that $v_k(\lambda) \in F_j\mathscr{N}_k$ and $v_k(\lambda) \notin F_{j-1}\mathscr{N}_k$. Let $\mathrm{gr}(v_k(\lambda)) (\neq 0)$ be its image in $F_j\mathscr{N}_k/F_{j-1}\mathscr{N}_k = \mathrm{gr}_j^F(\mathscr{N}_k)$. By our assumption, $v_k(\zeta_1,\dots,\zeta_k, \lambda_{k+1},\dots,\lambda_l) \in (\zeta_k - \lambda'_k)\mathscr{N}_k$ for any $\lambda'_k \in \mathbf{C}$. Hence

$$\Phi(\mathrm{gr}(v_k(\zeta_1,\dots,\zeta_k, \lambda_{k+1},\dots,\lambda_l))) \in (\zeta_k - \lambda'_k)\, gr^F(\mathscr{M}_k) \tag{4.4.1}$$

for any $\lambda'_k \in \mathbf{C}$. Note that

$$\mathrm{gr}^F(\mathscr{M}_k) = \mathbf{C}[\zeta_k] \otimes_{\mathbf{C}} \mathrm{gr}^F\left(\frac{\mathscr{E}[\zeta']f^{\zeta'}}{\sum_{i>k}(\zeta_i - \lambda_i)\mathscr{E}[\zeta']f^{\zeta'}}\right), \tag{4.4.2}$$

where the filtration F of $\mathscr{E}[\zeta']f^{\zeta'}$ etc. are defined in the same way as above. By (4.4.1) and (4.4.2), we get $\Phi(\mathrm{gr}(v_k)) = 0$. Since Φ is an isomorphism, $\mathrm{gr}(v_k) = 0$. Thus we get a contradiction. Hence

$$\varpi^{(k)}(A_0) \neq 0. \tag{4.4.3}$$

Define endomorphisms t_i of $\mathscr{N}_l = \mathscr{E}[\zeta]f^{\zeta}$ by $t_i(P(\zeta)f^{\zeta}) = P(\zeta + \varpi^{(i)})f^{\zeta + \varpi^{(i)}}$. Then t_i $(1 \leq i \leq j)$ induce endomorphisms of $\mathscr{N}_j$. By (2.14), (4.1.4) and (4.2),

$$n + k = \dim \operatorname{supp} \mathscr{N}_k \geq \dim \operatorname{supp} \mathscr{E}[\zeta, t_1,\dots,t_{k-1}]\, v_k \geq \dim \operatorname{supp} \mathscr{E} v_k = n + k.$$

(Note that the $\mathscr{E}[\zeta]$-module structure of $\mathscr{N} = \mathscr{N}_l$ induces that of $\mathscr{N}_k$.) Since the multiplicity of $\mathscr{N}_k$ along $\Lambda_k = \operatorname{supp} \mathscr{N}_k$ is one by (2.14),

$$\dim \operatorname{supp} (\mathscr{N}_k/\mathscr{E}[\zeta, t_1,\dots,t_{k-1}]v_k) < n + k.$$

Since $v_{k-1} = 0$, the natural morphism $\mathscr{N}_k \to \mathscr{N}_{k-1}$ induces a surjective morphism Ψ: $\mathscr{M} := \mathscr{N}_k/\mathscr{E}[\zeta, t_1,\dots,t_{k-1}]v_k \to \mathscr{N}_{k-1}$. Note that these modules can be naturally considered as $\mathscr{E}_X[\zeta_1,\dots,\zeta_{k-1}, t_1,\dots,t_{k-1}]$-modules. Let $E = \{(t_1,\dots,t_{k-1}) \in \mathbf{C}^{k-1}\}$. By the correspondence $\zeta_i \leftrightarrow -\partial_{t_i} t_i$, $\mathscr{E}_X[\zeta_1,\dots,\zeta_{k-1}, t_1,\dots,t_{k-1}]$ can be regarded as a subring of $\mathscr{E}_{X\times E}$. Let $\tilde{\mathscr{M}} = \mathscr{E}_{X\times E} \otimes \mathscr{M}$, $\tilde{\mathscr{N}}_j = \mathscr{E}_{X\times E} \otimes \mathscr{N}_j$ $(j =$

$k-1, k)$ and u_j be the section of $\mathcal{N}_j$ corresponding to $f^{\zeta} \in \mathcal{N}_l$.

Let us show that $\tilde{\mathcal{M}}$ is holonomic. Note that $u = 1 \otimes u_k$ $(\in \tilde{\mathcal{N}}_k)$ satisfies the equations

$$(4.4.4) \qquad P(B)u = 0 \ (B \in \mathfrak{g}_0), \quad (P(C_j) - \lambda_j)u = 0 \ (k \le j \le l)$$

and

$$(4.4.5) \qquad \left(P(C_j) + \frac{\partial}{\partial t_j} t_j\right) u = 0, \quad (t_j - f_j)u = 0 \ (1 \le j < k).$$

By the change of variables $t'_j = t_j - f_j$, (4.4.5) becomes

$$(4.4.6) \qquad P(C_j)u = 0, \quad t'_j u = 0 \ (1 \le j < k).$$

Hence $\tilde{\mathcal{N}}_k = \mathcal{E}_{X \times E}(1 \otimes u_k)$ is a quotient of

$$\tilde{\mathcal{N}}_k'' := \mathcal{E}_X \tilde{u}_k'' \hat{\boxtimes} \mathcal{E}_{E'} \delta(t'_1) \cdots \delta(t'_{k-1}) = \mathcal{E}_{X \times E}(\tilde{u}_k'' \delta(t_1 - f_1) \cdots \delta(t_{k-1} - f_{k-1})),$$

where $E' = \{(t'_1, \ldots, t'_{k-1}) \in \mathbf{C}^{k-1}\}$ and $\tilde{u}_j''$ is the section of $\mathcal{E}_X / (\sum_{B \in \mathfrak{g}_0} \mathcal{E}_X P(B) + \sum_{i \le k-1} \mathcal{E}_X P(C_i) + \sum_{i > k} \mathcal{E}_X (P(C_i) - \lambda_i))$ corresponding to $1 \in \mathcal{E}_X$. By (2.14), the support $\tilde{\Lambda}_1$ of $\mathcal{E}_{X \times E} \tilde{u}_k'' \delta(t'_1) \cdots \delta(t'_{k-1})$ is a non-singular variety of dimension $(n+1) + (k-1) = n + k$ and its multiplicity along $\tilde{\Lambda}_1$ is one. Hence the natural morphism $\tilde{\mathcal{N}}_k'' \to \tilde{\mathcal{N}}_k$ is an isomorphism or the support of its kernel is of dimension $n + k$ (cf. (4.3)). In the former case, we have dim supp $(\mathcal{E}_{X \times E}(1 \otimes v_k))$ $= n + k$ by (4.3), and $\tilde{\mathcal{M}}$ becomes holonomic (as a non-trivial quotient of the subholonomic module $\tilde{\mathcal{N}}_k$). In the latter case, dim supp $\tilde{\mathcal{N}}_k < n + k$, i.e., $\tilde{\mathcal{N}}_k$ is holonomic, and its quotient $\tilde{\mathcal{M}}$ is also holonomic.

Let us show that $\tilde{\mathcal{N}}_{k-1} \ne 0$ in a neighbourhood of $\Lambda_0 \times T^*_{E'} E'$ for a generic λ, where $T^*_{E'} E'$ denotes the zero section of $T^* E'$. (See (0.3) for Λ_0.) Assume the contrary. Since $\Lambda_0 \times T^*_{E'} E'$ is identified with $\Lambda_0 \times T^*_E E$ by the isomorphism $T^*(X \times E') \simeq T^*(X \times E)$ induced by $(x_i, t'_j) = (x_i, t_j - f_j)$, we have $\tilde{\mathcal{N}}_{k-1} | \Lambda_0 \times T^*_E E = 0$. Take a point $(p, q) \in \Lambda_0 \times T^*_E E$ so that every coordinate of q is non-zero. As is easily seen $\mathcal{E}_{X \times E, (p,q)}$ is faithfully flat over $A := \mathcal{E}_{X,p} \otimes_{\mathbf{C}} (\mathcal{O}_{E,q} \otimes_{\mathbf{C}[t'']} \mathbf{C}[t'', \partial''])$, where $t'' = (t_1, \ldots, t_{k-1})$ and $\partial'' = (\partial_{t_1}, \ldots, \partial_{t_{k-1}})$. By the correspondence $\zeta_i \leftrightarrow -\partial_{t_i} t_i$, we have $\mathcal{O}_{E,q} \otimes_{\mathbf{C}[t'']} \mathbf{C}[t'', \partial''] = \mathcal{O}_{E,q} \otimes_{\mathbf{C}[t'']} \mathbf{C}[t'', \zeta'']$, where $\zeta'' = (\zeta_1, \ldots, \zeta_{k-1})$. (Note that $t_i(q) \ne 0$.) Hence

$$0 = \tilde{\mathcal{N}}_{k-1,(p,q)} = \mathcal{E}_{X \times E, (p,q)} \otimes_{\mathcal{E}_{X,p}[t'', \zeta'']} \mathcal{N}_{k-1,p} = \mathcal{E}_{X \times E, (p,q)} \otimes_A (\mathcal{O}_{E,q} \otimes_{\mathbf{C}[t'']} \mathcal{N}_{k-1,p})$$

and we also get $\mathcal{O}_{E,q} \otimes_{\mathbf{C}[t'']} \mathcal{N}_{k-1,p} = 0$ for any $q \in (\mathbf{C}^\times)^{k-1}$ because of the faithful flatness. Thus we get $\mathcal{N}_{k-1,p}[t_1^{-1}, \ldots, t_{k-1}^{-1}] = 0$, i.e., $(t_1 \cdots t_{k-1})^N u_{k-1} = 0$ as an element of $\mathcal{N}_{k-1,p}$ for a sufficiently large N. Put $\delta' = \sum_{i<k} \varpi^{(i)}$. Then we get $f^{N\delta'} f^{\lambda} = 0$ as an element of $\mathcal{N}_0(\lambda)_p$. Put $L_0 = \mathbf{C}^l \setminus B_\mu^{-1}(0)$. Then for $\lambda \in L_0$, such an equality can not hold by (1.8). Hence $\tilde{\mathcal{N}}_{k-1} \ne 0$ for $\lambda \in L_0$. Henceforth in this proof, we assume that $\lambda \in L_0$.

Let us calculate the order of $1 \otimes u_{k-1} \in \mathcal{E}_{X \times E} \otimes \mathcal{N}_{k-1} = \tilde{\mathcal{N}}_{k-1}$. Since $u = 1 \otimes u_{k-1}$ satisfies (4.4.4), (4.4.5) and also $(P(C_k) - \lambda_k)u = 0$, $\tilde{\mathcal{N}}_{k-1}$ is a quotient of

$$\tilde{\mathcal{N}}''_{k-1} := \mathcal{E}_X \tilde{u}''_{k-1} \hat{\boxtimes} \mathcal{E}_{E'}\delta(t'_1)\cdots\delta(t'_{k-1}) = \mathcal{E}_{X\times E}\tilde{u}''_{k-1}\delta(t_1-f_1)\cdots\delta(t_{k-1}-f_{k-1}).$$

Since $\tilde{\mathcal{N}}''_{k-1}$ is a simple holonomic system in a neighbourhood of $\Lambda_0 \times T^*E'$ by (2.14), and $\tilde{\mathcal{N}}_{k-1} \neq 0$ there, the morphism $\tilde{\mathcal{N}}''_{k-1} \to \tilde{\mathcal{N}}_{k-1}$ is an isomorphism. Hence the order of $1 \otimes u_{k-1}$ $(\in \tilde{\mathcal{N}}_{k-1})$ on $\Lambda_2 := \operatorname{supp} \tilde{\mathcal{N}}_{k-1}$ is given by

$$\begin{aligned}\operatorname{ord}_{\Lambda_2} 1 \otimes u_{k-1} &= \operatorname{ord}_{\Lambda_2}(\tilde{u}''_{k-1}\delta(t_1-f_1)\cdots\delta(t_{k-1}-f_{k-1}))\\ &= \operatorname{ord}_{\Lambda_2}(\tilde{u}''_{k-1}\delta(t'_1)\cdots\delta(t'_{k-1})) = \operatorname{ord}_{\Lambda}\tilde{u}''_{k-1} + \operatorname{ord}_{\Lambda_3}\delta(t'_1,\dots,t'_{k-1}),\end{aligned}$$

where $\Lambda_3 = T^*_{\{0\}}E'$. Since $\mathcal{E}_X\tilde{u}''_{k-1} = \mathcal{E}_X(f_1^0\cdots f_{k-1}^0 f_k^{\lambda_k}\cdots f_l^{\lambda_l})$, we have

$$\text{(4.4.7)} \qquad \operatorname{ord}_{\Lambda_2} 1 \otimes u_{k-1} = \sum_{i=k}^{l} \lambda_i \varpi^{(i)}(A_0) - \operatorname{tr}(A_0|\Lambda_{q_0}) + \frac{1}{2}\dim \Lambda_{q_0} + \frac{1}{2}(k-1),$$

(cf. (3.3)).

By (1.11), we can show that, if two simple holonomic systems of the same support are isomorphic to each other, then the difference of the orders of the respective generators is an integer. (The converse also holds [10, chapter 2, Theorem 4.2.5]. But we do not need this deeper result.) Thus, by (4.4.3) and (4.4.7), moving λ_k continuously, we get infinitely many non-isomorphic quotients $\tilde{\mathcal{N}}_{k-1} = \tilde{\mathcal{N}}_{k-1}(\lambda)$ of $\tilde{\mathcal{M}}$. (Note that $\tilde{\mathcal{M}}$ is independent of λ_k.) But as we have seen, $\tilde{\mathcal{M}}$ is holonomic. Hence $\tilde{\mathcal{M}}$ can have only a finite number of quotients up to isomorphism. Thus we get a contradiction, and the proof is now complete.

Lemma 4.5. *Let $\alpha(\zeta)$ be a linear form in ζ, $a \in \mathbf{C}$, and $v(\zeta)$ a local section of $\mathcal{N} = \mathcal{E}[\zeta]f^\zeta$. Assume that the image of $v(\zeta)$ in $\mathcal{E}f^\lambda$ is zero whenever $\lambda \in \mathbf{C}^l$ satisfies $\alpha(\lambda) - a = 0$. Then $v(\zeta) \in (\alpha(\zeta) - a)\mathcal{N}$.*

Proof. We may assume that $\alpha(0,\dots,0,1) \neq 0$. Let $\mathcal{N}_k = \mathcal{N}_k(\varpi^{(1)},\dots,\varpi^{(l-1)}, \alpha; \lambda_1,\dots,\lambda_{l-1}, a)$ for $\tilde{\lambda} \in L := \sum_{i=1}^{l-1}\mathbf{C}\varpi^{(i)}$, and $v_k = v_k(\tilde{\lambda}) = v_k(\zeta_1,\dots,\zeta_k, \lambda_{k+1},\dots, \lambda_{l-1}, a)$ be the image of $v(\zeta)$ in $\mathcal{N}_k$. For any $\tilde{\lambda} = (\lambda_1,\dots,\lambda_{l-1},0) \in L$, there is a unique λ_l such that $\alpha(\lambda_1,\dots,\lambda_l) = a$. Then $\mathcal{N}_k = \mathcal{N}_k(\varpi^{(1)},\dots,\varpi^{(l)}; \lambda_1,\dots,\lambda_l)$ and $v_0(\tilde{\lambda}) = 0$ for any $\tilde{\lambda} \in L$. Assume that $v_k = v_k(\tilde{\lambda}) \neq 0$ for some $\tilde{\lambda} \in L$ and $v_{k-1}(\zeta_1,\dots,\zeta_{k-1}, \lambda'_k, \lambda_{k+1},\dots,\lambda_{l-1}, a) = 0$ for any $\lambda'_k \in \mathbf{C}$. Considering the sheaf homomorphism

$$P(\zeta)f_1^{\zeta_1}\cdots f_{l-1}^{\zeta_{l-1}}f^{\alpha(\zeta)} \longrightarrow (f_k^{-\zeta_k}P(\zeta)f_k^{\zeta_k})f_1^{\zeta_1}\cdots f_{k-1}^{\zeta_{k-1}}f_{k+1}^{\zeta_{k+1}}\cdots f_{l-1}^{\zeta_{l-1}}f^{\alpha(\zeta)}$$

modulo $\{(\zeta_i - \lambda_i)(k < i \leq l-1), \alpha(\zeta) - a\}$, we can show that $\varpi^{(k)}(A_0) \neq 0$ as in (4.4). We can follow also the remaining argument of (4.4) (with an obvious modification) and get a contradiction.

5. In this section, we prove the theorem stated in the introduction.

5.1. Fix an element $\mu \in \mathbf{N}^l$ and put $\hat{f}^\mu = \prod_{i=1}^l \hat{f}_i^{\mu_i}$ and $m = \sum_{i=1}^l \mu_i m_i$. Then $f^\mu = \hat{f}^\mu\sigma^m$ on W' and $\mu(A_0) = -m$ by (3.6). (See (0.4) for A_0, (3.5) for $\hat{f}_i$, m_i and $\sigma = \sigma(P(A_1))$.)

5.2. Put $L_0 = \mathbf{C}^l \setminus B_\mu^{-1}(0)$. (See (1.1) for B_μ.) As a consequence of (1.8), $f^\mu f^\lambda \neq 0$ on Λ for $\lambda \in L_0$.

5.3. For $\alpha \in \mathbf{N}^l$, put $\zeta^\alpha = \prod_i \zeta_i^{\alpha_i}$ and $|\alpha| = \sum_i \alpha_i$. For $T(\zeta) = \sum \zeta^\alpha T_\alpha$, let

$$\operatorname{ord} T(\zeta) = \max_\alpha (\operatorname{ord} T_\alpha), \text{ and}$$

$$\underline{\operatorname{ord}}\, T(\zeta) = \max_\alpha (\operatorname{ord} T_\alpha + |\alpha|).$$

Lemma 5.4 [11, Lemma 5.7]. *Let* $G(\zeta) = \sum_i \zeta^\alpha G_\alpha$ *be a microdifferential operator satisfying* $\operatorname{ord} G(\zeta) \leq d$, $\underline{\operatorname{ord}}\, G(\zeta) \leq e$ *and* $\sigma_d(G(\zeta))|\Lambda \equiv 0$. *Then there exists a microdifferential operator* $T(\zeta)$ *such that* $\operatorname{ord} T(\zeta) < d$, $\underline{\operatorname{ord}}\, T(\zeta) \leq e$ *and* $T(\lambda) f^\lambda = G(\lambda) f^\lambda$ *for any* $\lambda \in \mathbf{C}^l$.

Lemma 5.5 [11, Lemma 5.8]. *For* $\lambda \in \mathbf{C}^l$ *and* $G \in \mathscr{E}$ *such that* $G f^\lambda \neq 0$, *there exists a number* $r = r(\lambda)$ *such that* $\operatorname{ord} T \geq r$ *for any operator* T *satisfying* $T f^\lambda = G f^\lambda$.

(For the first three lines of the proof of [11, Lemma 5.8], see (1.11).)

5.6. Let $R_\mu(\zeta)$ be a microdifferential operator such that
(1) $R_\mu(\lambda) f^\lambda (\lambda \in \mathbf{C}^l)$ satisfies the same equations as $f^{\lambda+\mu}$, and
(2) $R_\mu(\lambda) f^\lambda \neq 0$ for any $\lambda \in L_0$.
For example, $R_\mu(\zeta) = f^\mu$ satisfies these conditions (cf. (5.2)).

Lemma 5.7. *Let* $\lambda \in L_0$. (1) *There exists an operator* Q *such that* $R_\mu(\lambda) f^\lambda = Q f^\lambda$ *and* $\sigma(Q)|\Lambda \not\equiv 0$. (2) $\operatorname{ord} Q = -m$. (*See* (5.1) *for* m.) (3) *If* $R_\mu(\lambda) f^\lambda = Q' f^\lambda$ *with an operator* Q', *then* $\operatorname{ord} Q' \geq -m$. (4) $\operatorname{ord}_\Lambda f^{\lambda+\mu} = -m + \operatorname{ord}_\Lambda f^\lambda$.

Proof. (1) follows from (1.11).
(2) We have

$$\begin{aligned} &\operatorname{ord} Q + \operatorname{ord}_\Lambda f^\lambda = \operatorname{ord}_\Lambda Q f^\lambda = \operatorname{ord}_\Lambda R_\mu(\lambda) f^\lambda = \operatorname{ord}_\Lambda f^{\lambda+\mu} \\ &= \langle \lambda + \mu, A_0 \rangle - \operatorname{tr}(A_0 | \Lambda_{q_0}) + \frac{1}{2} \dim \Lambda_{q_0} && \text{by (3.3)} \\ &= -m + \operatorname{ord}_\Lambda f^\lambda && \text{by (3.3) and (3.6).} \end{aligned}$$

We also get (4).

(3) If $\operatorname{ord} Q' < -m$, then $(Q - Q') f^\lambda = 0$ and $\sigma(Q - Q') = \sigma(Q) \not\equiv 0$ on Λ. This implies $f^\lambda = 0$ on Λ, which contradicts (2.8).

Lemma 5.8. *For* $R_\mu(\zeta)$ *as in* (5.6), *there exists an operator* $Q_\mu(\zeta) \in \mathscr{E}[\zeta]$ *such that* (1) $R_\mu(\lambda) f^\lambda = Q_\mu(\lambda) f^\lambda$ *for any* $\lambda \in \mathbf{C}^l$, (2) $\sigma(Q_\mu(\zeta))|\Lambda \not\equiv 0$, *and* (3) $\underline{\operatorname{ord}}\, Q_\mu(\zeta) \leq \underline{\operatorname{ord}}\, R_\mu(\zeta)$.

Moreover, any operator $Q_\mu(\zeta)$ *satisfying these conditions also satisfies* (4) ord $Q_\mu(\zeta) = -m$, *and* (5) $\sigma_{-m}(Q_\mu(\lambda))(p) \neq 0$ *for any* $\lambda \in L_0$ *and* $p \in \Lambda_0$.

Proof. Fix an element λ in L_0. By (5.6, (2)), $R_\mu(\lambda)f^\lambda \neq 0$. By (5.5), there exists a number r such that ord $T \geq r$ for any operator T satisfying $Tf^\lambda = R_\mu(\lambda)f^\lambda$. Put $R(\zeta) = R_\mu(\zeta)$. By (5.4), we get operators $R'(\zeta)$, $R''(\zeta)$,... such that

$$R(\lambda')f^{\lambda'} = R'(\lambda')f^{\lambda'} = \cdots \text{ for any } \lambda' \in \mathbf{C}^l$$
$$\text{ord } R(\zeta) > \text{ord } R'(\zeta) > \cdots, \text{ and}$$
$$\underline{\text{ort}}\ R(\zeta) \geq \underline{\text{ord}}\ R'(\zeta) \geq \cdots.$$

If $\sigma(R^{(i)}(\zeta))|\Lambda \equiv 0$ for any i, the sequence R, R',... can continue infinitely. But as we have shown above, the order of $T = R^{(i)}(\lambda)$ is at least r. Since ord $R^{(i)}(\zeta) \geq$ ord $R^{(i)}(\lambda)$, the sequence $\{R^{(i)}\}$ can not continue infinitely. Hence $\sigma(R^{(i)}(\zeta))|\Lambda \not\equiv 0$ for some i. Then $\tilde{Q}_\mu(\zeta) := R^{(i)}(\zeta)$ satisfies (1)–(3).

Let $Q_\mu(\zeta)$ be any operator satisfying (1)–(3). By (5.7), ord $Q_\mu(\lambda') = -m$ for a generic $\lambda' \in \mathbf{C}^l$, and we get (4).

To prove (5), first assume that $\sigma_{-m}(Q_\mu(\lambda))|\Lambda \equiv 0$ for the element $\lambda \in L_0$ fixed above. Applying (5.4) for $G = Q_\mu(\lambda)$ (an operator independent of ζ), we find an operator $T(\zeta)$ such that ord $T(\zeta) <$ ord G and $T(\lambda')f^{\lambda'} = Gf^{\lambda'}$ for any $\lambda' \in \mathbf{C}^l$. Applying (5.7, (3)) for $Q' = T(\lambda)$, we get ord $T(\lambda) \geq -m$. On the other hand, ord $T(\lambda) \leq$ ord $T(\zeta) <$ ord $G \leq$ ord $Q_\mu(\zeta) = -m$. Thus we get a contradiction. Hence $\sigma_{-m}(Q_\mu(\lambda))|\Lambda \not\equiv 0$, and

$$\sigma_\Lambda(f^{\lambda+\mu}) = \sigma_\Lambda(R_\mu(\lambda)f^\lambda) = \sigma_\Lambda(Q_\mu(\lambda)f^\lambda) = \sigma(Q_\mu(\lambda))\sigma_\Lambda(f^\lambda).$$

Since $\sigma_\Lambda(f^\lambda)$ and $\sigma_\Lambda(f^{\lambda+\mu})$ are relatively G-invariant, $\sigma(Q_\mu(\lambda))|\Lambda$ is also relatively G-invariant. Then $\sigma_{-m}(Q_\mu(\lambda))$ can not vanish at any point of the open G-orbit Λ_0.

Lemma 5.9. *Let* $R(\zeta) = R_\mu(\zeta)$ *be as in* (5.6), *and* $Q(\zeta) = Q_\mu(\zeta)$ *an operator satisfying the conditions* (1)–(3) *of* (5.8). *Then* (1) $\sigma_{-m}(Q_\mu(\zeta))|\Lambda = c_\mu(\zeta)\hat{f}'$ *with a polynomial* $c_\mu(\zeta) \in \mathbf{C}[\zeta]$ *and a function* $\hat{f}'$ *on* Λ *independent of* ζ. (2) deg $c_\mu(\zeta) \leq m$ + $\underline{\text{ord}}\ Q_\mu(\zeta) \leq m$ + $\underline{\text{ord}}\ R_\mu(\zeta)$, *and* (3) $c_\mu^{-1}(0) \subset \mathbf{C}^l \setminus L_0$. *Especially, if* $R_\mu(\zeta) = f^\mu$, *writing* b_μ *for* c_μ, *we have* deg $b_\mu \leq m$.

Proof. (1) Let $Q(\zeta) = \sum \zeta^\alpha Q_\alpha$. Let us show that the hypersurface

$$H(x, y) = \{\zeta \in \mathbf{C}^l | \sum_\alpha \zeta^\alpha \sigma_{-m}(Q_\alpha)(x, y) = 0\}$$

is independent of $(x, y) \in \Lambda_0$. Assume the contrary. Then $\bigcup_{(x,y)\in\Lambda_0} H(x, y)$ contains a non-empty open subset, say O, of $\mathbf{C}^l$. Since L_0 is a dense subset of $\mathbf{C}^l$, we can take an element $\lambda \in O \cap L_0$. Then $\sigma_{-m}(Q(\lambda))$ vanishses at some point p of Λ_0. This contradicts (5.8, (5)). Hence the hypersurface $H(x, y)$ is independent of (x, y). Thus we get

$$\sigma_{-m}(Q(\zeta))(x, y) = c_\mu(\zeta)\hat{f}'(x, y)$$

with a polynomial $c_\mu(\zeta) \in \mathbf{C}[\zeta]$ and a function $\hat{f}'(x, y)$ on Λ.

(2) Put $\underline{\mathrm{ord}}\, Q(\zeta) = m'$. Since $\mathrm{ord}\, Q(\zeta) = -m$ by (5.8, (4)), only the terms with $|\alpha| \leq m + m'$ appear in $\sigma_{-m}(Q(\zeta)) = \sum_\alpha \zeta^\alpha \sigma_{-m}(Q_\alpha)$. Hence $\deg c_\mu(\zeta) \leq m + m'$. The remaining inequality is nothing but (5.8, (3)).

(3) By (5.8, (5)), $\sigma_{-m}(Q_\mu(\lambda))|\Lambda \not\equiv 0$ for $\lambda \in L_0$. Hence $c_\mu(\lambda) \neq 0$.

5.10. Let s be a single complex variable. For an operator $T'(s) = \sum s^j T_j' \in \mathscr{E}[s]$, put

$$\mathrm{ord}\, T'(s) = \max(\mathrm{ord}\, T_j'), \text{ and}$$

$$\underline{\mathrm{ord}}\, T'(s) = \max(\mathrm{ord}\, T_j' + j).$$

Lemma 5.11. *There exists a microdifferential operator* $Q'(s) = Q_\mu'(s) \in \mathscr{E}[s]$ *and a polynomial* $b'(s) = b_\mu'(s) \in \mathbf{C}[s]$ *of degree* m *such that* (1) $f^\mu f^{a\delta} = Q'(a) f^{a\delta}$ *for any* $a \in \mathbf{C}$, (2) $\mathrm{ord}\, Q'(s) = -m$, (3) $\underline{\mathrm{ord}}\, Q'(s) \leq 0$, *and* (4) $\sigma_{-m}(Q'(s))|\Lambda = b_\mu'(s)\hat{f}^\mu$.

Proof. Let Q be a microdifferential operator of order $-m$ such that $\sigma_{-m}(Q)|W' = f^\mu \sigma^{-m}$. Since $\sigma_1(P(A_1)) = \sigma$, we have

$$f^\mu - QP(A_1)^m = \sum_j T_j P(B_j) + K$$

with some $B_j \in \mathfrak{g}_0'$ and operators T_j and K such that $\mathrm{ord}\, K \leq -1$. (See (2.5) for $\mathfrak{g}_0'$.) Applying both sides to $f^{a\delta}$ $(a \in \mathbf{C})$, we get

$$f^\mu f^{a\delta} - a^m Q f^{a\delta} = K f^{a\delta}.$$

By the same argument as in [11, 5.7–5.9], we can find an operator $G(s) \in \mathscr{E}[s]$ such that

$$(f^\mu - a^m Q) f^{a\delta} = G(a) f^{a\delta} \quad \text{for any } a \in \mathbf{C},$$

$$\mathrm{ord}\, G(s) \leq -1, \ \underline{\mathrm{ord}}\, G(s) \leq -1, \text{ and}$$

$G(a)$ is invertible at a generic point of Λ for a generic a.

If $\mathrm{ord}\, G(s) > -m$, then

$$\mathrm{ord}_\Lambda(a^m Q + G(a)) f^{a\delta} = \mathrm{ord}\, G(a) + \mathrm{ord}_\Lambda f^{a\delta} = \mathrm{ord}_\Lambda f^\mu f^{a\delta} = -m + \mathrm{ord}_\Lambda f^{a\delta}$$

for a generic $a \in \mathbf{C}$. Cf. (5.7, (4)). (Note that $B_\mu(a\delta) \neq 0$ for generic $a \in \mathbf{C}$ because of our special choice of $B_\mu(\zeta)$.) Hence $\mathrm{ord}\, G(s) = -m$, and we get a contradiction. Thus $\mathrm{ord}\, G(s) \leq -m$. Since $\underline{\mathrm{ord}}\, G(s) \leq -1$, $\sigma_{-m}(G(s)) = \sum_{j \lneq m} s^j g_j$ with some g_j. Put $Q'(s) = Q_\mu'(s) := s^m Q + G(s)$. Then

(5.11.1) $$\sigma_{-m}(Q'_\mu(s)) = s^m\sigma_{-m}(Q) + \sum_{j \lneq m} s^j g_j$$

and $Q'(s)$ satisfies (1)–(3). Moreover, by the same argument as in the proof of (5.9), we can show that

(5.11.2) $$\sigma_{-m}(Q(s))|\Lambda = b'_\mu(s)F(x, y)$$

with a polynomial $b'_\mu(s) \in \mathbf{C}[s]$ and a function $F(x, y)$ independent of s. Comparing (5.11.1) and (5.11.2), and recalling that $\sigma_{-m}(Q)|\Lambda = \hat{f}^\mu$, we get (4).

Lemma 5.12. *If $R_\mu(\zeta) = f^\mu$, then the function $\hat{f}'$ in* (5.9. (1)) *is $c\hat{f}^\mu$, and $b_\mu(s\delta) = c^{-1}b'_\mu(s)$ with some $c \in \mathbf{C}^\times$.*

Proof. For $a \in \mathbf{C}$, we get

$$Q'_\mu(a)f^{a\delta} = f^\mu f^{a\delta} = Q_\mu(a\delta)f^{a\delta},$$
$$\sigma_{-m}(Q(a\delta))|\Lambda = b_\mu(a\delta)\hat{f}', \text{ and}$$
$$\sigma_{-m}(Q'(a))|\Lambda = b'_\mu(a)\hat{f}^\mu.$$

Since $Q'_\mu(a) - Q_\mu(a\delta)$ annihilates $f^{a\delta}$, $\sigma_{-m}(Q'(a) - Q(a\delta))|\Lambda \equiv 0$ for any $a \in \mathbf{C}$, i.e., $b_\mu(s\delta)\hat{f}' = b'_\mu(s)\hat{f}^\mu$. Thus we get the desired assertion.

Lemma 5.13. *There exists an operator $Q_\mu(\zeta) \in \mathscr{E}[\zeta]$ and a polynomial $b_\mu(\zeta) \in \mathbf{C}[\zeta]$ such that* (1) $f^\mu f^\lambda = Q_\mu(\lambda)f^\lambda$ *for any* $\lambda \in \mathbf{C}^l$, (2) $\operatorname{ord} Q_\mu(\zeta) = -m$, $\underline{\operatorname{ord}}\, Q_\mu(\zeta) \le 0$, (3) $\deg b_\mu(\zeta) = m = -\sum_{i=1}^{l} \mu_i \varpi^{(i)}(A_0)$, (4) $\sigma_{-m}(Q_\mu(\zeta))|\Lambda = b_\mu(\zeta)\hat{f}^\mu$, *and* (5) $b_\mu^{-1}(0) \subset B_\mu^{-1}(0)$.

Moreover, there exist $c_\mu \in \mathbf{C}^\times$, a finite subset Δ of Δ_0, $n(\alpha) \in \mathbf{N}$ $(\alpha \in \Delta)$, and positive rational numbers $a_{\alpha,j}$ $(\alpha \in \Delta,\ 1 \le j \le n(\alpha))$ such that

$$b_\mu(\zeta) = c_\mu \prod_{\substack{\alpha \in \Delta \\ 1 \le j \le n(\alpha)}} (\alpha_1\zeta_1 + \cdots + \alpha_l\zeta_l + a_{\alpha,j}).$$

Proof. The assertions (1)–(5) follow from (3.6), (5.8), (5.9), (5.11) and (5.12). The last assertion follows from (5) and (1.1).

Lemma 5.14. *Suppose that $\lambda \in \mathbf{C}^l$, $\operatorname{ord} Q = -m$ and Qf^λ satisfies the same equation as $f^{\lambda+\mu}$. If $\sigma_{-m}(Q)|\Lambda \equiv 0$, then $Qf^\lambda = 0$.*

Proof. Let $\{A_j\}$ be a basis of $\mathfrak{g}$. Since $\sigma_{-m}(Q)|\Lambda \equiv 0$, we have $\sigma_{-m}(Q) = \sum_j F_j\sigma(P(A_j))$ for some $F_j \in \mathcal{O}_{T^*X}$ homogeneous of degree $-m-1$ in the fibre coordinates (cf. the proof of (1.10)), and hence $Q = \sum_j \Phi_j(P(A_j) - \lambda(A_j)) + K$ with Φ_j, $K \in \mathscr{E}$ such that $\operatorname{ord} K \le -m-1$. Thus we have $Qf^\lambda = Kf^\lambda$. If $Qf^\lambda \ne 0$, then there is an operator K' such that $Qf^\lambda = K'f^\lambda$ and $\sigma(K')|\Lambda \not\equiv 0$. If $\operatorname{ord} K' > \operatorname{ord} K$, then $\sigma(K' - K) = \sigma(K') \not\equiv 0$ on Λ. Since $(K' - K)f^\lambda = 0$, $f^\lambda = 0$. This is a contradiction. Hence

$$\operatorname{ord}_\Lambda f^{\mu+\lambda} = \operatorname{ord}_\Lambda Q f^\lambda = \operatorname{ord}_\Lambda K' f^\lambda = \operatorname{ord} K' + \operatorname{ord}_\Lambda f^\lambda$$
$$\leq \operatorname{ord} K + \operatorname{ord}_\Lambda f^\lambda \leq -m - 1 + \operatorname{ord}_\Lambda f^\lambda.$$

But this inequality contradicts (5.7, (4)). Hence $Qf^\lambda = 0$.

5.15. End of the proof. First, let $R_\mu(\zeta) = f^\mu$. By (5.13) and (5.14), $f^\mu f^\lambda = Q_\mu(\lambda) f^\lambda = 0$ whenever $b_\mu(\lambda) = 0$. Let $\alpha(\zeta) - a$ be a linear factor of $b_\mu(\zeta)$ and $b'_\mu(\zeta) = b_\mu(\zeta)/(\alpha(\zeta) - a)$. By (4.5), $f^\mu f^\zeta = (\alpha(\zeta) - a) R'_\mu(\zeta) f^\zeta$ with some $R'_\mu(\zeta) \in \mathscr{E}[\zeta]$. Then $R'_\mu(\zeta)$ satisfies the conditions of (5.6). Applying (5.8) to R'_μ, we get an operator $Q'_\mu(\zeta)$ satisfying (5.8, (1)–(3)). Since

$$Q_\mu(\lambda) f^\lambda = f^\mu f^\lambda = (\alpha(\lambda) - a) Q'_\mu(\lambda) f^\lambda \quad \text{for any } \lambda \in \mathbf{C}^l,$$

we have $\sigma_{-m}(Q_\mu(\lambda) - (\alpha(\lambda) - a) Q'_\mu(\lambda)) = 0$. Hence $\sigma_{-m}(Q'_\mu(\zeta)) = b'_\mu(\zeta) \hat{f}^\mu$ by (5.13, (4)). Thus we can repeat the same argument, and finally we get an operator $P_\mu(\zeta) \in \mathscr{E}[\zeta]$ such that

$$f^\mu f^\zeta = b_\mu(\zeta) P_\mu(\zeta) f^\zeta, \ \operatorname{ord} P_\mu(\zeta) = -m \text{ and } \sigma_{-m}(P_\mu(\zeta)) = \hat{f}^\mu.$$

These assertions together with (5.13) imply the assertions (1), (3) and (4) of Theorem in (0.5).

Let us prove (2). Assume that $\tilde{b}_\mu \in \mathbf{C}[\zeta]$ and $\tilde{P}_\mu(\zeta) \in \mathscr{E}[\zeta]$ also satisfy the conditions of (1). Then $(b_\mu(\lambda) P_\mu(\lambda) - \tilde{b}_\mu(\lambda) \tilde{P}_\mu(\lambda)) f^\lambda = 0$ for any λ. Hence $0 = \sigma_{-m}(b_\mu(\lambda) P_\mu(\lambda) - \tilde{b}_\mu(\lambda) \tilde{P}_\mu(\lambda)) = b_\mu(\lambda) \hat{f}^\mu - \tilde{b}_\mu(\lambda) \hat{f}^\mu$, and $b_\mu = \tilde{b}_\mu$.

6. In this section, we record some consequences which easily follow from our Theorem (see (0.5)) and (4.4).

Corollary 6.1. *The microdifferential operator $P_\mu(\zeta)$ and the polynomial $b_\mu(\zeta)$ of Theorem satisfy*

$$f^\mu f^\zeta = b_\mu(\zeta) P_\mu(\zeta) f^\zeta$$

as sections of $\mathscr{N} = \mathscr{E}[\zeta] f^\zeta$ on Λ_0. (See (0.3) *for Λ_0.)*

Corollary 6.2. *The polynomial $b_\mu(\zeta)$ of Theorem divides any $B_\mu(\zeta)$ as in* (1.1).

Proof. We have

$$b_\mu(\zeta) P'_\mu(\zeta) P_\mu(\zeta) f^\zeta = B_\mu(\zeta) f^\zeta$$

as sections of $\mathscr{N} = \mathscr{E}[\zeta] f^\zeta$ on Λ_0. Let $d = d(\zeta)$ be the greatest common divisor of b_μ and B_μ. If b_μ does not divide B_μ, then there exists $\lambda \in \mathbf{C}^l$ such that $d^{-1} b_\mu = 0$ and $d^{-1} B_\mu \neq 0$ for $\zeta = \lambda$. But then, $f^\lambda = 0$ on Λ_0, whcih contradicts the assumption (d) and (2.14).

Corollary 6.3. *Let $B_\mu(\zeta)$ be a polynomial as in* (1.1). *If* $\deg B_\mu(\zeta) = -\sum_{i=1}^{l} \mu_i \varpi^{(i)}(\Lambda_0)$, *then $B_\mu = b_\mu$.*

This assertion follows from (4) of Theorem.

Corollary 6.4. *Let $\mathscr{B}$ be the ideal of $\mathbf{C}[\zeta]$ consisting of B_μ's as in* (1.1). *If $b_\mu(\zeta)\in\mathscr{B}$, then $\mathscr{B}$ is the principal ideal generated by $b_\mu(\zeta)$.*

Acknowledgement. The author would like to express his thanks to Prof. M. Kashiwara for his comments.

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